8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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FROM CONCEPTS TO ALTERNATIVE ALGORITHMS AND MENTAL MATH STRATEGIES - SUBTRACTION OF WHOLE NUMBERS AND DECIMALS

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Greetings! What have you noticed about the different ways that students solve math problems? We have noticed that students are more confident to approach problems using addition strategies rather than subtraction. In fact, we've seen students eager to learn mental math addition strategies and be resistant to working with mental math subtraction strategies. What have you noticed?

In this issue, we will be engaging together with problems in whole and decimal number subtraction contexts. Your engagement might include solving these math problems in more than one way and comparing your solutions to the different ones we present to you. Consider taking time to jot down your own ideas about these problems, in terms of their mathematical detail. The mathematics inherent in these problems can be understood as being relative to a larger mathematics landscape of number quantity, relationships, and operations.

So, gather a few colleagues together to explore whole and decimal number subtraction. Some of these subtraction approaches are a little mind-twisting, so many heads together will be helpful. Enjoy!

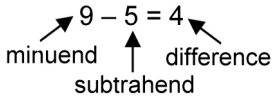
LINKS TO LITERATURE AND MANIPULATIVES



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Welcome! Have you used any of the picture books I presented in the last *Abacus* issue? For this issue, you will see base ten materials used to model subtraction in familiar and possibly unfamiliar ways. I hope to extend your experience in using a rekenrek in subtraction contexts. As well, I will be highlighting some picture books that are handy for modeling subtraction concepts. However, I don't have a picture book that models counting backwards by groups other than one. If you have found one, please share.

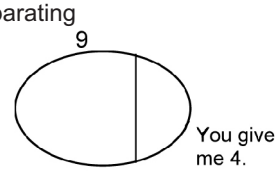
RESEARCH SUMMARY – ABOUT SUBTRACTION



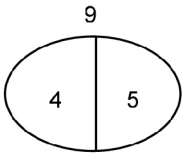
Kamii (2000) notes that from the perspective of Piaget's theory, subtraction is much harder than addition. In fact, her observations of first and second-grade classrooms reveal how unnatural subtraction is for young children. Yet, her research has shown that 1/3 to 1/2 of the children in kindergarten and first grade can solve subtraction word problems (e.g., I have 8 marbles and I give you 2 of my marbles. How many marbles do I have left?) without formal instruction. Yet, because children do not have to use subtraction to solve subtraction problems, they tend to use addition whenever possible.

In solving problems using subtraction, children initially carry out actions that directly model the mathematical distinctions of subtraction problems. Over time, such physical modelling strategies are transformed to more efficient counting strategies. According to Carpenter et al. (1999), the structure of a subtraction problem (like addition) influences the strategies that children use to solve such problems. In fact, they identified three types of subtraction problems: separate, part-part-whole, compare.

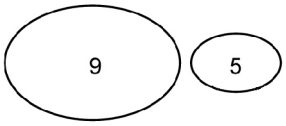
Separating



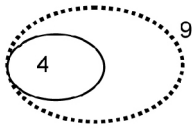
Part-Part Whole



Comparing



Equalizing



Kilpatrick, Swafford, and Findell (2001), Kamii (2000), and Carpenter et al. (1999) explain that in separate problems, three quantities can be the unknown; a starting quantity (minuend), a change quantity (subtrahend), or a result (difference). Part-part-whole problems involve static relationships among parts of a whole (subsets of a set). The problem either gives two parts and asks to determine the size of the whole, or it gives one of the parts and the whole and asks to determine the size of the other part. Kilpatrick, Swafford, and Findell (2001, p. 191) suggest that "for students in grades K to 2, learning to see the part-whole relations in addition and subtraction situations is one of their most important accomplishments in arithmetic." Further, compare problems are like part-part-whole problems as they involve relationships between quantities rather than a separating action. But, compare problems involve the comparison of two distinct, disjoint sets rather than a relationship between a set and its subset. One set is compared to the other, whereby one set is labelled the referent set and the other is the compared set. The third quantity in this type of problem is the difference or the amount by which one set exceeds the other.

Kamii (2000, p. 93) suggests that subtraction problems can be classified as four different kinds of part-whole relationships. Though her descriptions of separating, part-part-whole, and comparing are consistent with Kilpatrick, Swafford, and Findell (2001) and Carpenter et al. (1999), Kamii describes a fourth type of subtraction problem called equalizing; for example, "I have 4 marbles. 9 marbles are used for a game. How many more marbles do I need?"

Carpenter et al.'s (1999) research findings about the strategies that children use to solve different types of subtraction problems are summarized below.

SUBTRACTION PROBLEM TYPES	DIRECT MODELING STRATEGIES	COUNTING STRATEGIES	SYMBOLIC REPRESENTATION
Separate (Result Unknown) – Marg has 9 marbles. She gave 4 marbles to Jose. How many marbles does she have left?	A set of 9 objects is constructed. 4 objects are removed. The answer is the number of remaining objects.	*Counting Down* – A backward counting sequence is initiated from 9. The sequence continues for 4 counts. The last number in the counting sequence is the answer. (8, 7, 6, 5) is 5.	9 – 4 = □
Separate (Change Unknown) – Marg has 9 marbles. She gave some marbles to Jose. Then she had 5 marbles left. How many marbles did Marg give to him?	A set of 9 objects is counted out. Objects are removed from it until the number of objects remaining is equal to 5. The answer is the number of objects removed.	*Counting Down To* – A backward counting sequence is initiated from 9. The sequence continues until 5 is reached. The answer is the number of counts in the counting sequence. (8, 7, 6, 5) is 4 counts.	9 – □ = 5
Separate (Start Unknown) – Marg had some marbles. She gave 4 marbles to Jose. Then she had 5 marbles left. How many marbles did Marg have at the start?	Often trial and error; or 5 objects are counted out. 4 objects are joined onto the 5 objects. The answer is the total number of objects in the set.	*Counting On* – A forward counting sequence is initiated from 5 marbles left and continues 4 more marbles. The answer is the number of marbles at the end of the count. (6, 7, 8, 9) is 9.	□ – 4 = 5

SUBTRACTION PROBLEM TYPES	DIRECT MODELING STRATEGIES	COUNTING STRATEGIES	SYMBOLIC REPRESENTATION
Part-Part-Whole (Part Unknown) – 9 children were playing marbles. 4 were boys and the rest were girls. How many girls were playing marbles?	Often trial and error; or 4 objects are counted out. Objects are added to the starting set until 9 objects are counted in total.	*Counting On To* – A forward counting sequence is initiated from 4 and continues until 9 is reached. The answer is the number of counts in the forward counting sequence. (5, 6, 7, 8, 9) is 5 counts.	9 – 4 = □
Part-Part-Whole (Whole Unknown) – 5 girls and 4 boys were playing marbles. How many children were playing marbles?	5 objects and 4 objects are counted out and joined to make the whole set.	*Counting On* – A forward counting sequence is initiated from 4 and continues for 5 more. The answer is the final number count (5, 6, 7, 8, 9) is 9.	□ – 5 = 4 or □ – 4 = 5
Compare (Difference Unknown) – Mich has 4 marbles. Inez has 9 marbles. Inez has how many more marbles than Mich?	A set of 4 objects and a set of 9 objects are matched 1-to-1 until one set is used up. The answer is the number of unmatched objects remaining in the larger set.	*Counting On To* – Two sets of objects are counted out, one with 4 and one with 9. The elements in the two sets are matched 1 to 1. When one set is used up, the elements not matched in the second set are counted until 9 is reached. The answer is the number of counts in the forward counting sequence (5, 6, 7, 8, 9) is 5.	9 – 4 = □
Compare (Compared set Unknown) – Inez has 9 marbles. Inez has 5 more marbles than Mich. How many marbles does Mich have?	Two sets of 9 objects are counted out. From one set, 5 objects are removed, leaving 4 objects. 2 disjoint sets of 9 objects and 4 objects, shows the difference of 5 marbles.	*Counting Down* – Two sets of objects are counted out, one with 5 and the other with 9. The elements in the two sets are matched 1 to 1. When one set is used up, the elements not matched in the second set are counted until 9 is reached. The answer is the number of counts in the forward counting sequence (6, 7, 8, 9) is 4.	9 – □ = 5
Compare (Referent Unknown) – Inez has 5 more marbles than Mich. Mich has 4 marbles. How many marbles does Inez have?	Two sets of 4 objects are counted out. 5 objects are added to one set for a total count of 9 marbles.	*Counting On* – A forward counting sequence is initiated from 4 and continues for 5 more counts. The answer is the final number in the count. (5, 6, 7, 8, 9) is 9.	□ – 5 = 4

LET'S DO MATH!
CONCEPTUAL REPRESENTATIONS OF SUBTRACTION



About the Problem

Based on the story context of *100 Hungry Ants*, this problem is intended for grade 3 students to develop a conceptual understanding of subtraction of two-and-three-digit numbers using different representations. To solve this problem, both addition and subtraction strategies can be used. For grade 4 or 5 students, you could use this problem to activate their knowledge of whole number subtraction models which are foundational for decimal number subtraction. (Note: This problem is adaptable to a context appropriate for your students.)

Problem 1 **– How many ants are on these two pages? How many more ants are needed to make 100 ants?**



Getting Started

Ask students to explain the problem in their own words. If you have not used manipulatives or open number lines for subtraction recently, you will want to warm up using smaller numbers (e.g., 43 – 6 and 32 – 8) and have students represent their models using base ten blocks, open-number lines, bead strings, and/or rekenreks. Also, to prepare students for combining addition and subtraction operations, have them to model 28 + 7 – 9 and discuss the strategy that two different operations are used to solve the problem. Manipulatives to use include square tiles, counters, a 100 chart, 100 bead strings, base ten blocks, and open and closed number lines.

Anticipating a Range of Mathematical Solutions

Students could solve this problem by counting by ones, counting on from the smaller number, counting back from the greater number, separating/decomposing parts from a whole, and comparing two wholes. Listen to students' words and watch their material actions – the actions children do with their hands – to discern the way they are thinking about subtraction. Use formal mathematical terms to make explicit the mathematics inherent in the solutions. Some sample solutions are provided on the next page.

LET'S DO MATH! CONCEPTUAL REPRESENTATIONS OF SUBTRACTION ~ CONT'D...

Adding by compensating and subtracting by counting on

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

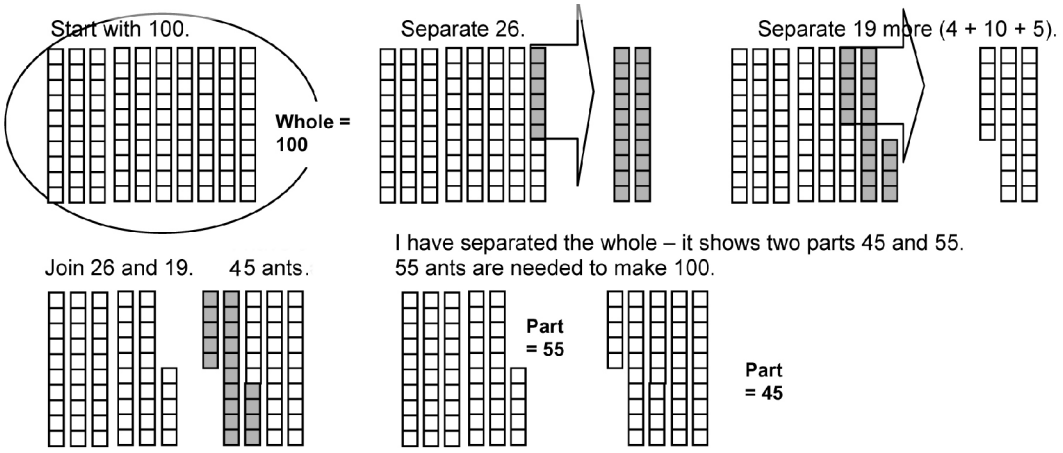
I counted 26 on the left page and 19 on the right page. I can remove 1 from 26 and put it with 19 to make 20. 26 + 19 = 25 +1 +19 = 25 + 20 = 45.

I am not finished because I need to determine the difference between 100 and 45.

I mark off 45 and count the unmarked squares to 100. From 45, I see there are 5 ones to get to 50 and 5 tens to get to 100. 5 tens + 5 ones = 55. So, 100 – 45 = 55.

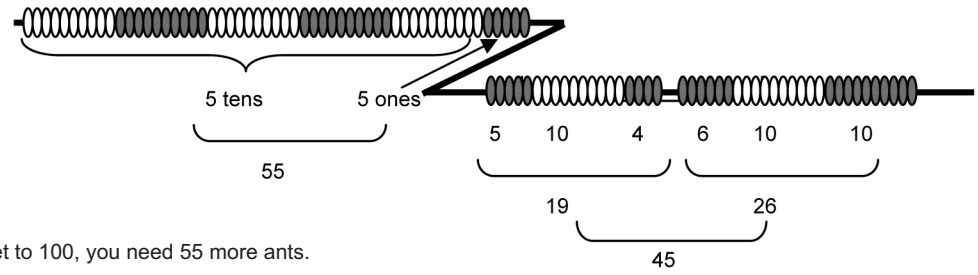
You need 55 more ants to get to 100.

Adding by joining and subtracting by separating



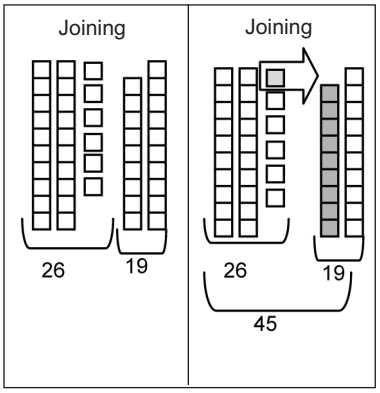
Counting back from 100

There are 100 beads on the bead string. There are 45 ants on these two pages. I will start at 100 and count back 26 and then 19 to represent the ants shown on the page. Then I will count the number of beads left to determine how many more to make 100.

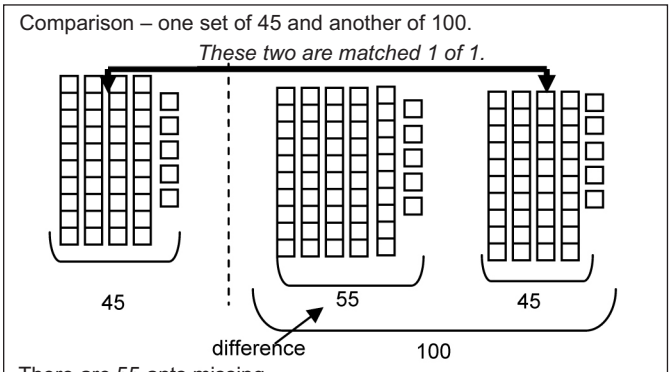


To get to 100, you need 55 more ants.

Joining for addition and comparison for subtraction



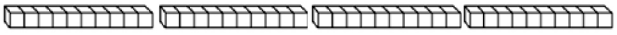
Comparison – one set of 45 and another of 100.



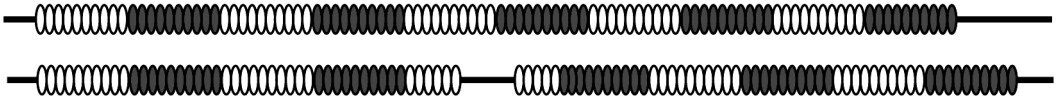
There are 55 ants missing.

LINKS TO MANIPULATIVES - LINEAR MODELS FOR SUBTRACTION

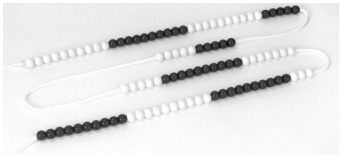
Base ten blocks and place value mats are often used to model subtraction with and without regrouping. Base ten blocks can also be used to make linear models of numbers.

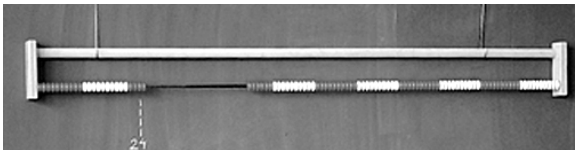


Linear models allow students to explore subtraction as decomposing a whole into parts, a separation of parts from a whole, and the comparison of the parts of a number to a whole. For example, modeling the number of ants (100) in the story, *One Hundred Hungry Ants* and the number of ants in the illustration (45), using linear models allows students to compare the difference between the two quantities (55).



Commercially available rekenrek bead strings or homemade bead strings already show one hundred so comparing the number of ants in the picture with 100 involves separating the beads into two parts. These manipulatives will be very useful during hundreds day celebrations.



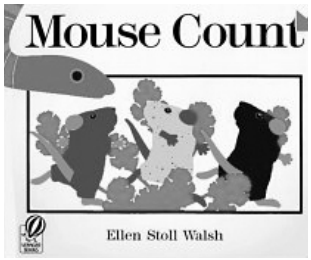


LINKS TO LITERATURE

The concept of subtraction can be developed using picture books.

The relationship between counting and subtraction is represented in books, such as *Next Please* and *Mouse Count,* which involve counting backwards. Point to the pictures of the mice as they escape from the jar while reading the text, *Ten, nine, eight, seven, six.* Ask students which number will come next. Ask them how they know. You can also cover the mice left in the jar with a paper and count the mice out of the jar backwards. Ask the students how many mice are still in the jar.





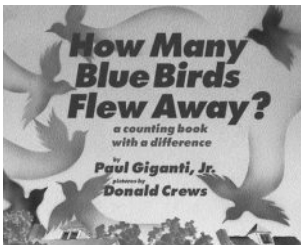
The inverse relationship between subtraction and addition can be made explicit through the books, *Little Quack* and *A Dozen Ducklings Lost and Found* In *Little Quack*, One by one, Mama Duck manages to convince her five ducklings to leave the nest and go into the water in the pond. Students can model the events in the story using counters for the ducks on a mat that is divided in half - one side is the nest the other side is the water. Ask students how they could represent what they have on their mat using numbers, e.g., 2 ducks in the nest and 3 ducks in the water could be 2 + 3 or 5 – 3. In *A Dozen Ducklings Lost and Found*, when Mother Duck takes her 12 babies for a walk, she counts up, “One, two, three, four,” and down, “Four, three, two, one.” How many of her babies are missing?

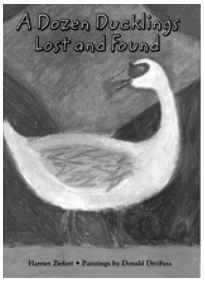




Stories about jumping such as *Boing*, naturally provide a context that allows students to use linear models to compare the lengths of jump. For example, the frog and the rabbit try to teach their friend the baby kangaroo how to jump. The rabbit jumped 640 cm. The rabbit jumped 300 cm. What is the difference? How far can you jump?

To learn about subtraction as comparison, the book, *How Many Blue Birds Flew Away?* explicitly draws attention to making comparisons between groups of objects. For example, in the book, questions can be asked like, How many apples are there? How many oranges are there? How many more apples than oranges are there?





LET'S DO MATH!
ALTERNATIVE ALGORITHMS FOR SUBTRACTION

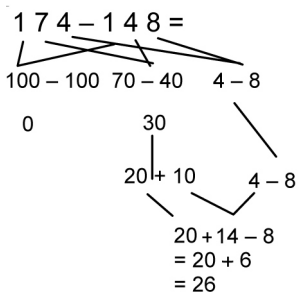
About the Problem

This problem provokes students to subtract decimal numbers to hundredths. To make the problem more personal, have the students measure their own heights and do comparisons using their own numbers. (Note: For a decimal subtraction problem, the students will need to record their heights in metres. Using their own data helps students judge the reasonableness of their answers.)

Getting Started

Students should be able to analyse and explain their reasoning used in whole number subtraction before they engage in subtracting decimal numbers. In preparation for this problem, some samples of student work on subtracting whole numbers in hundreds are shown. Consider reproducing these samples for students to analyse, as a way to activate their mathematical knowledge and experiences about subtracting whole numbers.

Decomposing numbers into hundreds, tens, and ones and subtracting the digits with the same place value



Subtract numbers represented in expanded form by renaming and regrouping

$$\begin{array}{rll} 174 = & 100 + 70 + 4 = & 100 + 60 + 14 \\ -\ 148 = & -\ (100 + 40 + 8) = & -\ (100 + 40 + 8) \\ & & 0 + 20 + 6 \end{array}$$

I rename 70 as 60 + 10 and regrouped the 10 with the 4 to make 14.

174 – 148 = 26

This makes sense because 174 can be regrouped to 170 and 480 to 150, to get 170 – 150 = 20. The answer is reasonable.

Left-to-right subtraction

$$\begin{array}{r} 174 \\ -\ 148 \\ \hline 0 \text{ -> } (100 - 100) \\ 30 \text{ -> } (70 - 40) \\ -\ \ 4 \text{ -> } (4 - 8) \\ \hline 26 \end{array}$$

Note: Add partial differences. This solution shows integer subtraction. Some students will be hesitant to do this until it is required in the curriculum, but it is an accurate move.

***Problem 2* – The tallest person on the jump rope team is 1.82 m tall. The shortest person is 1.49 cm tall. What is the difference in their heights?**

Anticipating a Range of Mathematical Solutions

Adding up subtraction

$$\begin{array}{r} 1.82 \\ -\ 1.49 \end{array}$$

1.49 + 0.01
1.50 + 0.30
1.80 + 0.02
1.82 0.33

The second person is 0.33 m or 33 cm taller than the other.

Traditional right-to-left subtraction with renaming and regrouping

$$\begin{array}{r} 1.82 \\ -\ 1.49 \end{array}$$

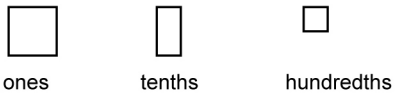
$$\begin{array}{r} {\scriptstyle 7\ 1} \\ 1.\not{8}2 \\ -\ 1.49 \\ \hline 0.33 \end{array}$$

There is a difference of 33 cm or 0.33 m. Note: This solution shows the traditional algorithm with renaming 8 tenths and 2 hundredths as 7 tenths and 12 hundredths. This representation works well when accompanied by base 10 blocks that have been renamed and regrouped.

Right-to-left subtraction with renaming and regrouping

$$\begin{array}{rl} 1.82 & \rightarrow \quad 1 \text{ and } 8 \text{ tenths} + 2 \text{ hundredths} \\ -\ 1.49 & \rightarrow \quad -\ (1 \text{ and } 4 \text{ tenths} + 9 \text{ hundredths}) \end{array}$$

$$\begin{array}{r} 1 \text{ and } 7 \text{ tenths} + 12 \text{ hundredths} \\ -\ (1 \text{ and } 4 \text{ tenths} + \ 9 \text{ hundredths}) \\ \hline 0 \text{ and } 3 \text{ tenths} + \ 3 \text{ hundredths} = 0.33 \end{array}$$



The shorter person is 0.33 m or 33 cm shorter than the tallest.

Note:This solutions involves renaming 8 tenths and 2 hundredths as 7 tenths and 12 hundredths which is something students are used to doing with whole numbers but resist with decimal operations. This representation works well when accompanied by base 10 blocks that have been renamed and regrouped.

LET'S DO MATH!
MENTAL MATH STRATEGIES FOR SUBTRACTION

About the Problem

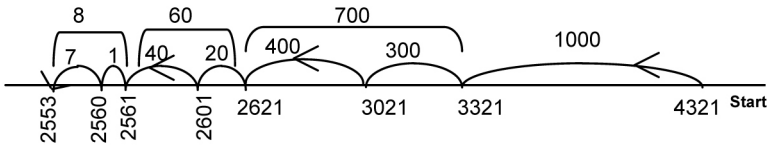
This is a multi-step problem. This years' EQAO recommendations focus on building strategies for solving multi-step problems. For example, to add 5 + 3 + 7 it is necessary whether consciously or unconsciously, that we do two steps. We can choose to operate on any two numbers, say add 5 + 3 first and then add 7 to get the total. Or we could calculate 3 + 7 first and then add 5. We arrive at the same answer. That is because when adding, the commutative and associative properties of whole numbers can be applied. Do these properties hold for subtraction? Multiplication? Division? Of whole numbers? Of decimal numbers?

Getting Started

Students need to be able to analyse and explain the methods used with single-step problems before they work with numbers in this problem. The samples below could be used as getting started work to remind them of ways to represent their work when doing mental math.

Partial subtraction and moving to friendly numbers on an open number line

4321 – 1768
4321 – 1000
3321 – 700 (300 + 400)
2621 – 60 (20 + 40)
2561 – 8 (1 + 7)
2553



***Problem 3* – The school jump-rope club was aiming for 84 members. They started the year with 53 jumpers and 19 new jumpers joined the team. How far are they from their target?**

Anticipating a Range of Mathematical Solutions

Some anticipated solutions would be those shown below.

Adding on addition with splitting and partial subtraction

Adding on addition with splitting

53 + 19 = 53 + 10 + 9
= 63 + 7 + 2
= 70 + 2
= 72

Partial subtraction

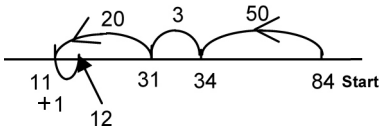
84 –72 = ?
84 –70 = 14
14 –2 = 12

The team needs 12 more players to reach their goal.

Friendly numbers on an open number line and counting backwards

84 – (53 + 19) =

I counted backwards from 84 using 50 and 3. Then I kept counting backwards 19 by moving 20 left and 1 right.



84 – 53 – 19 = 31 – 19= 12

They need 12 more members to total of 84 and reach their goal.

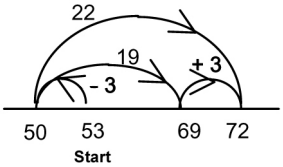
Compensation addition and partial difference subtraction

First for 53 +19, I want to make one number a multiple of 10 because I find it easier to add just 10s. So I will subtract 3 from 53 to make 50. So my sum will be 3 units less than the total so I have to add 3 to the other number or to the total after 1 calculation.

84 – (53+19)

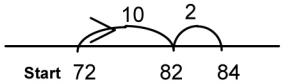
53 + 19 = (53 – 3) + (19 + 3)

= 50 + 22 = 72



72 + 10 = 82

82 + 2 = 84



When subtracting, I will calculate by adding on from 72 to 84.

They will need 12 more members to get to their goal of 84.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

Try these subtraction problems with your students. To suit your students' learning needs, you might adjust the numbers (e.g., larger or smaller whole or decimal numbers; fractions or integers for grades 7 and 8 students) and/or the context of the problem

Practise noticing the different strategies and the breadth of the mathematics your students use in their solutions. Take note of the number of solutions that show decomposition/separating or comparison, a set model or linear model of representation. In preparation for your class discussion of the solutions, decide which solution you will discuss first, second, and third. Think ahead of what mathematical comments you will make to render explicit key mathematical ideas, strategies, and terms, as the students share their solutions. As well, choose 4 student work samples to analyse further in terms of their mathematical details.

Suggested Readings

Some readings that we used in our Research Summary and in the development of the subtraction problems and solutions are listed below:

- Baroody, A. (1998). *Fostering children's mathematical power: An investigative approach to K-8 mathematics instruction.* Mahwah, NJ: Lawrence Erlbaum Associates, inc.
- Carpenter, T., Fennema, E., Franke, M., Levi, L., Empson, S. (1999). *Children's mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann.
- Fosnot, C. & Dolk, M. (2001). *Young mathematicians at work: Constructing number sense, addition, and subtraction.* Portsmouth, NH: Heinemann.
- Kamii,C. (2000). *Young children reinvent arithmetic: Implications of Piaget's theory* (2nd edition). New York, NY: Teachers College Press.
- Kilpatrick, J., Swafford, J., Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.
- Ma, L. (1999). *Knowing and teaching elementary mathematics.* Mahwah, NJ: Lawrence Erlbaum Associates.

OUR NEXT ISSUE

Our next issue will focus on key notions of whole and decimal number multiplication, how teachers need to know it for teaching, as well as, ways that teachers can use these mathematical ideas and pedagogical strategies through problem solving-based teaching and learning strategies.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@edu.yorku.ca or mkestell@sympatico.ca