Ontario

Number Sense and Numeration, Grades 4 to 6

Volume 1

The Big Ideas

A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6

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Number Sense and Numeration, Grades 4 to 6

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A Guide to Effective Instruction
in Mathematics,
Kindergarten to Grade 6

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INTRODUCTION

Number Sense and Numeration, Grades 4 to 6 is a practical guide, in six volumes, that teachers will find useful in helping students to achieve the curriculum expectations outlined for Grades 4 to 6 in the Number Sense and Numeration strand of *The Ontario Curriculum, Grades 1–8: Mathematics, 2005*. This guide provides teachers with practical applications of the principles and theories that are elaborated in *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6, 2006*.

The guide comprises the following volumes:

- Volume 1: The Big Ideas
- Volume 2: Addition and Subtraction
- Volume 3: Multiplication
- Volume 4: Division
- Volume 5: Fractions
- Volume 6: Decimal Numbers

This first volume of the guide provides a detailed discussion of the five “big ideas”, or major mathematical themes, in Number Sense and Numeration. The guide emphasizes the importance of focusing on the big ideas in mathematical instruction to achieve the goal of *helping students gain a deeper understanding of mathematical concepts*.

Volumes 2 to 6 of the guide focus on the important curriculum topics of addition and subtraction, multiplication, division, fractions, and decimal numbers, respectively. Each of these volumes provides:

- a description of the characteristics of junior learners and the implications those characteristics have for instruction;
- a discussion of mathematical models and instructional strategies that have proved effective in helping students understand the mathematical concepts related to the topic;
- sample learning activities, for Grades 4, 5, and 6, that illustrate how a learning activity can be designed to:
  - focus on an important curriculum topic;
  - involve students in applying the seven mathematical processes described in the mathematics curriculum document;
  - develop understanding of the big ideas in Number Sense and Numeration.

Each of the volumes also contains a list of the references cited throughout the guide. A glossary that includes mathematical and pedagogical terms used throughout the six volumes is included at the end of Volume 1.

The content of all six volumes of the guide is supported by "eLearning modules" that are available at www.eworkshop.on.ca. The instructional activities in the eLearning modules that relate to particular topics covered in this guide are identified at the end of each of the learning activities in Volumes 2 through 6.

Working Towards Equitable Outcomes for Diverse Students

All students, regardless of their socioeconomic, ethnocultural, or linguistic background, must have opportunities to learn and to grow, both cognitively and socially. When students see themselves reflected in what they are learning, and when they feel secure in their learning environment, their true potential will be reflected in their achievement. A commitment to equity and inclusive instruction in Ontario classrooms is therefore critical to enabling all students to succeed in school and, consequently, to become productive and contributing members of society.

To create the right conditions for learning, teachers must take care to avoid all forms of bias and stereotyping in resources and learning activities, which can quickly alienate students and limit their ability to learn. Teachers should be aware of the need to provide a variety of experiences and multiple perspectives, so that the diversity of the class is recognized and all students feel respected and valued. Learning activities and resources for teaching mathematics should be inclusive in nature, providing examples and illustrations and using approaches that reflect the range of experiences of students with diverse backgrounds, abilities, interests, and learning styles.

The following are some strategies for creating a learning environment that recognizes and respects the diversity of students, and allows them to participate fully in the learning experience:

- providing mathematics problems with contexts that are meaningful to all students (e.g., problems that reflect students' interests, home-life experiences, and cultural backgrounds);
- using mathematics examples that reflect diverse ethnocultural groups, including Aboriginal peoples;
- using children's literature that reflects various cultures and customs;
- respecting customs and adjusting teaching strategies, as necessary. For example, a student may come from a culture in which it is considered inappropriate for a child to ask for help, express opinions openly, or make direct eye contact with an adult;
- considering the appropriateness of references to holidays, celebrations, and traditions;
- providing clarification if the context of a learning activity is unfamiliar to students (e.g., describing or showing a food item that may be new to some students);
- evaluating the content of mathematics textbooks, children's literature, and supplementary materials for cultural bias;

- designing learning and assessment activities that allow students with various learning styles (e.g., auditory, visual, tactile/kinaesthetic) to participate meaningfully;
- providing opportunities for students to work both independently and with others;
- providing opportunities for students to communicate orally and in writing in their home language (e.g., pairing English language learners with a first-language peer who also speaks English);
- using diagrams, pictures, manipulatives, and gestures to clarify mathematical vocabulary that may be new to English language learners.

For a full discussion of equity and diversity in the classroom, as well as a detailed checklist for providing inclusive mathematics instruction, see pp. 34 – 40 in Volume 1 of *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6, 2006*.

An important aspect of inclusive instruction is accommodating students with special education needs. The following section discusses accommodations and modifications as they relate to mathematics instruction.

Accommodations and Modifications

The learning activities in Volumes 2 through 6 of this guide have been designed for students with a range of learning needs. Instructional and assessment tasks are open-ended, allowing most students to participate fully in learning experiences. In some cases, individual students may require *accommodations* and/or *modifications*, in accordance with their Individual Education Plan (IEP), to support their participation in learning activities.

The term accommodations *is used to refer to the special teaching and assessment strategies, human supports, and/or individualized equipment required to enable a student to learn and to demonstrate learning. Accommodations do not alter the provincial curriculum expectations for the grade.*

Modifications *are changes made in the age-appropriate grade-level expectations for a subject . . . in order to meet a student's learning needs. These changes may involve developing expectations that reflect knowledge and skills required in the curriculum for a different grade level and/or increasing or decreasing the number and/or complexity of the regular grade-level curriculum expectations.*

(Ontario Ministry of Education, 2004, pp. 25–26)

Providing accommodations

Students may require accommodations, including special strategies, support, and/or equipment to allow them to participate in learning activities. There are three types of accommodations:

- *Instructional accommodations* are adjustments in teaching strategies, including styles of presentation, methods of organization, or the use of technology or multimedia.
- *Environmental accommodations* are supports or changes that the student may require in the physical environment of the classroom and/or the school, such as preferential seating or special lighting.
- *Assessment accommodations* are adjustments in assessment activities and methods that enable the student to demonstrate learning, such as allowing additional time to complete tasks or permitting oral responses to test questions.

Some of the ways in which teachers can provide accommodations with respect to mathematics learning activities are listed in the table on pp. 8–9.

Instructional Accommodations

- Vary instructional strategies, using different manipulatives, examples, and visuals (e.g., concrete materials, pictures, diagrams) as necessary to aid understanding.
- Rephrase information and instructions to make them simpler and clearer.
- Use non-verbal signals and gesture cues to convey information.
- Teach mathematical vocabulary explicitly.
- Have students work with a peer.
- Structure activities by breaking them into smaller steps.
- Model concepts using concrete materials, and encourage students to use them when learning concepts or working on problems.
- Have students use calculators and/or addition and multiplication grids for computations.
- Format worksheets so that they are easy to understand (e.g., use large-size font; an uncluttered layout; spatial cues, such as arrows; colour cues).
- Encourage students to use graphic organizers and graph paper to organize ideas and written work.
- Provide augmentative and alternative communications systems.
- Provide assistive technology, such as text-to-speech software.
- Provide time-management aids (e.g., checklists).
- Encourage students to verbalize as they work on mathematics problems.
- Provide access to computers.
- Reduce the number of tasks to be completed.
- Provide extra time to complete tasks.

Environmental Accommodations

- Provide an alternative workspace.
- Seat students strategically (e.g., near the front of the room; close to the teacher in group settings; with a classmate who can help them).
- Reduce visual distractions.
- Minimize background noise.
- Provide a quiet setting.
- Provide headphones to reduce audio distractions.
- Provide special lighting.
- Provide assistive devices or adaptive equipment.

Assessment Accommodations

- Have students demonstrate understanding using concrete materials or orally rather than in written form.
- Have students record oral responses on audiotape.
- Have students' responses on written tasks recorded by a scribe.
- Provide assistive technology, such as speech-to-text software.
- Provide an alternative setting.
- Provide assistive devices or adaptive equipment.
- Provide augmentative and alternative communications systems.
- Format tests so that they are easy to understand (e.g., use large-size font; an uncluttered layout; spatial cues, such as arrows; colour cues).
- Provide access to computers.
- Provide access to calculators and/or addition and multiplication grids.
- Provide visual cues (e.g., posters).
- Provide extra time to complete problems or tasks or answer questions.
- Reduce the number of tasks used to assess a concept or skill.

Modifying curriculum expectations

Students who have an IEP may require modified expectations, which differ from the regular grade-level curriculum expectations. When developing modified expectations, teachers make important decisions regarding the concepts and skills that students need to learn.

Most of the learning activities in this document can be adapted for students who require modified expectations. The table on p. 10 provides examples of how a teacher could deliver learning activities that incorporate individual students' modified expectations.

It is important to note that some students may require both accommodations and modified expectations.

Modified Program	What It Means	Example
Modified learning expectations, *same* activity, *same* materials	The student with modified expectations works on the same or a similar activity, using the same materials.	The learning activity involves solving a problem that calls for the addition of decimal numbers to thousandths using concrete materials (e.g., base ten materials). Students with modified expectations solve a similar problem that involves the addition of decimal numbers to tenths.
Modified learning expectations, *same* activity, *different* materials	The student with modified expectations engages in the same activity, but uses different materials that enable him/her to remain an equal participant in the activity.	The activity involves ordering fractions on a number line with unlike denominators using Cuisenaire rods. Students with modified expectations may order fractions with like denominators on a number line using fraction circles.
Modified learning expectations, *different* activity, *different* materials	Students with modified expectations participate in different activities.	Students with modified expectations work on a fraction activities that reflect their learning expectations, using a variety of concrete materials.

(Adapted from *Education for All: The Report of the Expert Panel on Literacy and Numeracy Instruction for Students With Special Education Needs, Kindergarten to Grade 6*, p. 119)

THE "BIG IDEAS" IN NUMBER SENSE AND NUMERATION

The big ideas in Number Sense and Numeration in Grades 4 to 6 are as follows:

- quantity
- operational sense
- relationships
- representation
- proportional reasoning

The curriculum expectations outlined in the Number Sense and Numeration strand for each grade in *The Ontario Curriculum, Grades 1–8: Mathematics, 2005*[1] are organized around these big ideas.

The discussion of each big idea in this volume provides:

- an *overview*, which includes a general discussion of the big idea in the junior grades, along with an explanation of some key concepts inherent in the big idea;
- descriptions of the *characteristics of learning* that are evident among students in Grades 4–6 who have a strong understanding of the big idea (or, in the case of proportional reasoning, a developing understanding);
- descriptions of general *instructional strategies* that help students develop a strong understanding of the big idea.

In developing a mathematics program, it is important to concentrate on the big ideas and on the important knowledge and skills that relate to those big ideas. Programs that are organized around big ideas and focus on problem solving provide cohesive learning opportunities that allow students to explore mathematical concepts in depth. An emphasis on big ideas contributes to the main goal of mathematics instruction – to help students gain a deeper understanding of mathematical concepts.

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1. In the mathematics curriculum document, the specific expectations are grouped under subheadings that reflect particular aspects of the knowledge and skills students are expected to learn. For example, the subheadings in the Number Sense and Numeration strand for Grade 5 include "Quantity Relationships", "Operational Sense", and "Proportional Reasoning". The subheadings for the strand reflect the big ideas identified in *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 3 – Number Sense and Numeration, 2003*, and those identified here for Grades 4 to 6.

Teaching and Learning Mathematics: The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004 outlines components of effective mathematics instruction, including a focus on big ideas in student learning:

> *When students construct a big idea, it is big because they make connections that allow them to use mathematics more effectively and powerfully. The big ideas are also critical leaps for students who are developing mathematical concepts and abilities.*
>
> (Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004, p. 19)

Students are better able to see the connections in mathematics, and thus to learn mathematics, when it is organized in big, coherent "chunks". In organizing a mathematics program, teachers should concentrate on the big ideas in mathematics and view the expectations in the curriculum policy documents for Grades 4 to 6 as being clustered around those big ideas.

The clustering of expectations around big ideas provides a focus for student learning and for teacher professional development in mathematics. Teachers will find that investigating and discussing effective teaching strategies for a big idea is much more valuable than trying to determine specific strategies and approaches to help students achieve individual expectations. In fact, using big ideas as a focus helps teachers to see that the concepts presented in the curriculum expectations should not be taught as isolated bits of information but rather as a network of interrelated concepts.

In building a program, teachers need a sound understanding of the key mathematical concepts for their students' grade level, as well as an understanding of how those concepts connect with students' prior and future learning (Ma, 1999). Such knowledge includes an understanding of the "conceptual structure and basic attitudes of mathematics inherent in the elementary curriculum" (p. xxiv), as well as an understanding of how best to teach the concepts to students. Concentrating on developing this knowledge will enhance effective teaching and provide teachers with the tools to differentiate instruction.

Focusing on the big ideas provides teachers with a global view of the concepts represented in the strand. The big ideas also act as a "lens" for:

- making instructional decisions (e.g., choosing an emphasis for a lesson or set of lessons);
- identifying prior learning;
- looking at students' thinking and understanding in relation to the mathematical concepts addressed in the curriculum (e.g., making note of the ways in which a student solves a division problem);
- collecting observations and making anecdotal records;
- providing feedback to students;

- determining next steps;
- communicating concepts and providing feedback on students' achievement to parents[2] (e.g., in report card comments).

> *All learning, especially new learning, should be embedded in well-chosen contexts for learning – that is, contexts that are broad enough to allow students to investigate initial understandings, identify and develop relevant supporting skills, and gain experience with varied and interesting applications of the new knowledge. Such rich contexts for learning open the door for students to see the "big ideas", or key principles, of mathematics, such as pattern or relationship.*
>
> (Ontario Ministry of Education, 2005, p. 25)

2. In this document, *parent(s)* refers to parent(s) and guardian(s).

QUANTITY

> *Quantitative reasoning is more than reasoning about number and more than skilled calculations. It is about making sense of the situation to which we apply numbers and calculations.*
>
> (Thompson, 1995, p. 220)

Overview

Understanding the quantity represented by a number is an important aspect of having number sense. In the primary grades, students first explore quantities of numbers less than 10 (e.g., the meaning of “5”), and later learn to relate quantities to significant numbers, such as 10 and 100 (e.g., 12 is 2 more than 10; 20 is 2 tens; 98 is 2 less than 100). Towards the end of the primary grades, students gain an understanding of quantities associated with two- and three-digit whole numbers.

In the junior grades, students develop a sense of quantity of multidigit whole numbers, decimal numbers, fractions, and percents. The use of concrete and pictorial models (e.g., base ten blocks, fraction circles, 10 × 10 grids) is crucial in helping students develop an understanding of the “howmuchness” of different kinds of numbers, and provides experiences that allow students to interpret symbolic representations of numbers (e.g., understanding the quantity represented by “9/10”).

The following are key points that can be made about quantity in the junior grades:

- Having a sense of quantity involves understanding the “howmuchness” of whole numbers, decimal numbers, fractions, and percents.
- Experiences with numbers in meaningful contexts help to develop a sense of quantity.
- An understanding of quantity helps students estimate and reason with numbers.
- Quantity is important in understanding the effects of operations on numbers.

QUANTITY AS “HOWMUCHNESS”

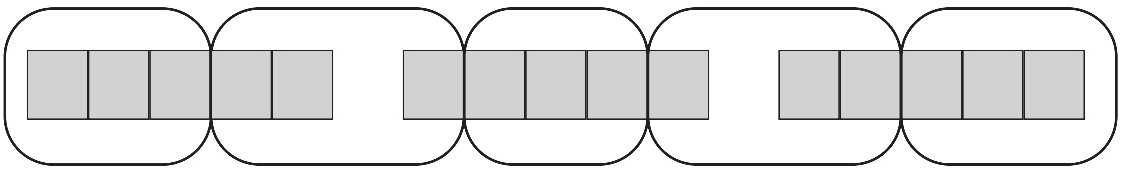
For students in the junior grades, possessing number sense depends on an understanding of quantities represented by whole numbers, decimal numbers, fractions, and percents. It is important that students have many opportunities to represent numbers using concrete and pictorial models, in order to gain a sense of number size. For example, experiences in using base ten blocks to represent multidigit whole numbers help students understand the base ten relationships in our number system, and the meaning of the digits in a number (e.g., in 3746 there are 3 thousands, 7 hundreds, 4 tens, 6 ones). Representing multidigit whole numbers also allows students to observe that place values increase by powers of 10 as they move to the left (ones, tens, hundreds, thousands, ten thousands, and so on).

Students learn that decimal numbers, as an extension of the base ten number system, can be used to represent quantities less than 1 (e.g., 0.2 represents 2 tenths; 0.54 represents 54 hundredths). Representing decimal numbers using materials (e.g., base ten blocks, fraction strips divided into tenths, 10×10 grids) helps students to visualize decimal quantities.

Representing fractions using a variety of models (e.g., fraction strips, fraction circles, grids) develops students' ability to interpret fractional quantities. Providing opportunities to solve problems using concrete materials and drawings helps students to understand the quantity represented by fractions. Consider the following problem.

> *"Five children share 3 submarine sandwiches equally. How much will each child receive?"*

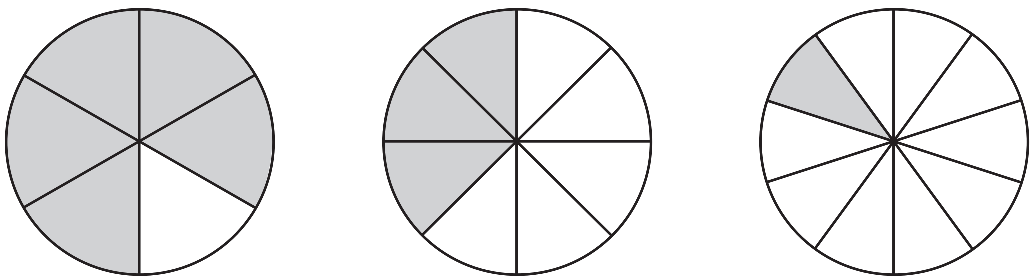
Fraction Strips Showing Five Equal Portions



Each child receives $\frac{3}{5}$ of a submarine sandwich.

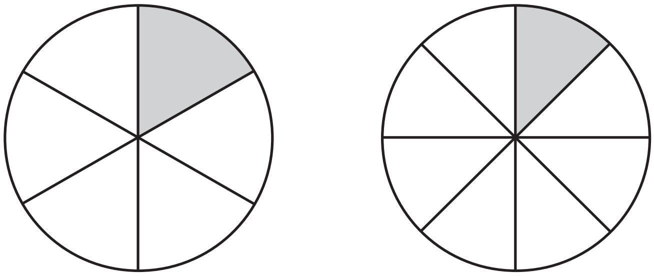
Using concrete materials and drawings also helps students to observe the proximity of a fraction to the numerical benchmarks of 0, 1/2, and 1 whole (e.g., 5/6 is a quantity that is close to 1 whole; 3/8 represents an amount that is close to 1/2; 1/10 is close to 0).

Fraction Circles Showing How Much of Each Whole Is Shaded



$\frac{5}{6}$ is close to 1. $\frac{3}{8}$ is close to $\frac{1}{2}$. $\frac{1}{10}$ is close to 0.

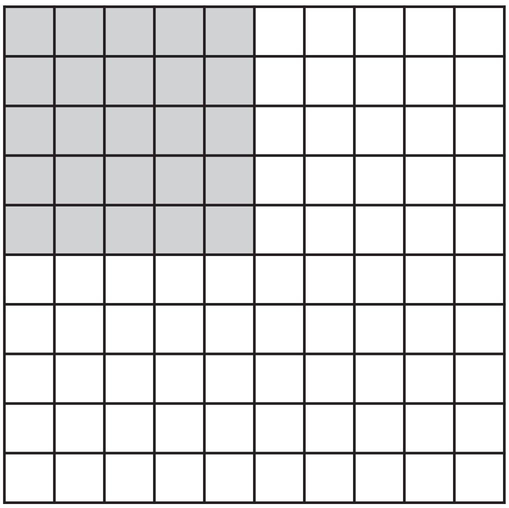
Representing fractions as parts of wholes also helps students to understand concepts about the size of fractional parts – the greater the numeral in the denominator, the smaller the fractional parts. For example, a whole pizza divided into sixths yields bigger slices than a pizza of equal size divided into eighths.



Sixths are larger than eighths, therefore, $\frac{1}{6} > \frac{1}{8}$.

In Grade 6, students investigate the meaning of percent and learn that percents, like decimal numbers and fractions, are used to represent quantities that are parts of a whole. Models, such as 10 × 10 grids, provide students with representations of percents and also demonstrate that quantities can be represented by different, but equivalent, number forms.

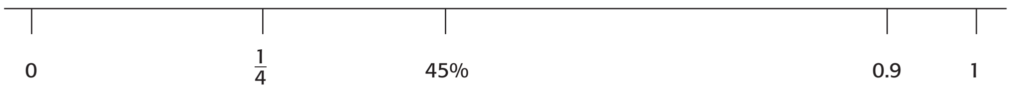
10 × 10 Grid Showing How Much of the Whole Is Unshaded



$75\% = 0.75 = \frac{3}{4}$

In the junior grades, students' understanding of quantity develops not only through experiences with modelling numbers concretely and pictorially but also through opportunities to represent numbers more abstractly. In particular, number lines provide a model for representing number quantities. Although a number line does not show the quantities associated with different numbers, it does provide a tool by which students can represent the relationships between number quantities. For example, given an open number line with only 0 and 1 indicated, students would need to reason about the relative quantities of numbers such as 1/4, 0.9, 45%, and their relationships to the benchmark numbers of 0, 1/2, and 1, in order to estimate their approximate positions on the line.

Open Number Line



EXPERIENCES WITH QUANTITY IN MEANINGFUL CONTEXTS

In the junior grades, there can be a tendency to focus on teaching computational procedures and to give less attention to the development of students' comprehension of quantity. It is important to provide students with opportunities to make sense of number quantities by relating them to meaningful contexts. For example, students might:

> *Numbers are ideas – abstractions that apply to a broad range of real and imagined situations.*
>
> (Kilpatrick, Swafford, & Findell, 2001, p. 72)

- discuss the reasonableness of numbers in various situations (e.g., "Could 1000 people stand in our classroom? Could 1000 people stand in the gym?");
- discuss how the same number can be used to express different notions of quantity (e.g., 58 students, 58 degrees, 58 kilometres, 58 minutes, 58 dollars, 58 hundredths);
- explore the magnitude of large and small numbers by placing them into meaningful contexts (e.g., to get an idea of how significantly greater 1 billion is than 1 million, students might explore these numbers in relation to time – 1 million seconds is a little more than 11 days, while 1 billion seconds is about 32 years!);
- solve numerical problems that are based on meaningful and interesting contexts;
- reflect on the size of an anticipated solution (e.g., "Will 5 × $175 be greater or less than $1000? How do you know?");
- justify their solutions to problems involving numbers (e.g., "If you need 3/8 of a metre of ribbon to make a bow, how many bows can you make with 1 metre of ribbon?").

QUANTITY AS IT RELATES TO ESTIMATING AND REASONING WITH NUMBERS

Having a sense of quantity is crucial in developing estimation skills. In the primary grades, students develop reasoning strategies to make appropriate estimates of quantities. For example, to estimate the number of marbles in a jar, students might think about what a group of 10 marbles looks like, and then imagine the number of groups of 10 in the large container. Students in the junior grades learn to apply similar reasoning to estimate larger quantities (e.g., imagining the size of a group of 20 students in order to estimate the number of students in a school assembly).

Students should have opportunities to observe quantities of 100, 500, and 1000, and to use these quantities to estimate larger amounts. Whole-class projects that involve collecting and keeping count of small objects (e.g., buttons, popcorn kernels, pennies) allow students to develop reference points or benchmarks for imagining large quantities. For example, "If a coffee cup holds 100 marbles, what size container would hold 1000 marbles? 10 000 marbles? 100 000 marbles?"

Students should also develop mental images of fractional amounts and use these images to estimate quantities. Learning activities in which students model fractions, decimal numbers, and percents using concrete materials (e.g., fraction circles, fraction strips, base ten blocks) and diagrams (e.g., number lines, 10 × 10 grids) allow students to visualize common fractional quantities, such as 1/4, 0.75, and 50%. Students can use these images to estimate other fractional amounts (e.g., the jar is about 1/4 full of paint; a little more than 50% of the magazine is filled with advertisements).

One of the goals of mathematics is to have students become fluent and flexible in their mathematical calculations and in the application of rules while at the same time continuing to understand what they are doing.

(Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004, p. 20)

QUANTITY AS IT RELATES TO UNDERSTANDING THE EFFECTS OF OPERATIONS ON NUMBERS

In the primary grades, students explore the effects of addition and subtraction on quantity: In addition, quantities increase as numbers are added together, and in subtraction, quantities decrease. In the junior grades, students should have opportunities to explore the effects of multiplication and division on number quantities, and they should investigate why:

- multiplying numbers greater than 1 results in a larger quantity (e.g., 4 × 1.5 = 6);
- multiplying a number by a decimal number less than 1 results in a smaller quantity (e.g., 4 × 0.5 = 2);
- dividing a number by a whole number greater than 1 results in a smaller quantity (e.g., 3.5 ÷ 7 = 0.5).

Students should also explore the effects of performing operations using particular numbers, such as multiplying or dividing a number by 10, 100, and 1000, and multiplying a number by 0.1, 0.01, and 0.001.

A solid understanding of fractional quantity will help students develop an understanding of operations with fractions in later grades, and will assist them in avoiding errors that are caused by poor number sense (e.g., adding fractions by adding the numerators and denominators, as in 1/2 + 1/3 and getting an incorrect answer of 2/5 instead of the correct answer of 5/6).

Characteristics of Student Learning

In general, students with a strong understanding of quantity:

- are able to represent quantities using concrete materials and diagrams, and explain how these representations show the "howmuchness" of numbers;
- have mental images of what quantities "look like" (e.g., have a mental image of 3/4);

- consider the size of numbers in various contexts, and judge whether the numbers are reasonable;
- understand the base ten relationships in our number system, and are able to compare whole numbers and decimal numbers by considering the place value of the digits in the numbers;
- relate numbers to benchmark numbers (e.g., recognize that 3/8 is close to 1/2);
- use common fractions, such as 1/4, 1/2, and 3/4, to estimate quantities (e.g., the jar is about 3/4 full of water);
- recognize the relationship between the denominator of a fraction and the size of the fractional parts (e.g., a circle divided into eighths has smaller fractional parts than a circle of equal size that is divided into sixths);
- have strategies for estimating large quantities (e.g., think about what 20 of an item would "look like", and use the smaller quantity as a reference point for estimating a larger quantity);
- are able to explain the effects of operations on quantity (e.g., multiplication of a whole number by a decimal less than 1 results in a quantity that is less than the whole number);
- are able to explain the effects of operations with particular numbers on quantity (e.g., the effect of adding or subtracting 100; the effect of multiplying or dividing a number by a power of 10);
- can explain whether numbers encountered in the media and on the Internet make sense (e.g., recognize the error in a newspaper report that states that 1 out of 25 people, or 25%, cannot read or write).

Instructional Strategies

Students benefit from the following instructional strategies:

- using a variety of concrete materials and diagrams to model the "howmuchness" of whole numbers, decimal numbers, fractions, and percents;
- locating numbers on an open number line by considering their proximity to other numbers on the line;
- discussing their mental visualizations of small and large quantities (e.g., "What does a hundredth look like?");
- discussing the reasonableness of numbers in various situations;
- discussing and representing (e.g., using concrete materials) the effects of operations on quantity;
- estimating quantities and discussing estimation strategies (e.g., visualizing what a small quantity looks like, and then using that quantity as a reference point for estimating a large quantity);
- providing opportunities to associate a large number with an observable quantity of an item (e.g., collecting and observing 10 000 pennies);
- solving numerical problems that are based on meaningful and interesting contexts.

OPERATIONAL SENSE

Algorithms – a structured series of procedures that can be used across problems regardless of the numbers – do have an important place in mathematics. . . . Algorithms should not be the primary goal of computational instruction, however. . . . Calculating with number sense means that one should look at the numbers first and then decide on a strategy that is fitting – and efficient.

(Fosnot & Dolk, 2001b, p.102)

Overview

Operational sense allows students to make sense of addition, subtraction, multiplication, and division, and to use these operations meaningfully in problem-solving situations. Students who possess a strong understanding of the operations see the relationships among them and develop flexible strategies for computing with numbers.

In the primary grades, students learn about addition and subtraction by using counting strategies and by combining and partitioning numbers. They also begin to understand that groups of equal size can be combined to form a quantity (a fundamental concept in multiplication). The development of operational sense, especially related to multiplication and division, is a focus of instruction in the junior grades. It is important for teachers to provide meaningful contexts, to help students develop an understanding of the operations, and to connect new concepts about the operations to what they already understand.

The following are key points that can be made about operational sense in the junior grades:

- Operational sense depends on an understanding of addition, subtraction, multiplication, and division, the properties of these operations, and the relationships among them.
- Efficiency in using the operations and in performing computations depends on an understanding of part-whole relationships.
- Students demonstrate operational sense when they can work flexibly with a variety of computational strategies, including those of their own devising.
- Solving problems and using models are key instructional components that allow students to develop conceptual and procedural understanding of the operations.

UNDERSTANDING THE OPERATIONS, PROPERTIES OF THE OPERATIONS, AND RELATIONSHIPS AMONG OPERATIONS

Instruction that focuses on the meaning of the operations, the properties of the operations, and the relationships among operations helps students to solve problems and develop strategies for computing with numbers.

Understanding the operations

In the primary grades, instruction about operations focuses on developing students' understanding of addition and subtraction. Students in the primary grades also explore multiplication and division, and are able to solve problems involving these operations by using strategies that make sense to them (e.g., counting out equal groups, using repeated addition, using repeated subtraction). Throughout the junior grades, students' multiplicative thinking – reasoning that involves ideas such as "three times as many" – continues to develop. Central to the ability to understand multiplication, and eventually division, is the concept of *unitizing* – the ability to recognize that a group of items can be considered as a single entity (e.g., in 5×8 or "5 groups of 8", each group of 8 represents a single entity).

In the past, the focus on developing conceptual understanding of the operations has not been prevalent in many classrooms. Research (Ma, 1999) indicates that many students are taught computational algorithms as a series of steps, with little focus on the understanding of underlying concepts and connections between operations. For example, many students are taught the standard division algorithm without first connecting the operation of division to multiplication and repeated subtraction. Students who rely solely on memorized procedures are often unable to use mathematical reasoning to solve problems. The time spent on promoting conceptual understanding does not hinder and does contribute to the development of computational efficiency (Fuson, 2003).

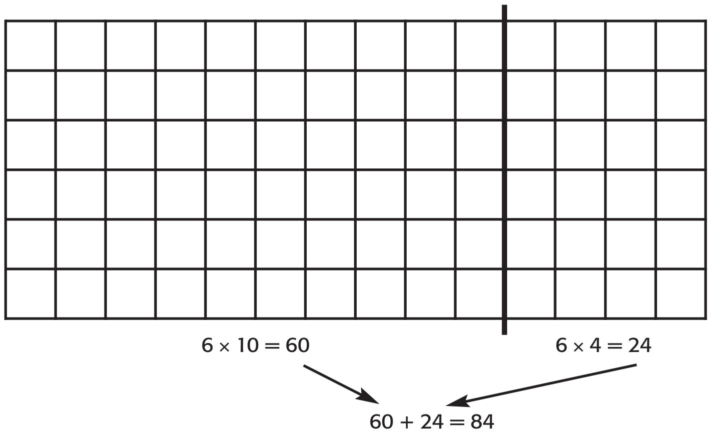
To develop a deep understanding of the operations, students should have experiences in solving a variety of problems that highlight various meanings of the operations. For example, students should solve division problems that involve both partitive situations and quotative situations. In *partitive* division (also called "distribution division" or "sharing division"), the whole amount and the number of groups are known, but the number of items in each group is unknown (e.g., "Matthew wants to share 42 marbles equally among his 7 friends. How many marbles will each friend get?"). In *quotative* division (also called "measurement division"), the whole amount and the number of items in each group are known, but the number of groups is unknown (e.g., "Sara is putting 42 marbles into bags. She put 6 marbles into each bag. How many bags does she need?"). Investigations with a variety of problem types provide students with meaningful contexts in which they can develop and consolidate their understanding of the operations.

> *When instruction begins with what children know, and the teacher works with their ideas and methods before the introduction of formal rules, then students understand the concepts more deeply. Beginning with student methods and working with these ideas to move toward more effective methods keeps students thinking about the mathematical concepts, trying to make sense of their work and eventually deepens their understanding. The mathematics does not end with the students' own methods but it should begin with them.*
>
> (Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004, p. 13)

Properties of the operations

Understanding the properties of the operations allows students to develop flexible and effective mental computation strategies. The properties should be introduced and emphasized in problem-solving situations, and should not be taught in isolation as rules to be memorized. The use of models (e.g., concrete materials, diagrams) helps students to visualize the properties and to recognize their utility as problem-solving strategies. For example, to introduce the distributive property and to demonstrate its use in multiplication, teachers might have students use grid paper to represent tiles arranged in a 6 × 14 array. To determine the number of tiles, students could partition the 6 × 14 array into a 6 × 10 array and a 6 × 4 array, calculate the number of tiles in each smaller array, and then add the partial products.

Array Modelling 6 × 14



After experiences with representing the distributive property using a grid, students can apply the property in performing mental computations. For example, 5 × 34 can be calculated by decomposing 34 into 30 + 4, then multiplying 5 × 30 and 5 × 4, and then adding 150 + 20.

Some of the properties that students should investigate and apply are the following:

- ***Identity property:*** In addition and subtraction, the identity element is 0, which means that adding 0 to or subtracting 0 from any number does not change the number's value (e.g., 36 + 0 = 36; 17 – 0 = 17). In multiplication and division, the identity element is 1 (e.g., 88 × 1 = 88; 56 ÷ 1 = 56).
- ***Zero property of multiplication:*** The product of any number and 0 is 0 (e.g., 98 × 0 = 0).
- ***Commutative property:*** In addition and multiplication, numbers can be added or multiplied in any order, without affecting the sum or product of the operation (e.g., 2 + 99 = 99 + 2; 25 × 4 = 4 × 25).
- ***Associative property:*** In addition and multiplication, the numbers being added or multiplied can be regrouped in any way without changing the result of the operations (e.g., (27 + 96) + 4 = 27 + (96 + 4); (17 × 25) × 4 = 17 × (25 × 4)).

- ***Distributive property:*** A number in a multiplication expression can be decomposed into two or more numbers. The distributive property can involve:
  - multiplication over addition (e.g., $6 \times 47 = (6 \times 40) + (6 \times 7)$, which gives $240 + 42 = 282$);
  - multiplication over subtraction (e.g., $4 \times 98 = (4 \times 100) - (4 \times 2)$, which gives $400 - 8 = 392$);
  - division over addition (e.g., $72 \div 6 = (60 \div 6) + (12 \div 6)$, which gives $10 + 2 = 12$);
  - division over subtraction (e.g., $4700 \div 4 = (4800 \div 4) - (100 \div 4)$, which gives $1200 - 25 = 1175$).

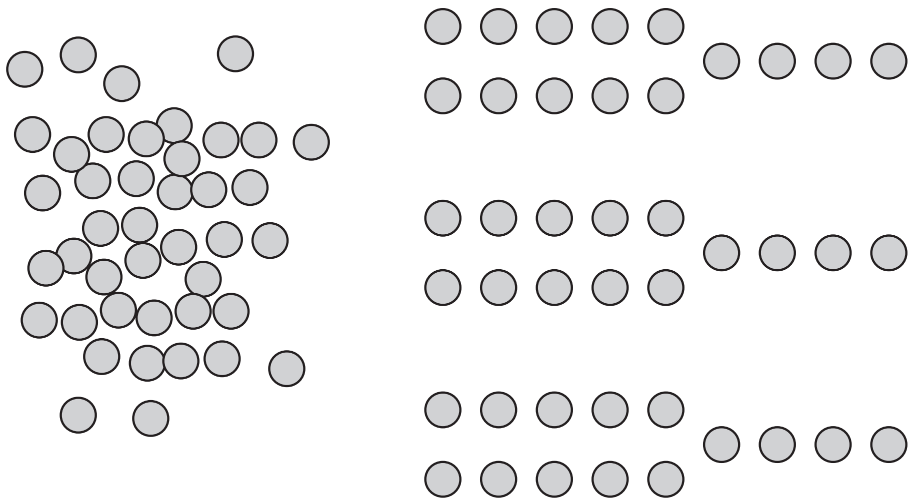
Note: It is not necessary for students to know the names of these properties (nor should they be required to memorize definitions of the properties), but it *is* important that they understand and recognize the usefulness of the properties in performing computations.

Relationships among the operations

Recognizing the relationships among operations allows students to develop a deeper understanding of the operations and helps them to develop flexible computation strategies. As the following example illustrates, students, in their early explorations of division problems, apply strategies and operations that they already understand.

Students are given a quotative division situation in which they are asked to put 84 marbles into bags to sell at a yard sale. They are to put 14 marbles in each bag and they need to determine how many bags they will need.

Using counting: Students might count out 84 counters, arrange them into groups of 14, and then count the number of groups. A counting strategy is commonly used by students who are beginning to explore division.



Using repeated addition: Students might repeatedly add 14 until they reach 84, and then go back to count the number of 14s they added.

$$\begin{array}{r} 14 \\ +\,14 \\ \hline 28 \end{array} \quad \begin{array}{r} 28 \\ +\,14 \\ \hline 42 \end{array} \quad \begin{array}{r} 42 \\ +\,14 \\ \hline 56 \end{array} \quad \begin{array}{r} 56 \\ +\,14 \\ \hline 70 \end{array} \quad \begin{array}{r} 70 \\ +\,14 \\ \hline 84 \end{array}$$

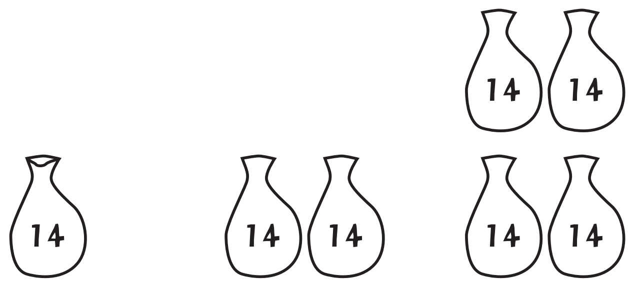
I added 6 14s.

Using repeated subtraction: Students might subtract 14 from 84 and then continue to subtract 14 until they reach 0.

$$\begin{array}{r} 84 \\ -\,14 \\ \hline 70 \end{array} \quad \begin{array}{r} 70 \\ -\,14 \\ \hline 56 \end{array} \quad \begin{array}{r} 56 \\ -\,14 \\ \hline 42 \end{array} \quad \begin{array}{r} 42 \\ -\,14 \\ \hline 28 \end{array} \quad \begin{array}{r} 28 \\ -\,14 \\ \hline 14 \end{array} \quad \begin{array}{r} 14 \\ -\,14 \\ \hline 0 \end{array}$$

I subtracted 14 6 times

Using doubling: Students might reason that 2 bags would contain 28 marbles, and that 4 bags would contain 56 marbles. They conclude that 6 bags are needed, since 28 + 56 = 84.



1 bag
14 marbles

2 bags
28 marbles

4 bags
56 marbles

28 + 56 = 84
2 bags + 4 bags = 6 bags

Using multiplicative thinking: Students might reason that 10 groups of 14 is 140 (which is greater than 84) but recognize that half of 140 is 70. Since 5 groups of 14 is 70 (which is close to 84), they might add 1 more group of 14 to 70 to get 84 and conclude that 6 bags are needed.

All the strategies described in the preceding example lead to a correct solution; however, some strategies are more efficient than others (i.e., the efficiency and sophistication of the methods increase as students move from simple counting, to additive strategies, and, eventually, to multiplicative strategies). It is important for teachers to remember that students need time, a variety of experiences with the operations, and modelling by teachers and peers before they can construct these relationships for themselves.

PART-WHOLE RELATIONSHIPS IN USING THE OPERATIONS AND PERFORMING COMPUTATIONS

In the primary grades, students explore part-whole relationships by composing (putting together) and decomposing (taking apart) numbers (e.g., 3 and 4 make 7; 25 is 20 and 5). These explorations allow students to develop an understanding of number quantity (e.g., the meaning of "7"), place value (e.g., 25 is 2 tens and 5 ones, or 1 ten and 15 ones), addition (combining numerical parts to create a whole), and subtraction (partitioning a whole into parts).

> *. . . just as understanding the connection between addition and subtraction is necessary to understanding the part/whole relationships in the structure of number, understanding the connection between multiplication and division is critical to understanding the part/whole relationships in the multiplicative structure.*
>
> (Fosnot & Dolk, 2001b, p. 51)

An understanding of part-whole relationships continues to be an important concept when students learn about multiplication and division. In multiplication, groups of equal size are combined to create a whole (e.g., 5 groups of 8 makes 40); and in division, the whole is partitioned into equal groups (e.g., 40 divided into groups of 8 results in 5 groups).

The knowledge of how numbers can be composed and decomposed is also fundamental in performing computations. For example, given a problem involving 47 + 26, students could take apart and combine numbers in different ways to generate a variety of mental computation strategies, including the following:

- increasing 47 to 50 by taking 3 from 26, to get 50 + 23 = 73; or
- adding the tens first to get 40 + 20 = 60; next, adding the ones to get 7 + 6 = 13; finally, adding the subtotals to get 60 + 13 = 73; or
- adding the first addend to the tens of the second addend to get 47 + 20 = 67; then adding the ones from the second addend to get 67 + 6 = 73.

WORKING FLEXIBLY WITH COMPUTATIONAL STRATEGIES

Students should develop confidence in their ability to solve problems involving operations and develop efficient strategies for solving such problems. Teachers need to encourage students to use strategies that make sense to them. However, when students' methods prove to be cumbersome, teachers should guide students in developing increasingly efficient strategies by providing carefully selected problems that lend themselves to learning new strategies. Sharing a variety of student solutions for problems, and modelling various strategies by teachers and students, as well as discussing the efficiency, relevancy, and accuracy of various strategies, will allow students to observe and learn new strategies.

Students should have opportunities to explore and develop a variety of strategies and algorithms before they are introduced to standard algorithms. When students solve problems using their own strategies, they develop contextual, conceptual, and procedural understandings of the operations; and this helps to bring meaning to the steps involved when performing standard algorithms. For example, the illustration below shows different strategies and algorithms for computing 224 ÷ 17.

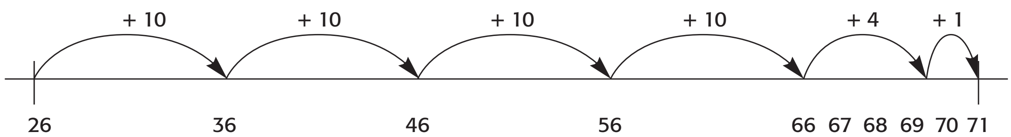
224	17)224		3	13 R3
− 17	170	10	10	17)224
207	54		17)224	17
− 17	34	2	170	54
190	20		54	51
− 17	17	1	51	3
173	3	13	3	
− 17				
156				
− 17				
…				

> *Problems should be the starting place for developing arithmetic understanding, thereby establishing the need and the context for computation skills.*
>
> (Burns, 2000, p. 13)

DEVELOPING UNDERSTANDING OF THE OPERATIONS THROUGH SOLVING PROBLEMS AND USING MODELS

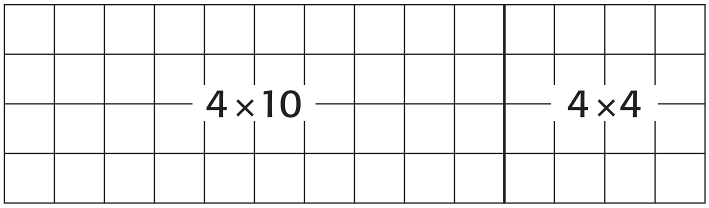
It is important for teachers to engage students in problem-solving experiences that allow them to explore, develop, practice, and consolidate operational concepts and procedures. Students should be encouraged to use both concrete materials and pictorial representations to model problem situations and strategies. Models serve as tools for students to analyse problem situations, to investigate possible strategies, and to conceptualize the role of operations in solving the problems. Usually, the selection of models should be made by students, although, at times, teachers may demonstrate the use of specific concrete materials or written representations (e.g., diagrams, graphic organizers). In the junior grades, number lines, arrays, and open arrays are important models that allow students to represent operations.

Number Line Modelling 26 + 45



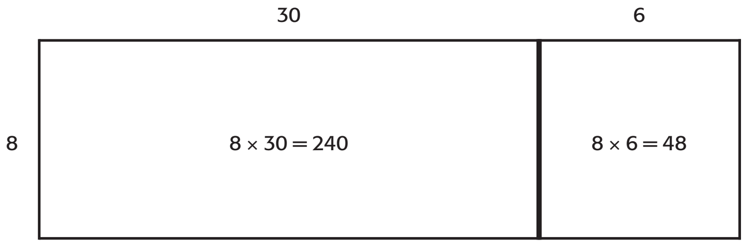
26 + 45 = 66 + 4 + 1

Array Modelling 4 × 14



4 × 14 = 40 + 16 = 56

Open Array Modelling 8 × 36



8 × 36 = 240 + 48 = 288

Teachers should be clear about their purposes for providing problems. For example, problems can be designed or selected with the following purposes in mind:

- conceptual understanding (e.g., to develop students' understanding of division);
- procedural understanding (e.g., to develop multiplication strategies);
- understanding of relationships (e.g., to help students explore how division is related to multiplication);
- understanding of properties (e.g., to demonstrate how the distributive property can be used to multiply).

Problems can address more than one purpose. For example, students might be asked to determine the number of windows on the side of a building if there are 24 rows of windows with 12 windows in each row. Students might create a 12 × 24 array using square tiles to represent the windows, and then determine the total number of windows. The problem-solving experience accomplishes the following purposes:

- conceptual understanding: students connect multiplication to an array;
- understanding of properties: students apply the distributive property by dividing the 12 × 24 array into a 10 × 24 array and a 2 × 24 array, then calculating the number of tiles in the small arrays (240 and 48), and then adding the sub-products (240 + 48) to determine the number of windows.

Characteristics of Student Learning

In general, students with strong operational sense:

- are able to explain the meanings of addition, subtraction, multiplication, and division in various contexts;
- understand the relationships among the operations (e.g., the inverse relationship between multiplication and division);
- apply appropriate operations and strategies in a variety of problem-solving situations;
- develop and apply a range of computational strategies and algorithms, including those of their own devising;
- use strategies and algorithms that make sense to them;
- are able to understand and apply properties of the operations when performing computations (e.g., use the distributive property to compute 8 × 43 by multiplying 8 × 40 and 8 × 3, and then adding the partial products to get 320 + 24 = 344);
- use a variety of strategies flexibly to perform mental computations;
- are able to demonstrate and explain the operations using models (e.g., base ten blocks, number lines, open arrays);
- determine an appropriate method of computation (e.g., estimation, mental computation, paper-and-pencil calculation, calculator) by considering the context and the numbers involved in the computation;
- use addition, subtraction, multiplication, and division flexibly (e.g., recognize that many division situations can also be solved using multiplication).

Instructional Strategies

Students benefit from the following instructional strategies:

- exploring concepts and procedures related to the operations through problem-solving situations with meaningful contexts;
- exploring various strategies for solving problems, and reflecting on the efficiency and accuracy of different strategies;
- posing and solving their own problems;
- solving a variety of problem types (e.g., solving both quotative and partitive division problems);
- using concrete materials and diagrams (e.g., number lines, open arrays) to model situations, concepts, and procedures related to the operations;
- estimating solutions to problems, and discussing whether estimates are reasonable;
- discussing various estimation strategies, and reflecting on the efficiency and accuracy of different strategies;

- exploring a variety of algorithms, and discussing the meaning of the procedures involved in the algorithms;
- applying and discussing the properties of the operations in problem-solving situations;
- exploring relationships among the operations (e.g., the inverse relationships between addition and subtraction; the inverse relationships between multiplication and division);
- providing opportunities to learn basic multiplication and division facts;
- providing opportunities to develop and practice mental computation strategies.

RELATIONSHIPS

> *At the heart of mathematics is the process of setting up relationships and trying to prove these relationships mathematically in order to communicate them to others. Creativity is at the core of what mathematicians do.*
>
> (Fosnot & Dolk, 2001a, p. 6)

Overview

To understand mathematics is to recognize the relationships it involves, which give it both its beauty and its utility. Seeing relationships between numbers helps students make powerful connections in mathematics, allowing them to discover new mathematics in flexible, efficient, and innovative ways.

In the junior grades, students' understanding of base ten relationships and place value expands as they begin to work with larger whole numbers and with decimal numbers. Recognizing number relationships also allows students to understand the operations of addition, subtraction, multiplication, and division, and helps them to develop strategies for computing with numbers in flexible ways.

Mathematical development in the junior grades is also marked by students' growing knowledge of fractions, percents, and ratios. Understanding these number forms depends on recognizing the relationships that are inherent in numerical expressions.

The following are key points that can be made about relationships in the junior grades:

- An understanding of whole numbers and decimal numbers depends on a recognition of relationships in our base ten number system.
- Numbers can be compared and ordered by relating them to one another and to benchmark numbers.
- An understanding of the relationships among the operations of addition, subtraction, multiplication, and division helps students to develop flexible computational strategies.
- Fractions, decimal numbers, and percents are all representations of fractional relationships.

RELATIONSHIPS IN OUR NUMBER SYSTEM

Having an understanding of relationships in our number system is crucial for making sense of multidigit whole numbers and decimal numbers, and for developing meaningful computational strategies. Students with a strong understanding of our number system comprehend the following concepts:

- Our number system is based on ten-relationships (e.g., 10 ones make 1 ten, 10 tens make 1 hundred, 10 hundreds make 1 thousand).
- Decimal numbers are an extension of the base ten system (e.g., 10 tenths make 1, 10 hundredths make 1 tenth, 10 thousandths make 1 hundredth).
- The position of a digit determines its value (e.g., in 5347, the 3 signifies 3 hundreds).
- Zero in a number indicates the absence of a place-value quantity (e.g., in 6.05, there are no tenths).
- A number can be decomposed according to the place value of its digits (e.g., 485 is 4 hundreds, 8 tens, 5 ones; 27.82 = 20 + 7 + 0.8 + 0.02).
- Number quantities can be regrouped. For example, 417 can be thought of as:
  - 4 hundreds, 1 ten, 7 ones;
  - 3 hundreds, 11 tens, 7 ones;
  - 3 hundreds, 10 tens, 17 ones; and so on.

It is important to note that although materials such as base ten blocks, strips of paper divided into tenths, and 10 × 10 grids can help students develop concepts about place value, the concepts are not inherent in the materials. Students must have opportunities to construct their own understanding of place value through interaction with the materials, using the materials as tools for representing ideas and for solving problems.

RELATIONSHIPS IN COMPARING AND ORDERING

Numbers, as representations of quantity, can be compared and ordered. The development of strategies for comparing and ordering numbers helps students to understand the relative size of numbers, and provides them with skills for solving real-life problems. It is important that students have opportunities to compare and order numbers using a variety of manipulatives (e.g., base ten blocks, fraction models). Concrete experiences help students to comprehend the processes involved in comparing and ordering, and allow them to visualize the relative size of numbers. These experiences also help students to develop comparing and ordering strategies that involve looking at written numbers and recognizing the quantities represented by their digits. For example, when comparing whole numbers and decimal numbers, students can consider the place value of the digits within the numbers (e.g., to compare 0.457 and 0.8, students might reason that in 0.457, there are 4 tenths – a value that is less than 8 tenths – and conclude that 0.457 is less than 0.8).

Relating numbers to benchmarks is a useful strategy for comparing and ordering fractions. For example, to order 3/8, 5/6, and 7/8 from least to greatest, students might consider the relative proximity of the fractions to the benchmarks of 0, 1/2, and 1:

- 3/8 is less than 1/2 (4/8);
- 5/6 is 1/6 less than 6/6 and therefore is close to 1;
- 7/8 is 1/8 less than 8/8, and is also close to 1.

Students might realize that 3/8 is the only fraction in the set that is less than 1/2 and therefore it is the least. To compare 5/6 and 7/8, students might reason that sixths are larger fractional parts than eighths, and so 5/6 is less than 7/8.

As discussed earlier on pp. 16–17, the open number line provides a model for representing the relative quantities of numbers. As such, it is also an effective tool for comparing and ordering numbers. As the following diagrams indicate, comparing and ordering numbers on an open number line helps students to think about the relative size of the numbers being compared and ordered, and allows students to reflect on the proximity of the numbers to other significant numbers (e.g., whole numbers that are close to decimal numbers or fractions).

Decimal Number Line

Fraction Number Line

Related to concepts about number comparisons is the notion of equivalency – that different numerical representations can signify the same quantity (e.g., 1/2 = 2/4 = 5/10; 1.2 = 1.20). An understanding of equivalency also involves knowing how different number types can represent the same quantity (e.g., 4/5 = 0.8 = 80%).

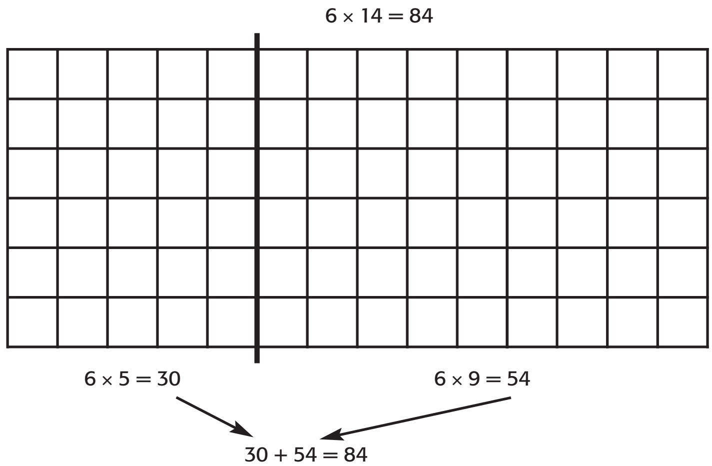
RELATIONSHIPS AMONG THE OPERATIONS AND IN PERFORMING COMPUTATIONS

When students understand how the operations are related, they find flexible ways of computing with numbers. Addition and subtraction are related as inverse operations, as are multiplication and division. Also, multiplication and addition are related (multiplication as repeated addition); and division and subtraction are connected in a similar way (division as repeated subtraction).

Knowing how numbers can be composed (put together) and decomposed (taken apart) is central to understanding the operations (addition, subtraction, multiplication, division) and to developing flexible computational strategies. In the primary grades, students learn that numbers can be decomposed in different ways (e.g., 8 can be broken down into 1 and 7, or 2 and 6, or 3 and 5, and so on). Understanding that a whole can be broken into parts allows students to develop strategies for addition and subtraction. For example, if students are unsure of the sum of 6 + 8, they might add 6 + 4 to get 10 and then add the remaining 4 from 8 to get 10 + 4 = 14. In the junior grades, students continue to apply composing and decomposing strategies to add and subtract larger numbers. To subtract 54 – 27, for example, students could decompose 27 into 20 and 7, subtract 20 from 54 (to get 54 – 20 = 34), and then subtract 7 from 34 (to get 34 – 7 = 27).

Composition and decomposition of number also play a significant role in students' understanding of multiplication and division. When students are given opportunities to represent multiplication (composing quantities by combining groups of equal size) and represent division (decomposing quantities by partitioning into equal groups), they observe number relationships in basic multiplication and division facts (e.g., 48 = 6 × 8 and 48 ÷ 8 = 6).

Number representations that use arrays also allow students to recognize how numbers are related in multiplication. The following diagram illustrates how 6 × 14 can be decomposed into 6 × 5 and 6 × 9. When students learn to decompose multiplication expressions (i.e., they understand the distributive property), they gain a powerful tool for computing with numbers mentally.

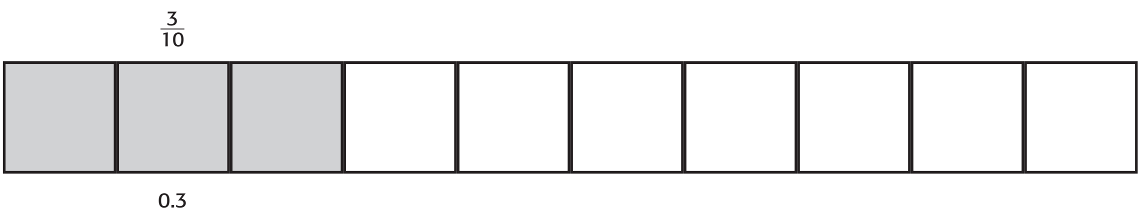


Development of concepts related to the operations influences the kinds of reasoning that students can bring to mathematical situations. Beyond seeing additive relationships (relationships involving addition and subtraction, such as 2 more or 3 less), students begin to recognize multiplicative relationships (relationships involving multiplication and division, such as 10 times more, 3 times smaller, half the size). Understanding multiplicative relationships allows students to develop skills in proportional reasoning.

RELATIONSHIPS AMONG FRACTIONS, DECIMAL NUMBERS, AND PERCENTS

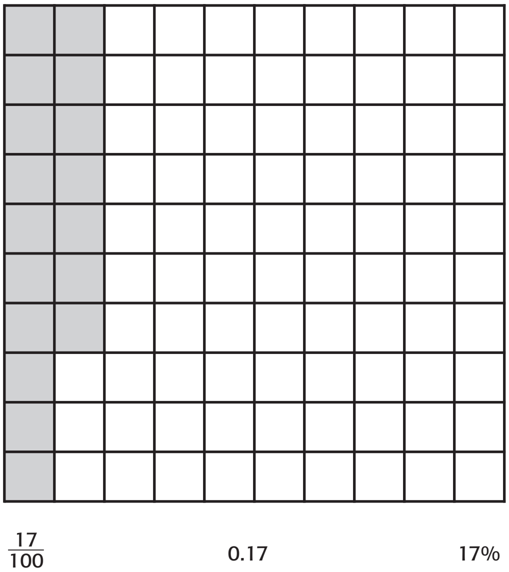
In the junior grades, students learn that fractions, decimal numbers, and percents are related to one another. They recognize decimal numbers as base ten fractional representations that make use of place value (i.e., decimal numbers represent fractions with denominators of 10, 100, 1000, and so on).

Fraction Strip (in Tenths) Showing How Much Is Shaded



Students also learn that percents are fractions that are based on a one-hundred-part whole. Models, such as 10 × 10 grids, allow students to recognize the relationship among fractions, decimal numbers, and percents.

10 × 10 Grid Showing How Much Is Shaded



As students establish these relationships, they are able to identify common decimal and percent equivalents for various fractions (e.g., 1/4 is equivalent to 25/100 and can be written as 0.25 or 25%), and select the representation that is most useful in a given situation. Consider the following problem.

"In a bag of 24 marbles, 25% of the marbles were red. How many red marbles were in the bag?"

To solve this problem, a student who is familiar with the relationship between percents and fractions might reason that 25% is the same as 1/4, and then determine that 1/4 of 24 is 6.

In the junior grades, the focus should be on the use of models to show the relationship among fractions, decimal numbers, and percents, rather than on learning rules for converting among forms.

Characteristics of Student Learning

In general, students with a strong understanding of relationships:

- understand base ten relationships in our number system (e.g., a 3 in the hundreds place is 10 times the value of a 3 in the tens place, and 100 times the value of a 3 in the ones place);
- use their understanding of base ten relationships in our number system to think about number size, compare numbers, and perform computations;
- apply effective strategies for comparing and ordering numbers (e.g., consider place value in whole numbers and decimal numbers);

- relate fractions to the benchmarks of 0, 1/2, and 1;
- identify and describe the relationships that exist among fractions, decimal numbers, and percents;
- use their understanding of number relationships when performing computations (e.g., to compute 37 + 49, students might add 37 + 50 first to get a sum of 87, and then take 1 from 87 to account for the difference between 49 and 50);
- compose and decompose numbers flexibly (e.g., to multiply 4 × 28, students might decompose 28 into 20 + 8, multiply 4 × 20 and 4 × 8, and then add the partial products to get 80 + 32 = 112);
- develop a sense of the relationships among operations (e.g., use the inverse relationship between multiplication and division to solve problems involving either operation);
- understand additive relationships (e.g., 10 more than, 25 less than), and begin to recognize multiplicative relationships (e.g., 10 times more, 3 times smaller, half the size).

Instructional Strategies

Students benefit from the following instructional strategies:

- representing whole numbers and decimal numbers using a variety of materials (e.g., base ten blocks, place-value charts, fraction strips divided into tenths, 10 × 10 grids), and discussing base ten relationships in numbers;
- using number lines, arrays, and place-value charts to develop an understanding of relationships between numbers;
- decomposing numbers in various ways (e.g., 1367 is 1 thousand, 3 hundreds, 6 tens, 7 ones, or 13 hundreds, 6 tens, 7 ones; 475 = 400 + 75, or 500 – 25);
- comparing and ordering whole numbers, decimal numbers, and fractions using a variety of methods (e.g., using concrete materials, drawing number lines);
- representing fractions using concrete materials and drawings, and discussing the proximity of fractions to the benchmarks of 0, 1/2, and 1;
- solving problems using a variety of strategies, and discussing the relationships between strategies (e.g., discussing how a problem might be solved using skip counting, repeated addition, and multiplication);
- providing opportunities to perform mental computations, and discussing various strategies used.

REPRESENTATION

> *We teach mathematics most effectively when we understand the effects on students' learning of external representations. To do this we need to be able to discuss how students are representing concepts internally – their assignments of meaning, the structural relationships that they develop, and how they connect different representations with one another.*
>
> (NCTM, 2001, p. 19)

Overview

Representation is an essential element in supporting students' understanding of mathematical concepts and relationships. It is needed when communicating mathematical understandings and recognizing connections among related mathematical concepts. Representations are used in applying mathematics to problem situations through modelling (National Council of Teachers of Mathematics [NCTM], 2000).

In the primary grades, students represent numbers up to 1000, use notation involving the four operations (addition, subtraction, multiplication, division), and are introduced to the symbols for fractions. In the junior grades, this knowledge is extended to include larger numbers up to 1 000 000 and new symbols (e.g., decimal point, percent sign, ratio). Students understand that making written representations of mathematical ideas is an essential part of learning and doing mathematics. They are encouraged to represent ideas in ways that make sense to them, even if these representations are not conventional ones. However, students also learn conventional forms of representation to help with both their own learning of mathematics and their ability to communicate with others about mathematical ideas.

The following are key points that can be made about representation in the junior grades:

- Symbols and placement are used to indicate quantity and relationships.
- Mathematical symbols and language, used in different ways, communicate mathematical ideas in various contexts and for various purposes.

USING SYMBOLS AND PLACEMENT TO INDICATE QUANTITY AND RELATIONSHIPS

Numerical symbols, and their position within a number or notation, are used to indicate quantity and relationships.

There is no doubt that one of the most significant mathematical concepts that students at all levels need to understand is that of place value and the role that place value plays in the way we represent numbers. It is this convention that allows us to infinitely extend our number system to include very large numbers and very small numbers. The positions of digits in whole numbers and decimal numbers determine what the digits represent (i.e., the size of group they count). For example, the 4 in 40 represents 4 tens; the 4 in 4 000 000 represents 4 millions; the 4 in 0.004 represents 4 thousandths, and so on. In particular, students in the junior grades need to recognize the decimal point as a convention that “announces” the separation between the whole-number part of a quantity and the part that is less than one whole.

The symbolic representation of a fraction is a convention that extends our number system to infinitely smaller parts. The bottom number (denominator) denotes the size of the part and describes the number of parts in the whole. The top number (numerator) counts the number of parts and tells how many are selected. The representation for fractions, however, can be very confusing for students. Fractions use whole numbers in the notation, and nothing in that notation or the words used to describe it conveys their meaning as “parts”. So, for example, the fraction 3/8 is read as “three eighths”, which doesn’t really convey the intended meaning that there are 8 equal parts and of these 8 parts, 3 are selected. To help students create meaning from fractional notations, teachers should provide them with many experiences that involve partitioning quantities into equal parts using concrete models, pictures, and meaningful contexts. Introducing the standard notation for fractions needs to be done in such a way as to ensure that students can connect the meanings already developed for these numbers with the symbols that represent them.

$\frac{3}{8}$ of the counters are black.

Although understanding that a fraction describes a relationship to a whole or a part of a group has already been discussed, it is also important to understand that a fraction is a quantity. A fraction can be thought of as a single entity, with its own unique place on the number line – just as 3 is a single entity. As such, fractions, like whole numbers, can be compared and ordered (and operated on).

Number Line Showing Whole Numbers, Proper Fractions, and Mixed Numbers

Although obviously related to fractions, the concept of ratio has its own representations. The conventional symbol for a ratio (:) describes two quantities in relation to each other. So, although the ratio of “three to four” (3 : 4) can also be represented symbolically using a fraction (3/4), a decimal number (0.75), a percent (75%), or words (3 out of 4), not all these representations are helpful or effective for *interpreting* what the ratio means. A ratio can express a comparison of a part to a whole, such as the ratio of boys to all the students in the class (e.g., there are 25 students and 15 boys, so the ratio of boys to students is 15 : 25). It can also express a comparison of one part of a whole to another part of the same whole, such as the ratio of boys in the class to girls in the class (e.g., there are 15 boys and 10 girls, so the ratio is 15 : 10).

Recognizing equivalent ratios (e.g., 3 out of 4 marbles being blue represents the same ratio as 9 out of 12 marbles being blue) is fundamental for developing proportional reasoning.

USING SYMBOLS AND LANGUAGE IN DIFFERENT WAYS TO COMMUNICATE MATHEMATICAL IDEAS

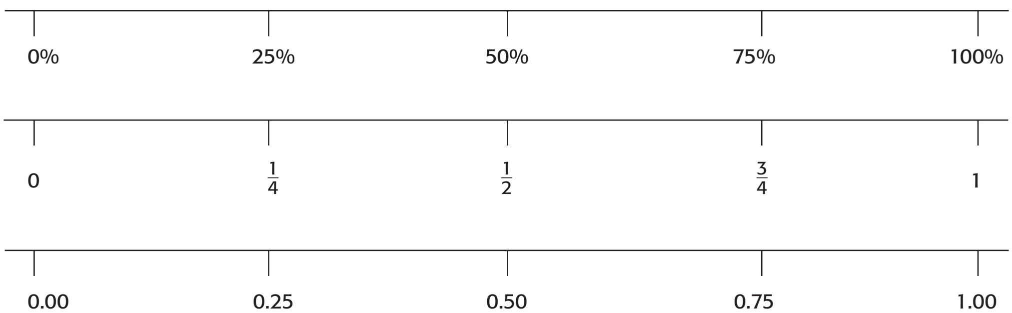
In the junior grades, students become more aware that a symbol can have different meanings, depending on the representation used (e.g., the meaning of "1" is different in 1, 100, 0.1, 1%, and so on).

What is not as commonly addressed, but is of significance in developing a deeper conceptual understanding of number, is that different representations can be used to more clearly communicate and emphasize particular relationships. Consider the following different representations for the number 144:

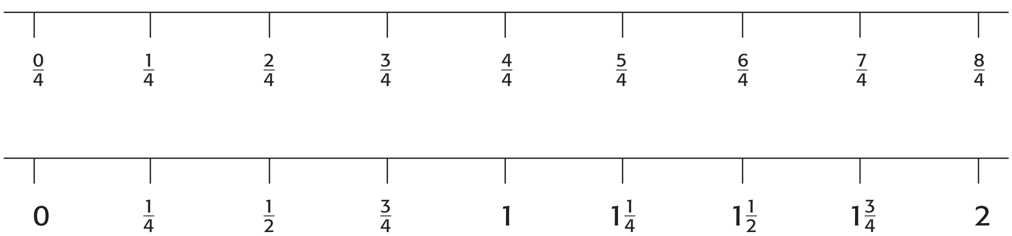
12^2 144 $2 \times 2 \times 2 \times 2 \times 3 \times 3$ 9×16

Teachers might choose one of these representations over the others to emphasize a certain characteristic of the number. For example, if teachers want to show that 144 is a perfect square, the first representation helps to make that clear. If teachers want to show that 144 is divisible by 9, the last representation might be selected. Students in the junior grades become aware not only that numbers may have different representations but that these representations might be used to illustrate or explain different characteristics/properties of the number.

Triple Number Lines Showing Different Number Forms for the Same Quantity



Double Number Lines Showing Proper Fractions, Improper Fractions, Whole Numbers, and Mixed Numbers



Students at this level realize that not only mathematical symbols but also mathematical words can assume new meanings and representations. For example, in the primary grades students learn that "sixth" means a "position" (I am the *sixth* person in the line) and it is an ordinal number; but, in the junior grades, that same word when used in the context of fractions might mean "1 of 6 equal parts or groups" (a *sixth* of the pie was eaten).

As students access mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically.

Characteristics of Student Learning

In general, students with a strong understanding of representation:

- are able to represent quantities using concrete materials and diagrams;
- recognize that a fraction is a representation whose denominator tells how many parts a whole is divided into, and whose numerator tells how many parts there are;
- know how to represent very large numbers but may be confused about how to read them. For example, 1 100 000 is one million, one hundred thousand. A common error is to forget that *and* means "decimal point" and say, "one million and one hundred thousand";
- understand the base ten relationships in our number system, and are able to recognize, for example, that 0.255 is less than 0.789;
- read large numbers correctly, such as 2 200 000, but become confused if the number is represented as 2.2 million;
- understand the representations of common fractions and recognize that the size of the whole is important to the representation (e.g., 1/8 of a large pizza is more than 1/8 of a small one);
- can link one representation of an amount to another representation (e.g., 1100 is the same as 11 hundreds, or 110 tens, or 1100 ones;
- can move flexibly among five different representations of mathematical ideas: pictures, written symbols, oral language, real-world situations, and manipulative models.

Instructional Strategies

Students benefit from the following instructional strategies:

- using concrete and pictorial representations and linking them to mathematical words, terminology, and symbolic notation (e.g., using double and triple number lines that show different representations of the same idea);
- providing opportunities to recognize that different number forms can represent the same quantity (e.g., 2/5 = 0.4 = 40%);
- examining many mathematical representations over time and in real-life situations (e.g., reading newspaper headlines and discussing what $1.2 million would look like if written with just words or numbers);

- providing opportunities to use self-initiated and teacher-suggested drawings of mathematical concepts and procedures (e.g., using a closed array and an open array to develop concepts and procedures related to multiplication and division);
- providing opportunities to connect standard notation for fractions (i.e., proper fractions, improper fractions, and mixed numbers) to the meanings already developed for these numbers and the symbols that represent them;
- providing many experiences to use a variety of tools (e.g., number lines, calculators, hundreds charts), to help students develop an understanding of how the movement of a digit to the right or left significantly alters its value, and to help students read the new numbers that each movement makes;
- providing experiences for students to orally say the names of symbols and to read numbers (e.g., saying "and" to distinguish between wholes and parts in reference to a decimal number; 2 756.032 is two thousand seven hundred fifty-six and thirty-two thousandths);
- providing opportunities to represent mathematical ideas in a variety of ways that make sense to them, even if these representations are not conventional ones.

PROPORTIONAL REASONING

Overview

The ability to compare an object or a set of objects to another helps students make sense of, organize, and describe their world. Young children begin by making qualitative comparisons (e.g., bigger, smaller). Through experience, students learn to make additive comparisons (e.g., 2 more than, 50 less than) and eventually progress to making multiplicative comparisons (e.g., three times as long as, half as much). Consider the following example.

Proportional reasoning is the capstone of the elementary curriculum and the corner-stone of algebra and beyond.

(Post, Behr, & Lesh, 1988)

200 cm tall

100 cm tall

Qualitative comparison: The adult is taller than the child.

Additive comparison: The adult is 100 cm taller than the child.

Multiplicative comparison: The adult is twice as tall as the child.

The ability to think about the multiplicative relationship between two ratios is fundamental to proportional reasoning. In the preceding example, knowing that the adult is twice as tall as the child involves recognizing that the ratio of 200 to 100 is equal to the ratio of 2 to 1 (the second value in each ratio is twice the first value).

Proportional reasoning is a significant development in students' mathematical thinking. It represents the ability to consider the relationship between two relationships – to understand not only the relationship between values in a ratio but also the relationship between two or more ratios. The following example illustrates the multiplicative relationship between two ratios.

> *"Four girls share 12 mini-pizzas equally, and 3 boys share 9 mini-pizzas equally. Who gets to eat more pizza?"*

The ratio of the number of girls to the number of pizzas is 4:12.

The ratio of the number of boys to the number of pizzas is 3:9.

In both cases, the number of mini-pizzas is three times the number of children.

Since the ratios are equal, both girls and boys get the same amount of pizza.

It is important that students in the junior grades be provided with experiences that help them begin to reason proportionally. These informal experiences provide the background for the more formal study, in later grades, of several topics including proportions, percent, similarity, and algebra.

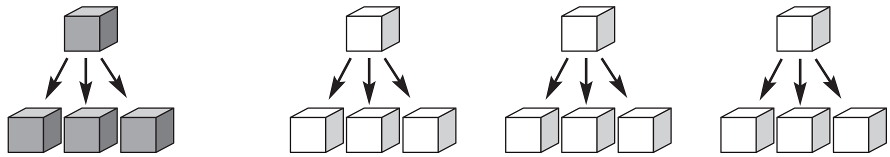
The following are key points that can be made about proportional reasoning in the junior grades:

- Proportional reasoning involves recognizing multiplicative comparisons between ratios.
- Proportional relationships can be expressed using fractions, ratios, and percents.
- Students begin to develop the ability to reason proportionally through informal activities.

PROPORTIONAL REASONING AS MULTIPLICATIVE COMPARISONS BETWEEN RATIOS

Proportions involve two or more equivalent ratios. For example, if orange juice is made by mixing 1 part juice concentrate to 3 parts water (a ratio of 1:3), then orange juice made with 3 parts concentrate needs to be mixed with 9 parts water (a ratio of 3:9). The concentrate-water ratios, 1:3 and 3:9, are equivalent – the concentrate and water parts in the first ratio (1 and 3) are both multiplied by 3 to generate the second, equivalent ratio.

Initially, students in the junior grades may make additive comparisons when trying to make sense of proportional situations. For example, students who are beginning to explore proportionality might conclude that 3 parts of concentrate need to be mixed with 5 parts of water (thinking that a concentrate-to-water ratio of 1:3 is equal to a ratio of 3:5 because the difference between the numbers in each ratio is 2). It is important that students have opportunities to represent proportional relationships using concrete materials and diagrams, so that they can observe the multiplicative relationships inherent in proportions.



Ratio tables can be very helpful in enabling students to identify and extend proportional relationships. For more on ratio tables, see the glossary.

Making Orange Juice					
Orange juice concentrate	1	3	4	6	?
Water	3	9	?	?	6

EXPRESSING PROPORTIONAL RELATIONSHIPS USING FRACTIONS, RATIOS, AND PERCENTS

Proportional relationships can be expressed using fractions, ratios, and percents: for example, about 1/3 of all homes have a pet dog; the ratio of the team's wins to losses is 2 to 3; sale items are 10% off the regular price. Students learn that the fractions, ratios, and percents used to express proportional relationships do not represent discrete quantities, but that they infer ideas such as "out of every", "for every", "compared with", and "per". In the preceding examples, the numbers do not refer to exactly 3 homes, or 2 and 3 games, or $10; but rather, 1 house out of every 3; 2 wins for every 3 losses; $10 per $100. With an understanding of such relationships, students can reason proportionally to determine exact quantities. In the examples, if about 1/3 of all homes have a pet dog, then about 20 out of 60 homes have a dog; if the ratio of a team's wins to losses is 2 to 3 and 15 games were played, the team won 6 games and lost 9; if items are $10 off, the discount on an item that regularly sells for $25 is $2.50.

An ability to reason using proportional relationships is a complex process that develops over an extended period of time. It takes many varied physical experiences to develop an understanding of what a proportional relationship is and then more time to gain the ability to deal with it abstractly. To begin, students need many opportunities to build understanding in real contexts where they can build models which directly compare the proportional attributes.

(Erickson, Cordel, & Mason, 2000, p.1)

DEVELOPING PROPORTIONAL REASONING THROUGH INFORMAL ACTIVITIES

Students in the junior grades should have opportunities to solve a variety of problems that involve proportional reasoning. Proportional-reasoning problems can involve:

- determining a missing value in a ratio. For example, Amy and Habib were biking at the same speed. It took Amy 1.5 hours to bike 24 km. How long did it take Habib to bike 20 km?;

- comparing two ratios. For example, Amy biked 12 km in 45 minutes, and Habib biked 8 km in 40 minutes. Who biked at a faster speed?;
- making qualitative comparisons. For example, Amy and Habib biked the same number of kilometres. It took Amy 35 minutes and it took Habib 30 minutes. Who biked faster?;
- determining unit rates. For example, Amy biked 18 km in 1.5 hours. What was her speed in kilometres per hour?;
- making conversions involving measurement units For example, It took Habib 15 minutes to bike 3550 m. What was his speed in kilometres per hour?;
- determining a fraction of a quantity. For example, Amy wants to bike a distance of 24 km. So far, she has biked 16 km. What fraction of the total distance has she biked?

Teaching mechanical procedures for determining equal ratios (e.g., using cross-multiplication, solving for the unknown variable) does little to promote the development of proportional reasoning. Conceptual understanding about proportional situations emerges when students apply strategies that they find personally meaningful. As illustrated below, students initially use concrete materials and diagrams to help them reason about proportional situations.

"A CD store allows you to trade in 3 of your CDs to get a different used CD. How many CDs could you get if you traded in 12 CDs?"

Students also make connections between proportional reasoning and mathematical processes that they already understand. For example, students recognize that determining equivalent ratios is similar to determining equivalent fractions. A ratio of 3:5 is equivalent to a ratio of 6:10, just as 3/8 = 6/10.

Note: A distinction between equivalent ratios and equivalent fractions must be made. Equivalent fractions represent the same quantity (e.g., 3/5 of a cord is the same length as 6/10 of a cord). Equivalent ratios, however, can represent different quantities (e.g., a bowl of 3 apples and 5 plums is a different quantity than a bowl of 6 apples and 10 plums; however, the ratios of apples to plums in both bowls are equivalent).

Students, with guidance from teachers, can devise methods to organize proportional thinking. For example, a ratio table is an effective tool for recording equivalent ratios.

"In a large aquarium, there is 1 goldfish for every 3 guppies. How many guppies are there if there are 12 goldfish?"

Goldfish	1	2	3	6	12
Guppies	3	6	9	18	36

If there are 12 goldfish, there are 36 guppies.

Characteristics of Student Learning

In general, students who are developing proportional reasoning:

- distinguish between additive relationships (e.g., 10 more than, 2 less than) and multiplicative relationships (e.g., 10 times more, half as much);
- describe multiplicative relationships between quantities (e.g., "You have three times as many stickers as I have");
- are able to solve problems involving proportional reasoning (e.g., determine the cost of 12 books if the cost of books is 3 for $5);
- solve problems involving unit rates;
- are able to make qualitative comparisons based on given ratios (e.g., a mixture of 3 parts white paint to 1 part black paint will be lighter than a mixture of 2 parts white paint to 1 part black paint);
- can compare ratios (e.g., a class with a girl-to-boy ratio of 2:3 has proportionally more girls than a class with a girl-to-boy ratio of 2:4);
- represent proportional relationships using concrete materials, drawings, and ratio tables;
- express and explain proportional relationships using fractions, ratios, and percents.

Instructional Strategies

Students benefit from the following instructional strategies:

- discussing everyday situations that involve proportional reasoning (e.g., determining the cost of multiple items using unit rates);
- using concrete materials and diagrams to model situations, concepts, and procedures related to proportional reasoning;
- providing opportunities to use and discuss students' informal strategies (e.g., using concrete materials, drawing diagrams, using ratio tables) for solving problems that involve proportional reasoning;
- exploring and discussing both proportional and non-proportional situations;
- solving a variety of different problem types (e.g., determining a missing value in a ratio, comparing two ratios, determining unit rates, making conversions involving measurement units);
- posing and solving their own proportional-reasoning problems.

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GLOSSARY

algorithm. A systematic procedure for carrying out a computation. An algorithm that has come into common usage over time is often called a *standard algorithm* (or *traditional algorithm*). Traditional algorithms focus on the digits (e.g., "Add the ones, then regroup ones to tens; add the tens, then regroup tens to hundreds; and so on"). An algorithm or strategy that focuses on the numbers involved in the computation is often called a *flexible algorithm (strategy)* or a *non-standard algorithm.*

Example:

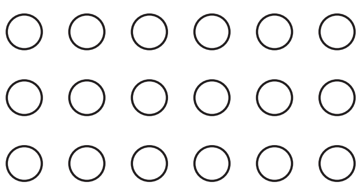
$$\begin{array}{r} {\scriptstyle 4} \\ 57 \\ \times\ 6 \\ \hline 342 \end{array}$$

The "standard algorithm" for multiplication begins by multiplying 6 times 7 to get 42, and recording the "regrouped" 4 tens above the 5; then multiplying 6 × 5 to get 30, and adding the 4 to get 34.

In applying a "flexible algorithm or strategy" for multiplying 6 × 57, a student might think of 57 as 50 and 7, and write:

$$\begin{array}{rl} 57 & \\ \times\ 6 & \\ \hline 42 & (6 \times 7) \\ 300 & (6 \times 50) \\ \hline 342 & \end{array}$$

array. A rectangular arrangement of objects into rows and columns, often used to represent multiplication. For example, 3 × 6 can be represented by an array showing 3 rows with 6 objects in each row. *See also* **open array**.



3 rows, 6 in each row, makes 18 altogether.

associative property. A property of addition and multiplication that allows the numbers being added or multiplied to be grouped differently without changing the result. For example, (77 + 99) + 1 = 77 + (99 + 1), and (7 × 4) × 5 = 7 × (4 × 5). Using the associative property can simplify computation. The property does not generally hold for subtraction or division.

base ten fraction. A fraction whose denominator is a power of 10 (e.g., 3/10, 29/100, 7/1000). *Also called a* **decimal fraction**.

base ten materials. Learning tools that can help students learn many number sense concepts (e.g., place value, operations with whole numbers, fractions, and decimals). Sets of base ten materials typically include small cubes called "units" that represent ones; "rods" or "longs" that represent tens (ten "units"), "flats" that represent hundreds (ten "rods" or "longs"), and large cubes that represent thousands (ten "flats").

benchmark. A number that is internalized and used as a reference to help judge other numbers. For example, 0, 1/2, and 1 are useful benchmarks when comparing and ordering fractions. *Also called a* **referent**.

big ideas. In mathematics, the important concepts or major underlying principles. For example, in this document, the big ideas that have been identified for Grades 4–6 in the Number Sense and Numeration strand of the Ontario curriculum are *quantity, operational sense, relationships, representation*, and *proportional reasoning*.

commutatitive property. A property of addition and multiplication that allows the numbers to be added or multiplied in any order without affecting the sum or product. For example, 3 + 99 = 99 + 3, and 23 × 2 = 2 × 23. Using the commutative property can simplify computation. The property does not generally hold for subtraction or division.

compensation. A mental mathematics strategy in which part of the value of one number is given to another number to make computation easier. For example, 26 + 99 can be thought of as 25 + 100; that is, 1 from the 26 is transferred to the 99 to make 100. Compensation sometimes takes place at the end of the computation. For example, 26 + 99 can be thought of as 26 + 100 = 126; and since 1 too many was added, take one away to get 125.

composite number. A number that has factors besides itself and 1. For example, 12 is a composite number – it has factors of 1, 2, 3, 4, 6, and 12. *See also* **prime number**.

composition of numbers. The putting together of numbers. For example, 4 thousands and 2 hundreds can be composed to make 4200. *See also* **decomposition of numbers** *and* **recomposition of numbers**.

computational strategies. Any of a variety of methods used for performing computations; for example, estimation, mental calculation, flexible and standard algorithms, and the use of technology (e.g., calculators, computer spreadsheets).

constant difference. In subtraction, the property that states that the same quantity can be added to (or subtracted from) each number in a subtraction computation without affecting the answer (difference). This property is often used to simplify subtraction computations. For example, 1001 – 398 is often regarded as a difficult subtraction computation because it involves regrouping across zeros. However, if 2 is added to each number in the computation, the answer (difference) will be the same, but the computation is much simpler: 1003 – 400 = 603, so 1001 – 398 = 603.

Cuisenaire rods. Commercially produced manipulatives that can help students learn about fractions, patterns, and so on. This set of rectangular rods of different lengths, in which each length is a different colour, was invented by Georges Cuisenaire (1891–1976).

decomposition of numbers. The taking apart of numbers. For example, the number 402 is usually taken apart as 400 and 2, but it can also be taken apart in other ways, such as 390 and 12. Students who can decompose numbers in many different ways develop computational fluency and have many strategies available for solving arithmetic questions mentally. *See also* **composition of numbers** *and* **recomposition of numbers**.

denominator. In fractions, the number written below the line. It represents the number of equal parts into which a whole or set is divided, or the divisor of a division sentence. For example, in 3/4, the denominator is 4 and might mean 4 equal parts, 4 objects in a group, or 3 divided by 4. *See also* **numerator**.

distributive properties. The properties that allow numbers in a multiplication or division expression to be decomposed into two or more numbers. These properties include:

- ***Distributive property of multiplication over addition***, for example, $6 \times 47 = (6 \times 40) + (6 \times 7)$, which results in $240 + 42 = 282$;
- ***Distributive property of multiplication over subtraction***, for example, $4 \times 98 = (4 \times 100) - (4 \times 2)$, which results in $400 - 8 = 392$;
- ***Distributive property of division over addition***, for example, $72 \div 6 = (60 \div 6) + (12 \div 6)$, which results in $10 + 2 = 12$;
- ***Distributive property of division over subtraction***, for example, $4700 \div 4 = (4800 \div 4) - (100 \div 4)$, which results in $1200 - 25 = 1175$.

division. The operation characterized by the equal sharing of a quantity into a known number of groups (partitive division), or by the repeated subtraction of an equal number of items from the whole quantity (quotative division). Division is the inverse operation of multiplication. The quantity to be divided is called the *dividend*. The number to be divided by is the *divisor*. The *quotient* is the result of a division problem. The *remainder* is the number "left over", which cannot be grouped or shared equally. For example:

67	÷	6	=	11	R 1
dividend		divisor		quotient	remainder

equivalent fractions. Different representations in fractional notation of the same part of a whole or group. For example, 1/3, 2/6, 3/9, and 4/12 are equivalent fractions.

estimation strategies. Mental mathematics strategies used to obtain an approximate answer. Students estimate when an exact answer is not required, and when they are checking the reasonableness of their mathematics work. Some examples of estimation strategies can be found in the introductory pages of Volume 3: Multiplication and Volume 4: Division.

factors. Natural numbers that divide evenly into a given natural number. For example, the factors of 12 are 1, 2, 3, 4, 6, and 12, because all these numbers divide evenly into 12. *See also* **multiplication**.

friendly numbers. Numbers that are easy to work with. For example, to calculate $192 \div 6$, students might think of 192 as 180 and 12, and calculate $180 \div 6$ and $12 \div 6$. Also called *compatible numbers*.

identity property (identity element). The identity property occurs when a number is combined with a special number (identity element) by using one of the operations, and the result leaves the original number unchanged. In addition and subtraction, the special number or identity element is 0; for example, $36 + 0 = 36$ and $17 - 0 = 17$. In multiplication and division, the identity element is 1; for example, $88 \times 1 = 88$ and $56 \div 1 = 56$.

improper fraction. A fraction whose numerator is greater than its denominator and whose value is greater than 1; for example, 7/3.

inverse operations. The opposite effects of addition and subtraction, and of multiplication and division. Addition involves joining sets; subtraction involves separating a quantity into sets. Multiplication refers to joining sets of equal amounts; division is the separation of an amount into equal sets.

manipulatives. Objects that students handle and use in constructing their own understanding of mathematical concepts and skills, and in demonstrating that understanding. Some examples are base ten materials, fraction circles, and tiles. Also called *concrete materials*.

mathematical conventions. Agreed-upon rules or symbols that make the communication of mathematical ideas easier.

mental math. Ways of computing mentally, with or without the support of paper and pencil. For example, to calculate 501 – 199 mentally, one might add 1 to both numbers and subtract 502 – 200 = 302.

mixed number. A number greater than 1 that is composed of a whole number and a fraction; for example, 8 1/4.

model. A concrete or pictorial representation of a mathematical idea, which students handle or use to construct their own understanding of mathematical concepts and skills, and to illustrate that understanding; for example, number line, array.

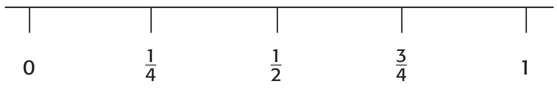
multiple. The product of a given whole number multiplied by any other whole number. For example, 7, 14, 21, 28, 35, …, are multiples of 7.

multiplication. An operation characterized by the combination of equal groups, by repeated addition, or by an array. Multiplication is the inverse operation of division. The multiplication of *factors* gives a *product*. For example:

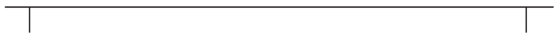
4	×	5	=	20
factor		factor		product

natural number. Any one of the counting numbers 1, 2, 3, 4, …

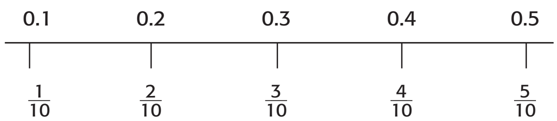
number line. A visual model that matches a set of numbers and a set of points one to one. For example:



- An ***open number line*** consists of a marked but unlabelled number line that can be used to represent various values depending on the starting point and interval selected.



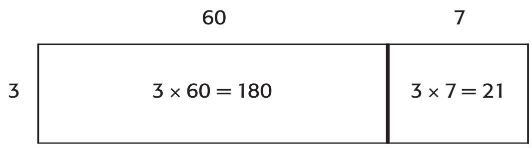
- A ***double number line*** is used to represent the equivalencies between two quantities. For example, a double number line that shows decimal numbers and their equivalent fractions might display the decimals sequentially along the top of the number line and display the corresponding equivalent fractions along the bottom of the number line.



number sense. The ability to interpret numbers and use them correctly and confidently.

numerator. In fractions, the number written above the line. It might represent the number of equal parts (or objects of a group) being considered or the dividend of a division sentence. For example, in 3/4, the numerator is 3 and might mean 3 of 4 equal parts, 3 of 4 objects in a group, or 3 divided by 4. *See also* **denominator**.

open array. A rectangular arrangement, used to represent multiplication or division, in which the rows and columns of individual objects are not represented. However, the factors of the multiplication expression (the number of implied rows and columns) are recorded on the length and width of the rectangle. An open array does not have to be drawn to scale. For example, 3 × 67 might be represented by an open array such as the following:



operational sense. An understanding of the mathematical concepts and procedures involved in operations on numbers (addition, subtraction, multiplication, and division) and of the application of operations to solve problems.

partitive division. In partitive division, the whole amount is known and the number of groups is known, while the number of items in each group is unknown. For example, "Daria has 42 bite-sized granola snacks to share among her 6 friends. How many does each friend get?" Also called *distribution division* or *sharing division*.

part-part-whole. The idea that a number can be made of two or more parts. For example, 57 can be separated into 50 and 7, or 30 and 20 and 7.

percent. A fraction or ratio, expressed by using the percent symbol, %, in which the denominator is 100. Percent means "out of 100". For example, 30% means 30 out of 100. A percent can be represented by a fraction with a denominator of 100; for example, 30% = 30/100.

prime number. A whole number greater than 1 that only has two factors, itself and 1. For example, 13 is a prime number – its only factors are 13 and 1. *See also* **composite number**.

proper fraction. A fraction whose numerator is less than its denominator and whose value is less than 1; for example, 2/3.

proportion. A statement or equation that two or more ratios are equal. For example, 2/3 = 6/9; and 1 : 7 = 2 : 14.

proportional reasoning. Reasoning that involves an understanding of the multiplicative relationship in the size of one quantity compared with another. Students express proportional reasoning informally by using phrases such as "twice as big as" and "a third the size of".

quantity. The "howmuchness" of a number. An understanding of quantity helps students estimate and reason with numbers and is an important prerequisite to understanding place value, the operations, and fractions.

quotative division. In quotative division, the whole amount is known and the number of items in each group is known, while the number of groups is unknown. For example, "Thomas is packaging 72 ears of corn into bags. If each bag contains 6 ears of corn, how many bags will Thomas need?" *Also called* **measurement division**.

rate. A comparison, or type of ratio, of two quantities with different units, such as distance and time; for example, a speed of 100 km/h, a gasoline consumption of 20 L/100 km.

ratio. A comparison of quantities with the same units. A ratio can be expressed in ratio form, fractional form, or words; for example, 3 : 4 or 3/4 or 3 to 4.

rational number. A number that can be expressed as a fraction in which the denominator is not 0.

ratio table. A model that can be used to develop an understanding of multiplication, equivalent fractions, division, and proportional reasoning. For example, the ratio table below would help students solve the following problem: "A grocer sells flour by the kilogram. If a 4 kg bag costs $6.60, how much would a 2 kg, a 3 kg, a 5 kg, and a 6 kg bag cost?" If students know that 4 kg is $6.60, they can easily halve that amount to find the 2 kg amount and halve it again for 1 kg. Students can then calculate 6 kg by adding the 2 kg and 4 kg amounts. 3 kg can be calculated by either halving the 6 kg amount or adding the 2 kg and 1 kg amounts (each student makes a choice based on what is fastest and easiest for him or her). 5 kg can be calculated by either adding the 4 kg and 1 kg amounts or calculating the 10 kg amount, which is easy, and then halving. The table would also help students extend the problem to other amounts.

recomposition of numbers. The putting back together of numbers that have been decomposed. For example, to calculate 298 + 303, a student might decompose the numbers as 298 + 300 + 3, and then recompose the numbers as 300 + 300 + 1 to get the answer 601. *See also* **composition of numbers** *and* **decomposition of numbers**.

relationship. In mathematics, a connection between mathematical concepts, or between a mathematical concept and an idea in another subject or in real life. As students connect ideas they already understand with new experiences and ideas, their understanding of mathematical relationships develops.

representation. The use of manipulatives, diagrams, pictures, or symbols to model a mathematical concept or real-world context or situation.

rounding. A process of replacing a number by an approximate value of that number to assist in estimation. Note that rounding does not refer to any set of rules or procedures (e.g., look to the number on the right – is it less than 5?). To estimate 5 × 27, for example, 27 might be rounded to 30 (to give an estimate of 5 × 30 = 150), but 27 could also be rounded to 25 (to give an estimate of 5 × 25 = 125).

Ratio Table

Size of bag	4 kg	2 kg	1 kg	6 kg	3 kg	5 kg	10 kg
Cost	$6.60	$3.30	$1.65	$9.90	$4.95	$8.25	$16.50

simplification. In division, the process of "reducing" or multiplying a given problem to make a friendlier problem. For example, 128 ÷ 32 has the same quotient as 64 ÷ 16 (halve both numbers), or 32 ÷ 8 (halve both numbers again); 80 ÷ 5 has the same quotient as 160 ÷ 10 (multiply both numbers by 2).

unitizing. The ability to recognize that a group of objects can be considered as a single entity. For example, 10 objects can be considered as one group of 10.

whole number. Any of the numbers 0, 1, 2, 3, 4, …

zero property of multiplication. The property that any number multiplied by 0 is 0.

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