8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Abacus

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PROPORTIONAL REASONING IN GEOMETRY AND MEASUREMENT

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Geometry and measurement provide many applications for proportionality. Proportional reasoning is used to determine the lengths, area and volume of 2-D and 3-D figures. Think about the use of proportionality in this context. "A rectangle has a length and width that are whole numbers of centimetres and the ratio of its length to its width is 4:5. Its area is 500 square centimetres." Could the problem be, "What are the rectangle's length and width?" or "If another rectangle has the same ratio of its length to its width, 4:5 and is double the area, what is the length and width of this other rectangle? In geometric contexts, proportional reasoning is related to the composition of similar 2D figures. For example, "What are least and greatest number of same-shape pattern blocks that compose 4 hexagons? 8 hexagons? 100 hexagons?"

In this issue, the Research Summary and problems (with multiple solutions) focus on the use of proportional reasoning within measurement and geometry contexts. The problems we provide are designed for use within a three-part problem-solving lesson structure.

LINKS TO LITERATURE AND MANIPULATIVES



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According to Douglas Clements almost 50% of every minute of children's play involves mathematics. This year I had the opportunity to observe students learning in play based kindergartens. Some of the students told me that long necked dinosaurs were plant eaters. Then they sorted the toy dinosaurs into two groups – long necked and not long necked. The dinosaurs were not all the same sized so students were applying the criteria of "long necks" by comparing the length of the necks to the height of the dinosaurs, proportionally. This experience reinforced the connection between measurement and proportionality. In this issue, I will look at using interlocking blocks such as LegoTM to continue exploring proportional relationships through play and a picture book that can be used to continue to investigate proportional relationships in bodies.

RESEARCH SUMMARY – PROPORTIONAL REASONING

According to Lesh, Post & Northern(1988), “proportional reasoning is a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information. Proportional reasoning is very much concerned with inference and prediction and involves both qualitative and quantitative methods of thought” (p. 93). A proportion is a relationship of equality between two ratios (Lobato & Ellis, 2010). In order to reason proportionally, students must not only reason with ratios, but they must also understand that the ratio of two quantities remains constant even as the corresponding values of the quantities change.

Proportional reasoning involves understanding that:

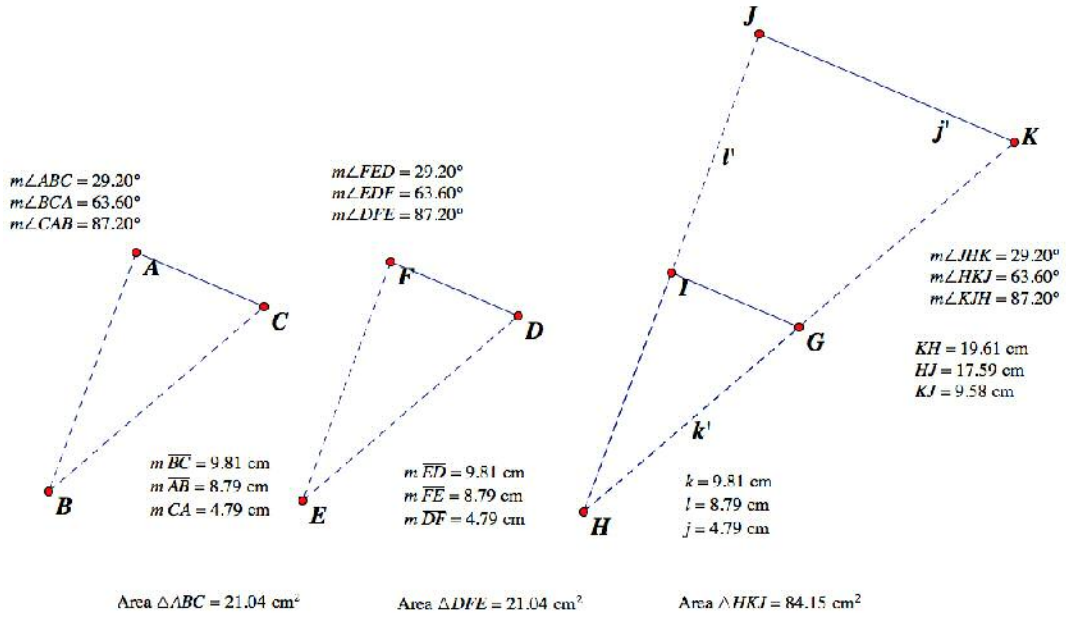
(a) equivalent ratios can be created by iterating and/or partitioning a composed unit and

(b) if one quantity in a ratio is multiplied or divided by a factor, then the other quantity must be multiplied or divided by the same factor to maintain proportional relationships.

So, proportional thinkers have a sense of **co-variation**. There is an understanding of proportional relationships as a change in one quantity, which causes a corresponding change in another quantity. In measurement and geometry contexts, Lobato, Ellis, Charles, & Zbiek (2010) interpret co-variation as a ratio-as-measure (ratio that measures a real-world attribute). A ratio-as-measure involves “isolating attributes and understanding the effect of changing a quantity on the attribute to be measured by the ratio” (p. 23). The attributes measured often include side lengths and angle measures.

Using Ratios to Compare the Attributes of Polygons

Polygons are often compared according to their corresponding side lengths, interior angles and interior area. In these ratios, we are comparing the measure of a side of the larger polygon to the corresponding measure of a side of the smaller polygon.



Similarity Ratio or Scale Factor describes a ratio between corresponding sides. Therefore, similar triangles have the same ‘shape’ but are just scaled differently.

- If two polygons are similar, then the ratio of the measures of any two corresponding sides equals the ratio of their corresponding perimeters. (perimeter of ΔABC is 23.39cm -> perimeter of ΔJKH is 46.78cm, which has a ratio of 1:2).
- Why is the area of the dilated triangle, ΔJKH 4 times the area of ΔABC?

LINKS TO MANIPULATIVES:

The German educator and founder of the Kindergarten movement Friedrich Froebel, developed a series of play materials that included geometric building blocks. In his book *Inventing Kindergarten*, Norman Brosterman argued that several major artists and designers such as Buckminster Fuller, Piet Mondrian, Paul Klee, Wassily Kandinsky, and Frank Lloyd Wright were greatly influenced by their experiences with Froebel's blocks in kindergarten.

While students play with blocks that use an arrangement of studs to connect such as LEGO™ blocks, they are also building their spatial and proportional awareness. Students can use a range of mathematics to compare two blocks.

Comparing the number of studs in the arrays numerically:



Red



Blue

There are 8 studs on the red block and 4 studs on the blue block.

The red block has more studs.

The red block has 4 more studs than the blue block.

The red block has 2 x as many studs as the blue block.

Comparing the blocks using measurement:

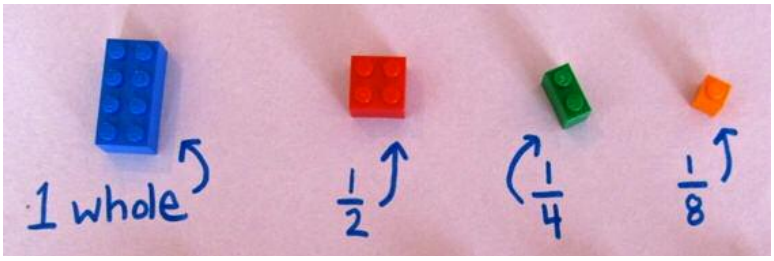
The blue block is half as long as the red block.

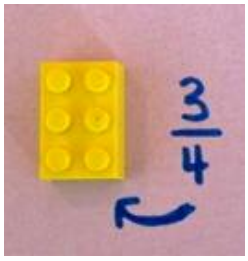
It would take two blue blocks to cover the red block

Comparing the blocks using geometry:

It would take two blue squares to make one red rectangle.

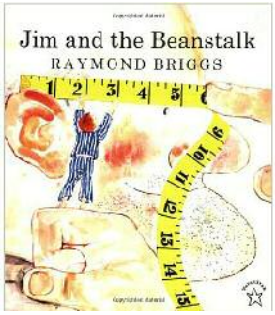
Students can use fractions of a whole, fractions of a set, and ratios to compare the blocks





http://www.scholastic.com/teachers/top-teaching/2013/12/using-lego-build-math-concepts

LINKS TO LITERATURE: *Jim and the Beanstalk*



In this sequel to *Jack and the Beanstalk*, Jim climbs a beanstalk and finds a bald, toothless, near –sighted giant. Jim measures the giant so that he can bring him a wig, dentures and glasses.

This story can be used to set the context for using mathematics to explore:

1. Measurement: Jim spends a lot of time measuring different parts of the giant's body. Find something in the room that is 3 times longer than your thumb. Find an object that has twice the circumference of your arm.
2. Proportional relationships in the body: How many heads tall are you?
3. Critical literacy skills: Measure some of the items in the illustrations from the book. If they were real, do you think they would actually be that size? Why or why not? Explain your thinking.

LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Tangram Runner Problem:

Explain how to make the image of the runner using 7 Tangram pieces.



Solution 1

The head is a square, which has the same area as 2 small triangles (feet). The 2 largest triangles are the shoulders (90^o angle points down) and bottom of the body (90^o angle points up). The left leg is the parallelogram and the right is the middle-sized triangle.

Solution 2

Shape	*# Pieces*	*Location on runner*	*Comparisons*
Small triangle (S)	*2*	*feet*	*2 S = 1 SQ*
"			*2 S = 1 M*
"			*4 S = 1 L = 2 M*
Medium triangle (M)	*1*	*right leg*	*1 M = 1 P*
Large triangle (L)	*2*	*body*	*1 L = 2 M = 4 S*
Square (SQ)	*1*	*head*	*1 SQ = 1 M*
Parallelogram (P)	*1*	*left leg*	*1 P = 2 S*
Total	*7*		

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to make a plan to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Tangram Problem 1:

Examine the relationships of size and shape in a set of Tangrams. If the medium triangle has an area of 4 square units, what is the area of the total Tangram?



What information are we going to use to solve the problem?

- *7 pieces in a Tangram set; Area of all 7 tangram pieces?*
- *Medium tangram -> area of 4 sqare units*

AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Proportional Quantities:

1 medium triangle = 1 square = 1 parallelogram
Ratio i-> 1 : 1 : 1 or M : Sq : P

1/2 medium triangle = 1 small triangle
1 medium triangle = 2 small triangle
Ratio -> 1/2 : 1 = (2 x $\frac{1}{2}$) : (2 x 2)
or $\frac{1}{2}$ M : 1 S = 2x1/2 M: 2 x 1 S

1 medium triangle = $\frac{1}{2}$ large triangle
2 medium triangles = 1 large triangle
Ratio -> 1 : $\frac{1}{2}$ = 2x1 : 2 x 1/2
or 1M : $\frac{1}{2}$ L = 2 x 1M : 2 x 1/2 L

Solution 1

1 Medium triangle (M)
= 4 sq units

2 Medium triangles (M)
= 1 L (large triangle)
=2 x 4 sq units =8 sq units

4 Small triangles (S) = 1 L
=4 x 2 sq units =8 sq units

2 S = 1 P (parallelogram)

2 S = 1 SQ (square)

Identifying Area Proportional Relationships Relative to 1 Shape

Solution 2

M + P + SQ
= (4 + 4 + 4) sq units
= 12 sq units
(or M=P=SQ=4 sq units)

2 S = 2 x 2 sq units
= 4 sq units
2 L = 2x 2 x 4 sq units
= 16 sq units

Total area of Tangram
M+P+SQ+2S+2L =
(12+4+16) sq units
= 32 sq units

Area Proportional Relationships Relative to Several Shapes

Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of students' mathematical discussion.

Grade 4	Grade 5	Grade 6
- describe relationships that involve simple whole number multiplication	- describe multiplicative relationships between quantities by using simple fractions	- represent ratios found in real-life contexts, using concrete materials, drawings and standard fractional notation

Proportional Relationships

As a change in one quantity, which causes a corresponding change in another quantity (co-variation)

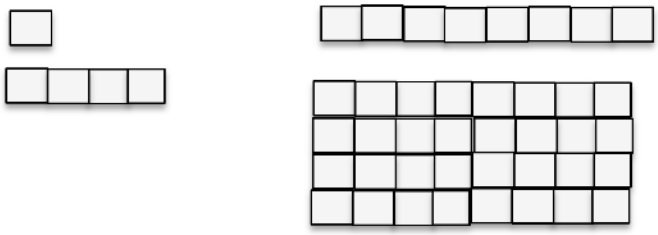
If 1 triangle is 4 square units,
then 8 triangles is 32 square units } *related by a multiple of 8 (scale factor)*

comparing ratio within and between objects

➔ *equivalent ratio* $1 : 4 = 8 : 32$
$1 : 4 = 8x1 : 8x4$

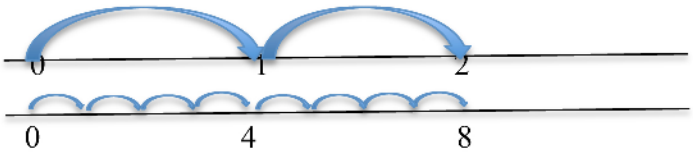
➔ *equivalent fraction* $\frac{1}{4} = 8/32$
$\frac{1}{4} \times 8/8 = 8/32$

If *Then*



1 : 4 = 8 : 32

equivalent ratios by a scale factor of 8



Solution 3

Whole Tangram	*L*	*M*	*S*	*SQ*	*P*
part	*1/4*	*1/8*	*1/16*	*1/8*	*1/8*
sq units	*8*	*4*	*2*	*4*	*4*

2 L + 1 M + 2 S + SQ + P
= (2 x ¼) + 1/8 + (2 x 1/16) + 1/8 + 1/8
= ½ + 1/8 + 1/8 + 1/8 + 1/8 = 1
➔ *8 + 8 + 4 + 2 + 2 + 4 + 4*
➔ *(2 x 8) + (3 x 4) + (2 x 2)*
➔ *16 + 12 + 4 = 32 sq units*

Solution 4

M (Medium Triangle) = 4 sq units and it is 1/8 of the whole trangram

Whole Tangram has units, so 8 x 4 sq units or 32 square units

Fractions of a Whole Proportional to 1 Shape

Unit Fraction Proportionality

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed

Highlights/Summary:

A proportional relationship shows a multiplicative relationship; that is, a change in one quantity which causes a corresponding change in another quantity using a scale factor (co-variation).
(e.g., If 1 triangle is 4 square units, then 8 triangles is 32 square units → scale factor of 8)

Compare quantities using multiplicative relationships by:

- *equivalent ratios (1:4 = 8x1 : 8x4 = 8 : 32)*
- *equivalent fractions: 1/4 = 8x1/8x4 = 8/32*

AFTER (Practice) – 5 to 10 minutes

- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Cat Tangram Problem:

If the tail of the cat is worth 100 points, what is the value of the whole cat?



Solution 1

Medium triangle (M), parallelogram (P), square (SQ) is 100 points; small triangle (S) is ½ of M; large triangle is 2xM

P	*SQ*	*M*	*2xS*	*2xL*
1	*1*	*1*	*2x1/2*	*2x2*
100	*100*	*100*	*100*	*400*

Whole cat tangram is worth 800 points

Solution 2

The parallelogram is 1/8 of the whole Tangram

1/8 Tangram is worth 100 points

Whole tangram has 8 1/8; so 8 x 100 points = 800 points

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.

LET'S DO MATH –



BEFORE (GETTING STARTED)

3 Generations Problem:

In the picture, you can see Josh, his father Jon, and Jon's father Jim.

When you compare their heights they are in the ratio 5 : 6 : 7.

If Jon is 150 cm tall, how tall are the others?

Solution 1:

5 : 6 : 7 = josh height : jon height : jim height
5 : 6 : 7 = josh height: 150 cm : jim height

josh height = (150 ÷ 6) x 5 = 125 cm
jim height = (150 ÷ 6) x 7 = 175 cm

Josh is 125 cm tall. Jon is 150 cm tall.
Jim is 175 cm tall.

Solution 2:

Josh Jon Jim

1 2 3 4 5 6 7

20 40 60 80 100 120 140 ←jumps of 20 too small
25 50 75 100 125 150 175 ←jumps of 25 just right

Grandfather Jim is 175 cm tall, father Jon is 150 cm and son Josh is 125 cm tall.



DURING (WORKING ON IT)

Enlarging Picture Problem:

The Photo Shop prints photos from negatives, smart phones or computers. They offer many sizes of prints. The chart was destroyed by a coffee spill. If a magnification of the small horse photo, 12 x 18 units, takes the length to 30 units, what will be the width and the magnification factor?

Width	8	12	16	20
Length	12	18	24	30

AFTER (CONSOLIDATION)

Solution 1:

I'm going to work in a ratio table to get the bottom number to be 30.

12 ÷ 2 | 6 ÷ 3 | 2 x 10

12	6	2	20
18	9	3	30

18 ÷ 2 | 9 ÷ 3 | 3 x 10

So the dimensions could be shrunk by a factor of 6 (2x3) and then enlarged by a factor of 10. The 12 x 18 photo will now be 20 x 30 units.

Solution 2:

The length changes from 18 units to 30 units.
18 units X factor of change = 30 units
18 units X _____ = 30 units
30 units ÷ 18 = 1.666 This is the magnification factor.

The width will be multiplied by 1.666 too.
l = 1.666 x 12 = 20

The 12 x 18 photo will be enlarged to 20 units x 30 units

Solution 3:

The pictures are similar shapes. Angles are equal and the sides are proportional.

$$\frac{\text{small length}}{\text{large length}} = \frac{\text{small width}}{\text{large width}}$$

$$\frac{18}{30} = \frac{12}{W}$$

OR

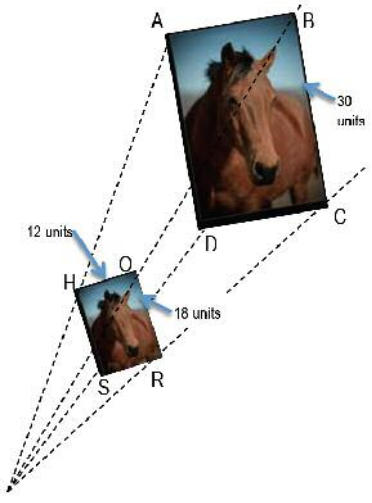
$$\frac{\text{large length}}{\text{small length}} = \frac{\text{large width}}{\text{small width}}$$

$$\frac{30}{18} = \frac{w}{12}$$

$$w = \frac{30x12}{18}$$

$$w = 20$$

The dimension of the blown up photo will be 20 x 30 units.



Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solutions 1 and 3?

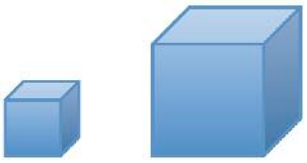
- *Solution 2* – This method honours the similarity of the figures and determines the scale factor necessary to take a length of 18 to a length of 30. Then the factor is used to determine the width using the constant scale factor.
- *Solution 1* – The ratio table sets up equivalent ratios with both width and length represented. Dividing and then multiplying the top and bottom units by the same number lets you determine the width when you know the length.
- Solution 3 – The solution shows equivalent ratios of side lengths because the polygons are similar. Once the ratios are created, the proportion can be solved using equation-solving strategies – whatever you do to one side, you do the same to the other.

LET'S DO MATH –

BEFORE (GETTING STARTED)

Stretching Volumes Problem:

Box A, a cube, has a volume of 1 cm^3. If the lengths of the sides are increased, to make Box B, what is the relationship between the volume of Box A and Box B, which is a box made by increasing the lengths of the sides?



Box A Box B

Solution 1:

Box A Let side length be 4 cm.

Volume (A) = l x w x h

= 4 x 4 x 4 = 64

Box A = 64 cm^3.

Box B is a cube with side lengths of 12 cm (4 x 3)

Volume (B) = 12 x 12 x 12 = 1728

Relationship $= \frac{V(A)}{V(B)} = \frac{64}{1728} = \frac{1}{27}$

So, Box B has 27 times greater volume that Box A.

Solution 2:

Box A = 1 x 1 x 1 = 1 cm^3

Box B = 2 x 2 x 2 = 8 cm^3

When you double the length of sides the area is 2 x 2 x 2 or 8 times bigger.

Solution 3:

Box A = s x s x s = s^3 cm^3

Box B = ns x ns x ns = n^3s^3 cm^3

When you multiply the length of the sides of a cube by any number, n, the volume increases by n^3 cm^3.

DURING (WORKING ON IT)

Suitcase Capacity and Mass Problem:

The ratio of dimensions between the smallest suitcase and the largest suitcase is about 1.4. The mass of an empty suitcase increases proportionally as the capacity of the suitcase increases.

Approximately what is the mass of the largest suitcase if the smallest suitcase has a mass of 3 kg?



Dimensions	L	W	D
small	50	33	20
medium	60	40	24
large	70	46	28

AFTER (CONSOLIDATION)

Anticipating Student Responses:

Solution 1:

V (small) = 50 x 33 x 20 = 33 000 cm^3

V (large) = 70 x 46 x 28 = 90 160 cm^3

Comparison of capacity (ratio) $= \frac{90\,160 \text{ cm}^3}{33\,000 \text{ cm}^3} = 2.7$

Comparing mass $= \frac{large}{small} = \frac{2.7}{1} = \frac{M}{3}$

M = 8.1 kg

Solution 2:

I know that if the side lengths on a polyhedron increase by a factor of n, then the volume / capacity increase by that number multiplied by itself 3 times.

So the capacity increases proportionally in terms of the ratio of each linear dimension:

1.4 x 1.4 x 1.4 = 2.74

So the mass should be 2.74 x 3 = 8 kg

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solutions 2?

Solution 1 – This solution shows the calculation of volume / capacity of both bags and confirms the increase will be factor of 1.4 for each length, then multiplies that factor by the mass.

Solution 2 – This student simply uses the comparison of 1.4 x 1.4 x 1.4 = 2.74 (an increase for length, increase for width, and increase for height) to multiply the mass of the smaller case by 2.74. (3 x 2.74 = about 8 kg).

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Try these proportional reasoning problems yourself first and then offer them to your students.
- To suit your students' learning needs, vary the types of measurements (length, area, volume) in relation to different polygons (e.g., triangle, rectangle).
- Consider also having students build similar polygons using single shapes and combinations of shapes to see proportional relationships.
- Practise noticing the different strategies and the breadth of the mathematics your students use to describe, show and make sense of proportional relationships, in terms of ratios of measurement attributes of 2D and 3D figures.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit key mathematical ideas and strategies for determining proportional ratios of linear, area and volume measurements as the students discuss and analyze each other's solutions to co-construct success criteria for the lesson learning goal.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below:

Cai, J. & Sun, W. (2002). Developing students' proportional reasoning: A Chinese perspective. In B. Litwiller and G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 195-204). Reston, VA: NCTM.

Kent, L., Arnosky, J., & McMonague, J. (2002). Using representational contexts to support multiplicative reasoning. In B. Litwiller and G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 145-152). Reston, VA: NCTM.

Lesh, R., Post, T., & Norther, M. (1988). Proportional Reasoning. In J. Hiebert & M. Behr (Eds.) *Number Concepts and Operations in the Middle Grades* (pp. 93-118). Reston, VA: Lawrence Erlbaum & National Council of Teachers of Mathematics.

Lobato, J, Ellis, A.B. (2010). *Developing essential understanding of frations, proportiosn and proportional reasoning for teaching mathematics, Grades 6-8*. Reston, VA: National Council of Teachers of Mathematics.

Lobato, J., Ellis, A. Charles, R., & Zbiek, R.M. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics in grades 6-8*. Reston, VA: National Council of Teachers of Mathematics.

Small, M. (2010). *Big ideas from Dr. Small, Grades K-3.* Toronto: Nelson Education.

Ontario Ministry of Ontario. (2012). *Paying attention to proportional reasoning, K-12.* Toronto, Ontario: Queen's Printer for Ontario.

Small, M. (2010). *Big ideas from Dr. Small, Grades 5-8.* Toronto: Nelson Education.

CLOSING

Our next issue will focus on proportional reasoning within the patterning and algebra and data management and probability strands. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a problem solving-based teaching and learning lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or mkestell@sympatico.ca