TIPS4RM

Leading MATH Success

TIPS4RM Targeted Implementation and Planning Supports for Revised Mathematics

Calculus and Vectors (MCV4U)
Final Evaluation

Introduction

A course performance task and a bank of questions for a written examination are provided to support teachers in assessing the overall expectations of this course through the lens of the mathematical processes. They allow for a balance of demonstration of the four categories of the Achievement Chart. As well, the specific expectations are identified for both the task and questions. The questions focus on the Thinking and Communication categories and are not intended to be exhaustive. Teachers should use their professional judgment to determine which questions to select or what additional questions they should create, noting that not all specific expectations must be evaluated and that expectations addressed in the performance task do not need to be duplicated on a written examination.

Students should have had multiple opportunities to participate in similar instructional and assessment tasks/questions and to be assessed using rubrics based on the mathematical processes.

In creating the performance task for this course, the following criteria were considered:

- Focus on the big ideas:
  - connecting rate of change to applications;
  - solving problems involving applications of derivatives, e.g., optimization;
  - representing 2D:3D vectors, lines, and planes.
- Focus on critical thinking.
- Explicitly reference the mathematical processes.
- Address several overall expectations and connections between and among them.
- Choose a context relevant to the student, e.g., environment, social justice.
- Provide students multiple entry points for solving the problem.
- Allow for four levels of performance.
- Assess with a rubric which clearly articulates distinguishing features.
- Involve modelling, synthesizing, analysing and interpreting data.
- Allow access to a variety of resources and tools, e.g., open book, formula sheets, group brainstorm, technology.
- Engage students in actively doing and using mathematics.
- Include anticipation opportunities, e.g., prior reading, research, brainstorming.
- Ensure equity and balance – geographic, cultural, learning styles, and resources.
- Include an opportunity to reflect on the process they used.
- Include choice.

The task should be a SMART problem:

- Specific
- Measurable
- Attainable
- Realistic
- Timely

Calculus and Vectors (MCV4U)
Performance Task

TIPS4RM Targeted Implementation and Planning Supports for Revised Mathematics

MCV4U Performance Task

Grade 12

Lesson Outline

Big Picture

Students will be evaluated on:

- recognizing the relevance of calculus when applied to real-world problems;
- solving a problem in a real-world context by applying derivatives and properties of functions;
- making appropriate recommendations from generated solutions;
- reflecting on their thought processes and proposing future modifications.

Day	Lesson Title	Math Learning Goals	Expectations
1	Getting Ready for the Summative Performance Task	• Analyse the use of numeracy in media to support arguments. • Recognize the significance of minimal changes in a real context. • Identify calculus techniques of relevance to a real-world situation.	A1.1, 1.3 CGE 3c
2	MSeaV 4You Environmental Design *TI-Nspire File:* **Bottle Design.tns**	• Solve optimization problems involving surface area in a real context by applying derivative techniques. • Verify an algebraic solution by using a numeric and/or graphical representation. • Make appropriate recommendations after discussing and reflecting on generated solutions. • Connect algebraic, numeric, and graphical representations.	A3.3, 3.5 B1.4, 2.2, 2.3, 2.4, 2.5 CGE 2c
3	MSeaV 4You Economic Design *TI-Nspire File:* **Bottle Cost Design.tns**	• Solve optimization problems involving economic and environmental factors by applying derivative techniques. • Verify an algebraic solution by using a numeric and/or graphical representation. • Make appropriate recommendations after discussing and reflecting on generated solutions. • Compare and contrast the two scenarios considering additional factors to make and justify an informed decision for the best design • Reflect on their solution and propose modifications based on their reflections.	A3.3, 3.5 B1.4, 2.2, 2.3, 2.4, 2.5 CGE 4f

Note: Days 2 and 3 should not be consecutive days to allow teachers to provide meaningful and appropriate feedback so that students experience success on Day 3.

Unit 7: Day 1: Getting Ready for the Summative Performance Task

Grade 12



75 min

Math Learning Goals

- Analyse the use of numeracy in media to support arguments.
- Recognize the significance of minimal changes in a real context.
- Identify calculus techniques of relevance to a real world situation.

Materials

- highlighter
- large envelopes or folders

Assessment Opportunities

Minds On…

Whole Class → Discussion

Pose and solve the Fermi problems: How many bottles of water are consumed by our class, our school, our town in a week? How many water bottles in a week end up in your landfill site in a year?

Action!

Individual → Reading and Responding

Review protocols and considerations for accessing information on the Internet, e.g., source, perspective. Students record information that they find interesting regarding the ecological impact that water bottle usage is having in our current world.

One source indicates that Americans send 30 million water bottles to landfills every day.

Ask:
If a recycled water bottle can be compacted to a volume of $8cm^3$ and that the average classroom measures 12.5 m by 3 m, how many 'classrooms' of compacted water bottles would this represent annually for Americans?

What would be the impact if the bottle was not crushed?

Students complete the task.

Search and guide students to sources about the environmental impact or ask students to research this information in preparation for this task.

Show students what $8cm^3$ looks like by showing them one linking cube to help them visualize the overall impact.

Consolidate Debrief

Small Groups → Sharing

Students share their solutions. They discuss what the long-term and global impacts would be, justifying their response with information from their reading.

Whole Class → Sharing

Groups share their discoveries with the class.

Students place their work in their folders/envelopes.

Home Activity or Further Classroom Consolidation

Complete the practice questions.

Select appropriate practice questions involving calculating derivatives, solving for critical points, and using graphical and algebraic methods in preparation for the paper/pencil exam portion of Units 1-4.

7.1.1: Impact of Bottled Water on Our Ecosystem (Teacher)

If a recycled water bottle can be compacted to a volume of $8cm^3$ and that the average classroom measures 12.5 m by 12.5 m by 3 m, how many "classrooms" of compacted water bottles would this design represent annually for Americans?

Possible Solution

Saved Volume = Number of Bottles x Crushed Volume x number of days in a year

= 30 000 000 x 8 cm^2 x 365

= 80 760 000 000

Classroom Volume $= 12.5\,m \times 12.5\,m \times 3\,m$

$= 468.75\,m^3$

$= 468\,750\,000\,cm^3$

Number of Classrooms = Saved Volume ÷ Classroom Volume

= 80 760 000 000 ÷ 468 750 000

= 186.88

Unit 7: Day 2: MSeaV 4You Environmental Design

Grade 12



75 min

Math Learning Goals

- Solve optimization problems involving surface area in a real context by applying derivative techniques.
- Verify algebraic solution by using a numeric and/or graphical representation.
- Make appropriate recommendations after discussing and reflecting on generated solutions.
- Connect algebraic, numeric, and graphical representations.

Materials

- BLM 7.2.1–7.2.6
- graphing technology (optional)

Assessment Opportunities

Minds On…

Small Groups → Brainstorming

Ask: What concepts in this course would be relevant and could contribute to making a more eco-friendly world?
(*Possible Responses*: Product designs that minimize surface area. Studying rates of change involving deterioration or accumulation of waste and pollutants.)

BLMs 7.2.3–7.2.6 provide graphing, numeric and CAS solutions.

Whole Class → Introduction of Task

Redistribute envelopes or folders.

Introduce task (BLM 7.2.1) while making connections to the ideas raised through the brainstorming activity.

Bottle Design.tns demonstrates CAS solutions.

Reinforce that one of the methods to solve the problem is to be algebraic using calculus techniques and to verify the solution using either a numeric or graphical approach using technology. Identify that students should round off their calculations to 6 decimal places.

Review key information on the rubric (BLM 7.2.2), clarify expectations, and answer any questions.

Students who have been unsuccessful in creating an appropriate model could be provided with an algebraic model.

Pairs → Discussion

Assign partners Design A or Design B (BLM 7.2.1).

Partner A and Partner B discuss task and possible approaches to solve the problem.

Students do not write while discussing.

Action!

Individual → Problem Solving

Students solve problem outlined by Design A or B on BLM 7.2.1.

Students submit their solution in their folder/envelope and submit to the teacher. They can work quietly on reviewing materials from Units 1–4 in preparation for the paper/pencil exam.

Curriculum Expectations/Rubric: Evaluate students' performance on the expectations identified in the Unit Outline.

Consolidate Debrief

Pairs → Whole Class → Discussion

Students compare their results with their partner, then discuss as a class which design they would recommend, based on minimum surface area as well as other considerations, such as shape.

Home Activity or Further Classroom Consolidation

Review materials from Units 1–4 for paper/pencil exam.

Research the impact of using water bottles (optional).

Select appropriate practice questions involving calculating derivatives, solving for critical points, and using graphical and algebraic methods in preparation for the paper/pencil exam portion of Units 1-4.

Practice Application

7.2.1: Enviro Design A: Cone-topped Cylinder

Instructions

MSeaV has hired you to design a new prototype for an environmentally friendly plastic water bottle. The water bottle is to have a volume of 750cm^3. You are to consider a cylindrical model that is topped by a cone. The design would have a snip off top so no cap would be needed.



For ergonomic purposes, the radius should be at least 3 cm and at most 6 cm and the height of the conical piece should be equivalent to its radius. Determine the dimensions and resulting surface area for a cone topped cylindrical plastic water bottle that minimizes the amount of plastic used.

Provide a solution to this problem that includes:

- evidence of a plan to solve an optimization problem that includes an algebraic solution using calculus techniques (differentiation, solving for extreme values, testing interval endpoints, etc.);
- at least one other method to verify your result (graphical or numeric);
- supporting sketches;
- a reflection on the reasonableness of your answer with justification.

7.2.1: Enviro Design B: Hemispherical-topped Cylinder

Instructions

MSeaV has hired you to design a new prototype for an environmentally friendly plastic water bottle. The water bottle is to have a volume of 750 cm^3. You are to consider a cylindrical model that is topped by a hemisphere. The design would have a puncture hole so no cap would be needed.



For ergonomic purposes the radius should be at least 3 cm and at most 6 cm. Determine the dimensions and resulting surface area for a hemispherical-topped, cylindrical, plastic water bottle that minimizes the amount of plastic used.

Provide a solution to this problem that includes:

- evidence of a plan to solve an optimization problem that includes an algebraic solution using calculus techniques (differentiation, solving for extreme values, testing interval endpoints, etc.);
- at least one other method to verify your result (graphical or numeric);
- supporting sketches;
- a reflection on the reasonableness of your answer with justification.

7.2.2: MSeaV Packaging Design Evaluation Rubric

Thinking					
Problem Solving					
Criteria	**Below Level 1 Specific Feedback**	**Level 1**	**Level 2**	**Level 3**	**Level 4**
Selects, sequences and applies appropriate mathematical processes to solve the problem.		- selects, sequences and applies mathematical processes to the assigned problem with significant prompting	- selects, sequences and applies mathematical processes to the assigned problem with minimal prompting	- selects, sequences and applies mathematical processes to the assigned problem independently	- selects, sequences and applies mathematical processes to the assigned problem with a broader view of the problem
Uses critical thinking skills to solve the problem algebraically and uses another model to verify the solution.		- uses minimal logic and precision in mathematical reasoning between the two approaches	- uses logic in both approaches but lacks precision in mathematical reasoning	- both approaches are logical and precise in mathematical reasoning	- demonstrates a sophisticated level of mathematical reasoning and precision in both approaches
Reasoning and Proving					
Makes and supports an appropriate design recommendation, based on calculations.		- makes limited connection to the problem solving process and models presented when justifying the design recommendation	- makes some connections to the problem solving process and models presented when justifying the design recommendation	- makes direct connections to the problem solving process and models presented when justifying the design recommendation	- makes direct and insightful connection to the problem solving process and models presented when justifying the design recommendation
Reflecting					
Uses metacognitive skills to determine which mathematical processes to revisit in order to reach the goal.		- applies metacognitive skills with significant prompting in determining which mathematical process to revisit in order to reach the goal	- applies metacognitive skills with minimal prompting in determining which mathematical process to revisit in order to reach the goal	- applies metacognitive skills independently in determining which mathematical process to revisit in order to reach the goal	- applies metacognitive skills independently in determining which mathematical process to revisit in order to reach the goal with a broader view of the goal
Reflects on the reasonableness of answers and the selection of scales in graphical models.		- makes inadequate or inappropriate justifications for reasonableness of answers - selects inappropriate scales for graphical models, but adequate for the function	- makes some appropriate justifications for reasonableness of answers - selects scales for some graphical models that show an appropriate range	- makes good justification for reasonableness of answers - selects scales for most graphical models that show an appropriate range	- makes thorough justification for reasonableness of answers - selects scales for all graphical models that show an appropriate range

7.2.2: MSeaV Packaging Design Evaluation Rubric (continued)

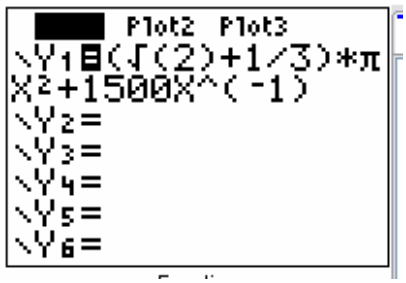
Communication					
Representing					
Criteria	**Below Level 1 Specific Feedback**	**Level 1**	**Level 2**	**Level 3**	**Level 4**
Creates an algebraic model to represent and solve the two problems.		- creates algebraic models that represent both problems with limited effectiveness	- creates algebraic models that represent both problems with some effectiveness	- creates algebraic models that represent both problems with considerable effectiveness	- creates algebraic models that represent both problems with a high degree of effectiveness
Makes connections between numeric, graphical, and algebraic representations.		- makes limited connections between representations	- makes some connections between any two representations	- makes appropriate connections between the algebraic representations and either the graphical or numeric representations	- makes strong and insightful connections between the algebraic representations and either the graphical or numeric representations
Communicating					
Uses clear language to make presentations and to explain and justify solutions when reporting for various purposes and different audiences.		- uses unclear language to explain and justify solutions for different audiences	- uses language that is somewhat unclear to explain and justify solutions for different audiences	- uses clear language to explain and justify solutions for different audiences	- uses clear and precise language to explain and justify solutions for different audiences
Uses of mathematical symbols, labels, units, and conventions correctly.		- sometimes uses mathematical symbols, labels, and conventions correctly	- usually uses mathematical symbols, labels and conventions correctly	- consistently uses mathematical symbols, labels and conventions correctly	- consistently uses mathematical symbols, labels and conventions, presenting novel or insightful opportunities for their use
Uses mathematical vocabulary appropriately.		- uses common language in place of mathematical vocabulary or uses key mathematical terms with major errors	- uses mathematical vocabulary with minimal errors or uses some common language in place of vocabulary	- uses mathematical vocabulary appropriately	- consistently uses mathematical vocabulary appropriately, presenting novel or insightful opportunities for its use

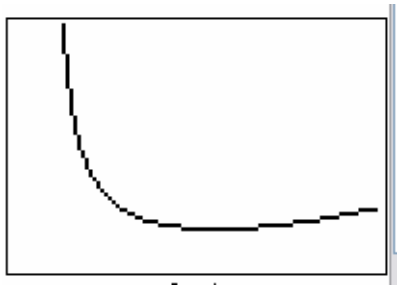
7.2.3: Bottle Design Problem: Graphing Calculator Solution

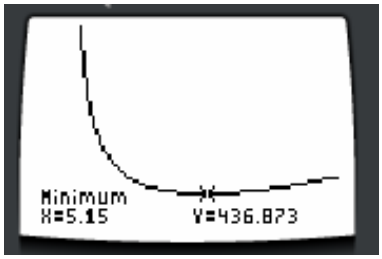
Window settings for both shapes is:



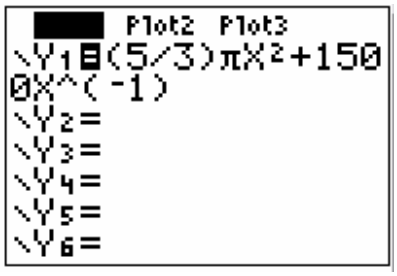
Cone-topped Cylinder

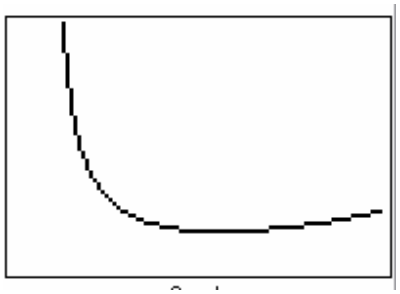


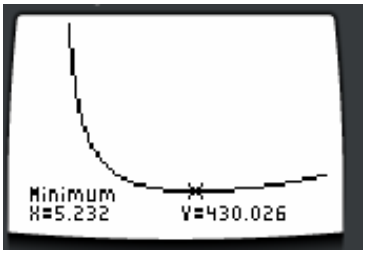




Hemispherical-topped Cylinder







7.2.4: Bottle Design Problem: Numeric Solution

Cone-topped Cylinder

- Identify what the lowest surface area is for the radius as a whole number 3 through 6, based on constraints stated in the problem.

Radius (cm)	**Surface Area** (cm^2)
3	549.41
4	462.84
5	437.25
6	447.64

- Identify what the lowest surface area is for the radius decreasing by tenths from 5 and increasing by tenths from 5 (values shown until a lowest value between two points is evident).

Radius (cm)	**Surface Area** (cm^2)
4.8	438.99
4.9	437.94
5	437.25
5.1	436.91
5.2	436.91
5.3	437.24

- Since the lowest surface area is apparent for a radius of 5.1 and 5.2 which are equivalent, thus the table identifies the lowest surface area around 5.15.

Radius (cm)	**Surface Area** (cm^2)
5.13	436.88
5.14	436.87
5.15	436.87
5.16	436.87
5.17	436.88

Since the lowest surface area is 436.87 for three different radius values, then the average radius is what is considered to provide the lowest surface area. Thus, the optimal size of the cone-topped, cylinder bottle has a surface area of 436.87 cm for a radius of 5.15 cm.

7.2.4: Bottle Design Problem: Numeric Solution (continued)

Hemispherical-topped Cylinder

- Identify what the lowest surface area is for the radius as a whole number 3 through 6, based on constraints stated in the problem.

Radius (cm)	**Surface Area** (cm^2)
3	547.12
4	458.78
5	430.03
6	438.50

- Since the lowest surface area is apparent for a radius of 5 cm. Thus, the table identifies what the lowest surface area is for the radius decreasing by tenths from 5 and increasing by tenths from 5 (values shown until a lowest value between two points is evident)

Radius (cm)	**Surface Area** (cm^2)
4.8	433.14
4.9	431.84
5	430.90
5.1	430.31
5.2	430.04
5.3	430.10

- Since the lowest surface area is apparent for a radius of 5.2. Thus, the table identifies what the lowest surface area is for the radius decreasing and increasing by hundredths around 5.2.

Radius (cm)	**Surface Area** (cm^2)
5.18	430.07
5.19	430.05
5.2	430.04
5.21	430.03
5.22	430.03
5.23	430.03
5.24	430.03
5.25	430.03
5.26	430.04

- Since the lowest surface area is 430.03 for five different radius values, then the average radius is what will be considered to provide the lowest surface area. Thus, the optimal size of the hemispherical-topped cylinder bottle has a surface area of 430.03 cm for a radius of 5.23 cm.

7.2.5: Bottle Design Problem: WinPlot Solution

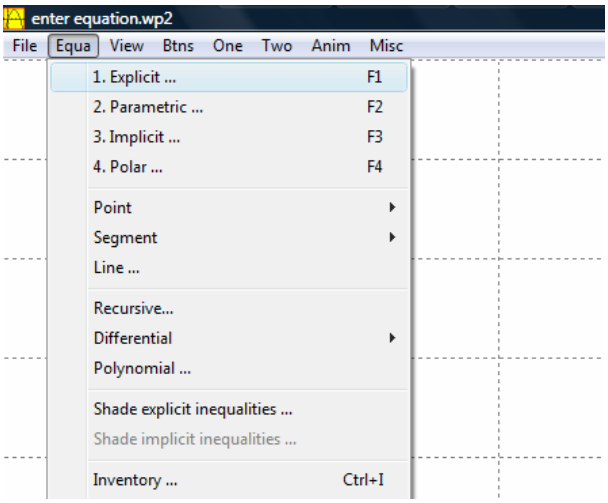
Students determine the dimensions of a water bottle with volume 750 ml that minimizes the total surface area. Students are assigned one of two bottle designs.



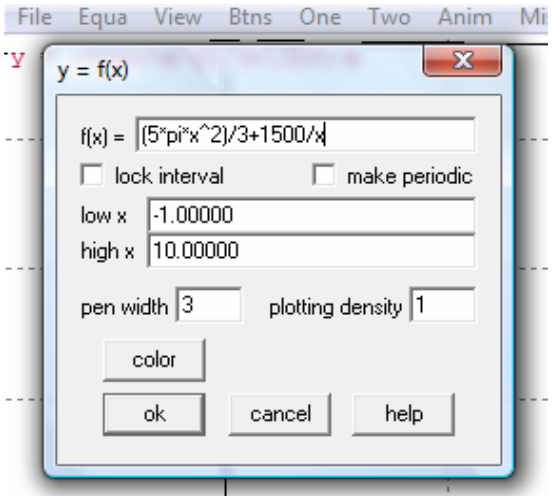
The solution provided, shows the development of the surface area function, a graphical solution for the optimum surface area, and a CAS solution, using calculus techniques.

The following table describes the key steps necessary in graphing the functions relating Surface Area and Radius for the two bottle options.

Each equation is entered by selecting **Explicit** under the **Equa** (Equation) menu. Note that the independent variable must be called x, not r.

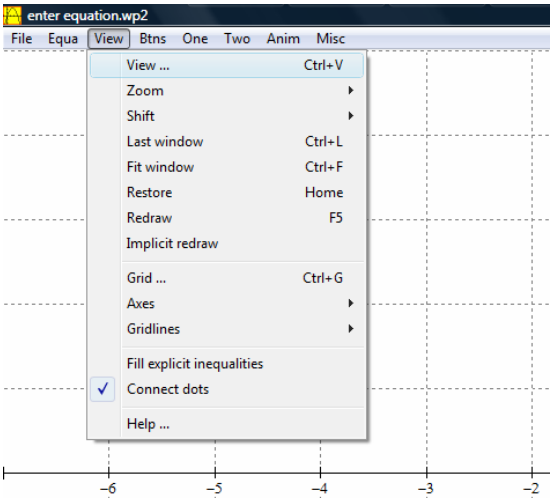


In the Explicit equation window, students may want to increase the pen width and choose appropriate colours for the graphs.



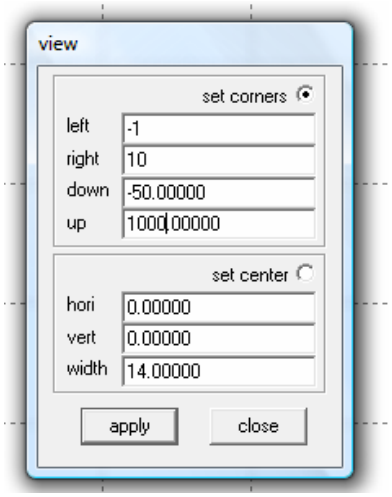
7.2.5 Bottle Design Problem: WinPlot Solution (continued)

To set an appropriate domain and range, select **View** under the **View** menu.

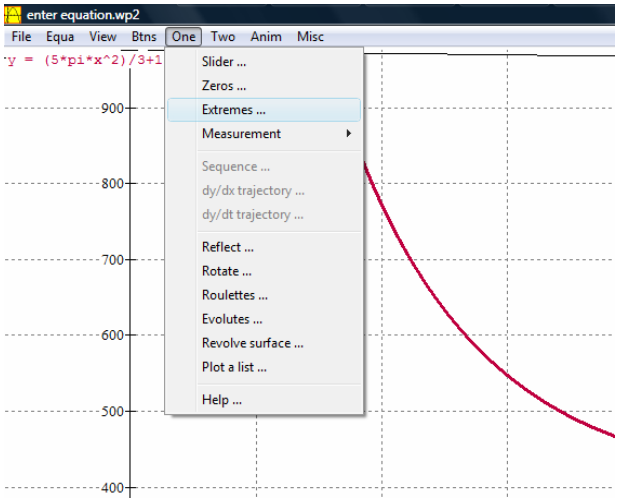


Select **set corners** and enter values for x (radius) and $f(x)$ (S.A.).

Students may need to experiment with values to view the necessary interval for each function.

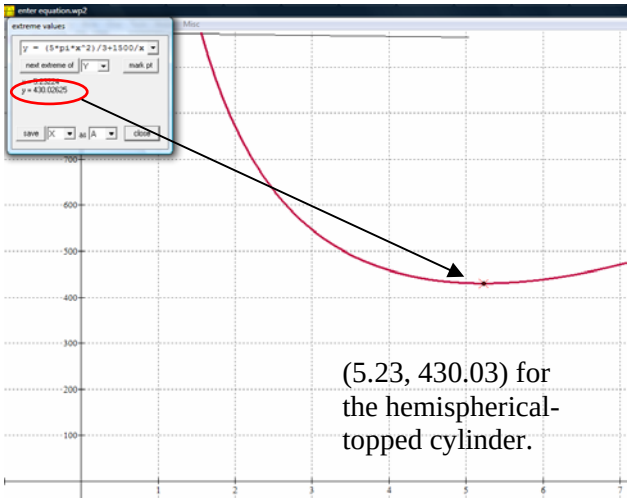


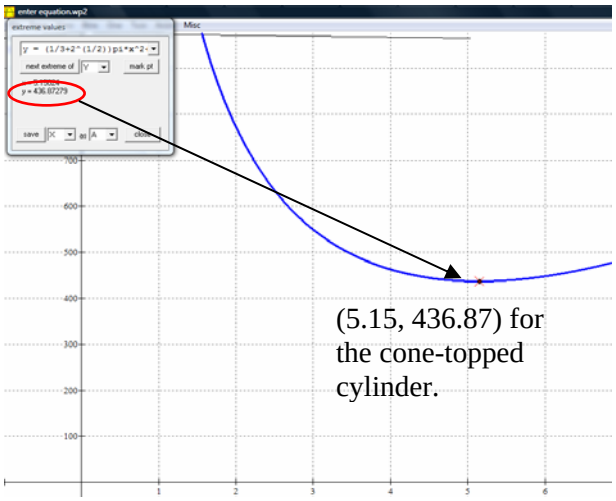
Determine the extreme values by selecting **Extremes** under the **One** menu.



7.2.5 Bottle Design Problem: WinPlot Solution (continued)

The Extreme Values menu states maximum/minimum points and allows students to mark the point on the graph.





The dimensions of the model of a hemispherical-topped cylinder with minimum surface area, has radius of 5.23 cm and overall height of 10.46 cm.

The dimensions of the model of a cone-topped cylinder with minimum surface area, has radius of 5.15 cm and overall height of 12.43 cm.

7.2.6: Bottle Design Problem: TI-Nspire CAS Solution

Students determine the dimensions of a water bottle with volume 750 ml that minimizes the total surface area. They are assigned one of two bottle designs.



The solution shows the development of the surface area function, a graphical solution for the optimum surface area, and a CAS solution, using calculus techniques.

To follow the solution, download the solution file onto a TI-Nspire handheld calculator.

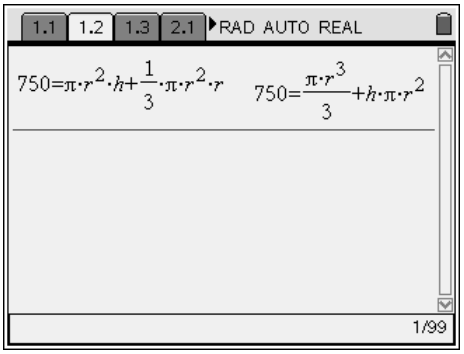
Turn on the TI-Nspire calculator.

From the **Home** screen choose **My Documents** and open the file: Bottle Design.tns

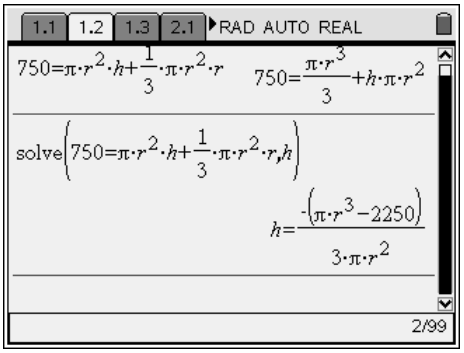


On page 1.2, the function for the surface area is developed. The first bottle model is a cone-topped cylinder. The cone has equal height and radius.

First, enter an equation for the volume with radius of the cone equal to its height.



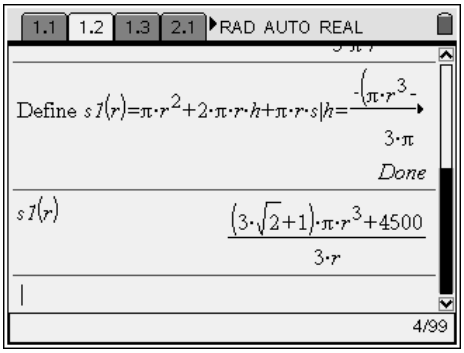
Solve for h.



7.2.6: Bottle Design Problem: TI-Nspire CAS Solution (continued)

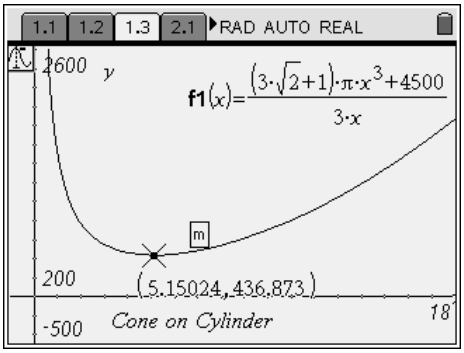
Substitute the expression for h and $s = \sqrt{2r}$ into the surface area function

Note: Since the radius and height of the cone are equal, the slant height $s = \sqrt{2r}$.



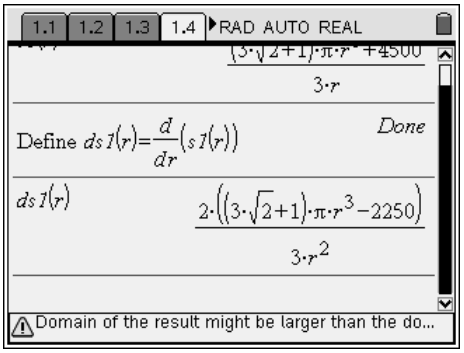
On page 1.3, determine the optimum radius that minimizes the surface area by tracing along the function graph. The graph shows the minimal value for the radius (5.15 cm) and the corresponding surface area (436.88).

Note: The small m denoted that a minimum value has been reached.



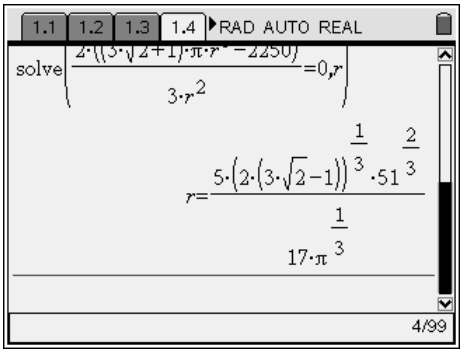
On page 1.4, use CAS to perform the calculus techniques to find the optimum radius that minimizes surface area.

Define the derivative of the surface area.



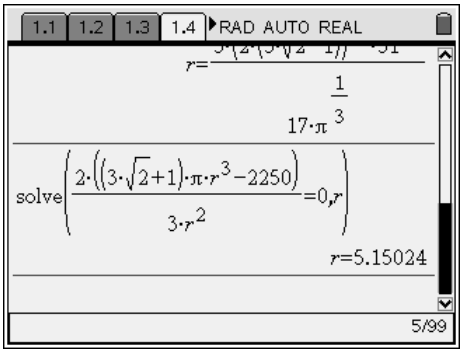
The optimum radius is the solution that makes the derivative equal to zero.

The exact radius is shown.

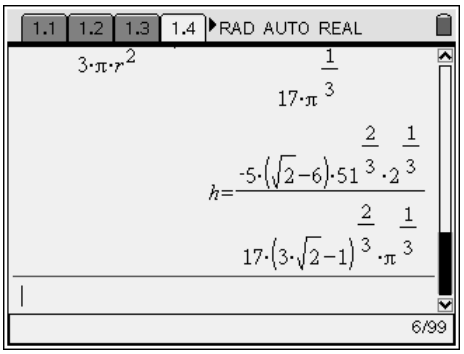


7.2.6: Bottle Design Problem: TI-Nspire CAS Solution (continued)

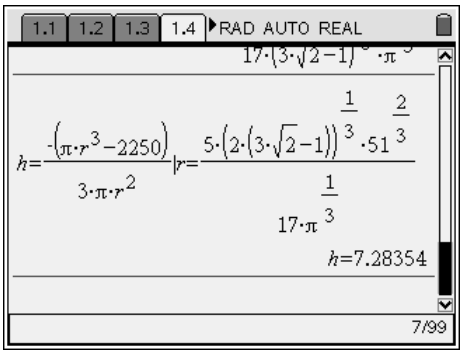
The numeric approximation is also shown.



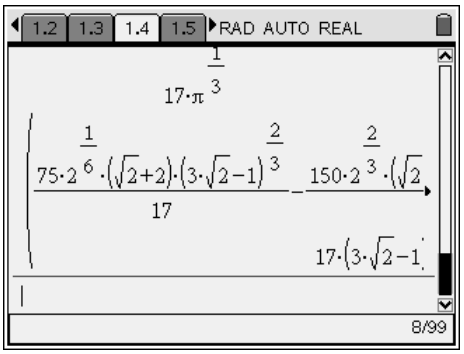
The exact height of the cylinder is calculated and shown.



The numeric approximation of the height is also shown.

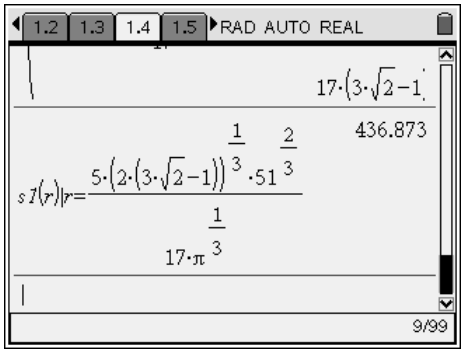


The final screen shot shows the calculation of the exact minimum surface area.

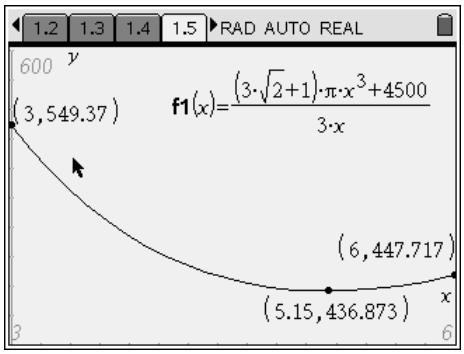


7.2.6: Bottle Design Problem: TI-Nspire CAS Solution (continued)

The numeric approximation is also shown.



On page 1.5, the graph confirms that this radius provides the minimum surface area by considering the surface areas at the end points of the given interval.

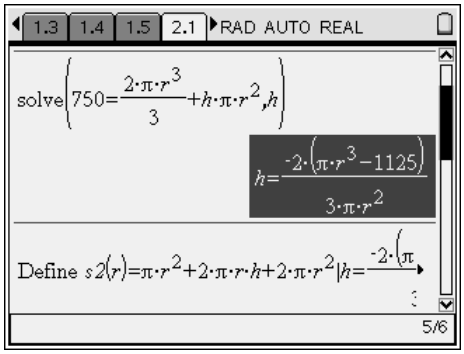


Thus, the dimensions of the bottle model of a cone-topped cylinder with minimum surface area, has radius of 5.15 cm and overall height of 12.43 cm.

The solution to the hemispherical-topped cylinder follows a similar approach in the next few screen shots. Only the highlights are shown here. For more detail refer to the TI-Nspire application Bottle Design.tns

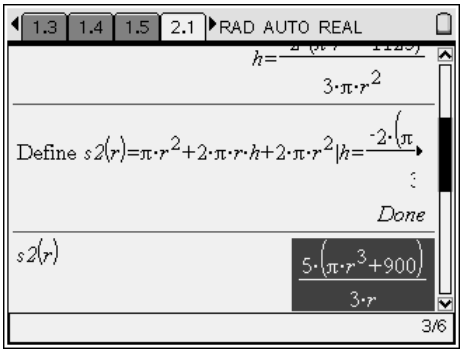


On page 2.1, enter an equation for the volume of the hemispherical-topped cylinder and solved for h.

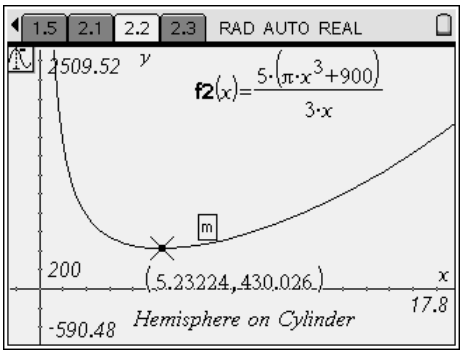


7.2.6: Bottle Design Problem: TI-Nspire CAS Solution (continued)

Substitute the expression for h into the surface area formula and find the surface area function.



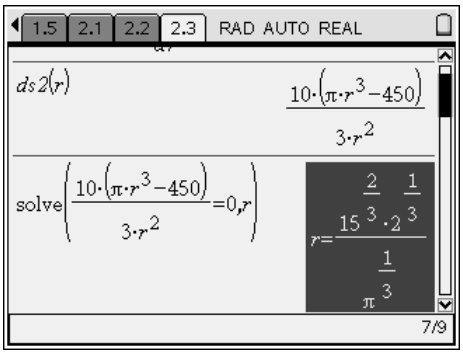
On page 2.2, determine the optimum radius that minimizes the surface area by tracing along the function graph. The graph shows the minimal value for the radius (5.23 cm) and the corresponding surface area (430.03).



On page 2.3, minimize the surface area by finding the derivative and solving for r.

The decimal equivalent of the optimum radius is 5.23 cm and the corresponding height is also 5.23 cm.

This produces the minimum surface area of 430.03 cm^2



Thus, the dimensions of the model of a hemispherical-topped cylinder with minimum surface area has radius of 5.23 cm and overall height of 10.46 cm.

Unit 7: Day 3: MSeaV 4You Economic Design

Grade 12



75 min

Math Learning Goals

- Solve optimization problems involving economic and environmental factors by applying derivative techniques.
- Verify an algebraic solution by using a numeric and/or graphical representation.
- Make appropriate recommendations after discussing and reflecting on generated solutions.
- Compare and contrast the two scenarios when considering additional factors to make and justify an informed decision for the best design.
- Reflect on their solution and propose modifications based on their reflections.

Materials

- BLM 7.3.1–7.3.6
- graphing technology (optional)
- folders/envelopes with students' work

Assessment Opportunities

Minds On…

Individual → Anticipation Guide

Students complete the "Before" column of the Anticipation Guide (BLM 7.3.1).

Students do not write while discussing.

Whole Class → Discussion

Discuss the first statement on the Anticipation Guide to ensure that students determined that the surface area of a cone-topped cylindrical bottle is less than the surface area of the hemispherical-topped cylindrical bottle with the same volume.

Encourage students to convert to thousandths of a cent for calculations.

Introduce the costs factors that the MSeaV 4You Company needs to consider in the design for the water bottle (BLM 7.3.2). Review key information on the rubric (BLM 7.2.2) distributed on Day 2 and answer any questions.

Pairs → Discussion

Assign partners to work on Design A or B.

Pairs Sharing should not exceed 5 minutes.

Partners discuss the task and possible approaches to solve the problem.

See **Bottle Cost Design.tns** for a TI-Nspire CAS solution.

Action!

Individual → Problem Solving

Students solve the problem outlined in Design A or B (BLM7.3.2).

BLM 7.3.4–7.3.6 provides graphing calculator, numeric, and TI-Nspire solutions.

They place their solution in their envelope and return it to the teacher.

Students complete the right side of the anticipation guide to activate their thinking for which design they will recommend: Environmental or Economic.

Curriculum Expectations/Rubric: Evaluate students' performance on the expectations identified in the Unit Outline.

Establish length and type of report you want students to submit.

Consolidate Debrief

Whole Class → Sharing

Partners share the results of their investigation followed by a whole class discussion. Discuss with students the expectations for their written report (BLM 7.3.3) and highlight key information on the rubric (BLM 7.2.2).

Students may need class time beyond Day 3 to complete the report.

Individual → Report Writing

Students write their report and submit all work in their folders/envelopes.

Home Activity or Further Classroom Consolidation

Practice

Complete the practice questions.

Select appropriate practice questions involving calculating derivatives, solving for critical points, and using graphical and algebraic methods in preparation for the paper/pencil exam portion of Units 1-4.

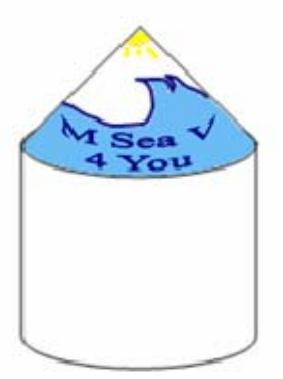
7.3.1: Anticipation Guide

Before		Statement	After		
Agree	Disagree		Always	Sometimes	Never
		The surface area of a cone-topped, cylindrical bottle is less than the surface area of the hemispherical-topped, cylindrical bottle with the same volume.			
		The cone-topped, cylindrical bottle will always be cheaper to produce than the hemispherical-topped, cylindrical bottle with the same volume.			
		Manufacturing costs are the largest portion of the costs involved in water bottle production.			

7.3.2: Economic Design A: Cone-topped Cylinder

Instructions

MSeaV is concerned about the cost elements associated with the manufacturing of an environmentally friendly plastic water bottle.



Design criteria for the cylindrical model topped by a cone are:

- volume of 750 cm^3;
- radius should be at least 3 cm and at most 6 cm;
- height of the conical piece is to be equivalent to its radius.

Associated costs are:

- cylindrical lateral surface 0.002¢/cm^2 (plastic);
- conical surface 0.004¢/cm^2 (plastic and graphic);
- circular base 0.003¢/cm^2 (thicker plastic).

Determine the dimensions, resulting cost, and resulting surface area for a cone-topped cylindrical, plastic water bottle that minimizes the overall **cost** to produce a bottle.

Provide a solution to this problem that includes:

- evidence of a plan to solve an optimization problem that includes an algebraic solution using calculus techniques (differentiation, solving for extreme values, testing interval endpoints, etc.);
- at least one other method to verify your result (graphical or numeric);
- supporting sketches;
- a reflection on the reasonableness of your answer with justification.

7.3.2: Economic Design B: Hemispherical-topped Cylinder

Instructions

MSeaV is concerned about the cost elements associated with the manufacturing of an environmentally friendly plastic water bottle.

Design criteria for the hemispherical topped cylinder are:

- volume of 750 cm^3;
- radius should be at least 3 cm and at most 6 cm.



Associated costs are:

- cylindrical lateral surface 0.002¢/cm^2 (plastic);
- hemispherical surface 0.005¢/cm^2 (plastic and graphic);
- circular base 0.003¢/cm^2 (thicker plastic).

Determine the dimensions, resulting cost, and resulting surface area for a hemispherical-topped, cylindrical, plastic water bottle that minimizes the overall **cost** to produce a bottle.

Provide a solution to this problem that includes:

- evidence of a plan to solve an optimization problem that includes an algebraic solution using calculus techniques (differentiation, solving for extreme values, testing interval endpoints, etc.);
- at least one other method to verify your result (graphical or numeric);
- supporting sketches;
- a reflection on the reasonableness of your answer with justification.

7.3.3: MSeaV Water Bottle Design Report

Instructions

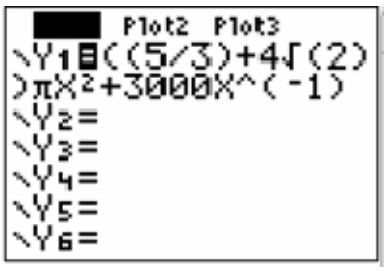
You are to complete a report that makes an overall recommendation including rationale as to which of the four water bottle prototypes MSeaV Design Department should manufacture.

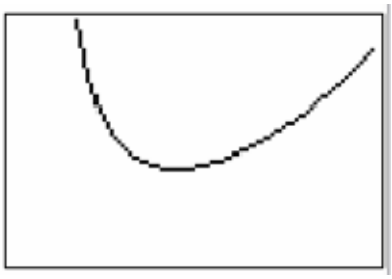
Include the following items in the report:

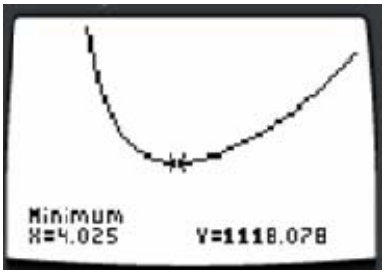
- a clear statement of your recommendation, rationale and assumptions with key algebraic expressions in simplified form, and calculations and neatly labelled illustrations and graphs;
- summary sheets;
- rough work.

7.3.4: Bottle Cost Problem: Graphing Calculator Solution

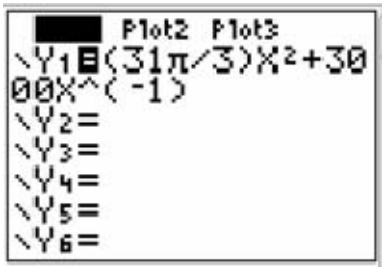
Cone-topped Cylinder

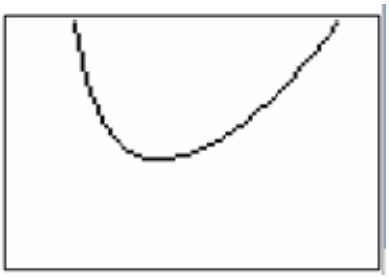


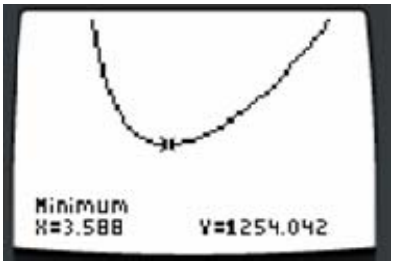




Hemispherical-topped Cylinder







7.3.5: Bottle Cost Problem: Numeric Solution

Cone-topped Cylinder

- Identify what the lowest cost is for the radius as a whole number 3 through 6 based on constraints stated in the problem.

Radius (cm)	**Cost** (thousandths of a cent)
3	1207.1 → 1.2071¢
4	1118.1
5	1175.2
6	1328.3

- Identify what the lowest cost is for the radius decreasing by tenths from 4 and increasing by tenths from 4 (values shown until a lowest value between two points is evident).

Radius (cm)	**Cost** (thousandths of a cent)
3.8	1121.7
3.9	1119.2
4	1118.1
4.1	1118.5
4.2	1120.1

- Since the lowest cost is apparent for a radius between 3.9 and 4.1, the next table identifies the lowest cost around 4 plus or minus hundredths.

Radius (cm)	**Cost** (thousandths of a cent)
3.99	1118.2
4	1118.1
4.01	1118.1
4.02	1118.1
4.03	1118.1
4.04	1118.1
4.05	1118.1
4.06	1118.2

Since the lowest cost is 1.1181¢ for six different radius values, then the average radius is considered to provide the lowest cost. The optimal cost of the cone-topped, cylinder bottle is 1.1181¢ for a radius of 4.025 cm.

7.3.5: Bottle Cost Problem: Numeric Solution (continued)

Hemispherical-topped Cylinder

- Identify what the lowest cost is for the radius as a whole number 3 through 6 based on constraints stated in the problem.

Radius (cm)	**Cost** (thousandths of a cent)
3	1292.2
4	1269.4
5	1411.6
6	1668.7

- Identify what the lowest cost is for the radius decreasing by tenths from 4 and increasing by tenths from 4 (values shown until a lowest value between two points is evident).

Radius (cm)	**Cost** (thousandths of a cent)
3.5	1254.8
3.6	1254.1
3.7	1255.2
3.8	1258.2
3.9	1263.0
4	1269.4
4.1	1277.4

- Since the lowest cost is apparent for a radius between 3.5 and 3.6, the next table identifies the lowest cost around 3.5 plus or minus hundredths.

Radius (cm)	**Cost** (thousandths of a cent)
3.57	1254.1
3.58	1254.0
3.59	1254.0
3.60	1254.1

Since the lowest cost is 1.254¢ for two different radius values, the average radius is considered to provide the lowest cost. The optimal cost of the hemispherical-topped cylinder bottle is 1.254¢ for a radius of 3.585 cm.

7.3.6: Bottle Cost Problem: TI-Nspire CAS Solution

Students determine the dimensions of a water bottle with volume 750 ml that minimizes the total production costs given the unit costs for the circular-based, lateral, cylindrical surface and the cone surface. Students are assigned one of two bottle designs.

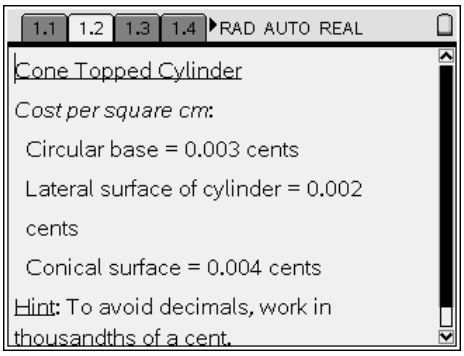


The solution shows the development of the cost function, a graphical solution for the optimum cost, and a CAS solution using calculus techniques.

Download the solution file onto a TI-Nspire calculator.

Turn on the TI-Nspire calculator.

From the Home screen choose **My Documents** and open the file: Bottle Cost Design.tns.

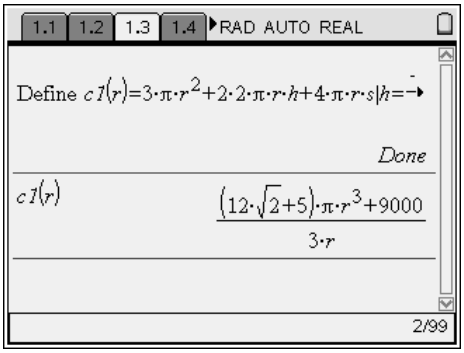


On page 1.2, review the unit costs for a cone-topped cylinder.

On page 1.3, develop the function for the production cost. The first bottle model is a cone-topped cylinder. The cone has equal height and radius.

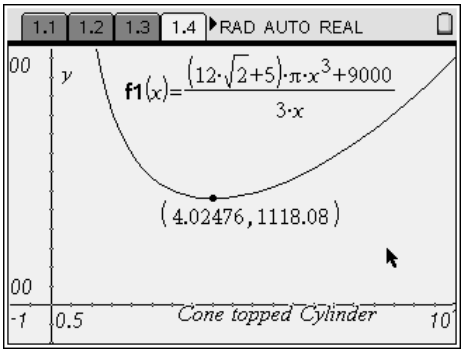
The development follows similar steps as before by substituting expressions for h and s and using the appropriate unit costs.

The cost function $c1(r)$ is displayed.



On page 1.4, determine the optimum radius that minimizes the production costs by tracing along the function graph. The graph shows the minimal value for the radius (4.02 cm) and the corresponding cost (1118.08).

Note: The production cost is measured in thousandths of a cent.



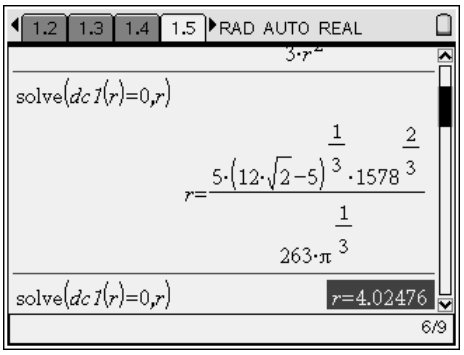
7.3.6: Bottle Cost Problem: TI-Nspire CAS Solution (continued)

On page 1.5, use CAS to perform the calculus techniques to find the optimum radius that minimizes production cost.

Define the derivative of the production cost function $c1(r)$.

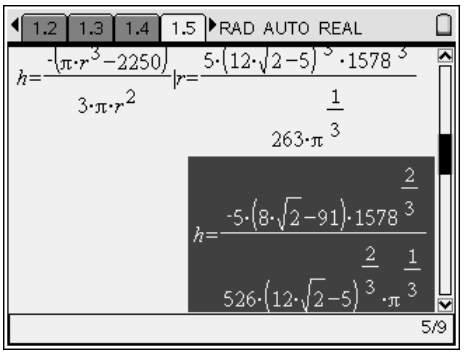
The optimum radius is the solution that makes the derivative function equal to zero.

The exact radius and numerical equivalent are shown.



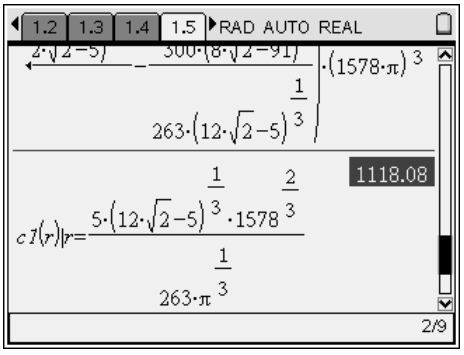
Calculate the exact height of the cylinder.

The numeric equivalent is 13.3961 cm.



Calculate the exact minimum production cost. The numeric equivalent shown as the exact calculation is too complex to fit on the screen without scrolling.

Note: The minimum cost is 1.11808 cents.



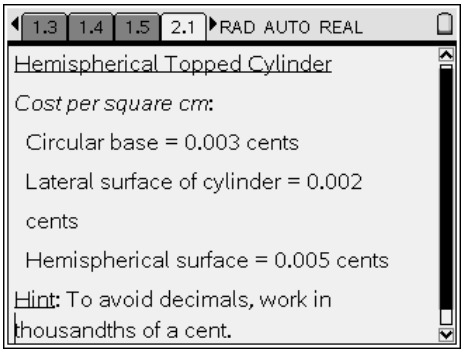
The dimensions of the model of a cone-topped cylinder with minimum production cost has radius of 4.02 cm and overall height of 17.42 cm.

7.3.6: Bottle Cost Problem: TI-Nspire CAS Solution (continued)

The solution to the hemisphere-topped cylinder follows a similar approach. Only the highlights are shown here. For more detail refer to the TI-Nspire application Bottle Cost Design.tns



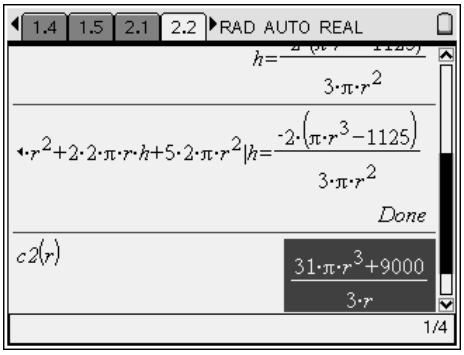
On page 2.1, review the unit costs for a cone-topped cylinder.



On page 2.2, develop the function for the production cost. The second bottle model is a hemispherical-topped cylinder.

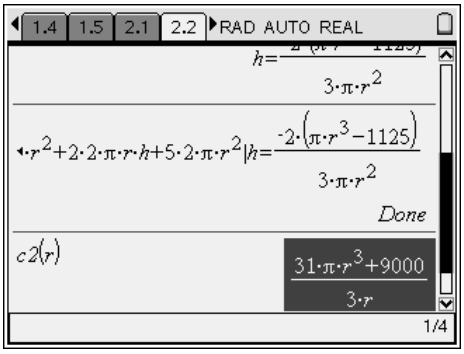
The development follows similar steps as before by substituting expressions for h and using the appropriate unit costs.

The cost function $c2(r)$ is displayed.



On page 2.3, determine the optimum radius that minimizes the production cost by tracing along the function graph. The graph shows the minimal value for the radius (3.588 cm) and the corresponding surface area (1254.04).

Note: The production cost is measured in thousandths of a cent.



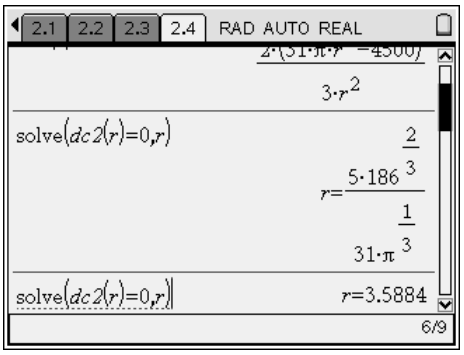
7.3.6: Bottle Cost Problem: TI-Nspire CAS Solution (continued)

On page 2.4, use CAS to perform the calculus techniques to find the optimum radius that minimizes production cost.

Define the derivative of the production cost function $c2(r)$.

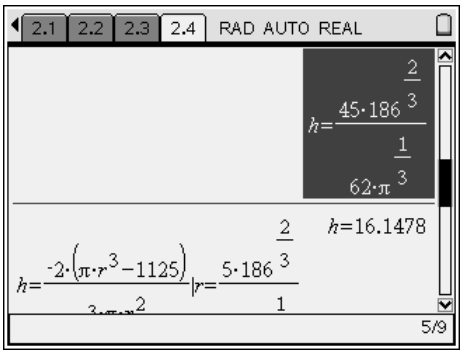
The optimum radius is the solution that makes the derivative function equal to zero.

The exact radius and numerical equivalent are shown.



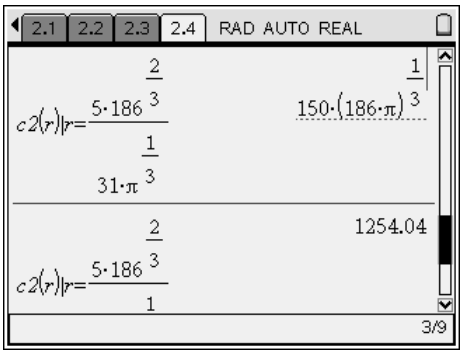
Calculate the exact height of the cylinder.

The numeric equivalent is 16.1478 cm



Calculate the exact minimum production cost. The exact cost and the numeric equivalent are shown.

Note: The minimum cost is 1.25404 cents



The dimensions of the model of a hemispherical-topped cylinder with minimum production cost, has radius of 3.59 cm and overall height of 19.74 cm.

Calculus and Vectors (MCV4U)
Question Bank

TIPS4RM
Targeted Implementation and Planning Supports for Revised Mathematics

MCV4U Question Bank

The questions address the thinking and communication categories in the Achievement Chart. Each question is categorized by an overall expectation and linked to specific expectations.

Overall Expectation – A1
Demonstrate an understanding of rate of change by making connections between average rate of change over an interval and instantaneous rate of change at a point, using the slopes of secants and tangents and the concept of the limit.

1. Given $f(x) = x^2 - 4x + 5$ and $g(x) = x^3 - 3x^2$

A1.2
A1.3

Mickey and Minnie are given the task of determining the instantaneous rate of change for each function at $x = 2$.
Mickey presents the following solution for $f(x)$:

Solution

Select close points the same distance on either side of $P(2, f(2))$, say $A(1, f(1))$ and $B(3, f(3))$ and determine the slopes of the secants AP and BP

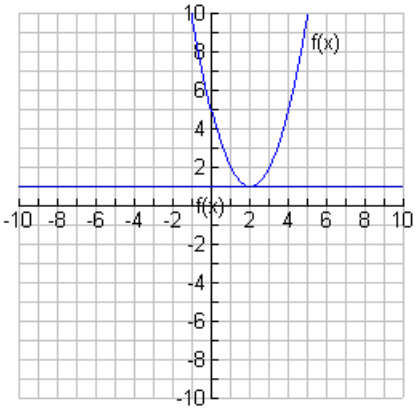
$$m_{AP} = \frac{f(2) - f(1)}{2 - 1} = \frac{1 - 2}{1} = -1$$

$$m_{BP} = \frac{f(3) - f(2)}{3 - 2} = \frac{2 - 1}{1} = 1$$

The average of these two slopes would give the exact instantaneous rate of change of $f(x)$ at $x = 2$

$$IRC = \frac{-1 + 1}{2} = \frac{0}{2} = 0$$

A sketch of the graph supports this solution in that there is a horizontal tangent at $(2, f(2))$.



Minnie challenges his solution claiming that his strategy is flawed. Do you agree or disagree? Explain referencing both $f(x)$ and $g(x)$.

MCV4U Question Bank

Overall Expectation – A2
Graph the derivatives of polynomial, sinusoidal, and exponential functions, and make connections between the numeric, graphical, and algebraic representations of a function and its derivative.

1. Points $A(1,-3)$ and $B(4,3)$ lie on the curve $y = x - \frac{4}{x}$. Find the coordinates of the point(s) on the curve between A and B such that the tangent line to the curve is parallel to the line through AB. Provide a supporting sketch.

A2.2 A3.3

2. Given functions $f(t) = \cos(2t)$ and $g(t) = (t-\pi)\left(t-\frac{\pi}{2}\right)^2$ that describe the position of two moving objects, where t is measured in seconds and $f(t)$ and $g(t)$ are measured in m, graph both functions and their derivatives. Determine the intervals when both objects are moving in the same direction. Provide a graphical analysis that includes supporting details and appropriate calculations.

A2.1 A2.4 B2.1

Overall Expectation – A3
Verify graphically and algebraically the rules for determining derivatives; apply these rules to determine the derivatives of polynomial, sinusoidal, exponential, rational, and radical functions, and simple combinations of functions; and solve related problems.

1. Verify that the line tangent to the curve $y = x + 2x^2 - x^4$ at the point $(-1,0)$ is also tangent to the curve at $(1,2)$.

A3.3

2. In Calculus, the Quotient Rule can be used to determine the derivative of a quotient such as $h(x) = \frac{2x+3}{(x+4)^3}$. It is fairly complex and looks like this:

A3.5

$$\text{If } h(x) = \frac{f(x)}{g(x)}, \text{ then } h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

In reality, the rule is not necessary.

a) Suggest a way to differentiate the above function using the rules already established in class.

b) Determine what you believe the derivative to be using your idea.

c) Show that you get the same result using the Quotient Rule.

3. Find the derivative of $f(x) = (\sin(x) + \cos(x))^2 - 2\sin(x)\cos(x)$ using the Chain Rule, Product Rule, and the Derivative of a Sum or Difference Rule. Verify the result using at least two other techniques. **A3.5**

4. The curves with equations $y = \frac{4}{x}$ and $y = ax^2 + bx + c$ have the following properties: **A3.4**
   - a common point where $x = 2$.
   - a common tangent line where $x = 2$.
   - both curves pass through the point $(1,4)$

   Find the values of a, b and c.

Overall Expectation – B1
Make connections, graphically and algebraically, between the key features of a function and its first and second derivatives, and use the connections in curve sketching.

1. Consider the following statement. **B1.3**

   If $y = f(x)$ is continuous and differentiable for all $x \in \Re$ and if $f''(a) = 0$ then $(a, f(a))$ is a point of inflection. This statement is:

   a) always true
   b) never true
   c) sometimes true
   d) none of the above

   Justify your answer.

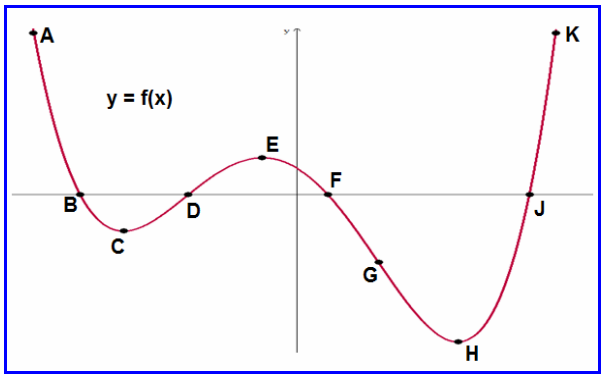
2. A continuous function, f, is defined for $x > -12$, and has all the properties listed below. For each piece of information state, in one sentence, its implication on the graph of $y = f(x)$. **B1.4**

   (i) $f(-8) = 2, f(1) = 5$
   (ii) $f'(x) < 0$ if $-12 < x < -8, f'(-8) = 0, f'(x) > 0$ if $x > -8$
   (iii) $f''(x) > 0$ if $x < 1, f''(x) < 0$ if $x > 1$

   Using the above information, sketch a possible graph of $y = f(x)$

MCV4U Question Bank

3. **B1.4**



For the above graph:

i) D and G are inflection points

ii) E is a relative maximum

iii) C and H are relative minima

State the point(s) or interval(s) where:

a) $f'(x) = 0$ and $f''(x) = 0$ _____

b) $f'(x) < 0$ and $f''(x) > 0$ _____

c) $f'(x) = 0$ and $f''(x) < 0$ _____

Overall Expectation – B2
Solve problems, including optimization problems, that require the use of the concepts and procedures associated with the derivative, including problems arising from real-world applications and involving the development of mathematical models.

1. The population of Narnia is modelled by $P(t) = \dfrac{5t+8}{3t+2}$, where, the population is measured in thousands and is the time in years since 1950. **B2.3 B2.5**

a) Determine the rate of change of the population in 1987. What does the value of this rate of change tell you about what is happening in Narnia?

b) Is the population of Narnia ever expected to start increasing? If so when. Explain.

c) Narnia will lose its status as a town if its population drops below 1 500. According to this model will this ever happen? Explain.

2. Consider the problem below. A partial solution is given.

B2.1 B2.3 B2.5

A person on an ATV is travelling on a tree lined pathway. Rounding a corner, he sees a large stump directly in his path. If he applies the brakes immediately the position of the ATV is defined by $s(t) = 12t - t^3, t \geq 0,$ where t is in seconds and $s(t)$ is in metres. Will a crash occur?

Solution:

Line 1: $s'(t) = 12 - 3t^2$

Line 2: $12 - 3t^2 = 0$

Line 3: $t^2 = 4$

Line 4: $t = \pm 2$

$\therefore s(2) = 12(2) - 2^3$

$= 16$

or $s(-2) = -16$

a) Explain the meaning and significance of $s'(t)$ in line 1 and explain what is happening in line 2.

b) Explain:

(i) the significance of $s(2)$

(ii) why it was not necessary to evaluate $s(-2)$?

c) Answer the question posed in the problem, explaining your reasoning.

3. The position function of an object moving along a straight line is given by the function $s(t) = t^3 - 15t + 48t - 10$, where s is in metres and t is in seconds and $0 \leq t \leq 15$.

B2.1 B2.2 C1.1

a) When is the **velocity** of the object greater than 21 m/s?

b) When is the **speed** of the object less than 21 m/s?

c) Illustrate the graphical representation for each of the above.

MCV4U Question Bank

Overall Expectation – C1
Demonstrate an understanding of vectors in two-space and three-space by representing them algebraically and geometrically and by recognizing their applications.

1. a) Explain why it is not enough to tell a person that requires directions to your house that you live 5 km from school.

   b) Create a vector diagram with reasonable scaling showing the route you take home from school each day.

   c) Create a vector diagram with reasonable scaling showing the route that a homing pigeon might likely take to your home if you released it at school.

   Note: The difference in level of thinking would be apparent if the student uses a 3-space representation for Part c.

C1.1 C1.2

Overall Expectation – C2
Perform operations on vectors in two-space and three-space, and use the properties of these operations to solve problems, including those arising from real-world applications.

1. For each of the following expressions, place brackets to make it "meaningful." Justify your answer. If this is impossible, explain why.

   a) $\vec{a} \bullet \vec{b} \bullet \vec{c}$

   b) $\vec{a} \bullet \vec{b} \times \vec{c}$

   c) $\vec{a} \times \vec{b} \bullet \vec{c}$

   d) $\vec{a} \times \vec{b} \times \vec{c}$

C2.5 C2.7

2. Explain whether the quantity $\dfrac{\vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{a} \times \vec{b}}{\left|\vec{b}\right|}$ is a **vector**, **scalar**, or **neither**.

C2.5 C2.7

3. If $\vec{a} = (3, 1, 5)$ and $\vec{b} = (-2, 1, 1)$

   a) Evaluate: $\vec{a} \bullet \left(\vec{a} \times \vec{b}\right)$

   b) Explain why this result will be the same for any two vectors in $\Re^3$

C2.4 C2.5 C2.6 C2.7

4. $\vec{u}$ and $\vec{v}$ are both non-zero vectors. How are $\vec{u}$ and $\vec{v}$ related if:

C2.4 C2.6

a) $\vec{u} \bullet \vec{v} < 0$?

b) $\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|$?

c) $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|$?

5. The town of Newton is 2 000 kilometres due east of Euclid. An aircraft that flies at a constant velocity of 480 km/h requires a bearing of 85° when it sets out from Euclid to compensate for wind conditions. If the trip takes 4 hours, what is the velocity and direction of the wind?

C2.3

Overall Expectation – C3

Distinguish between the geometric representations of a single linear equation or a system of two linear equations in two-space and three-space, and determine different geometric configurations of lines and planes in three-space.

1. Explain or justify each of the three steps in the following solution and make a concluding statement about the information found in each step.

C3.3 C4.3

Question:

Show that the line $(x, y, z) = (-5 - 3t, 3 - 4t, 1 + 5t)$ lies in the plane $2x + y + 2z + 5 = 0$.

Solution:

$2x + y + 2z + 5 = 0 \Rightarrow \vec{n} = (2,1,2)$

$(2,1,2) \bullet (-3,-4,5) = 0$

Therefore,

Also, for the plane $2x + y + 2z + 5 = 0$

$\text{L.S.} = 2(-5) + (3) + 2(1) + 5$

$= 0$

$= \text{R.S.}$

Therefore,

Describe another method for solving this same question.

MCV4U Question Bank

Overall Expectation – C4
Represent lines and planes using scalar, vector, and parametric equations, and solve problems involving distances and intersections.

1. For what values of k will the planes
$2x-6y+4z+3=0$ and $3x-9y+6z+k=0$

C4.7

   a) not intersect?

   b) intersect in a line?

   c) intersect in a plane?

2. Assume that you are given the equations of three planes. **Without solving the system**, describe a "test," with rationale, that you could use to determine whether or not the planes intersect at a single point.

C4.3

3. Given the equations of three lines in $\Re^3$,
$I_1: (x,y,z)=(1,4,1)+t(3,3,-2)$
$I_2: (x,y,z)=(-3,-5,8)+r(5,0,1)$
$I_3: (x,y,z)=(3,-5,8)+s(-5,0,-1)$

C3.3
C4.7

   a) Can all three lines be concurrent (i.e., *intersect in one point*)?
   Explain. An algebraic solution is not required.
   Draw a sketch of the relationships among these lines.

   b) Determine the point of intersection of I_1 and I_2.

   c) Use the information of part (a) to help in determining the distance between I_3 and I_2.

4. a) Under what conditions would three points in determine a line?

   b) Given the points $A(1,-1,5)$, $B(-2,1,3)$, and $C(7,-5,9)$, state (*with reason*) whether these points define a line or a plane.

C4.2
C4.5

   c) Find the **vector** equation of that line or plane.

   d) If A, B, and C determine a line, state a fourth point D that is not on this line. Determine the vector equation of the plane defined by A, B and D.

   e) If A, B, and C determine a plane, state a fourth point D that would be on the line determined by A an B.