MCV4U
Calculus and Vectors
University Preparation
Unit 2

To be updated the week of August 27 and further edited in September 2007

Unit 2: Exploring Derivatives

Grade 12

Lesson Outline

Big Picture

Students will:

- make connections between functions (polynomial, sinusoidal, exponential) and their corresponding derivative functions.
- determine, using limits, the algebraic representation of the derivative of polynomial functions at any point.
- use patterning and reasoning to investigate connections between the graphs of functions and their derivatives
- make connections between the inverse relation of $f(x) = ln(x)$ and $f(x) = e^x$.

Day	Lesson Title	Math Learning Goals	Expectations
1	Key Characteristics of Instantaneous Rates of Change *(Sample Lesson Included)*	• Determine intervals in order to identify increasing, decreasing, and zero rates of change using graphical and numerical representations of polynomial functions • Describe the behaviour of the instantaneous rate of change at and between local maxima and minima	A2.1
2	Patterns in the Derivative of Polynomial Functions *(Sample Lesson Included)*	• Use numerical and graphical representations to define and explore the derivative function of a polynomial function with technology • Make connections between the graphs of the derivative function and the function	A2.2
3	Derivatives of Polynomial Functions *(Sample Lesson Included)*	• Determine, using limits, the algebraic representation of the derivative of polynomial functions at any point	A2.3
4	Patterns in the Derivative of Sinusoidal Functions	• Use patterning and reasoning to investigate connections graphically and numerically between the graphs of $f(x) = \sin(x)$, $f(x) = \cos(x)$, and their derivatives using technology	A2.4
5	Patterns in the Derivative of Exponential Functions	• determine the graph of the derivative of $f(x) = a^x$ using technology • Investigate connections between the graph of $f(x) = a^x$ and its derivative using technology	A2.5
6	Identify "*e*"	• investigate connections between an exponential function whose graph is the same as its derivative using technology and recognize the significance of this result	A2.6
7	Relating $f(x) = ln(x)$ and $f(x) = e^x$	• Make connections between the natural logarithm function and the function $f(x) = e^x$ • Make connections between the inverse relation of $f(x) = ln(x)$ and $f(x) = e^x$	A2.7
8	Verify derivatives of exponential functions	• Verify the derivative of the exponential function $f(x) = a^x$ is $f(x) = a^x\, lna$ for various values of a, using technology	
9	Jazz Day/ Summative Assessment	•	

Day	Lesson Title	Math Learning Goals	Expectations
10–11	Power Rule	• Verify the power rule for functions of the form $f(x) = x^n$ (where n is a natural number) • Verify the power rule applies to functions with rational exponents • Verify numerically and graphically, and read and interpret proofs involving limits, of the constant, constant multiple, sums, and difference rules	A3.1, A3.2 A3.4
12	Solve Problems Involving The Power Rule	• determine the derivatives of polynomial functions algebraically, and use these to solve problems involving rates of change	A3.3
13–15	Explore and Apply the Product Rule and the Chain Rule	• verify the chain rule and product rule • Solve problems involving the Product Rule and Chain Rule and develop algebraic facility where appropriate	A3.4 A3.5
16–17	Connections to Rational and Radical Functions	• Use the Product Rule and Chain Rule to determine derivatives of rational and radical functions • Solve problems involving rates of change for rational and radical functions and develop algebraic facility where appropriate	A3.4 A3.5
18–19	Applications of Derivatives	• Pose and solve problems in context involving instantaneous rates of change	A3.5
20	Jazz Day	•	
21	Summative Assessment	• Added a day	

Math Learning Goals

- Identify increasing and decreasing rates of change using graphical and numerical representations of polynomial functions
- Investigate connections graphically and numerically between the graph of a polynomial function and the graph of its derivative using technology

75 min

Materials

- BLM 2.1.1, 2.1.2
- flashlight

Assessment Opportunities

Minds On…

Whole Class → Discussion

A teacher provide students with tables of values (numerical representation) of quadratic, linear, exponential functions and ask student to identify when the rate of change is positive and when it is negative and describe what that means in terms of the behaviour of the rate of change for each example. Teachers may wish to make connections to finite differences over constant intervals to je;[students assess the behaviour of the rate of change.

Teacher then uses BLM 2.1.1 to show how we can describe the rate of change of a function when given the graphical representation of the function.

BLM 2.1.1

Flashlight

Action!

Small Groups → Experiment

Teachers use the examples on BLM 2.1.2 to help students describe the behaviour of the rate of change of a function given its graphical representation.

Teachers may ask students to generate graphs of functions for other students in the group to analyze.

Learning Skills/Teamwork/Checkbric:

Mathematical Process Focus: Representing, Connecting

BLM 2.1.2

Consolidate Debrief

Whole Class → Discussion

Functions are described using the information about the sign of the rate of change and students are asked to generate examples of tables of values and graphs of functions that model the information provided. Eg., Generate an example of a table of values or a graph of a function that has a negative rate of change for $x < -3$, a rate of change of zero from $x = -3$ to $x = 3$ and a positive rate of change for $x > 3$.

Application

Home Activity or Further Classroom Consolidation

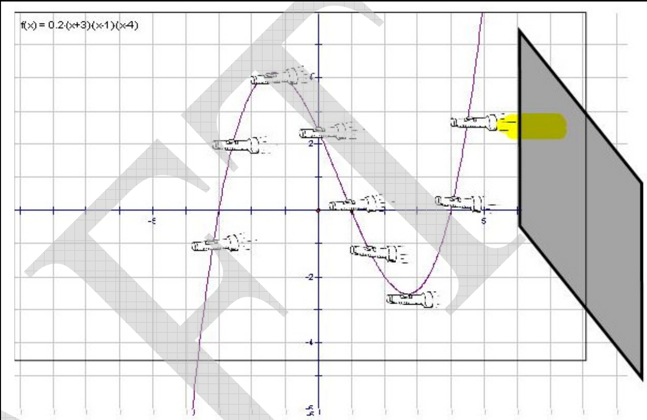
Students reinforce the skills developed in this lesson.

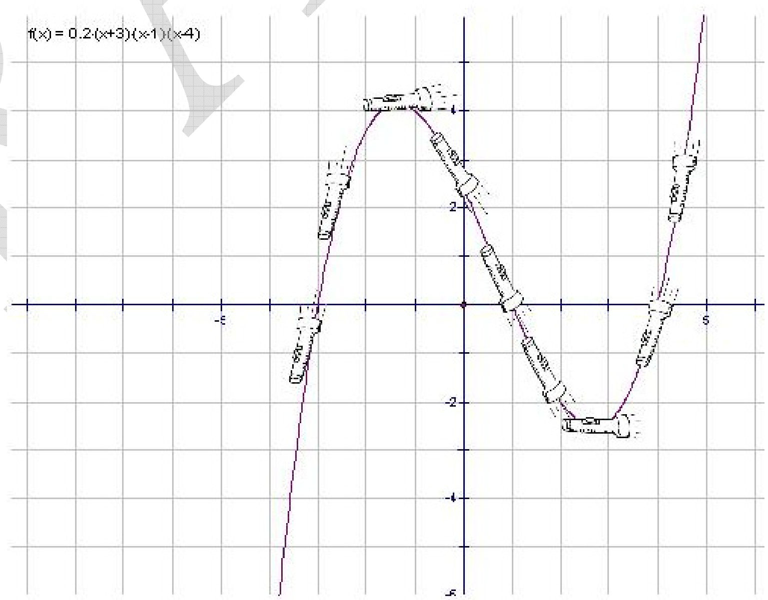
2.1.1: Illuminating Rates of Change (Teacher)

Use a flashlight in two different ways to demonstrate rates of change of a function over intervals between local maxima and minima.

Set-up

Display a graph of a smooth function on the board (IWB, board, white board, ...). As an image surface, there should be a wall or other flat surface perpendicular to the board.

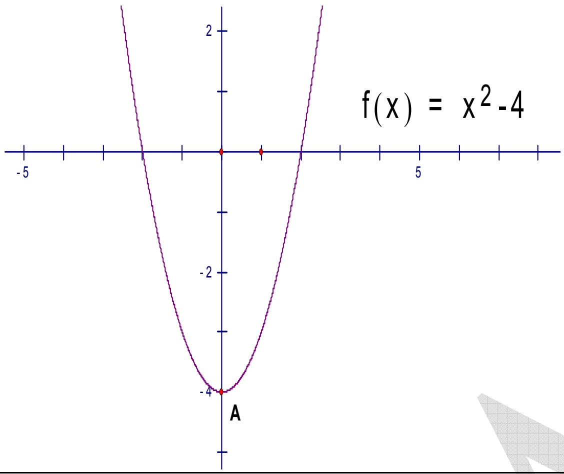
Part 1 Use a flashlight to trace the curve at a constant speed, keeping the flashlight horizontal against the board. Have students watch the beam of light as it strikes the image surface. The light will move quickly up or down the image surface when the rate of change is a high value and will move slowly when the rate of change is a low value. Demonstrate with different familiar functions as needed.	

Part 2 Instantaneous rate of change: Students have seen that the instantaneous rate of change is the slope of the tangent at a specific point. Using one of the functions from the first part of the activity and the flashlight, demonstrate the tangents at various points on the graph. The graph can again be traced using the flashlight (a ruler may be a better choice for some in order to easily find the slope of the tangent from the graph) This time the light must remain tangent to the graph at all times. Complete a table like the following table, to create a set of points that can be plotted *(x, y')* where *x* is still *x* and *y'* is now the slope of the tangent.	

2.1.2: Recognizing Changes in the Instantaneous Rates of Change

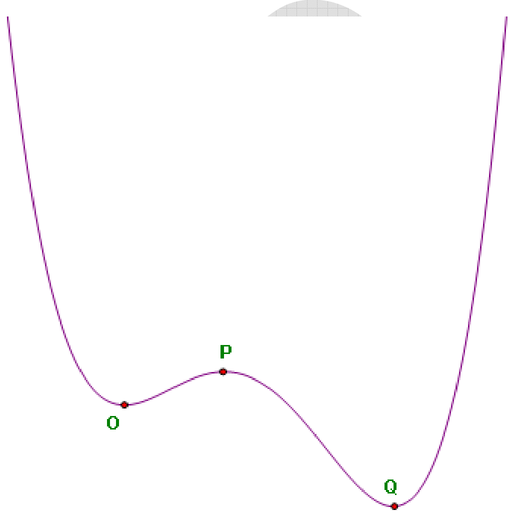
For each graph, divide the domain into intervals using local minima and maxima and determine the intervals that where the instantaneous rate of change is negative or positive. The first one is done for you.

a)



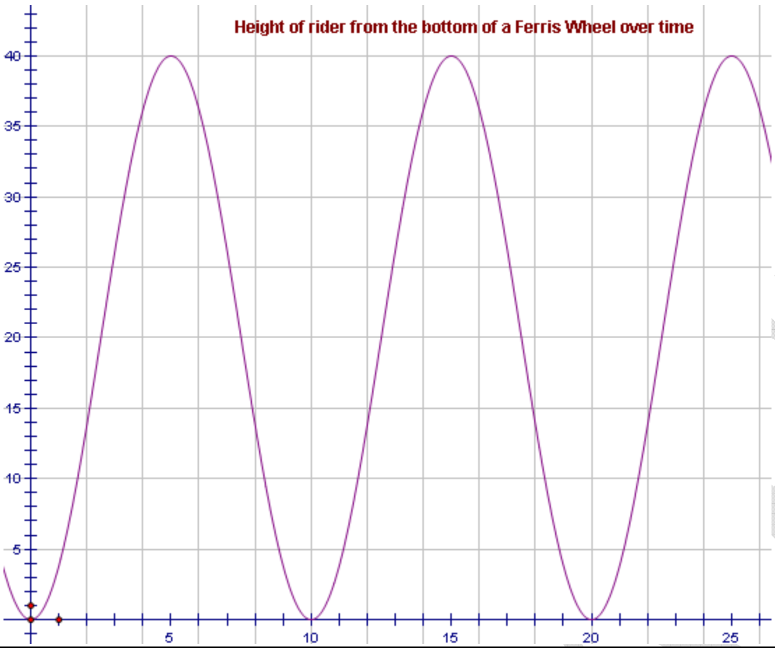
Domain	Sign of Instantaneous Rate of Change
< A	Negative
At A	Zero
> A	positive

b)



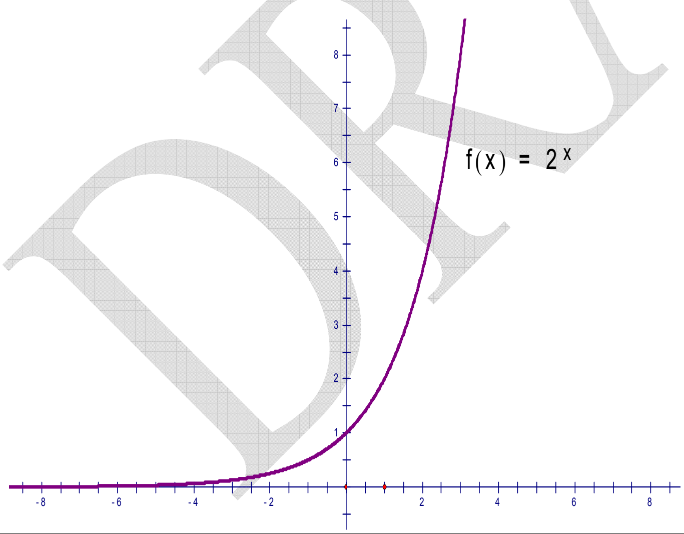
Domain	Sign of Instantaneous Rate of Change
< O	
At O	
Between O and P	
At P	
Between P and Q	
At Q	
Q >	

b)



Domain	Sign of Instantaneous Rate of Change

d)



Domain	Sign of Instantaneous Rate of Change

Unit 2: Day 2: The Instantaneous Rate of Change Functions — Grade 12

75 min

Math Learning Goals

- Generate, through investigation using technology, a table of values showing the instantaneous rate of change of a polynomial function, $f(x)$, for various values of x, graph the ordered pairs, recognize that the graph represents the instantaneous rate of change function called the derivative function and represented using $f'(x)$
- Make connections between the graphs and equations of $f(x)$ and $f'(x)$

Materials

- BLM 2.2.1, 2.2.2, 2.2.3
- Appendix 2.2.4
- graphing technology

Assessment Opportunities

Minds On…

Pairs → Exploration

Description of activity: Students describe the behaviour of the average rate of change for the function $f(x) = x^2$. They use first and second differences to investigate how the average rate of change for the function $f(x) = x^2$ changes and conjecture what type of function could model this behaviour.

Assessment opportunity for learning skills available when students are working in their groups.

GSP® files:
Group One: ax^2
Group Two: ax^3
Group Three: x^2+ax
Group Four: $x^3 + ax^2$

Action!

Whole Class → Teacher Directed Investigation

This activity will help students describe algebraically, the behaviour of the instantaneous rate of change for the function $f(x) = x^2$ Slopes of tangents are used to investigate how the instantaneous rate of change for the function $f(x) = x^2$ changes and how to model this behaviour algebraically.

Small Groups → Jigsaw → Investigation

In this activity, expert groups use patterning to investigate the relationship between the equations of a group of similar polynomial functions and the equations of their instantaneous rate of change function. The instantaneous rate of change function is described as the ***derivative function*** or simply the ***derivative*** and is represented using the notation $f'(x)$.

Name of Mathematical Process of Lesson Focus: Connecting, Selecting Tools and Strategies

Consolidate Debrief

Small Groups → Jigsaw

Students regroup to allow share their expertise with students from other groups. They work together to conjecture what the graphical and algebraic connections are between a polynomial function and its derivative.

Whole Class → Journal: Representing

Teacher consolidates the observations and provides some examples for students to work on.

Application
Skill Drill

Home Activity or Further Classroom Consolidation

Students work on exercises that allow for more practice and applications that reinforce the meaning of the derivative.

2.2.1: Behaviour of Average Rate of Change for $f(x) = x^2$

This activity will help you describe the behaviour of the average rate of change for the function $f(x) = x^2$. You will use first and second differences to investigate how the average rate of change for the function $f(x) = x^2$ changes.

Complete the table below for $f(x) = x^2$

x	$f(x)$	First Differences	Second Differences
–4			
–3			
–2			
–1			
0			
1			
2			
3			
4			

What connection can you make between the first difference when $x = 1$ and $x = 2$, the secant with endpoints (1, 1) and (2, 4) and the average rate of change of $f(x) = x^2$ between $x = 1$ and $x = 2$?

For the function $f(x) = x^2$, what do the second differences suggest about how the average rate of change varies?

What type of function (e.g., linear, quadratic, exponential, trigonometric) best models the changes in the average rate of change?

2.2.2: The Instantaneous Rate of Change Function for $f(x) = x^2$

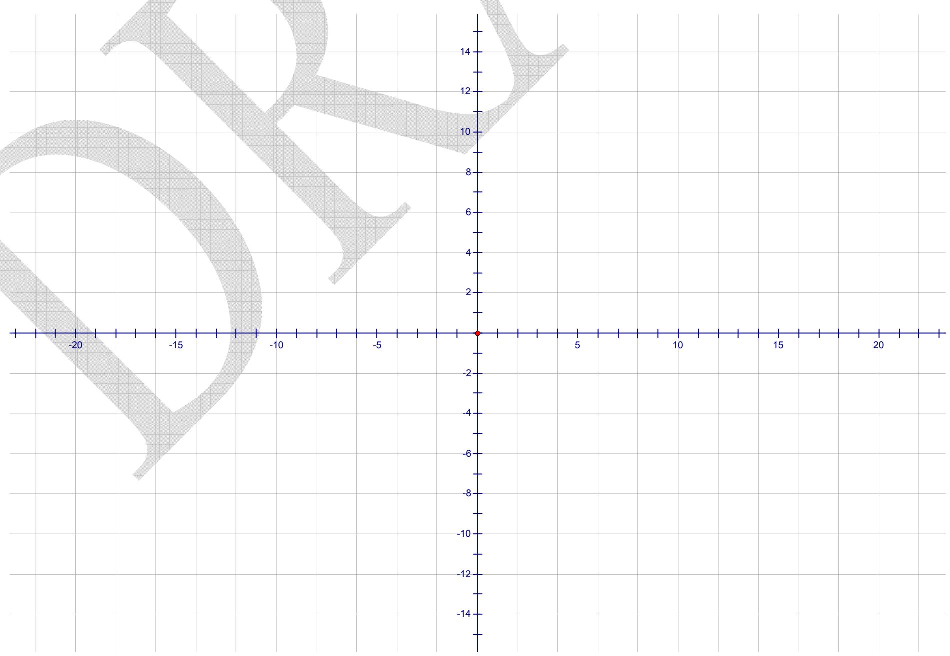
This activity will help you describe the behaviour of the instantaneous rate of change for the function $f(x) = x^2$ algebraically. You will use the slopes of tangents to investigate how the instantaneous rate of change for the function $f(x) = x^2$ changes and how to describe that relationship algebraically using a function.

Use graphing technology (e.g., graphing calculators, Geometer's Sketchpad, Winplot) to graph the function $f(x) = x^2$. Create a movable tangent to the function and show the slope of the tangent for any point on the function (see 2.2.3: Appendix or use GSP sketch ax^2 and select tab one).

Determining the equation of the instantaneous rate of change function: Move the tangent to the function $f(x) = x^2$ over to the point where $x = –4$. Use the slope of the tangent to record the instantaneous rate of change of the function when $x = –4$. Repeat for the other values of x in the table. To distinguish the function $f(x) = x^2$ from the instantaneous rate of change function, the symbol we will use to describe the instantaneous rate of change function will be $f'(x)$.

Graph the rate of change function (plot the points $(x, f'(x))$) on the graph below and determine the algebraic equation of the instantaneous rate of change function.

Function: $f(x) = x^2$		**Instantaneous Rate of Change Function**	
x	$f(x)$	x	$f'(x)$
–4	16	–4	
–3	9	–3	
–2	4	–2	
–1	1	–1	
0	0	0	
1	1	1	
2	2	2	
3	4	3	
4	16	4	



Algebraic Equation of Rate of Change Function: $f'(x) =$

2.2.3: The Instantaneous Rate of Change Function for Simple Polynomials

In this activity, patterning will be used to investigate the relationship between the equation of a function and the equation of its instantaneous rate of change function. The instantaneous rate of change function is called the ***derivative function*** or simply the ***derivative*** and is represented using the notation $f'(x)$.

You will be assigned one of the following groups of functions to work with.

Group One: $f(x) = x^2$, $f(x) = 2x^2$, $f(x) = 3x^2$, $f(x) = 4x^2$

Group Two: $f(x) = x^3$, $f(x) = 2x^3$, $f(x) = 3x^3$, $f(x) = 4x^3$

Group Three: $f(x) = x^2 + x$, $f(x) = x^2 + 2x$, $f(x) = x^2 + 3x$, $f(x) = x^2 + 4x$

Group Four: $f(x) = x^3 + x^2$, $f(x) = x^3 + 2x^2$, $f(x) = x^3 + 3x^2$, $f(x) = x^3 + 4x^2$

Use graphing technology (e.g., graphing calculators, Geometer's Sketchpad®, Winplot) to graph each function in the group. Create a movable tangent for each function and use the slope of the tangent to the function to determine the instantaneous rate of change of the function at that point. Record your data in the tables provided and determine the equation of the derivative function:

$f(x) =$		**Derivative Function**	
x	$f(x)$	x	$f'(x)$
–4		–4	
–3		–3	
–2		–2	
–1		–1	
0		0	
1		1	
2		2	
3		3	
4		4	

Use the numerical values of the derivative to determine the equation of the derivative function

$f(x) =$		**Derivative Function**	
x	$f(x)$	x	$f'(x)$
–4		–4	
–3		–3	
–2		–2	
–1		–1	
0		0	
1		1	
2		2	
3		3	
4		4	

Use the numerical values of the derivative to determine the equation of the derivative function.

2.2.3: The Instantaneous Rate of Change Function for Simple Polynomials

$f(x) =$		Derivative Function	
x	$f(x)$	x	$f'(x)$
–4		–4	
–3		–3	
–2		–2	
–1		–1	
0		0	
1		1	
2		2	
3		3	
4		4	

Use the numerical values of the derivative to determine the equation of the derivative function

$f(x) =$		Derivative Function	
x	$f(x)$	x	$f'(x)$
–4		–4	
–3		–3	
–2		–2	
–1		–1	
0		0	
1		1	
2		2	
3		3	
4		4	

Use the numerical values of the derivative to determine the equation of the derivative function.

Summary:

Record the equations from this investigation in the summary table below:

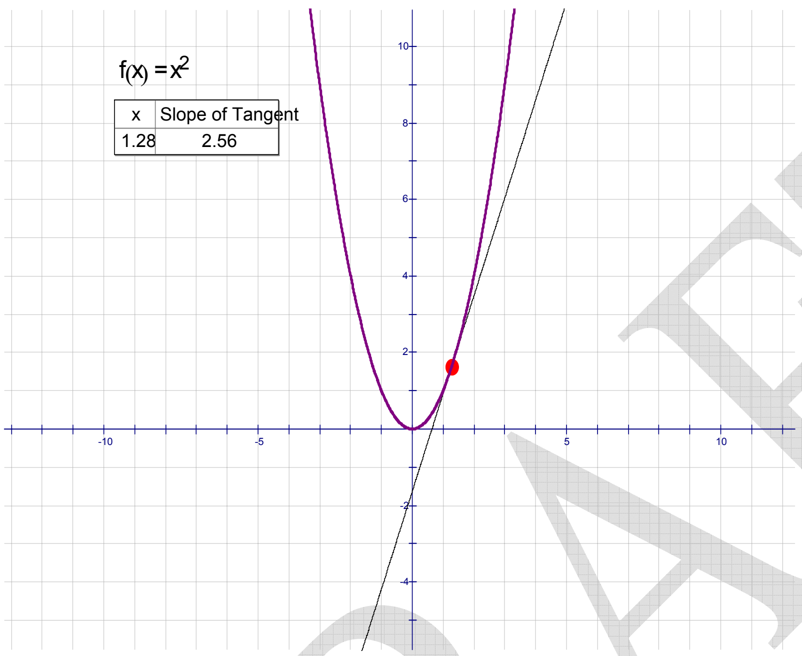
Function	Derivative Function

What relationship can you observe between the graphs of the polynomial functions in your group and the graphs of their derivatives? Between the equations of the polynomial functions in your group and the equations of their derivatives? Bring your expertise to a group with representatives from each of the function groups. Share your results with the class.

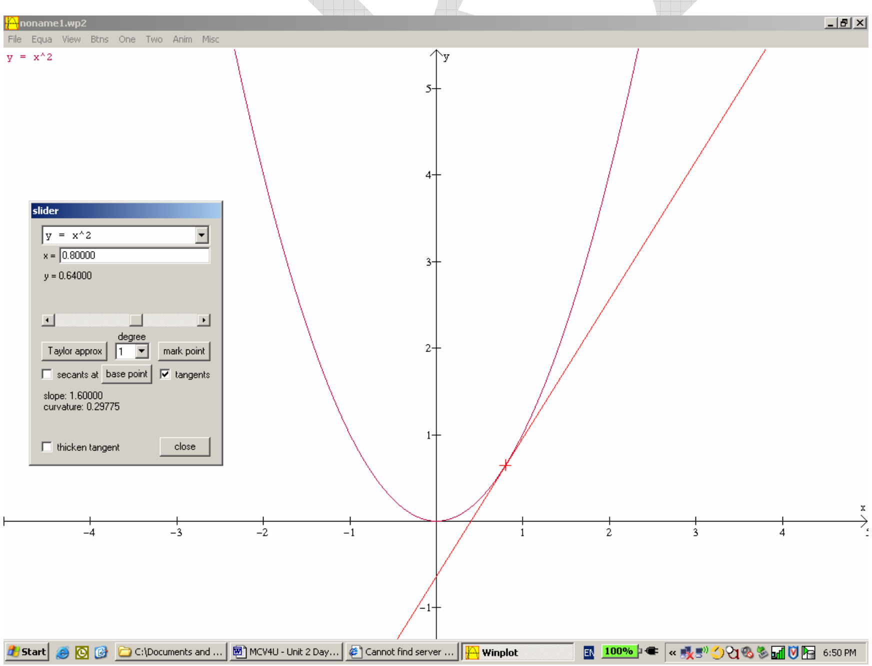
2.2.4: The Instantaneous Rate of Change Function for $f(x) = x^2$ (Teacher)

The directions below show how to use graphing technology Geometer's Sketchpad® and Winplot to graph the function $f(x) = x^2$, to create a movable tangent to the function and to show the slope of the tangent for any point on the function.

Geometer's Sketchpad® Version:



- Select **Graph** then **Show Grid**.
- Select **New Function** then enter x^2 and **OK**.
- Select **Graph** then **Plot Function**.
- Click on the graph then click on the graph of the function. Select **Construct** and **Point on Object**.
- Repeat the step above to create a second point on the function.
- Deselect all then click the two points on the graph then select **Construct** and **Line**.
- Select **Measure** and **Slope**.
- Move the two points together to approximate the tangent.
- Grab the tangent and move it along the graph.



- Select **Equa** and **1. Explicit** and enter x^2 for $f(x)$ and **OK**.
- Select **Equa** and **Inventory** and **Equa**.
- Select **One** then **Slider**.
- A pop-up box appears. Check off the **Tangents** box. Move the pop-up box to the side so it does not cover your graph.
- Slide the slider and watch the tangent move along the function.
- Note the slope of the tangent in the lower half of the pop up box.