8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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FROM CONCEPTS TO ALTERNATIVE ALGORITHMS AND MENTAL MATH STRATEGIES - MULTIPLICATION OF WHOLE NUMBERS AND DECIMALS

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LINKS TO LITERATURE AND MANIPULATIVES

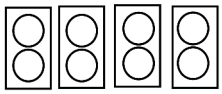


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Welcome back! In the two previous issues of the Abacus, I made suggestions for using the 20 bead rekenrek for addition, and the 100 beads string for subtraction. If you think that you are detecting a trend in the featured manipulatives, you are right. In this issue, I will continue to explore the use of rekenreks by looking at the 10 bar rekenrek for visualizing multiplication facts.

RESEARCH SUMMARY – ABOUT MULTIPLICATION

Kyle has 4 bags. There are 2 cookies in each bag.



How many cookies does Kyle have all together?

The solution to this problem can be symbolically written as 2 x 4 = 8

multiplicand multiplier product

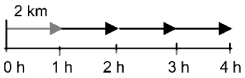
Carpenter, Fennema, Franke, Levi, and Empson (1999) explained that typical multiplication problems are comprised of two known numbers that represent different objects, such as bags and cookies. In the problem above, the bags represent the number of groups, the number of cookies represents the number of objects in each group, and the unknown number is the total number of objects, or this case, cookies. This distinction of objects is important as they are represented in the direct modelling and the use of more efficient strategies (e.g., counting adding, number facts) that children use to solve problems involving multiplication.

More specifically, for multiplication problems, children model the number in each of the groups (using counters, tally marks, or other representations) and then count the total number of objects. Counting strategies usually involve some form of skip-counting. Generally, children are more proficient in skip-counting by numbers like two, three, and five, than other numbers, such as seven. As well, rather than skip-counting, some children may repeatedly add the number representing the objects (cookies) in each group (bags). Though skip-counting is essentially repeated addition, some children tend to think of their solution more in terms of counting, and others think of it more in terms of adding.

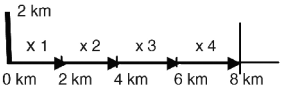
According to Carpenter, Fennema, Franke, Levi, and Empson (1999) there are seven types of multiplication problems:



- Grouping – Matt has 4 tomato plants. There are 2 tomatoes on each plant. How many tomatoes are there all together?



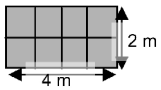
- Rate – Krysten walks 2 km in an hour. How many kilometres does she walk in 4 hours?



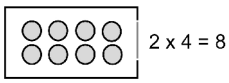
- Multiplicative Comparison – Joyce walks 2 km. Pat walks 4 times farther than Joyce. How far does Pat walk?



- Price – Ice cream cones cost 2 dollars each. How much do 4 ice cream cones cost?



- Area – A school flower garden measures 2 m x 4 m. How many square metres of garden are there?

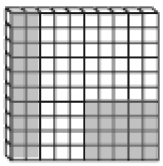


- Array – There are 2 rows of cookies with 4 cookies in each row on a cookie sheet. How many cookies are there?



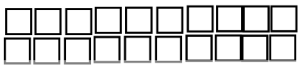
- Combination – Heather has 4 different tops and 2 different skirts? How many different outfits can she make by combining the tops and skirts?

According to Carpenter, Fennema, Franke, Levi, and Empson (1999), grouping, rate, price, and multiplication comparison problems are not symmetric; that is, the numbers in them are related to specific referents and the referents are not interchangeable. Area, array, and combination problems are symmetric; that is, the two numbers in the multiplication problems can be interchanged. Also, young children can solve problems that introduce multiplication in area contexts before they have a complete understanding of area concepts by working directly with rectangular regions partitioned into square units. For example, a rectangle made of base ten blocks has an area of 20 square units. It can be partitioned with 4 x 5 unit cubes or 2 x 10 unit cubes (which is the same as 2 ten rods).



partition of 4 x 5 square units

partition of 10 x 2 square units



2 x 10 array of discrete objects

In addition, determining the number of objects in a rectangular array involves the same basic concept of multiplication as measuring the area of a rectangle. The primary difference is that arrays may be made up of discrete objects. As well, array problems can be directly modeled in a systematic way by constructing a given number of rows of counters with the same number of

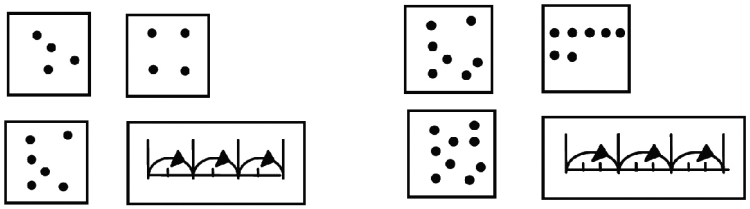
counters in each row. While array and area problems are central in developing concepts of multiplication, combination problems are not central to such development.

Furthermore, Davis, Simmt, and Sumara (2006) describe multiplication as a complex conceptual blend or a recursive elaboration of numerous interpretations consisting of images and analogies. Such interpretations are knitted together as a web of interconnections that constitute the concept of multiplication. Some interpretations include:

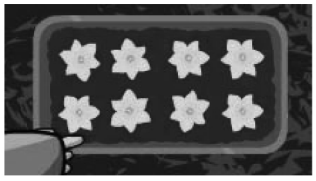
- repeated addition (e.g., 2 x 4 = 2 + 2 + 2 + 2 or 4 x 2 = 4 + 4)
- equal grouping (e.g., 2 x 4 means 4 groups of 2)
- number-line hopping (e.g., 2 x 4 means make 4 jumps of length 2)
- sequential folding (e.g., 2 x 4 refers to the action of folding a page into two parts, then folding again to have a result of 4 parts, and again to have a total of 8 parts)
- many-layered (e.g., 2 x 4 means 4 layers, each of which contains 2 layers)
- ratios and rates (e.g., 4 L at $2/L costs $8 or $2/L x 4L = $8)
- array-generating (e.g., 2 x 4 array has 2 rows of 4 columns or 2 x 4 = 8 objects)
- area-producing (e.g., a 2 linear units x 4 linear units rectangle has an area of 8 square units or 2 linear units x 4 linear units = 8 square units)
- number-line stretching or compressing (e.g., 2 x 4 = 8 can mean that 4 corresponds to 8 when a number line is stretched by a factor of 2)

LET'S DO MATH!
CONCEPTUAL REPRESENTATIONS OF MULTIPLICATION

Getting Started



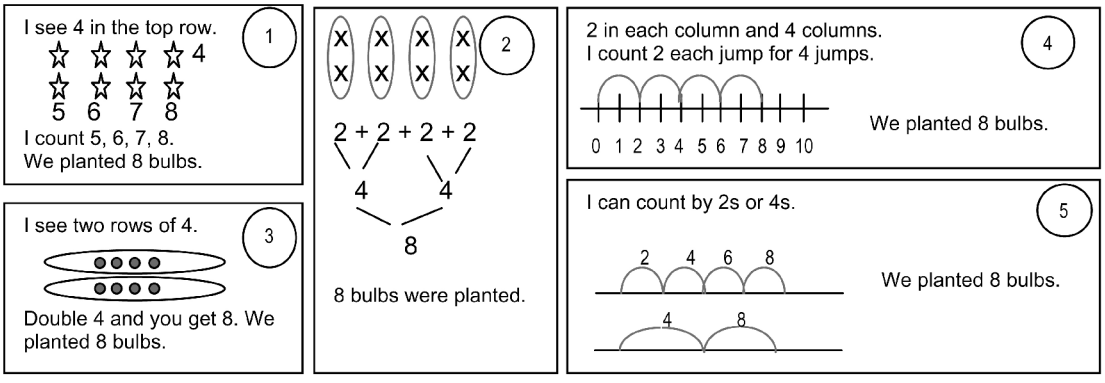
The use of arrays and number lines as visual organizers can facilitate more efficient counting of numbers. Provide quick images of dots that represent numbers and ask the students to decide and explain which number representations are easier to count.



Problem #1: **Gina and Tosh planted some flower bulbs. Each bulb grew into one flower. How many bulbs did they plant? Show 2 different solutions.**

Anticipating a Range of Mathematical Solutions

What mathematics do you see in each student sample work?



Sample Responses:

- *solution 1 shows straight counting-on by ones starting with the total number (4) in the first row*
- *solution 2 shows two rows of equal number, 4 and doubling shows the use of a number fact of 4 x 2 = 8 to get the total number in the two rows*

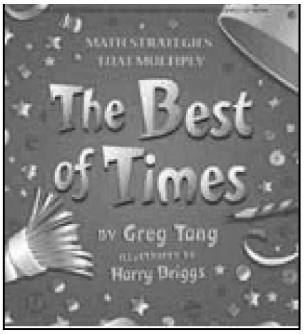
- *solution 3 shows four groups of 2 re-grouped to two groups of 4 that are added to get 8*
- *solution 4 shows skip counting by 2s on a number line with a scale unit of 1*
- *solution 5 shows skip counting by 2s and 4s on an open number line*

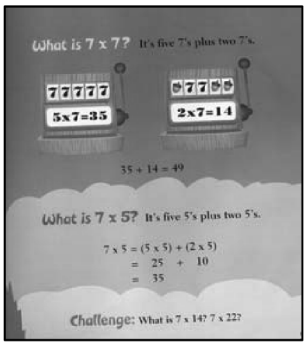
Why might solution 1 be chosen first for student discussion, followed by solutions 2, 3, 4, and 5?

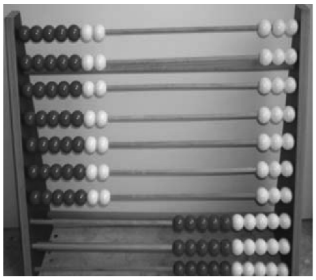
Sample responses:

- *The developmental sequence of solutions 1 to 5 range from counting, grouping, using an array (rows x columns), to skip counting on a scaled and open number line*
- *Models of representation develop from a discrete set of objects to number lines (scaled to open)*

LINKS TO MANIPULATIVES - VISUALIZING MULTIPLICATION FACTS WITH A 10 X 10 REKENREK





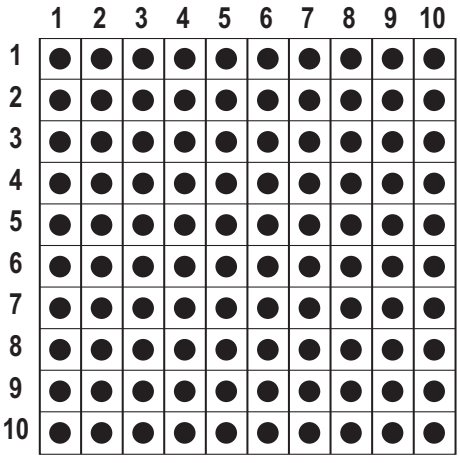


There are strategies for making connections between (a) developing an understanding of multiplication using models of representation, (b) having procedural fluency with single digit multiplication facts (or combinations), and (c) developing flexible thinking about numbers.

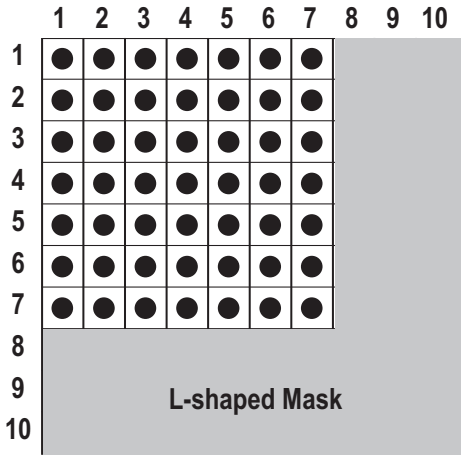
For example, think of 7 x 5 as being half of 7 x 10. In Greg Tang's book,"The Best of Times: Math Strategies That Multiply," he shows a strategy for using multiplication facts that you know to help you know new facts. He illustrates a distributive relationship for 7 x 7 which can also be modeled using a 10 x 10 rekenrek. This manipulative makes it possible for visual learners to "see" the symbolic representation of this distributive relationship, 7 x 7 = (7 x 5) + (7 x 2) and its corresponding visual representation of (7 rows of 5 red beads) + (7 rows of 2 white beads). Alternatively, some students may notice that 7 x 7 = (7 x 10) – (7 x 3) or (7 rows of 10 red and white beads) – (7 rows of 3 white beads). Consequently, the visual and symbolic representations of such distributive relationships provide a conceptual foundation for order of operations.

As well, multiplication facts modeled on the rekenrek can be re-presented on a dot array.

Using an "L" shaped mask, slide it to show one of the factors on the vertical side and the other factor on the horizontal side. At the left of the L-shaped mask, you will be able to see the 7 x 7 array on the dot array.



10x10 dot array

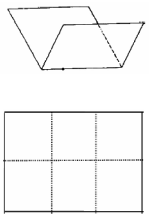


The rekenrek can also be used to provide "quick images" of multiplication facts for whole class practice. The teacher covers a multiplication fact (e.g., 8 x 8) modeled on the rekenrek. When the rekenrek is revealed, the students have a few seconds to record and show the corresponding multiplication fact on a 10 x 10 dot array or as a multiplication sentence on individual dry erase boards or laminated square grid paper.

LINKS TO LITERATURE

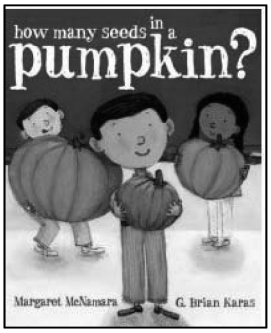
I chose these two books because they challenge and deepen children's understandings of multiplication within life contexts, such as the relationship between the linear dimensions and area of a rectangular quilt.





In this humorous story, *Grandpa's Quilt*, children try to re-design a coverlet for their grandpa. He has a square coverlet, made of 36 quilt blocks in a 6 x 6 array. But, it is a problem for grandpa because the coverlet does not cover his feet. The children cut apart the quilt, rearrange the quilt blocks to increase its length, and re-sew it into a different rectangle. Though the new coverlet covers grandpa's feet, a new problem of coverage arises. It is puzzling for the children that the area of the coverlet cannot be increased by rearranging the 36 quilt blocks.

Prompt your students to model the multiplication facts illustrated in the book (e.g., 6 x 6, 4 x 9) by creating arrays using square tiles, and then by paper folding. Paper folding helps students to experience a different way to create arrays, as well as to understand and visualize the concept of doubling and tripling. Challenge the students to model 36 in different arrays through paper folding (i.e., 6 x 6, 9 x 4, 12 x 3, 2 x 18, 1 x 36).



In this story, *How Many Seeds in a Pumpkin?*, the students in Mr. Tiffin's class estimate and count the number of seeds in a big, medium, and small pumpkin. Then they count the seeds by grouping them by 2s, 5s, and 10s. The students are surprised when the pumpkin with the fewest groups has the most seeds.

Show the students two plastic bags: one bag has 6 groups of 3 seeds and the other bag has 2 groups of 8 seeds. Ask the students, "Which bag has more? How do you know?" Have the students create and share "groups of seeds" challenges for each other.

LET'S DO MATH!
ALTERNATIVE ALGORITHMS FOR MULTIPLICATION

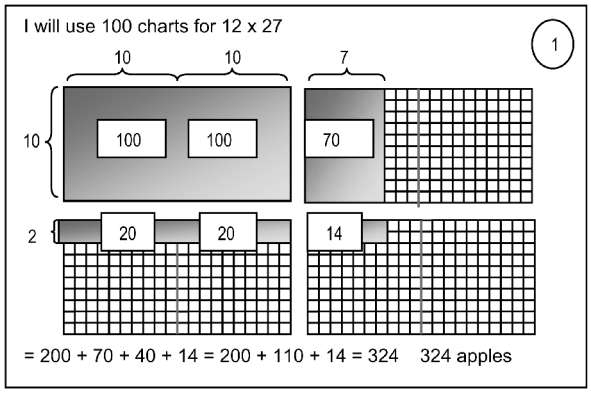
About the Problem

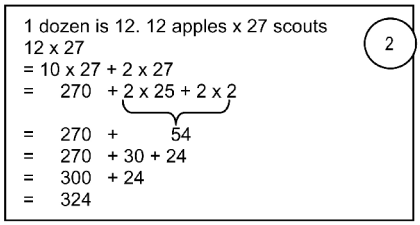
For this problem, conceptual models of representation and alternative algorithms for multiplication will be provided as solutions. According to the Ontario curriculum, students must be able to demonstrate the operation, multiplication using concrete materials, pictures, student-generated algorithms, and the standard algorithm. It is important for students to understand the concept of multiplication, before they learn to use alternative and standard algorithms.

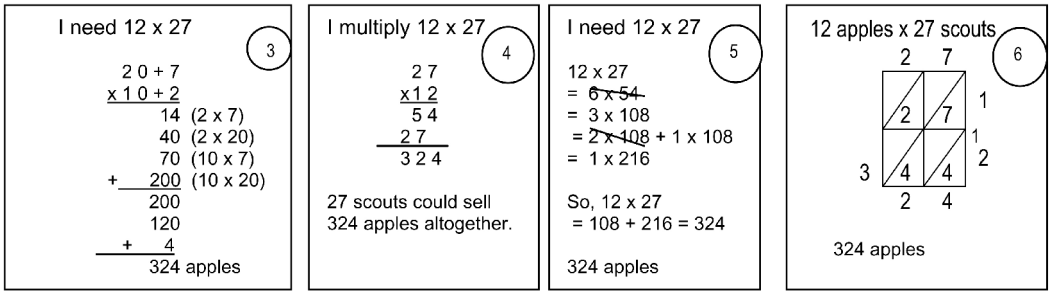
Problem #2: **On apple day, a boy scout troop sells apples in the community. The leaders estimate that each boy will sell about one dozen apples. How many apples could the troop with 27 scouts sell?**

Anticipating a Range of Mathematical Solutions

Solutions 1 to 3 show conceptual models of multiplication. Solutions 4 to 6 show alternative algorithms for multiplication. What mathematics do you see in each of the solutions? How would you organize these for classroom discussion?



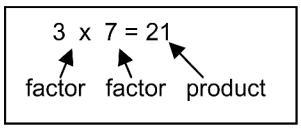




Sample Responses:

- *solution 1 shows a representation on a 100 grid. The 2-digit numbers are decomposed into their component parts: 27 = 10 + 10 + 7 and 12 = 10 + 2. An area model of the product is shown as 2 rectangles with area 100 (or 10 x 10) square units each; 2 rectangles with area of 20 square units each; 1 rectangle with area of 14 square units for a total area of 324 square units.*
- *solution 2 shows the distributive property as a way to get friendly number factors. So, 12 can be thought of as 10 and 2 or 12 x 27 = [(10 x 27) + (2 x 27)] = 270 + 2 x 27. And then 27 is written as 25 and 2 so 270 + (2 x 27) = 270 +[(2 x 25) + (2 x 2)].*
- *solution 3 shows partial products multiplication. Each number is written in expanded form – 27 is 20 + 7 and 12 is 10 + 2. The partial products of 2 x 7 = 14, 2 x 20 = 40, 10 x 7 = 70 and 10 x 20 = 200 are added together to get the product for 12 x 27.*
- *solution 4 shows the "standard" Ontario algorithm with which many people demonstrate procedural application, but lack conceptual understanding about why it works.*
- *solution 5 shows halving one factor and doubling the other factor, so each multiplication expression is equivalent to the line above – unless there is an odd number to be halved – then one factor is not crossed out and is carried to the end where partial products are summed.*
- *solution 6 shows a lattice multiplication where each cell is broken into two parts. The factors are written on the top and right side of the rectangular cells of the table. The multiplication is performed digit-by-digit and each product's digits are placed in the separate triangular cells (representing adjacent place values) below and to the left of the factors. The product is determined by adding digits on the diagonals running from upper right to lower left and written around the table on the left and bottom.*

LET'S DO MATH!
MENTAL MATH STRATEGIES FOR MULTIPLICATION



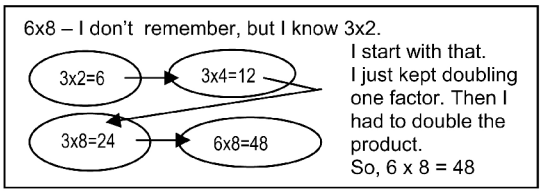
Getting Started

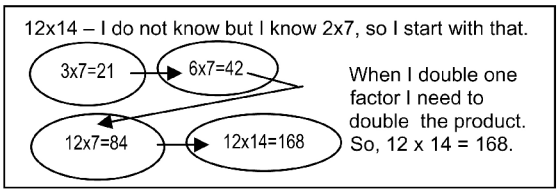
Mental math strategies for multiplication can be developed through work with the multiplication table. When 2 is a factor, we can double.

For the factors 3 and 5, we can skip count. Square numbers have two same number factors, like 6 x 6 = 36. Learning about the relationships between factors and factor pairs will give students flexibility in calculating more complicated products. For example:

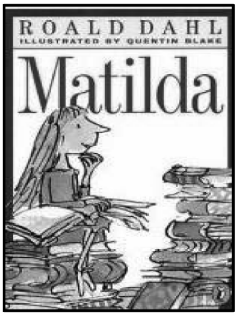
- 6 x 7 = 6 x 6 + 6 x 1 = 36 + 6 = 42 (squares plus 1 factor)
- 5 x 9 = 5 x 10 – 5 x 1 = 50 – 5 = 45 (x multiple of 10 minus 1 factor)
- 8 x 6 = 6 x 8 = 6 x 6 + 2 x 6 = 36 + 12 = 48 (squares plus 2 factors)

Developing number webs (like word webs) is a useful strategy for practising multiplication facts and can be developed as a strategy on a day when your memory fails you or when the facts are too big to simply be recalled from memory.





Problem #3



"Miss Honey said, "… multiply fourteen by nineteen."

"It's two hundred and sixty-six," Matilda said softly.

Miss Honey stared at her. Then she picked up a pencil and quickly worked out the product on a piece of paper. "Now tell me exactly what goes on inside your head when you arrive at an answer."

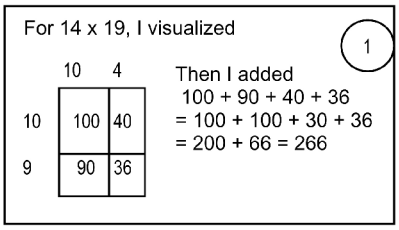
How might Miss Honey use a pencil and paper to figure out 14 x 19?

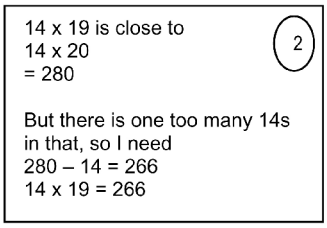
How might Matilda use mental math to figure out the answer?

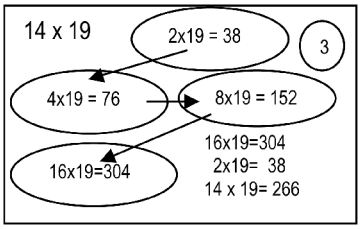
Anticipating a Range of Mathematical Solutions

Mental math means calculating in your head. But, you can have a pencil and paper by your side to record partial products.

Which methods would you be able to do in your head and which methods would you need to do with pencil and paper?







Sample Responses:

- *solution 1 shows the multiplication algorithm done efficiently with, what some call, an area method. The 4 boxes represent the place value multiplications that must be done when you multiply two-digit numbers together. It can be expanded to work with three-digit whole number, decimal, fraction, and algebraic multiplication.*
- *solution 2 shows compensation where 14 x 20 was considered instead of 14 x 19 and it was realized that after the product, the extra product of 14 x 1 would have to be subtracted.*
- *solution 3 shows the use of doubling in a number web to build up from 2x19 to 16x19 and then subtracting the 2x19 extra from the 16x19 to get 14x19.*

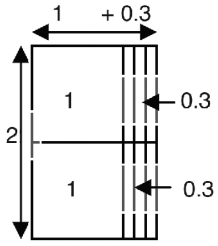
NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

Try these multiplication problems with your students. To suit your students' learning needs, you might adjust the numbers (i.e., larger or small whole or decimal numbers, using fractions or integers) and/or the context of the problem.

Practise noticing the different strategies and the breadth of the mathematics your students use in their solutions. Take note of the number of solutions that show groups of, a set model (groups of, arrays), a region model (area) or a linear model of representation. In preparation for your class discussion of the solutions, decide which solution you will highlight first, second, and third. Think ahead of what mathematical comments you will make to render explicit key mathematical ideas, strategies, and terms, as the students share their solutions. As well, choose 4 student work samples to analyse further in terms of its mathematical details related to multiplication.

What if your students need to learn decimal multiplication? Using similar strategies when making connections between whole and decimal number addition and subtraction, how will you adjust the problems and anticipated solutions for decimal number multiplication?



1.3 x 2
= 1 + 1 + 0.3 + 0.3
= 2.6

Sample Responses:

- *Conceptual representations using arrays - think of the number of groups to be a decimal number and the number of objects in one group to be a decimal number*
- *Conceptual representations using an area model (with linear dimensions) - use decimal numbers to be the linear dimensions.*
- *Mental math strategies - decompose the decimal numbers into products of whole numbers and decimal numbers of tenths, hundredths, and/or thousandths (e.g., 0.3 x 0.6 = 3 x 0.10 x 6 x 0.10 = 3 x 6 x 0.10 x 0.10 = 18 x 0.01= 0.18)*

Suggested Readings

Some readings that we used in our Research Summary and in the development of the multiplication problems and solutions are listed below:

- Carpenter, T., Fennema, E., Franke, M., Levi, L., and Empson, S. (1999). *Children's mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann.
- Davis, B., Simmt, E., and Sumara, D. (2006). Mathematics-for-teaching: The cases of multiplication and division. In Novotna, J., Moraova, H., Kratka, M., & Stehlikova, N. (Eds.). *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol 2, pp. 385-392. Prague, PME.
- Donovan, S. and Bransford, J. (2005). *How students learn: History, mathematics, and science in the classroom.* Washington, D.C. The National Academies Press.
- Fosnot, C. & Dolk, M. (2001). *Young mathematicians at work: Constructing multiplication and division.* Portsmouth, NH: Heinemann.
- Kamii,C. (2000). *Young children reinvent arithmetic: Implications of Piaget's theory* (2nd edition). New York, NY: Teachers College Press.
- Kilpatrick, J., Swafford, J., Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.
- Ma, L. (1999). *Knowing and teaching elementary mathematics.* Mahwah, NJ: Lawrence Erlbaum Associates.

CLOSING

Our next issue will focus on key notions of whole and decimal number division, how teachers need to know it for teaching, as well as, ways that teachers can use these mathematical ideas and pedagogical strategies through problem solving-based teaching and learning strategies.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@edu.yorku.ca or mkestell@sympatico.ca