MCV4U
Calculus and Vectors
University Preparation

Calculus and Vectors: Content and Reporting Targets

Mathematical Processes across all strands:
Problem Solving, Reasoning and Proving, Reflecting, Selecting Tools and Computational Strategies, Connecting, Representing, and Communicating

Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6
Explore rate of "flow" problems using non-algebraic means • Explore contexts and solve problems where one needs to know rate of change at a specific point, using verbal and graphical representations of the function. Include examples where mechanical tools are not readily available (e.g., income flow, garbage accumulation rate) • Analyse rates of change and provide qualitative solutions to problems (e.g., increase, decrease, tend towards something) **Standardize the process of finding instantaneous rate of change at a particular point** • Apply a standard process for determining instantaneous rate of change of a function at a specific point on its graph • For polynomial, simple rational, and radical functions, form, evaluate, and interpret the first principles definition of the derivative, using a fixed (numerical) value	**Derivative Functions from First Principles** • Recognize numerical and graphical representations of increasing and decreasing rates of change • Use patterning and reasoning to determine that there is a function that describes the derivative at all points • For polynomial, rational and radical functions, determine, using limits, the algebraic representation of the derivative at any point **Application of Derivatives of Polynomial Functions** • Graph, without technology, the derivative of polynomials with given equations • Given the graph of the derivative, sketch the original polynomial **Derivative Functions Through Investigation** • Through investigation, determine the algebraic representation of the derivative at any point for exponential, logarithmic and sine/cosine functions	**Derivative Functions: Properties and their Applications** • Investigate properties of derivatives (power rule, chain rule as change of scale and as patterning, no quotient rule use product rule, Sample Problem: Examine the relationship between the derivative of a function and the derivative of its inverse. Generalize the power rule for all rational powers) • Apply these properties to form derivatives of functions and simple combinations of functions (no simplification of derivatives formed outside of problem-solving contexts)	**Applications of Derivatives in Rate of Change and Optimization Problems, Including Those Requiring Modelling** • Solve rate of change and optimization problems given algebraic models • Solve rate of change and optimization problems requiring the creation of an algebraic model (more variety in problems to get at various types of algebraic simplification and analysis) • Solve problems calling for the modelling of the rate of change flow problems), not necessarily finding the original function but just a property of it e.g., pt of inflection	**Representing Vectors** • Introduce vectors in 2-D and 3-D • Represent vectors geometrically and algebraically • Operate with vectors • Solve problems involving vectors	**Representing Lines and Planes** • Parametric equations of functions • Represent lines and planes in a variety of ways • Find intersections of two planes • Find intersections of three planes

Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6
of "a" (e.g., What is the graphical significance of $\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$)	**Applications of Derivatives, Given Algebraic Representations** • Pose and solve problems that require identifying conditions that result in a desired rate of change				

Rationale

Teaching Calculus before Vectors

- Provides a natural flow from Advanced Functions to this course and students build on prior knowledge
- Calculus problems are situated in a two-dimensional context while vector problems progress from two-dimensions to three-dimensions
- The introduction of parametric equations can help make connections

Focusing Unit 1 on rates of change problems:

- Provides an opportunity for students to investigate a variety of real-world contexts involving change; develops an appreciation of the need to analyse rates of change
- Establishes a need for algebraic representations of rates of changes (e.g., the need for precision, for information at many different data points)

Separating Units 1, 2, and 3:

- Introduces abstract concepts at a developmentally appropriate pace
- Provides opportunities to connect each abstract concept to problem solving situations
- Provides the time for students to investigate and consolidate conceptual understanding of rates of change, derivatives and limits, prior to combining these concepts with algebraic procedures

Graph analysis within Unit 2

- Curriculum revisions focus curve sketching on polynomials only
- Graph analysis can be one of the strategies students use to confirm the reasonableness of solutions to problems in Unit 4

Problems requiring modelling congregated in Unit 4

- These problems require students to choose from amongst all possible function types when formulating a mathematical model.
- The problem solving in Unit 4 provides a segue from calculus to vectors

Numbers of Units

- It is recommended that calculus concepts be taught in $\frac{2}{3}$ of the time available, and that vectors be taught in $\frac{1}{3}$ of the time available

Calculus and Vectors
Year Outline – Planning Tool

P Number of pre-planned lessons (including instruction, diagnostic and formative assessments, summative assessments other than summative performance tasks)

J Number of jazz days of time (instructional or assessment)

T Total number of days

SP Summative performance task (see Assessment – Grade 9 Applied)

Unit	Cluster of Curriculum Expectations	Overall and Specific Expectations	P	J	T	SP
1	• Explore rates of change in context to consolidate their understanding from Advanced Functions • Connect instantaneous rates of change with the derivative • Connect the characteristics of the instantaneous rate of change with the characteristics of the function	**B1** demonstrate an understanding of rate of change by making connections between average rate of change over an interval and instantaneous rate of change at a point, using the slopes of secants and tangents and the concept of the limit **B2** graph the derivatives of polynomial, sinusoidal, and exponential functions, and make connections between the numeric, graphical, and algebraic representations of a function and its derivative	8	1	9	
2	• Investigate connections graphical and numerically between the graph of a function and its derivative • Determine, using limits, the algebraic representation of derivatives • Determine and apply the power, chain and product rules • Apply power, product and chain rules to rational and radical functions • Develop the derivatives of $f(x) = e^x$, $f(x) = \sin(x)$ and $f(x) = \cos(x)$ • Explore the relationship between $f(x) = e^x$ and $f(x) = ln(x)$ • Solve problems involving instantaneous rates of change	**B2** graph the derivatives of polynomial, sinusoidal, and exponential functions, and make connections between the numeric, graphical, and algebraic representations of a function and its derivative **B3** verify graphically and algebraically the rules for determining derivatives, apply these rules to determine the derivatives of polynomial, sinusoidal, exponential, rational, and radical functions, and simple combinations of functions, and solve related problems	18	2	20	
3	• Examine the relationship between first and second derivatives and the original polynomial or rational function • Sketch curves of polynomial or rational functions given information or equations • Apply the properties of derivatives to real-world problems	**C1** make connections, graphically and algebraically, between the key features of a function and its first and second derivatives, and use the connections in curve sketching **C2** solve optimization problems that require the use of the concepts and procedures associated with the derivative, including problems arising from real-world applications and involving the development of mathematical models	8	1	9	

Unit	Cluster of Curriculum Expectations	Overall and Specific Expectations	P	J	T	SP
4	• Solve rate of change and optimization problems in a wide variety of contexts using properties of derivatives • Collect data, create mathematical models and solve problems arising from real-world contexts	**C2** solve optimization problems that require the use of the concepts and procedures associated with the derivative, including problems arising from real-world applications and involving the development of mathematical models	11	2	13	
5	• Introduce vectors in two-space and three-space • Represent vectors geometrically and algebraically • Determine vector operations and properties • Solve problems involving vectors	**A1** demonstrate an understanding of vectors in two-space and three-space by representing them algebraically and geometrically and by recognizing their applications **A2** perform operations on vectors in two-space and three-space, and use the properties of these operations to solve problems, including those arising from real-world applications	16	3	19	
6	• Represent equations of lines in two-space and three space using a variety of forms • Investigate intersections of planes • Solve problems involving planes arising from real-world contexts	**A3A** distinguish between the geometric representations of a single linear equation or a system of two linear equations in two-space and three-space, and determine different geometric configurations of lines and planes in three-space **A3B** represent lines and planes using scalar, vector, and parametric equations, and solve problems involving distances and intersections				
	Summative Performance Tasks					**14**
	Total Days		**64**	**7**	**72**	**85**

The number of prepared lessons represents the lessons that could be planned ahead based on the range of student readiness, interests, and learning profiles that can be expected in a class. The extra time available for "instructional jazz" can be taken a few minutes at a time within a pre-planned lesson or taken a whole class at a time, as informed by teachers' observations of student needs.

The reference numbers are intended to indicate which lessons are planned to precede and follow each other. Actual day numbers for particular lessons and separations between terms will need to be adjusted by teachers.

Unit 1: Rates of Change

Grade 12

Lesson Outline

Big Picture

Students will:

- connect slopes of secants to average rates of change, and slopes of tangents to instantaneous rates of change in a variety of contexts.
- approximate rates of change graphically and numerically.

Day	Lesson Title	Math Learning Goals	Expectations
1	Rates of Change Revisited *(Sample Lesson Included)*	• Describe real-world applications of rate of change (e.g., flow) problems using verbal and graphical representations (e.g., business, heating, cooling, motion, currents, water pressure, population, environment, transportations) • Describe connections between average rate of change and slope of secant, and instantaneous rate of change and slope of tangent in context	A1.1, A1.2
2	Determine Instantaneous Rate of Change using Technology *(Sample Lesson Included)*	• With or without technology, determine approximations of and make connections between instantaneous rates of change as secant lines tends to the tangent line in context	A1.3
3	Exploring the Concept of a Limit *(Sample Lesson Included)*	• Explore the concept of a limit by investigating numerical and graphical examples and explain the reasoning involved • Explore the ratio of successive terms of sequences and series (use both divergent and convergent examples) • (e.g., Explore the nature of a function that approaches an asymptote (horizontal and vertical)	A1.4
4–5	Calculating an instantaneous rate of change using a numerical approach *(Sample Lesson Included)*	• connect average rate of change to $\frac{f(a+h)-f(a)}{h}$ and instantaneous rate of change to $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$	A1.5, A1.6
6–7	Jazz/ Summative	•	

Unit 2: Exploring Derivatives

Grade 12

Lesson Outline

Big Picture

Students will:

- make connections between functions (polynomial, sinusoidal, exponential) and their corresponding derivative functions.
- determine, using limits, the algebraic representation of the derivative of polynomial functions at any point.
- use patterning and reasoning to investigate connections between the graphs of functions and their derivatives
- make connections between the inverse relation of $f(x) = ln(x)$ and $f(x) = e^x$.

Day	Lesson Title	Math Learning Goals	Expectations
1	Key Characteristics of Instantaneous Rates of Change *(Sample Lesson Included)*	• determine intervals in order to identify increasing, decreasing, and zero rates of change using graphical and numerical representations of polynomial functions • Describe the behaviour of the instantaneous rate of change at and between local maxima and minima	A2.1
2	Patterns in the Derivative of Polynomial Functions *(Sample Lesson Included)*	• Use numerical and graphical representations to define and explore the derivative function of a polynomial function with technology • Make connections between the graphs of the derivative function and the function	A2.2
3	Derivatives of Polynomial Functions *(Sample Lesson Included)*	• Determine, using limits, the algebraic representation of the derivative of polynomial functions at any point	A2.3
4	Patterns in the Derivative of Sinusoidal Functions	• Use patterning and reasoning to investigate connections graphically and numerically between the graphs of $f(x) = \sin(x)$, $f(x) = \cos(x)$, and their derivatives using technology	A2.4
5	Patterns in the Derivative of Exponential Functions	• determine the graph of the derivative of $f(x) = a^x$ using technology • Investigate connections between the graph of $f(x) = a^x$ and its derivative using technology	A2.5
6	Identify "e"	• investigate connections between an exponential function whose graph is the same as its derivative using technology and recognize the significance of this result	A2.6
7	Relating $f(x) = ln(x)$ and $f(x) = e^x$	• Make connections between the natural logarithm function and the function $f(x) = e^x$ • Make connections between the inverse relation of $f(x) = ln(x)$ and $f(x) = e^x$	A2.7
8	Verify derivatives of exponential functions	• Verify the derivative of the exponential function $f(x) = a^x$ is $f'(x) = a^x \, lna$ for various values of a, using technology	
9	Jazz Day/ Summative Assessment	•	

Day	Lesson Title	Math Learning Goals	Expectations
10–11	Power Rule	• Verify the power rule for functions of the form $f(x) = x^n$ (where n is a natural number) • Verify the power rule applies to functions with rational exponents • Verify numerically and graphically, and read and interpret proofs involving limits, of the constant, constant multiple, sums, and difference rules	A3.1, A3.2 A3.4
12	Solve Problems Involving The Power Rule	• determine the derivatives of polynomial functions algebraically, and use these to solve problems involving rates of change	A3.3
13–15	Explore and Apply the Product Rule and the Chain Rule	• verify the chain rule and product rule • Solve problems involving the Product Rule and Chain Rule and develop algebraic facility where appropriate	A3.4 A3.5
16–17	Connections to Rational and Radical Functions	• Use the Product Rule and Chain Rule to determine derivatives of rational and radical functions • Solve problems involving rates of change for rational and radical functions and develop algebraic facility where appropriate	A3.4 A3.5
18–19	Applications of Derivatives	• Pose and solve problems in context involving instantaneous rates of change	A3.5
20	Jazz Day		
21	Summative Assessment		

Unit 3: Applying Properties of Derivatives

Grade 12

Lesson Outline

Big Picture

Students will:

- investigate using technology the key features of the graph of the function and those of the first and second derivatives.
- connect the key properties of the second derivative to the first derivative and the original polynomial or rational function.
- determine algebraically the equation of the second derivative of a polynomial or simple rational function.
- sketch and verify graphs of polynomial functions from given key features.

Day	Lesson Title	Math Learning Goals	Expectations
1–3	The Second Derivative	• Define the second derivative • Investigate using technology to connect the key properties of the second derivative to the first derivative and the original polynomial or rational function (increasing and decreasing intervals, local maximum and minimum, concavity and point of inflection) • Determine algebraically the equation of the second derivative $f(x)$ of a polynomial or simple rational function $f(x)$, and make connections, through investigation using technology, between the key features of the graph of the function and those of the first and second derivatives	B1.1, B1.2, B1.3
4	Curve Sketching from information	• Describe key features of a polynomial function and sketch two or more possible graphs of a polynomial function given information from first and second derivatives – explain why multiple graphs are possible.	B1.4
5–6	Curve Sketching from an Equation	• Extract information about a polynomial function from its equation, and from the first and second derivative to determine the key features of its graph • Organize the information about the key features to sketch the graph and use technology to verify.	B1.5
7	Jazz Day	•	
8	Unit Summative	•	

Unit 4: Rate of Change Problems

Grade 12

Lesson Outline

Big Picture

Students will:

- connect concepts of motion and derivatives.
- solve rates of change and optimization problems in a wide variety of contexts and interpret the results.

Day	Lesson Title	Math Learning Goals	Expectations
1–3	Rate of Change Problems	• Make connections between the concept of motion and the concept of the derivative in a variety of ways. • Make connections between graphical and algebraic representations and real-world applications • Solve problems in wide variety of contexts and interpret the results	B2.1, B2.2, B2.3
4–8	Optimization Problems	• Solve a variety of optimization problems given an algebraic model • Solve a variety of optimization problems requiring the creation of an algebraic model	B2.4
9	Solve problems from Data	• Solve problems arising from real-world applications by applying a mathematical model and the concepts and procedures associated with the derivative to determine mathematical results, and interpret and communicate results. • Revisit some of the rate of change and rate of flow problems from Unit 1	B2.5
10	Jazz Day	•	
11–13	Summative Assessment for Units 3–4	•	

Unit 5: Representing Vectors

Grade 12

Lesson Outline

Big Picture

Students will:

- introduce vectors in two-space and three-space.
- represent vectors geometrically and algebraically.
- determine vector operations and properties.
- solve problems involving vectors including those arising from real-world applications.

Day	Lesson Title	Math Learning Goals	Expectations
1		• Explore connections between calculus and vectors	
2		• Represent vectors geometrically and algebraically in two-space. • Develop an understanding of equivalent vectors • Use geometric vectors to interpret information arising from real- world applications (Use applets described in Appendix A)	C1.1, 1.2
3		• Determine methods for changing from geometric (directed line segment) to algebraic (Cartesian) forms of a vector in two-space and vice versa.	C1.3
4		• Add, subtract, and multiply vectors by a scalar in two-space, both geometrically and algebraically • Solve problems including problems arising from real-world applications involving vector operations in two-space	C2.1, 2.3
5		• Determine the dot product of vectors in two-space geometrically and algebraically • Describe applications in two-space of the dot-product including projections	C2.4
6	Jazz Day	(Use applets described in Appendix A)	
7	Summative Assessment		
8		• Represent both points and vectors algebraically in three-space • Determine the distance between points and the magnitude of vectors in three-space both geometrically and algebraically • Solve problems including problems arising from real-world applications involving vector operations in three-space	C1.4, 2.1, 2.3
9		• Investigate, with and without technology, the commutative, associative and distributive properties of the operations of addition, subtraction and multiplication by a scalar in two and three-space (Use Vector Laws applet described in Appendix A)	C2.2
10		• Determine the dot product of vectors in three-space geometrically and algebraically • Describe applications in three-space of the dot-product including projections	C2.4

Day	Lesson Title	Math Learning Goals	Expectations
11		• Determine through investigation the properties of dot product in two- and three-space	C2.5
12		• Determine the cross product of vectors in three-space algebraically including magnitude and describe applications	C2.6
13		• Through investigation, determine properties of the cross product of vectors	C2.7
14		• Solve problems arising from real-world applications that involve the use of dot products, cross products, including projections	C2.8
15	Jazz Day		
16	Summative Assessment		

Unit 6: Representing Lines and Planes

Grade 12

Lesson Outline

Big Picture

Students will:

- represent lines and planes in a variety of forms and solve problems involving distances and intersections.
- determine different geometric configurations of lines and planes in three-space.
- investigate intersections of and distances between lines and/or planes.

Day	Lesson Title	Math Learning Goals	Expectations
1		• Recognize that a linear equation in two-space forms a line and represent it geometrically and algebraically • Represent a line in two-space in a variety of forms (scalar, vector, parametric) and make connections between the forms	C3.1, C4.1
2		• Recognize that a line in three-space cannot be represented in scalar form • Represent a line in two-space in a variety of forms (vector and parametric) and make connections between the forms	C4.2
3		• Recognize that a linear equation in three-space forms a plane and represent it geometrically and algebraically • Determine through investigation geometric properties of planes including a normal to a plane • Determine using the properties of the plane the scalar, vector and parametric equations of a plane	C3.2, C4.3, C4.5
4		• Determine the equation of a plane in its scalar, vector, or parametric form given another of these forms. • Represent a line in three-space by using the scalar equations of two intersecting planes	C4.6, C4.2
Refer to Smart Ideas file (Overview.ipr) for a flowchart of the concepts covered in Lessons 5 through 10.			
5	Lots of Lines *(Sample Lesson Included)*	• Recognize that a linear equation in two-space forms a line represent it geometrically and algebraically • Recognize that the solution to a system of two linear equations in two-space determines a point in two space if the lines are not coincident or parallel • Solve and classify solutions to systems of equations in two-space in vector and parametric forms and understand the connections between the graphical and algebraic representations	C3.1, C4.1
6	Concrete Critters *(Sample Lesson Included)*	• Determine through investigation different geometric configurations of combinations of up to three lines and/or planes in three space • Classify sets of lines and planes in three space that result in a common point, common line, common plane or no intersection	C3.3
7	Interesting Intersections I *(Sample Lesson Included)*	• Determine the intersections of two lines, and a line and a plane in three space given equations in various forms and understand the connections between the geometric and algebraic representations	C3.3, C4.7

Day	Lesson Title	Math Learning Goals	Expectations
8	Interesting Intersections II *(Sample Lesson Included)*	• Determine the intersections in three-space of 2 planes and 3 planes intersecting in a unique point given equations in various forms and understand the connections between the graphical and algebraic representations of the intersection	C3.3, C4.3, C4.4, C4.7
9	Interesting Intersections III *(Sample Lesson Included)*	• Determine the intersections of 3 planes in three space given equations in various forms and understand the connections between the graphical and algebraic representation of the intersection • Recognize that if $\vec{a} \bullet \vec{b} \times \vec{c} \neq 0$ is true then the three planes intersect at a point • Solve problems involving the intersection of lines and planes in three-space represented in a variety of ways	C4.4, C4.7
10	How Far Can it Be? *(Sample Lesson Included)*	• Calculate the distance in three-space between lines and planes with no intersection • Solve problems related to lines and planes in three-space that are represented in a variety of ways involving intersections	C3.3, C4.3, C4.7
11	Jazz Day		
12–14	Summative Assessment Units 5 and 6		

Appendix A: Electronic Learning Objects to Support MCV4U

E-Learning Ontario Web Site: MGA4U Unit 3 Vectors

http://www.elearningontario.ca/eng/Default.aspx

Activity 2: Vector Laws

The last applet on Vector Laws allows the user to investigate the commutative, associative, distributive properties of 2-space vectors in geometric form.
At the bottom of activity 2 is a link to the University of Guelph's Physics department where a tutorial for vectors is provided.

Activity 3: Applications of Geometric Vectors

The second applet in the Velocity Java Applets allows the user to investigate the resultant vector for a boat crossing a river. The user controls 2-space vectors in geometric form for the boat's velocity and the current.

Activity 5: Algebraic Vectors

The first applet allows users to interactively explore the connections between geometric and algebraic forms of vectors in 2-space.

At the end of this activity is a link to a 3-space Graphing Tool that allows students to graph points, lines, and planes in various forms.

Activity 6: Operations with Algebraic Vectors

There are four applets on addition of vectors, scalar multiplication, unit vectors, and position vectors. They allow the user to interactively manipulate 2-space vectors.

E-Learning Ontario Web Site: MGA4U Unit 5 Vector Methods with Planes and Lines

Activity 1: Equations of Lines in 2-space

There are five guided and three interactive applets on forms of vector equations, how to convert between forms, distance from a point to a line.

Activity 3: Intersection of Lines

There are two guided applets on intersection of lines in 2-space and 3-space.

Activity 5: Equations of Planes

There are four guided applets on the forms of equations of planes and how to convert between forms.

Activity 6: Intersection of a Line and a Plane

There is one guided applet.

Appendix A: Electronic Learning Objects to Support MCV4U (continued)

Activity 7: Intersection of Planes

There is one guided applet on solving systems of planes algebraically.

Activity 8: Task: *X*, *Y*, and *Z* Factor

An open ended task using the 3-space Graphing Tool allows students to consolidate vector concepts.

Vector Applets on the Web

NCTM
http://standards.nctm.org/document/eexamples/chap7/7.1/index.htm

> This site has two applets. The first illustrates the components of a vector to control a car. The user interactively controls the speed and direction. The second illustrates vector addition for an aircraft flying that is acted upon by wind. The user controls the speed and direction of both the aircraft and wind.

Syracuse University
http://physics.syr.edu/courses/java-suite/crosspro.html

> This applet demonstrates cross product of two vectors in 3-space. It allows users to interactively change the vectors and see the resulting cross-product. The two vectors are limited to one plane but the plane can be moved to different viewing angles.

International Education Software
http://www.ies.co.jp/math/products/vector/menu.html

> This Japanese site has a collection of applets that cover a wide variety of 2-space and 3-space vector topics. The controls are not very user-friendly but there are topics covered here like vector forms of lines in 2-space and 3-space that are not covered on other sites.

Professor Bob's Physics Lab (Rob Scott)
http://www.after4.ca/SchoolStuff/PhysicsLab/roomnojpgtest.html

> This interactive site has flash applets on various Physics topics. Some topics such as Milliken and Momentum labs allow students to apply vector concepts.

B.Surendranath Reddy (Physics Teacher in India)
http://surendranath.org/Applets.html

> This site has several applets that can be used in MCV4U. For vectors there are applets for addition, cross product of vectors, converting between Cartesian and directed line segment forms and several kinematics applets. For calculus there are applets for instantaneous speed and velocity.