8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

# Abacus

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# FROM CONCEPTS TO ALTERNATIVE ALGORITHMS AND MENTAL MATH STRATEGIES ADDITION OF WHOLE NUMBERS AND DECIMALS

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***Greetings!*** *Through the Abacus, we'd like to invite you to engage fully in mathematical thinking and doing as a means of furthering your experiences and insight into mathematics for teaching (Ball, 2000, 2005).*

*Mason (2005) explains that full mathematical engagement includes actively doing such things as jotting your own ideas, doing tasks and constructing your own examples, trying to make connections, getting involved in detail, standing back to get the big picture, explaining to someone else what you are doing, being prepared to struggle, and acknowledging feelings.*

*Also, by doing mathematics yourself, rather than reading and hearing about mathematics, you will have the immediate experience upon which to consider your own mathematics content pedagogical knowledge needed for teaching mathematics to students.*

*So, get some colleagues together from your division, and explore aspects of teaching and learning whole and decimal number addition. So, let's do math!*

## LINKS TO LITERATURE AND MANIPULATIVES



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*Greetings! What's your favourite math picture book? Which manipulatives do you use most often? Have you been thinking about extending your resource kit of books and manips? Through this issue of the Abacus, I will highlight a new manipulative and some picture books that can be used to teach whole and/or decimal number addition.* ▲

**A Publication of the OAME/AOEM**

# RESEARCH SUMMARY – ABOUT ADDITION

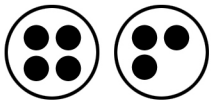
REPRESENTATIONAL MODELS FOR ADDITION

Examples:

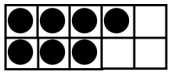
Numbers

4 + 3 = 7

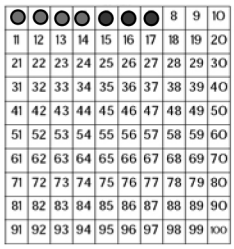
Dot plates



Five and ten frames



Hundred chart



| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

Number line




Carpenter et al. (1999) explains that "children enter school with a great deal of informal or intuitive knowledge of mathematics that can serve as a basis for developing understanding of the mathematics of the primary school curriculum. Without formal or direct instruction on specific number facts, algorithms or procedures, children can construct viable solutions to a variety of problems. Basic operations of addition, subtraction, multiplication, and division can be defined in terms of these intuitive problem-solving processes, and symbolic procedures can be developed as extensions of them." (p. 4)

Kamii (2000) says that adding single-digit numbers is natural for young children. When children construct number concepts, addition is part of this construction because all numbers are created by the repeated addition of one. Further, Kamii describes addition as a mental action young children can be expected to invent arithmetic out of everyday living. For example, children count and add cookies, as well as people, chairs, crayons, to name a few. In fact, Kamii describes addition as a mental action (constructive abstraction) of combining two wholes to create a higher-order whole in which the previous wholes become two parts. Therefore, her perspective, aligned with Piaget's, is that the source of logico-mathematical knowledge is inside children and that children can be expected to construct number concepts and invent arithmetic through constructive abstraction.

Research by Carpenter et al. (1999) shows that children think about and solve different types of addition problems depending on the way they are worded. For the most basic strategies, children use physical objects (counters) or fingers to directly model the action and/or relationships described in each problem.

Carpenter et al. (1999) further explains that the structure of an addition problem influences the strategies that children use to solve it. Initially, children model the action and relations in problems, reflecting the distinctions portrayed in the analysis of problem types. Over time, these physical, direct modeling strategies give way to more efficient counting strategies which are generally more abstract ways of modeling a problem. Eventually children come to rely on the number facts but the learning of numbers facts is not necessarily a rote skill. It can build upon an understanding of number relationships.

| ADDITION - JOIN PROBLEM TYPE | DIRECT MODELING STRATEGIES | COUNTING STRATEGIES | SYMBOLIC REPRESENTATION |
|---|---|---|---|
| **Result Unknown** – John has 5 tomatoes. He picked 3 more tomatoes. How many tomatoes does he have now? | **Joining All** – Sets of objects are modeled. The sets are joined and the union of the 2 sets is counted. | **Counting On From First** – The counting sequence begins with 5 and continues on 3 more counts. The answer is the last number in the counting sequence.<br>**Counting On From Large** – The counting sequence begins with 5 and continues on 3 more counts. The answer is the last number in the counting sequence. | 5 + ☐ = |
| **Change Unknown** – Marc had 5 peanuts. Beth gave him some more peanuts. Now Marc has 8 peanuts. How many peanuts did Beth give to him? | **Joining To** – One set of objects is constructed. Objects are added to this set until there the total is reached. The answer is found by counting the number of objects added. | **Counting On To** – A forward counting sequence starts from 5 and continues until 8 is reached. The answer is the number of counting words in the sequence after 5. | 5 + ☐ =8 |
| **Start Unknown** – Joy had some toy cars. She went to the toy box and got 3 more cars. Now she has 8 cars. How many cars did she have at the start? | **Trial and Error** – A set of 3 objects is constructed. Objects are added to this set until the total of 8 is reached. The total of 8 is sorted into two sets: 3 in one set and the rest in a second set. The second set is counted again. | **Trial and Error** – A set of objects is constructed. Another set of objects is added to the set and the resulting set is counted. If the final count is 8, then the number of objects in the initial set is the answer. | ☐ + 3 = 8 |

Fosnot (2001) explains that some common addition strategies include:

- doubles plus or minus (e.g., 6 + 7 = 6 + 6 + 1 = 13 or 7 + 7 – 1 = 13)
- working with the structure of five (e.g., 6 + 7 = 5 + 1 + 5 + 2 = 10 + 3)
- making tens (e.g., 9 + 7 = 10 + 6 = 16)
- using compensation (e.g., 6 + 8 = 7 + 7 = 14)
- using known factors (e.g., 6 + 8 = 14, so 7 + 8 must be 14 + 1 = 15)

# LET'S DO MATH:
# CONCEPTUAL REPRESENTATIONS OF ADDITION

***About the Problem***

This problem is posed for students in grade 2 classrooms to build conceptual understanding of addition of one- and two-digit numbers and to show their related representations. For a grade 4 or 5 classroom, you might use this problem to activate and build knowledge of models of whole number addition which is foundational for decimal number addition in the junior grades

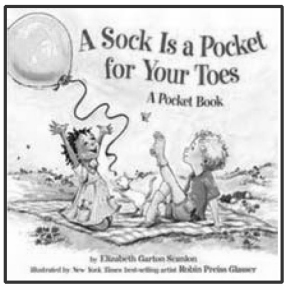
***Getting Started***

The *Getting Started* suggests you revisit models of addition of whole numbers. To get the students started, you might revisit addition of whole numbers by having the students represent addition expressions, such as 2 + 3, 5 + 2 using different models. Use a story context about pockets, such as *A Pocket for Corduroy, Plenty of Pockets, A Sock is a Pocket for Your Toes* to introduce the context.










***Problem 1* – How many pockets are there on the clothes of all the children in our classroom?**

Sample prompts:

1a. How many on the children in our group? Put one cube in each pocket to keep track of those you have counted.

1b. How many altogether in the class?

2a. If you fill your pockets with cubes, how many cubes can you hold in all of your pockets?

2b. Do all pockets have the same capacity? How do you know?

3. Do boys have greater or fewer numbers of pockets than girls? Are all pockets same size? How do you know?

***Anticipating a Range of Mathematical Solutions***

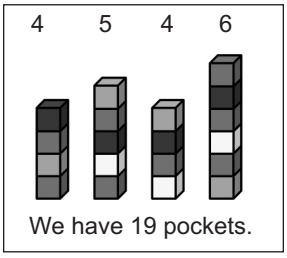
Some anticipated solutions would be counting by ones, counting on from any number or the greatest number, and splitting or decomposing numbers to make friendly pairs. The materials used are base-ten block, interlocking cubes, alternating 2 colour bead strings, and counters.

MATERIALS:

- 100 bead strings (2 colour)
- 2 cm x 2 cm x 2 cm cubes
- number lines,

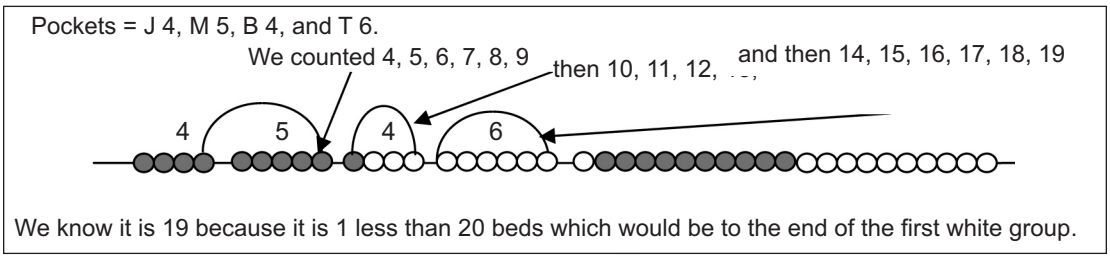
***Here are 4 sample solutions for problem 1a:***

**Counting altogether by 1's**




Student's work sample shows 4 groups. One each of 4, 5, 4, and 6. Students explained they counted 1, 2, 3, 4, 5, ..., 18, 19.

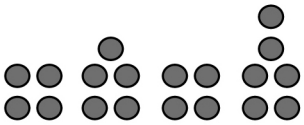
**Counting altogether by 1's starting with the first number**




Students explained, "We counted by ones too but we started with the first number 4 and counted on 5 more, then 4 more, then 6 more."

## LET'S DO MATH: CONCEPTUAL REPRESENTATIONS OF ADDITION – CONT'D...

**Renaming and adding doubles**

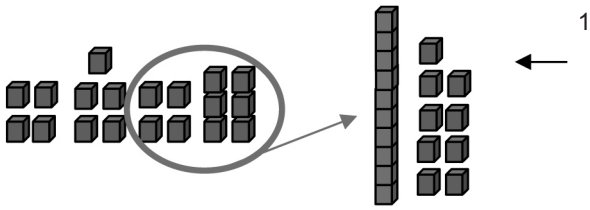


Four people's pocket counts

4 + 5 + 4 + 6
= 4 + 4 + 4 + 4 +1 + 2
= 8 + 8 + 3
= 16 + 3 =19

Students' work sample shows representing and renaming numbers in to show doubles. (5 = 4 + 1 and 6 = 4 + 2) then shows adding numbers using doubles.

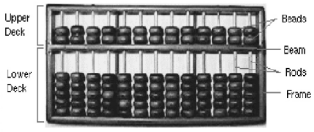
**Joining blocks and regrouping**



10 + 9 = 19

Students' work sample shows using base ten material to represent groups of 4, 5, 4, and 6 and re-grouping tens and ones to show 19.

## MATH MANIPULATIVES - ABACUS



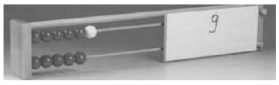

**What does it look like?**

1. 16 + 7
2. 456 + 567
3. 4.56 + 5.67

Each bead in the upper deck has a value of 5; each bead in the lower deck has a value of 1. Beads are considered counted, when moved towards the beam that separates the two decks. When 5 beads are counted in the lower deck, the result is re-presented in the upper deck; after both beads in the upper deck are counted, the result (10) is then re-presented in the adjacent column to the left.

The right-most column is the ones column; the next adjacent to the left is the tens column; the next adjacent to the left is the hundreds column, and so on. Decimal number calculations are performed by designating a space between 2 columns as the decimal-point and all the rows to the right of that space represent fractional portions while all the rows to the left represent whole number digits.

## MATH MANIPULATIVES - REKENREK



Teacher's Rekenrek

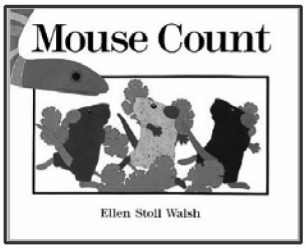


Student Rekenrek

It is appropriate to welcome this updated version of the *Abacus* with a focus on manipulatives that are updated versions of an abacus. Tools with sliding beads on wires have existed for about 3000 years. Many cultures have developed variations of this tool. The versions currently available can support students' mathematical thinking in the following ways:

- Developing the use of anchors of 5 and 10 to understand number relationships – Five beads are red and five beads are white so students can readily compare the numbers represented on the beads to 5 or 10. (e.g., 7 is 2 more than 5 or 3 fewer than 10)
- Understanding of place value – Organizing the beads to in 2 rows of 10 in the rekenrek enables students to represent numbers from 10 to 20. (e.g., 13 is represented as 10 + 3)
- Developing one-to-one correspondence – When students slide the beads while counting, they they model each object being counted as getting one count and one count only. It is useful in early stages of use to have children actually move a bead physically (to the left or right depending on task) as it is counted.
- Developing conceptual understanding of addition as combining – Moving beads together can help students understand that addition involves putting parts together to make a whole.
- Developing multiplicative thinking through doubling and halving numbers – Show how 8 is a double of 4 and half of 16 on a rekenrek.

## LINKS TO LITERATURE FOR ADDITION




When students retell stories in which there are characters that move, they also have an opportunity to practice math skills. For example, in the story *Mouse Count* by Ellen Stoll Walsh, a hungry snake fills an empty jar with sleepy mice for his dinner. “First I will count them and then I will eat them up,” said the snake.

“Mouse Count! One . . . two . . . three.”

Students can retell the story using props such as a rubber snake, plastic mice, and an empty jar and count the mice as they are placed in the jar. Then students can use the rekenrek to retell the story. The teacher can support students’ retelling and counting by taping a picture of a mouse on each bead.



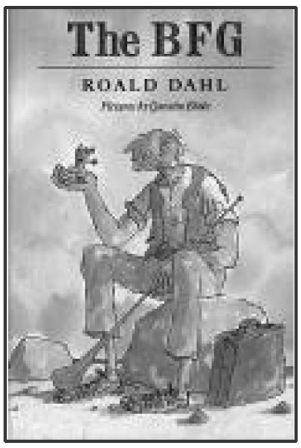

*The Mitten* by Jan Brett is another story which can be retold using a rekenrek as each animal moves into the mitten. A BLM of the animals in the story is available from Jan Brett’s website: **http://www.janbrett.com/put_the_animals_in_the_mitten.htm**




The book, *Madeline* by Ludwig Bemelmans provides a context for using the rekenrek to model doubling and halving. Madeline lives in a boarding school in Paris. “In an old house in Paris that was covered with vines lived twelve little girls in two straight lines.”

How many girls are in each line? Use the rekenrek to model numbers that can be represented in two straight lines. Use the terms double and half to talk about the numbers e.g., 6 is double 3, 3 is half of 6.

## LET’S DO MATH: ALTERNATE ALGORITHMS OF ADDITION



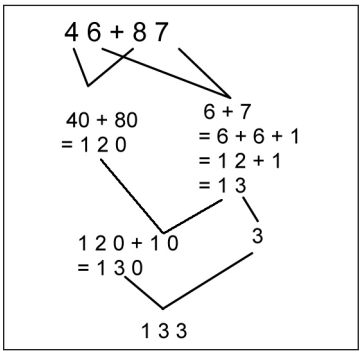

### *About the Problem*

This problem is posed as if the class has read or is reading *The BFG* by Roald Dahl. The Big Friendly Giant takes a hop, skip, and a jump. How far does he travel? The students are directed to use dice to determine how far he goes on a hop, a skip, and a jump. This gives a quick start to the task of adding decimal numbers. First the BFG hops; the one die (maybe red) will represent the number of metres and the other die (maybe blue) will represent the number of tenths of a metre he hops. (e.g., red 4 and blue 2, then the hop was 4.2 m long). Some sample student work involving addition of whole numbers is below. You might reproduce these samples and have students analyse them as a way to activate their prior knowledge about adding whole numbers before they begin adding decimal numbers.

### *Getting Started*

Students should be able to analyse and explain the reasoning used in whole number addition algorithms before they engage in adding decimal numbers in tenths.

**Decomposing numbers into tens and ones and adding the digits in same place value**




**“Partial Sums” is adding, from left, all the digits with the same place value**

$$\begin{array}{r} 128 \\ +\,437 \\ \hline 500 \\ 50 \\ +\,15 \\ \hline 565 \end{array}$$

**Decomposing numbers and renaming them to make doubles**

43 + 35 + 87
= 40 + 30 + 3 + 5 + 80 + 7
= (40 + 40 – 10) + (3 + 3 + 2) + 80 + 7
= (80 – 10) + (6 + 2) + 80 + 7
= 70 + 8 + 80 + 7
= (70 + 70+ 10) + (8 + 8 – 1)
= (140 + 10) + (16 – 1)
= 150 + 15 = 165

# LET'S DO MATH:
# ALTERNATE ALGORITHMS OF ADDITION CONT'D...

**MATERIALS:**

dice: 2 dice of different colours – red to represent ones and blue to represent tenths

***Problem 2:*** **The BFG takes a hop, skip, and a jump. How far can he travel on a hop, skip and a jump?**

Have students generate distances using 2 dice of different colours (e.g. red for the ones digit and blue for the tenths digit – so if the 1st student rolls 4 on red and 1 on blue the distance represented is 4.1 m or four and one tenth of a metre).

To make the experience more personal, have the students gather their own data. They might carry out a few experiments, for the purpose of establishing rules first – such as, after a hop you must land on the same foot you took off on; after a skip, you land on the opposite foot; and after a jump, you land on both feet. Then they could do a hop, skip, and a jump and measure their own distances.

***Anticipating a Range of Mathematical Solutions***

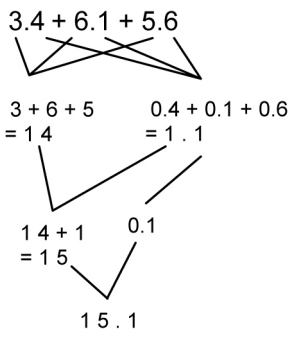
Assuming the students are rolling dice to determine the distance the BFG traveled on a hop (H), skip (S), and a jump (J), the total distances will be between 3.3 m and 19.8 m.

**"Partial Sums" method is used to add ones then tenths**

H 3.4, S 6.1, J 5.6

$$\begin{array}{r} 3.4 \\ 6.1 \\ +\,5.6 \\ \hline 14.0 \\ +\,1.1 \\ \hline 15.1 \end{array}$$

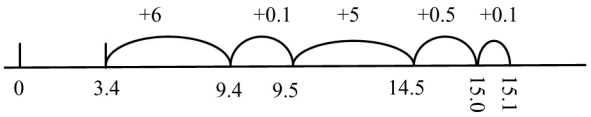
**Decompose each number into its ones and tenths and add the digits with the same place value then re-compose to a single number**




Decompose numbers into ones and tenths and add each separately.

**Decimal numbers are added to make friendly numbers using a number line**

3.4 m + 6.1 m + 5.6 m



# LET'S DO MATH:
# MENTAL MATH STRATEGIES FOR ADDITION

### *About the Problem*

The strategies used below can be used by students as early as grade 2, given that the size of the numbers are related to grade-specific expectations and appropriate to the context of the problem.

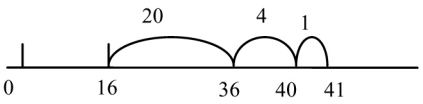
### *Getting Started*

Students should be able to analyse and explain the methods used with single- and double-digit numbers before they work with the numbers in this problem. The samples below could be used as getting started work. Have students analyse and describe the solutions.

MATERIALS
- pencil and paper
- open number lines

**"Adding On" addition using friendly numbers**

16 + 25

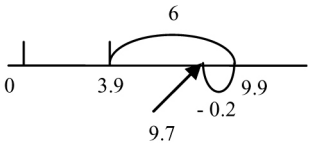



16 + 25 = 41

Students says, "We started with the first number 16 and added the 2 tens part of the second number, 25. Then we needed to add the ones part, 5 to 36. We split 5 into 4 + 1, added 36 + 4 to get to 40 and then added 1 to get 41."

**"Compensation" addition**

3.9 + 5.8




3.9 + 5.8 = 9.7

Students says, "We started with the first number 3.9. We looked at the second number 5.8 and added 0.2 to make a whole number, 5.8 + 0.2 = 6. Then we added 6 and 3.9 to get 9.9 but remembered that we had a number that was 0.2 units greater than what we wanted so we subtracted 0.2 to get 9.7."

***Problem 3:*** **Add, using mental math strategies.**

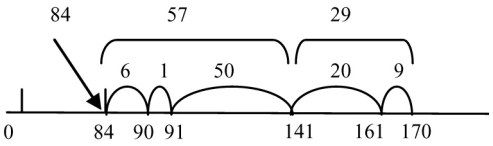
a) 84 + 57 + 29

b) $3.45 + $91.75

Some sample solutions are shown below.

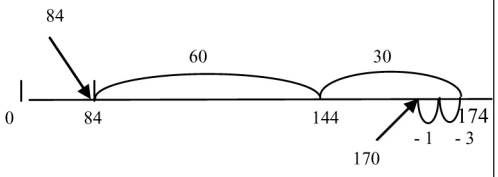
**"Adding On" addition to get friendly numbers**

84 + 57 + 29




**"Compensation" addition**

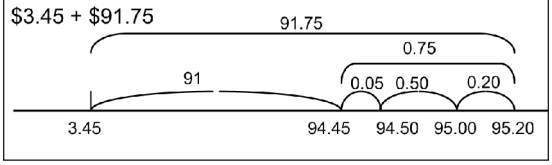
84 + 57 + 29 = (84 + 60 + 30) – 3 – 1




84 + 57 + 29 = 170

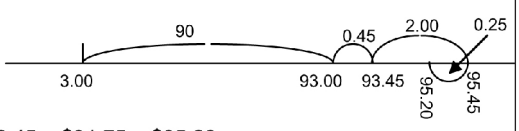
**"Adding On" addition to get friendly numbers**

$3.45 + $91.75




**"Compensation" addition**

$3.45 + $91.75 = 3 + 90 + (0.45 + 2 – 0.25)




$3.45 + $91.75 = $95.20

## NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

***Application to Your Classroom***

Practise noticing the variety of problem solving strategies and the breadth of mathematics that students use in their solutions.

During class discussions, make explicit comments about the mathematics students are showing in their solutions.

Choose 4 student work samples to analyse and describe in terms of:

a) the mathematics evident in their work,

b) the addition strategies used to develop their solutions.

## SUGGESTED READINGS


Ball, D. (2000). Bridging practices: intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education (51)*3, pp. 241-247.

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## CLOSING

The next issue will focus on key notions of whole and decimal number subtraction, how you need to know it for teaching, as well as, ways that you can learn to use these ideas and strategies through classroom-based inquiries.

**CALL FOR *ABACUS* SUBMISSIONS**

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@edu.yorku.ca or mkestell@sympatico.ca