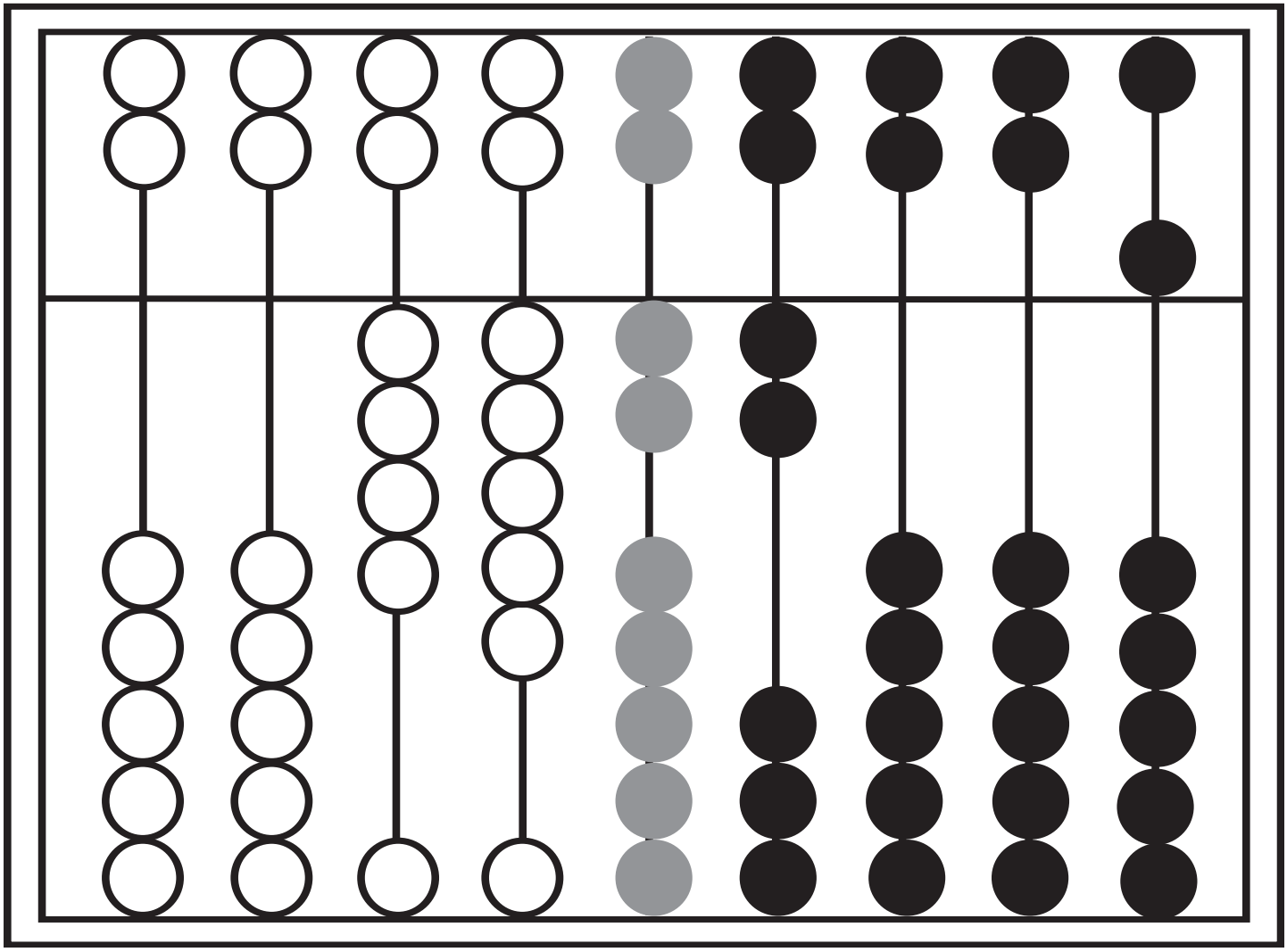
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# Abacus

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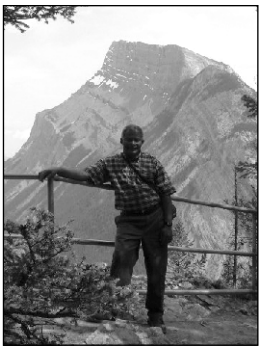
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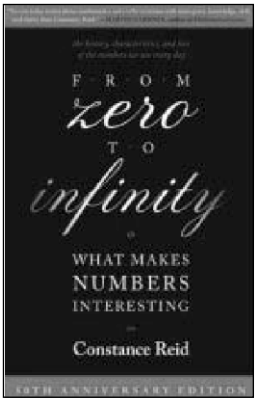
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# ▲ From the Editor's Desk

***Trevor Brown***



There are many books written about the nature and discipline of mathematics. Authors like W.W. Sawyer, Ivars Peterson, Martin Gardiner, Keith Devlin, Ian Stewart and others come to mind. A book that has tried to accomplish this objective has recently re-surfaced and is receiving many positive reviews from mathematicians and mathematics educators. The following review is one of many that I have seen and gives an excellent summary of the contents of the book and why it is worth 'reading'.




## *From zero to infinity* reviewed by Phil Wilson

*From zero to infinity: what makes numbers interesting*

By Constance Reid

What makes numbers interesting? The subtitle of this beautiful book is the motivation, map, and message of its 188-page journey from zero to infinity. With concise insight, Reid takes the digits from 0 to 9 as chapter titles and starting points of voyages into the history and deep concepts of modern mathematics. This new edition marks the 50th anniversary of the book's publication and contains a new chapter on the transfinite cardinal $\aleph_0$ (aleph zero), complimenting an earlier addition of a chapter on Euler's number, *e*.

You may be disappointed if you only want a book telling you interesting things about the numbers themselves. The number lore, for example "The Romans attributed the number six to the goddess of love", is incidental to the main themes: patterns, problems, and propositions associated with numbers, and the passion with which we have been attracted to them for thousands of years. (Reid stresses the antiquity of mathematical thought with frequent praise for the ancient Greeks.) Each digit, as well as the two numbers $\aleph_0$ and *e*, is used to introduce at least one new mathematical concept. Thus armed, a pleasing number of theorems can be introduced. Some are proved in the book, some require an undergraduate degree in mathematics, and some are as yet unproved and here stated with a tone of challenge to the keen reader.

This book is challenging: no spoon feeding of easy concepts and fashionable analogies here. The book conveys the beauty of numbers and the theories we build around them with as much detail as possible, while encouraging the enthused reader to learn more themselves. Each line must be read carefully because not a word is used needlessly. This effort is repaid: each chapter could be the equivalent of a great mathematics teacher in its potential to excite a new generation of people about mathematics.

Chapter "Four", for example, begins with the simple but startling sentences "Two times two is four. This is the most interesting fact about the number four, and it is very interesting indeed". Four is the first perfect square (excluding the trivial 0 and 1), and so begins the reader's rapidly ascending journey. Despite that speed and height, one never feels rushed or dizzy because Reid's writing is clear and clean. After spending some time with the geometric thinking of the Greeks (subject for a book in itself), we fast forward to Galileo Galilei, the famous Italian astronomer and co-inventor of the telescope, working around the turn of the 17th century.

A towering figure of scientific history, Galileo also pioneered the writing of fun scientific articles for the general public. (He probably only did so to avoid, unsuccessfully, the wrath of the Church but popular science certainly took off.) Reid points out

that Galileo was a mathematician who founded the basis of what would become a theory of infinite sets centuries later. Galileo observed that every square number can be attached to a positive whole number: we attach 1 to 1, 4 to 2, 9 to 3, and so on. Each square number is attached to its positive square root so that each square is attached to a single number, and no number is attached to more than one square. This process of attaching positive whole numbers to things in sequence is what we call "counting"; Galileo had realised that you can count the square numbers. But this is very strange, even worrying. Our common sense says that there are many more positive whole numbers than there are square numbers, and yet they pair up perfectly, meaning that there must be the same number of each!

Galileo side-stepped the issue by suggesting that the notion of equality does not apply to the infinite, and it isn't until the $\aleph_0$ chapter that we learn that the real issue is that there are different sizes of infinity, and that it makes sound mathematical sense to say that there are the same number of positive whole numbers as there are square numbers, odd numbers, or even fractions.

At this point, "Four" has barely got going. We will encounter the great names of Pythagoras, Diophantus, Fermat, and Gauss. The latter is probably the greatest mathematician who has yet lived, and his influence is felt throughout the book. This influence is matched only by Euler, whose number *e* is a chapter title. Fermat was probably the first deep thinker about number theory, despite a full-time job as a lawyer in 17th century France. His annoying habit of not writing down proofs means that his propositions have kept mathematicians busy for the past few centuries. Like all chapters, "Four" concludes with problems for the reader; not repetitive examples of applying a rule, but real tests and encouragements to think. This is, after all, what mathematicians do: think.

Where is the infinity of the title? It is everywhere in the book. We see Euclid's exciting proof of the infinity of primes, and infinitudes of other types of numbers are discussed, as are still-unanswered questions about whether, for example, the number of perfect numbers is infinite. Perfect numbers are the sum of all their divisors apart from themselves and are introduced in "Six", the first perfect number. The final two chapters take us explicitly to infinity (and beyond). Euler's number is irrational: it cannot be written as the ratio of two whole numbers. Its decimal expression therefore not only never terminates, but we have no way of knowing what the nth decimal place is before we work it out. This chapter discusses the Prime Number Theorem, one of the most beautiful and amazing theorems in mathematics, linking this strange beast *e* with the way the primes are scattered amongst the whole numbers. The final chapter treats $\aleph_0$, a measure of the size of one infinite set, and opens the door to previously unimaginable ideas such as different sizes of infinities.

I was surprised to find one or two spelling mistakes in the fifth edition of a book, including one in the middle of a formula, but this is my best attempt to present a balanced review of this wonderful book. As the numbers increase, the complexity of the mathematics does also, but never does the clarity or excitement decline. The enthusiasm and erudition of the author, not a trained mathematician, penetrate past the sparkly but more minor issues to the real core of mathematical thought. Densely-written, insightful books which reward careful reading with a sense of the beauty, vastness, and mystery of mathematics are rare, but this is one.

## Book details:

*From zero to infinity: what makes numbers interesting* Constance Reid. Paperback — 200 pages (2006). A K Peters ISBN: 1568812736

## About the Author

Phil Wilson uses mathematics to understand how the skin of red blood cells works. The people he works with at the University of Tokyo hope that they can use this understanding to invent better ways of delivering medicine to the body.

# ▲ Focus Issue: Data Management and Probability

The following tasks, some old and well-known, and others maybe not so well-known, are designed to enrich and deepen your students' understanding of probability. We would like to see how your students attempted to solve each of the following tasks. We would certainly appreciate hearing from you and/or students!

We would like to express our appreciation to NRICH for allowing us to use and modify some of their tasks for this issue of the *Abacus*.

## ▲ Task One: Bipin's Choice

Bipin is playing in a game show.

The host tells Bipin that there are 10 balls hidden in a box - 5 are Red, 3 are Yellow and 2 are Blue.

Bipin pulls out his first ball. It's Red and he wins a large sum of money. Now his choices are:

He can stop playing and take his prize.

He can pull out another ball - if it's Red he doubles his money, if it's Yellow he leaves with nothing, if it's Blue he can pick one last ball.

What should Bipin do?

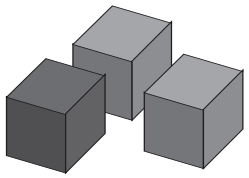
Why?



## ▲ Task Two: Same or Different

Anna and Becky were playing a game.

They put one purple cube and two yellow cubes into a bag.



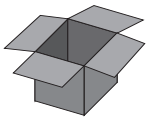
First Anna picked a cube out of the bag without looking, then Becky picked one out.
If the two cubes the girls picked out were the same colour, Anna won the game.

If they picked out two differently coloured cubes, then Becky was the the winner.

Is this a fair game?
Explain your answer.

## ▲ Task Three: In a Box

Chris and Jo decide to play a game.



They put two red and four blue ribbons in a box.

They each pick a ribbon from the box without looking (and without replacing them).

Jo wins if the two ribbons are the same colour and Chris wins if the two ribbons are a different colour.

Is the game fair?

If not, how many ribbons of each colour would you need in the box to make it a fair game?

Is there more than one way to make a fair game?

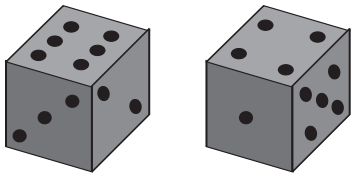
*This problem is based on one offered by Doug Williams at the 2003 ATM conference in Bath UK.*

## ▲ Task Four: A Bit of a Dicey Problem

When you throw two regular, six-faced number cubes you have more chance of getting one particular result than any other.

What result would that be?

Why is this?



What if you were to use a pair of decahedron dice.

What would be the most likely total you would throw then?

How would the probable results change for both types of dice if two pairs of each rather than two dice were used?

## ▲ Task Five: Game of PIG — Sixes

### Piggy Sixes

You can play this game with your family and friends.

It is a game, played with two number cubes, for any number of people but it is best for 2, 3 or 4 players.

The winner is the first player to accumulate 100 or more points.

Players all start with zero points, they take turns and they can throw the dice as many times as they like, adding the total at each throw to their cumulative total.

Throwing a double 6 ends the turn and sets the score back to zero.

Throwing one 6 ends the turn and nothing is added to the score for that turn.

The art is to decide when to toss again and when to stick.

Can you formulate an optimal strategy for the game?

What is the probability of throwing a double six?

What about the probability of throwing a double six in two successive throws, or three...?

What is the expected score if you throw the pair of dice once, twice, three times...?



## ▲ Task Six: The Lady or the Lions

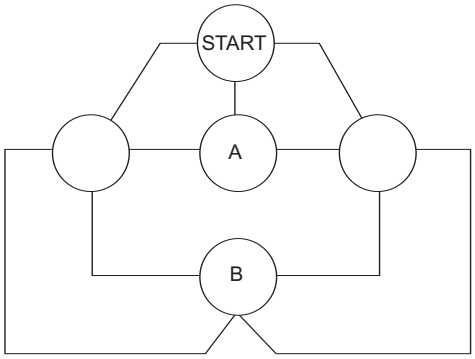
The King of a distant kingdom arranged a marriage for his only daughter to a prince from a nearby kingdom. However, the Princess had already fallen in love with a handsome, clever but unfortunately poor peasant. The king upon learning of the Princess's relationship with the peasant, ordered the suitor to be thrown into the lions' den. His daughter pleaded with the king for mercy and the king offered a compromise. Her lover could walk through a maze, with each path leading to one of two rooms. In one room there would be



the Princess, but in the other would be a pride of very hungry lions. If the peasant entered the room with the Princess, they would be allowed to marry. If the peasant entered the room with the lions, well, that would be that.

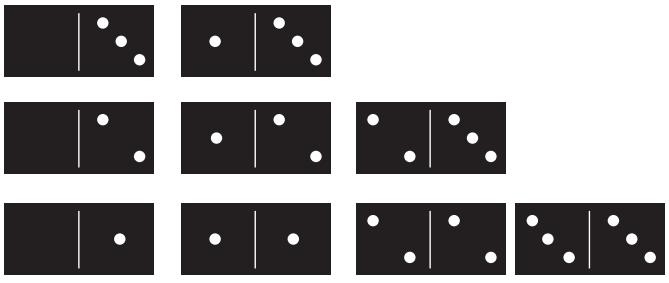
The King showed the Princess a map of the maze and the Princess was allowed to decide which room she would wait in, either A or B. She was not allowed to send a copy to her lover, so she knew he would have to guess which path to follow. Which room should she wait in to give her lover the greatest chance of finding her? What is the probability that the story will end happily ever after?

The map of the maze is reproduced below, where the heavy lines represent corridors of the maze each of which end in a door to a chamber. The peasant will not know his fate until he opens the soundproof door and steps into the chamber either to the arms of his Lady or the jaws of the Lions. Also, the Princess will not hear him knock and hence open the door.




## ▲ Task Seven: Domino Pick

Erik and Sanjay have these dominoes laid out:



They pick a domino in turn.

Erik can only choose dominoes which have an odd number of spots in total.
Sanjay chooses the dominoes which have an even number of spots.

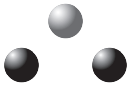
How many will they each be able to pick up? Is this fair? Why?

Now they change the rules. They lay out the dominoes again.

Erik can only choose dominoes where the total number of dots is a multiple of 3.
Sanjay chooses the dominoes where the total number of dots is not a multiple of 3.
Who is able to pick up the most dominoes now? Is this fair? Explain your answer.

## ▲ Task Eight: Two and One

Two children are playing a game with three balls, two blue ones and one red one. Terry is red and Ali is blue.



They toss up the balls, which run down a slope so that they land in a row of three, like this:



The colour in the middle wins that round. Is this a fair game?

Who do you think will win?

Why?

## ▲ Task Nine: Trick or Treat

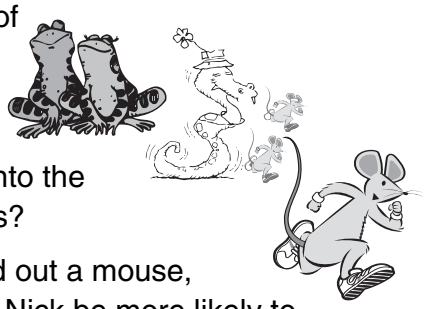
Mrs. Smith had emptied packets of chocolate covered mice, plastic frogs and gummi-worms into a cauldron for treats.

There were 9 mice, 5 frogs and 4 worms.

Trixie was allowed to dip into the cauldron for a treat but had to do it without peeping. What treat

is Trixie most likely to pick out?

Trixie wanted to be certain of pulling out at least one of each type of the treats. How many times would Trixie need to be able to dip into the cauldron to do this?



If Trixie had pulled out a mouse, would her brother Nick be more likely to pick out a mouse too, or a frog or worm instead?

## ▲ Task Ten: Let's Make a Deal

Marilyn is on the game show, "Let's Make a Deal," where she can win a brand new car if she selects the right door in a game.

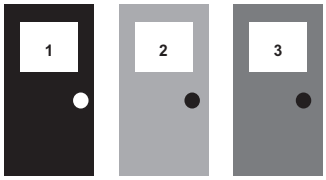
One of the doors contains the new car, and the other two doors each contain a goat. Marilyn picks one of doors.

Monty, the game show host, shows Marilyn a goat behind one of the doors that she did not pick.

He asks her, "Of the two doors remaining, would you like to keep your first door, or switch to the other remaining door?"

What would you do in this situation?

Design a simulation that will help you answer this question.






## ▲ Task Eleven: Adding Marbles

A bag contains five red marbles, three green marbles, and four blue marbles. How many green marbles must be added so that the probability of drawing a green marble is 3/4? (Source: Mathematics Teaching in the Middle School, Mar-Apr 1997)

## ▲ Task Twelve: Photo Exchange

If each student in a class of 25 students exchanges photos with every other student in the class, how many photos are exchanged?

## ▲ Task Thirteen: Job Interview

Zach is going on a job interview and has to decide what to wear.

He needs to decide between a black suit and a blue suit, black shoes and brown shoes, a red tie and a blue tie, and a plain shirt and a striped shirt.

How many different outfits are possible?

## ▲ Task Fourteen: Lining Up

Four people line up single file to board a roller coaster.

One person refused to be first in line.

In how many different ways is it possible for the people to line up?

## ▲ Task Fifteen: Tying the Knot

A civilization in a far off land has an unusual custom about marriage. The leader holds six pieces of string in her fist so that six pieces stick out of the top and six pieces stick out of the

bottom (and you can't see the middle of the string). The chief tests the potential for marriage by involving the couple in a knot tying ceremony. Each knot is formed by tying the ends of two pieces of string. The ceremony includes the following events:



- the first partner ties three different knots using the top parts of all six pieces of string,
- the second partner ties three different knots using the bottom parts of all six pieces of string,
- the chief releases her fist to show all of the strings and shows the number of loops that were created.

If the chief, shows exactly one loop, then the couple can be married.

What are the chances that a couple can get married under this custom?

Design a simulation that will help you answer this question.

Explain your experimental design and results.

## ▲ Task Sixteen: Mystery Envelopes

You are offered a choice to pick between two envelopes.




One of the envelopes contains twice as much money as the other.

After you pick one of the envelopes, you will have the opportunity to switch to the other one if you want.
You pick one of them and find ten dollars.
Without knowing the contents of the other envelope, should you keep your envelope or switch and ask for the other one? Why?

## ▲ Task Seventeen: Matching Socks

You have a drawer with 10 red socks and 20 blue socks.

It is very dark so you can't see anything.

How many socks should you pull out in order to guarantee that you will have a pair?.



## ▲ Task Eighteen: River Crossing

This is a game for two players.

Each player has twelve counters which he puts on his bank of the river.

They take it in turns to throw two number cubes (dice) and the sum of the spots shows the position from which one counter may cross the river.

The player who gets his team across first is the winner.

