8 PAGE INSERT ELEMENTARY MATH

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PATTERNING AND ALGEBRA, DATA MANAGEMENT AND PROBABILITY

TABLE OF CONTENTS

Abacus Co-Editors Greetings. . . .
RESEARCH SUMMARY 2
- Proportional Reasoning with Patterning and Algebra
LINKS TO MANIPULATIVES . . 3
- Congruent Paper Circles – a manipulative to show parts of a whole
LET'S DO MATH: 5
- Quilt Planning and Good Deed Counts
LINKS TO LITERATURE 6
- *Eggs and Legs*, *Lion's Share* and *The King's Chessboard*
LET'S DO MATH: 6
- Dicey Problems
LET'S DO MATH: 7
- Survey Problems
NEXT STEPS FOR YOUR PROFESSIONAL LEARNING. 8
- Application to Your Classroom
- Suggested Readings

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ABACUS CO-EDITORS



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Proportional reasoning is the underpinning of pattern relationships. Think about the use of proportionality in this context. What is the 5th figure? How does the pattern grow? What changes as the pattern grows? What remains the same as the pattern grows? These kinds of questions provoke students to think about multiplicative relationship between the figure (term) number and the number and type of shapes in each figure (term).



In this issue, the Research Summary and problems (with multiple solutions) focus on the use of proportional reasoning within data management and patterning and algebra contexts. The problems we provide are designed for use within a three-part problem-solving lesson structure.

LINKS TO LITERATURE AND MANIPULATIVES



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As we come to the end of another school year, I thought that I would share some criteria for choosing books so I began by searching the Internet for some of Marilyn Burns' articles about using picture books to teach math. I came across this article on Scholastic's website, "Teaching Math With Picture Books, Part 1" by Alycia Zimmerman (2012). In this article, the author classifies books and sorts them into three categories: (a) fundamental math picture books in which the math content is the primary purpose of the book (b) embedded math picture books in which the plot has deliberate connections to math, but the story stands on its own and (c) connected math picture books in which the connections to math are not explicit or obvious, but can be made by problem posing and inquiry. In this issue, I will use graphs to represent the math in some "fundamental" and "embedded" math picture books. See if you can tell which titles belong to which category. I will also share a math manipulative that appears in the English translation of a Japanese textbook.

RESEARCH SUMMARY – PROPORTIONAL REASONING WITH PATTERNING AND ALGEBRA

Proportional reasoning involves the deliberate use of multiplicative relationships to compare quantities and to predict the value of one quantity based on the value of another. The development of proportional reasoning is a gradual process, underpinned by the multiplicative thinking and the capacity to compare two quantities in relative (multiplicative) rather than absolute (additive) terms (Small, 2010a, 2010b). More specifically, "proportional reasoning [is] a form of mathematical reasoning that involves co-variation sense and multiple comparisons, as well as the aptitude to mentally gather and process several groups of information … it is related with inference and prediction and it involves qualitative and quantitative thought (Lesh, Post, & Northern, 1988, p. 93).

Types of Proportional Relationships

There are three main types of proportional relationships:

(a) *Simple Direct Proportion (or Direct Variation)*

A direct proportion occurs when two quantities change in the same direction, that is, as one quantity increases (or decreases), the other quantity also increases (or decreases), but the relative size of the change stays the same (Cai & Sun, 2002). For example, as the number of cans increases and the price of cans increases with a relative size of change, or the c*onstant of proportionality,* is $0.52

Number of cans	1	2	5	10	100
Price of cans	$0.52	$1.04	$2.60	$5.20	$52.00

(b) *Inverse Proportion (or Inverse Variation)*

Inverse proportion is when one quantity increases and the other quantity decreases and the relative size of the change stays the same (Cai & Sun, 2002). For example, as speed increases, the travel time decreases and as speed decreases, the travel time increases or speed is inversely proportional to travel time. In another example, suppose it takes 6 students, 4 days to complete the stuffing of school flyers in envelopes. Suppose it takes 4 days to complete the stuffing of 1000 envelopes. If we double the number of students, the stuffing should be completed in half the time. Another way of looking at this situation is to look at the product: as one factor increases (or decreases), the other factor decreases (or increases), but the product remains the same (e.g., number of students/days is # students x # days = 24)

Number of days	2	3	4	6
Number of students	12	8	6	4
Number of students/day	24	24	24	24

(c) *Complex Proportion (or Joint Variation)*

A complex proportion situation occurs when proportions involve more than two variables, and one of the variables may change simultaneously, in the same direction as one and in the opposite direction from the other.

Types of Representations for Proportional Relationships

Proportional reasoning includes explaining, interpreting and recognizing proportionality between two or more quantities that are being compared. Such proportional relationships are often represented as:

(a) *Equations* - represent a proportion by expressing two ratios as equal. An example is comparing 2 quantities such as 4 boys : 8 children = 1 boy : 2 children or 4/8=1/2.

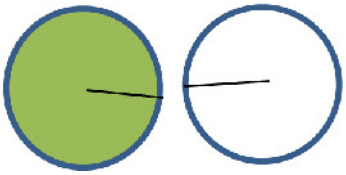
(b) *Ratio Tables* - are a convenient way of keeping track of the quantities within proportional situations, and for supporting proportional thinking strategies (e.g., the use of halving, doubling, multiplying by 10) and number strategies. For example, if there is 1 sports team per 14 children, how many children can are there on 14 sports teams?

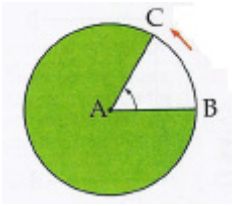
Number of teams	2	10	5	15	14
Number of children	28	140	70	210	196

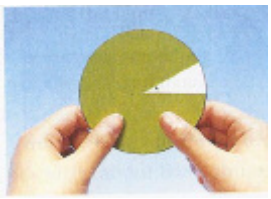
(c) *Table of Values and Graphs* - enable students to see both direct and inverse proportional relationships. To see proportional relationships between 2 quantities, students record data in a table and graph the results on a Cartesian coordinate plane. When coordinate pairs of a simple direct proportion are plotted, the points all lie on the same line, either with an increasing trend (simple direct proportion) or decreasing trend (inverse proportion). In fact, all direct proportional relationships are linear relationships. More specifically, Lobato, Ellis, Charles and Zbiek (2010) suggest that students consider ordered pairs to be ratios, that is, multiplicative comparisons, and think about how many times greater each term value (*y*-value) is than the corresponding term number (*x*-value). This constant of proportionality (or constant of variation) is referred to as the slope of the line.

LINKS TO MANIPULATIVES: CONGRUENT PAPER CIRCLES

This manipulative is an easy to make, inexpensive yet powerful tool. Cut two congruent circles from two different coloured paper (the size doesn't matter). Then, cut a slit in each circle from the circumference of the circle to the centre. Join the two circles together using the slits.







Provide each student with congruent paper circles and challenge them to describe the mathematics they see:

- I see fractions – one fourth, one half
- I see angles – right angle, obtuse angle, acute angle, straight angle
- I see how I could make a circle target to use as a spinner for probability

The manipulative is useful in the "before" part of a lesson as each student can respond to challenges (e.g., show what could a spinner could look like if it had two colours (green and white) and represents a situation that it is less likely to land on green than on white.) Students can share and compare responses and analyse the solutions (e.g., each solution is more than half white) Use the manipulative to make connections between proportional reasoning and spinners and circle graphs.

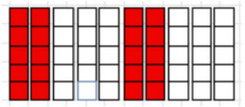
LESSON LEARNING GOALS
Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

Grade 5	Grade 6	Grade 7
-describe multiplicative relationships between quantities using fractions/decimals -simple multiplicative relationships involving whole number rates	-represent relationships using unit rates -identify quantities in an equation that vary and those that remain constant	-demonstrate understanding of rate as a comparison or ratio of two measurements with different units; solve problems involving calculations of unit rates

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Quilting Planning Problem A:

A quilt has 60 pieces. Every 5 pieces in a quilt consists of 2 red pieces and 3 white pieces. How many pieces are there of each colour?



Solution 1
2 red ; 3 white → 5 total
4 red : 6 white → 10 total
6 red : 9 white → 15 total
8 red : 12 white → 20 total
20 red : 30 white → 50 total

24 red
36 white
60 total

same

Solution 2

Sets	1	2	3	4	5	6	7	8
Red	2	4	6	8	10	12	14	16
White	3	6	9	12	15	18	21	24
Total	5	10	15	20	25	30	35	40

Total of 20, 12 white, 8 red
Total of 30, 18 white, 12 red
Total of 60, 18x2 white, 12x2 red (36+24=60)

x2

x6

1:6
2:12
3:18
5:30

5 groups of (2+3) -> 25
10 groups of (2+3) -> 50
12 groups of (2+3) -> 60

Multiplicative Relationship *– How many times larger is the y value (deeds) than the x-value (days)*
-> x-value X 1.5 = y-value (between 2 quantities)

Equivalent (Part-Part) Ratios *are related by ratio constant or constant of proportionality … X 1.5*

for 4 days -> 6 deeds
4x 1.5 = 4x1+4x.5 = 4+2 = 6
for 8 days-> 12 deeds
8x1.5 = 8.1+8x.5 = 8+4 =12
for 12 days -> 18 deeds
…
for 32 days -> 48 deeds
32x1.5 =32x1+32x.5 =32+16= 48

 # days x 1.5 good deeds

4:6
8:12
12:18
16:24
20:30
24:36
28:42

part-part equivalent ratios x:y

Solution 1
days
4 8 12 16 20 24 28 **32**
6 12 18 24 30 36 42 **48**
deeds

Solution 2
x-value y-value

days	deeds
4	6
8	12
12	18
16	24
20	30
24	36
28	42
32	48

In-Between 2 Quantities
Using Double Number Line
4 X 1.5 = 6

In-Between 2 Quantities
Using Table of Values
x-value X 1.5 = y-value

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to make a plan to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Tangram Problem 1:



Travis does 6 good deeds every 4 days. How many good deeds will he do in 32 days?

What information are we going to use to solve the problem?
- *4 days with 6 good deeds*
- *32 days – How many total good deeds to do?*

AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Proportional Relationship
As a change in one quantity, which causes a corresponding change in another quantity with size of change staying same (co-variation – across sets of quantities)
→ 4 days/6 deeds -> X 8/8 -> (4x8)/(6x8) = 32/48

Simple Direct Proportion (or Direct Variation)
- two quantities change in the same direction
- as one quantity (# of days) increase, the 2nd quantity increases (# of good deeds)
→ 4:6 (X4/4) … 16:24 (X2/2) … 32:48
X 8/8

1 = 2/2 = 4/4
= 8/8

Does the order of the ratio matter to predict?
6:4 (x4/4) … 24:16 (X2/2) … 48:32
→ # good deeds : # days … y-value : x-value

Equivalent Ratios
-multiplicative relationship between x:y is X 1.5
4:6, is 4x1.5= 6
8:12 is 8x1.5=12
32:48 is 32x1.5=48

Predict Quantity Across Sets of Quantities
4 : 6
4(÷4) : 6 (÷4)
1 : 1.5 *unit rate*
1(X32) : 1.5 (X32)
32 : 48
x : y

Rate *is a special ratio – comparing 2 quantities of different units (days/deeds)*

Equivalent Rates
4 days / 6 deeds (÷4/4)
1 day / 1.5 deeds (X32/32)
32 days / 48 deeds

Solution 3

	÷4		X32	
Days	4	1	16	32
Deeds	6	1.5	24	48
	÷4		X32	

Divide both by 4/4 and multiplied up by 32/32

Solution 4
equivalent rates
4 days / 6 deeds = 32 days / ??
X 8
4 days / 6 deeds = 32 days / 48 deeds
X 8

Across Sets of Quantities
Ratio Table
Using Unit Rates

Across Sets of Quantities
Predicting Unknown Rate
Using Equivalent Rates

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed

Highlights/Summary:

A simple direct proportion shows a change of 2 quantities in the same direction with size of change staying the same.

Relate and predict rates using proportional relationships by comparing:
- *in-between quantities (part-part equivalent ratios related by ratio constant/constant of proportionality)*
  - *double number line (4 days x 1.5 = 6 deeds)*
  - *table of values (x-value x 1.5 = y-value)*
- *across sets of quantities (multiplicative relationships between equivalent rates)*
  - *ratio table – all rates are equivalent – multiply and divide by 1 (1=4/4=32/32)*
  - *equivalent rates - 4 days/6 deeds x8/8 =32 days/48 deeds*

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Good Deeds Problem B:

Clifford does 6 good deeds every 4 days (same as Travis). How long will it take Clifford, working at the same rate, to complete 21 good deeds?

Solution 1

	÷2		x6	
Deeds	6	3	18	21
Days	4	2	12	14
	÷2		x6	

3 + 18 = 21
2 + 12 = 14

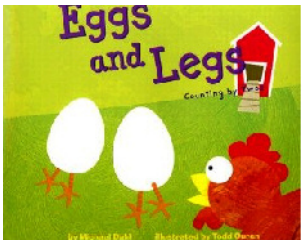
Solution 2

6 deeds	18 deeds	3 deeds
4 days	12 days	2 days

3/2 x 7/7
14 days he could do 21 good deeds.

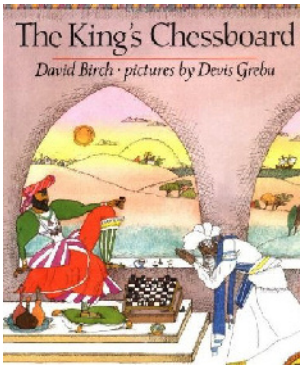
Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.



Michael Dahl has a collection of counting books, by ones, twos, fives and tens in his "Know Your Numbers Series" (e.g., titles for counting by twos; Pie for Piglets, Footprints in the Snow and Eggs and Legs.) The book Eggs and Legs introduces counting by twos by counting the number of legs coming out of the eggs and then create a bar graph of counting by twos to visualize the +2 growing pattern.



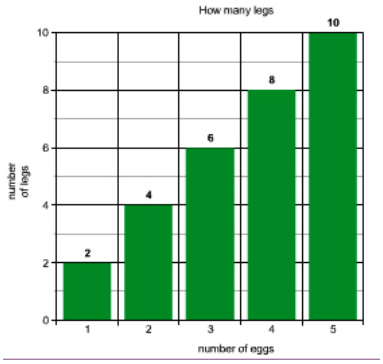


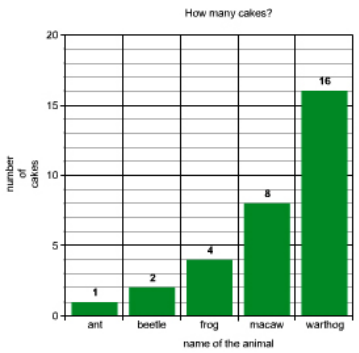
LINKS TO LITERATURE:

Representing the changes in the numbers, in bar graphs, in picture books is a strategy that can be used to compare growing patterns.

According to the expectations in the patterning and algebra strand, students in grade 2 are expected to "represent a given growing or shrinking pattern in a variety of ways (e.g., using pictures, actions, colours, sounds, numbers, letters, number lines, **bar graphs**)."

Compare bar graphs for counting by twos (from the book, *Eggs and Legs*) to a bar graph that shows doubling (from the book, *The Lion's Share*). *The King's Chessboard* also has a growing pattern that is based on doubling.





Students in junior grades can use equivalent ratios to describe the counting by twos pattern (e.g., 1 egg: 2 legs; 2 eggs: 4 legs; 3 eggs: 6 legs) and fractional language to describe the proportional relationships in the doubling pattern (e.g., the frog is making one fourth as many cakes as the warthog.)

LET'S DO MATH – DICEY PROBLEMS

Before (Getting Started)



Dicey Problem 1:

What do you notice about the number cube? If you roll it 10 times, how many times do you think a 5 will land face up?

Solution 1

The number cube has 6 sides numbered 1, 2, 3, 4, 5, and 6.

If you roll it 6 times each number should come up once. If you roll it 10 times a 5 might come up twice.

Solution 2

Probability (5) = 1/6
If you roll it 12 times, you should get 5 two times.
If you roll it 18 times, three 5s should come up.

So, rolling the cube 10 times suggests you will get a 5 one or two times.

During (Working On It)

Dicey Problem 2:

If you win when you roll an even number less than 5 on a die (same as number cube), what is the probability that you will win in a game involving 100 rolls of the die?

Scaffolding Questions

- *How many times should you win when the die is rolled 6 times?*
- *How many times should you win when the die is rolled 60 times?*



After (Consolidation)

Anticipating Student Responses:

Solution 1

I win when 2, or 4 are rolled. So, in 6 rolls I win 2 times or 2/6 or 1/3 of the time.

In 60 rolls, I should win 20 times.
(60 x 2/6 = 20)

In 100 rolls, I should win about 33 times.
(100 x 0.33 about 33 times)

Solution 2

# rolls	12	24	36	72	96	4	100
# times get even # < 5	4	8	12	25	32	4/3	33.3

In 100 rolls, I should win 33-34 times times.

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solution 2?

- *solution 1* – based on proportional reasoning as a winning roll comes up 1 in 3 times or in the ratio of 1 : 2. So for 100 rolls the ratio would be ((1 x 33.3) : (2 x 33.3) = 33.3 : 66.7)
- *solution 2* – ratio table shows the relationships between the numbers growing or shrinking at the same rate. So when we know 96 rolls + 4 rolls makes 100 rolls, we know 32 + 4/3 = 33.3 successful rolls.

LET'S DO MATH – SURVEY PROBLEMS

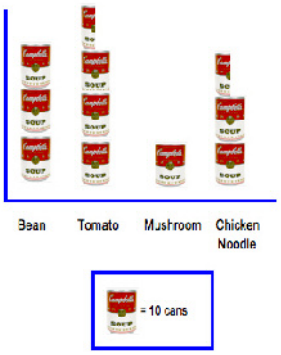
Before (Getting Started)

Survey Data Problem 1:

Examine the graph. What do you notice?

Compare the numbers of different kinds of soup. Give evidence.

Favourite Kinds of Soup



Solution 1

- I see there are more cans of Tomato soup than any other kind (35). (1 picture = 10 cans; 3 pictures = 30 cans and Half a picture = 5 cans)
- The fewest number of cans are mushroom (10). (1 picture = 10 cans)
- 25% of the cans are chicken noodle. (10 whole pictures = 100 cans The ratio of (chicken noodle) : (all kinds) = 25 : 100 or 25/100 or 25% so 25% of the choices are chicken noodle.

Solution 2

- I see a legend that says a picture of a can of soup represents 10 cans. So, the total can be considered 100% (per cent)
- So, I think the image of the half-cans must represent 5 cans
- In order, the least popular to most is: mushroom 10; chicken noodle 25; bean 30; and tomato 35.
- The ratio of bean : chicken noodle = 30 : 25. If this ratio is maintained, and there were 100 chicken noodle cans then there would be 120 cans of bean.

During (Working On It)

Survey the Survey Data Problem 2:

Students at DeGrassi Middle School collected cans and packages of food for the food drive for Second Harvest. The chart to the right reports the results of their month-long effort.

Food	Number of items
Can fish	5
Can beans	45
Can soup	25
Rice bags	30
Can fruits	10
Peanut butter jars	35
Can vegetables	30
Macaroni & cheese	20

Two other schools joined the food drive. Northern school matched DeGrassi's total and East Meadows school had twice as many of each item than DeGrassi. How many items, in total did the 3 schools collect?

If Second Harvest estimates they need 100 bags of rice for the weekend, would they have enough from the 3 schools?

After (Consolidation)

Anticipating Student Responses:

Solution 1

The total number of items that DeGrassi collected:

In pairs: 30 + 40 + 75 + 55 = 200 items

Northern total = 200 items
East Meadows = 2 x 200 = 400 items
Total 200+200+400 = 800 items

Rice : 30 + 30 + (30 + 30) = 120
So yes, Second Harvest has enough rice for the weekend.

Solution 2

DeGrassi : Northern : East Meadows = 1 : 2 : 1 So, altogether there would be 4 times what DeGrassi collected.

Total items = 200 x 4 = 800 items
Or 1 : 2 : 1 = 200 : 400 : 200 with a total of 800 items

Rice: Well it says that Northern's total was the same as DeGrassi's but not that each item was matched so for sure 3 x 30 is 90 bags of rice but we can assume that there would be at least 10 compared to 30 bags of rice in the collection at Northern so Second Harvest would have enough rice.

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion, followed by solution 2?

- *solution 1* – shows the amounts taken for each school and added together.
- *solution 2* – shows the ratios comparing the 3 schools' totals as a proportion 1 : 2 : 1 = 200 : 400 : 200.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Try these proportional reasoning problems yourself first and then offer them to your students.
- To suit your students' learning needs, choose growing and shrinking (linear relations) problems that are relative to the grade specific curriculum expectations in the Patterning and Algebra or Data Management and Probability strands.
- Provide problems that have students explain, extend, predict and create growing patterns in order to use multiplicative reasoning in order to see proportional relationship between the term number and term value in terms of its expanded arithmetic structure.
- Practise noticing the different strategies and the breadth of the mathematics your students use to describe, show and make sense of proportional relationships, from ratio tables, table of values (term number, term value, arithmetic structure of term value), graphs (relationship between x and y values).
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit the multiplicative relationships between term number and term value of a growing or shrinking linear pattern as the students discuss and analyze each other's solutions to co-construct success criteria for the lesson learning goal.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below:

- Cai, J. & Sun, W. (2002). Developing students' proportional reasoning: A Chinese perspective. In B. Litwiller and G. Bright (Eds.), Making sense of fractions, ratios, and proportions (pp. 195-204). Reston, VA: NCTM.
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- Small, M. (2010b). Big ideas from Dr. Small, Grades 5-8. Toronto: Nelson Education.

CLOSING

Our next issue will focus on spatial reasoning within geometric and measurement contexts. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within learning and teaching mathematics within a three-part problem solving lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or mkestell@sympatico.ca