8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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SPATIAL REASONING WITHIN GEOMETRY CONTEXTS

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A Publication of the OAME/AOEM

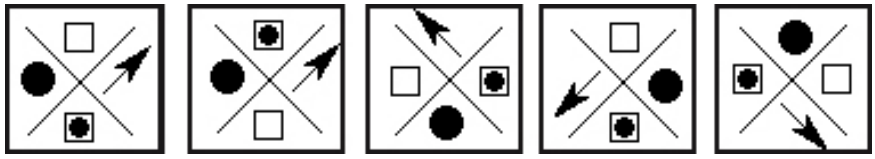
ABACUS CO-EDITORS



MARY LOU KESTELL works on K to 12 mathematics professional learning at the provincial level. She has spent more than 30+ years working in all aspects of Ontario mathematics education and is a past-president of both OAME and OMCA. **mkestell@sympatico.ca**



KATHY KUBOTA-ZARIVNIJ currently focuses on K to 6 mathematics professional learning at the provincial level. She is a long-time OAME board member and OAME past president. In relation to her work in mathematics education, she uses complexity thinking to interpret mathematics teaching and learning for students and teachers. **kathkubo@rogers.com**



Which diagram is a rotation of another? How did you decide?

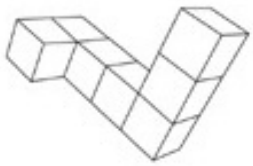
- Did you choose one shape and compare the shape to the right and left of it for each diagram?
- Did you imagine spinning each diagram about its centre or the intersection of the 2 line segments to see which diagrams look the same?
- Did you imagine, in your mind, a rotation of 1 or 2 turns counter-clockwise or clockwise (i.e., 90˚ or 180˚ CW or CCW) to match it with a diagram?
- Did you move your body around the diagram?

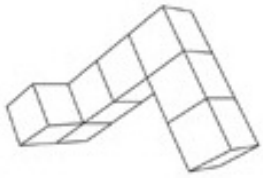
Any of the strategies listed above constitute the use of spatial reasoning, such as: moving one's body in space, perspective-taking, comparing object attributes, imagining moving objects in space. More specifically in the geometry and spatial sense curriculum, spatial reasoning includes: analyzing attributes, properties and geometric relationships for two- and three-dimensional geometric shapes and geometric relationships; describing locations and spatial relationships using coordinate geometry; recognizing and applying transformations (i.e., translations, rotations, reflections, dilatations); recognizing and creating shapes containing line and rotational symmetry; visualize mental images of geometric shapes; and recognize and represent two- and three-dimensional shapes from different perspectives.

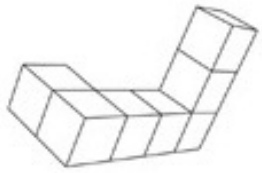
In this issue, the Research Summary and problems (with multiple solutions) focus on the use of spatial reasoning within geometry contexts. The problems we provide are designed for use with a three-part problem-solving lesson structure.

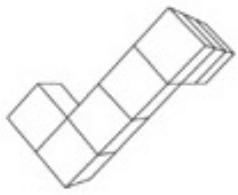
RESEARCH SUMMARY – SPATIAL REASONING WITHIN GEOMETRY CONTEXTS

Bahr and de Garcia (2010) describe spatial reasoning as thinking and reasoning by comparing, manipulating and transforming mental pictures through spatial visualization (i.e., comparing shapes that have changes in position on the plane or in space) and spatial orientation (i.e., the capacity to look at a fixed figure from several different points of view).









According to Linn and Peterson (1985), spatial thinking is not a unitary construct; it is a collection of specific skills, such as representing, transforming, generating and recalling symbolic, nonlinguistic information. More specifically, spatial thinking consists of: (1) mental rotation of two and three-dimensional figures; (2) spatial perception which includes determining spatial relationships with respect to the orientation of one's own body; (3) spatial visualization which includes multiple manipulations of spatially presented information derived from the analysis of the relationship between different spatial representations.

Further, Hegarty and Waller (2004) identified a dissociation of abilities involved with spatial orientation from those involved with spatial visualization, specifically mental rotation. They suggest that the dissociation between tests of perspective taking and mental rotation reflects a distinction between ability to make egocentric spatial transformations (i.e., to imagine the results of changing one's egocentric frame of reference with respect to the environment) and making object-based transformations (i.e., to imagine the results of changing the positions of objects in the environment, while maintaining one's current orientation in the environment).

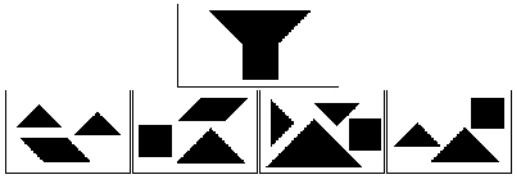
The National Research Council (2006) states that "spatial thinking is a collection of cognitive skills … it is a constructive amalgam of three elements: concepts of space, tools of representation, and processes of reasoning … It is the concept of space that makes spatial thinking a distinctive form of thinking" (p. 12).

1) *Space* – Properties of space include dimensionality, continuity, proximity and separation. In a geometry context, the nature of spaces includes the number of dimensions (two-or three-dimensional) and the kind of coordinate system, such as the Cartesian plane.
2) *Representation* – We use different spatial structures (e.g., maps, multidimensional scaling models (e.g., house floor plan, elevation house plan, orthogonal or perspective maps), dynamic geometry software like Geometer's Sketchpad, and computer-assisted design (CAD)) to help us to perceive, analyse static (two-or-three-dimensional) and dynamic (transformational) properties of objects and the relationship between objects. "These structures help us to remember, understand, reason and communicate about the properties and relations between objects represented in space, whether or not those objects themselves are inherently spatial." (p. 27)
3) *Reasoning* – Reasoning processes include: (a) extracting spatial structure (i.e., boundary, pattern, cluster, centre, path, surface, network, sequence, hierarchy, region); (b) performing spatial transformations (e.g., perspective, rotation, 3D, 2D, scale, interpolation, generalization); (c) drawing functional inferences (e.g., spatial correlation, spatial dependence heterogeneity, extrapolation, prediction, causation).

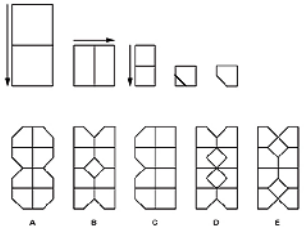
Examples of Spatial Reasoning Tasks or Problems with a Geometry Context

Wright, Thompson, Ganis, Newcombe and Kosslyn (2008) state that the performance of spatial tasks can be improved through practice and training to maximize learning.

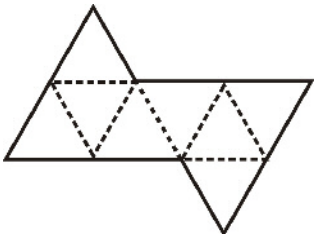
Two Dimensional and Three-Dimensional Figures



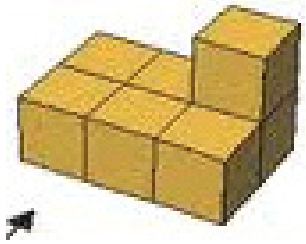
Which set of shapes can make the hexagon?



Which shape is a result of the cut-out sections of the 4 layer square?



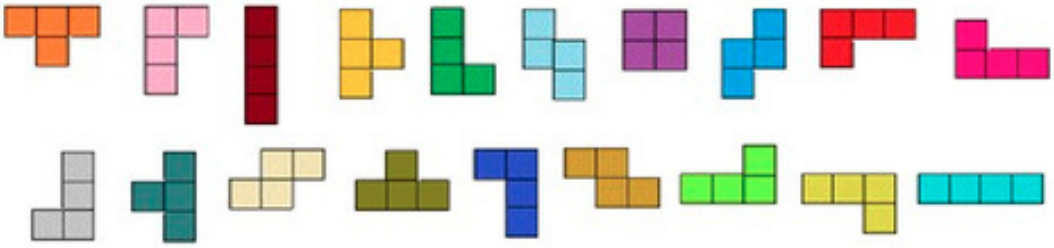
When this net is folded up, what does the 3D object look like?



Sketch 3 different views of this cube structure.

Transformations

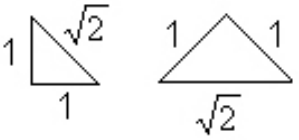
Which shapes are different? Which shapes are the same? Use transformations to explain your choices.



Position in Space

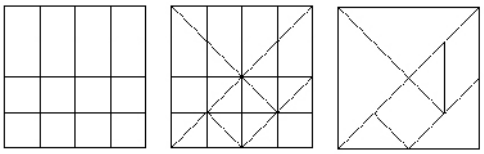
B is on the right of A. C is on the left of B. D is in front of C. E is in front of B. What is the relationship between D and E?

LINKS TO MANIPULATIVES: TANGRAMS



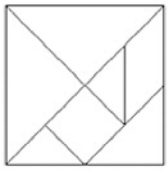
The tangram (Chinese: 七巧板; or *Qi Qiao Ban*) literally means "seven boards of skill or cunning" is a dissection puzzle consisting of 7 flat (2-D) shapes, called *tans*. All tans are made using squares and isosceles triangles (made from half a square). The 7 tangram pieces (or tans) comprise a dissection puzzle of 5 triangles, 1 square and 1 parallelogram that can be put together to form various shapes given only an outline or silhouette. So, 7 tangram pieces are composed of 32 half squares from 16 whole squares.

Tangram pieces (tans) are made by decomposing a 4x4 square into pieces. Some possible spatial reasoning tasks include:



- Explain how to make a tangram from a 4x4 square, by cutting paper or by paper folding.
- If paper folding, what are the least number of folds needed to make the 7 tans?

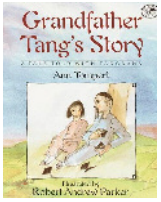
LINKS TO LITERATURE: TANGRAM STORIES

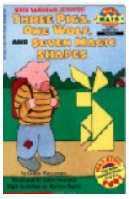


There are several kinds of stories that include the use of tangrams to recognize and apply transformations which require spatial reasoning (e.g., spatial memory and spatial visualization skills) and to become more precise in their use of vocabulary about space. For any of these stories, students can create tangram shapes (e.g., objects, animals, people) and explain the transformations they used

to move the tansfrom being a square to the new tangram shape.

- *Logicville* has tangrams stories with narratives that include different tangram illustrations, such as *The Story of a Crown and Bottle, The Boy Who Cried Wolf, The First Christmas, Weighing an Elephant, Twelve Days of Christmas* (**http://www.logicville.com/puzTS-A1.htm**)
- *Legend of the Tangram* (**http://www.jumpintoabook.com/2013/01/grandfather-tangs-story-storytelling-with-tangrams/**)
- *Grandfather Tang's Story* by Ann Tompert is about a grandfather who is telling a story using tangrams as illustrations.
- *Three Pigs, One Wolf, Seven Magic Shapes* by Grace Maccaroneis about 3 pigs, which use the 7 magic shapes to form solutions to problems they encounter with Big Brad Wolf.





LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

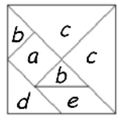
Grade 5	Grade 6	Grade 7
-create and analyse designs by translating and/or reflecting a shape or shapes using a variety of tools)	-create and analyse designs made by reflecting, translating, and/or rotating a shape or shapes by 90^0 or 180^0	-create and analyse designs involving translations, reflections, dilatations and/or simple rotations of two-dimensional shapes using a variety of shapes

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

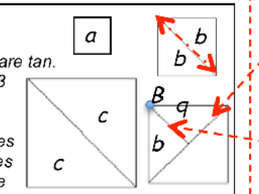
Tangram Shape Problem A:

What different squares can be made using different number of tans.
Explain your strategies.



Solution 1

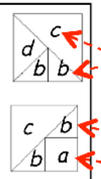
Start with 1 square tan. Choose 2 then 3 tans to make a square
- *2 small triangles*
- *2 large triangles*
- *1 small triangle and 1 large triangle.*



Visualize:
-cut a square in half along the diagonal.
-rotate 1 small triangle b about point, B, 90^0 CW to make 1 square

Solution 2

-2 same size triangles make a larger triangle; rearrange 4 tans to make groups of triangles;
-2 right triangles make a square.
-choose triangle and square tans only to make squares



Imagine how:
-4 small triangles make 1 large triangle
-2 small triangles make 1 square

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to make a plan to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Tangram Shape Problem B:

What different rectangles (not including squares) can be made using different number and combination of tans. Explain your strategies.

What information are we going to use to solve the problem?
- *rectangles have opposite same side lengths*
- *rectangle is an area enclosed by 4 straight sides with 90^0 corner angles*

AFTER (Consolidation) – 20 to 25 minutes

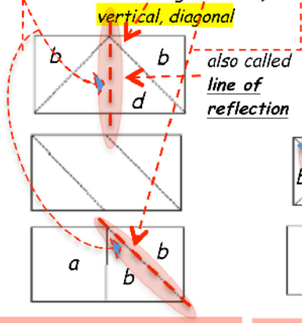
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1^{st}, 2^{nd}, 3^{rd}, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Visualize (form a mental image)

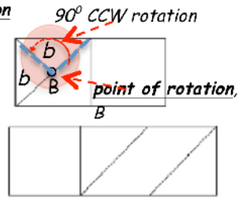
then check by modelling with tangrams.

Reflection *- **flip** a 2-D shape over a line of symmetry and the 2-D shape remains the same size and facing the opposite direction like a mirror image (or the orientation of image is reversed from the pre-image), like a mirror image*

Line of Symmetry
(divides a 2-D shape into two mirror-image halves):



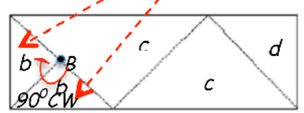
Rotation *– is a circular movement of 2-D shape around a centre (or point) of rotation. Track a rotation (CW or CCW) by measure of **turn** or angle measure by a side length.*



3 tans
4 small triangular units

4 tans
6 small triangular units

Transformations *include reflections (flip), rotations (turn) and translation (slide) of 2-D shapes, where the **pre-image** (shape prior) and **image** (shape after) of the 2-D shape is congruent (i.e., same size, area, angles and polygon slide lengths).*

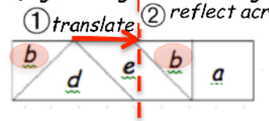


*270^0 CCW or 90^0 CW **rotation** about point, B*



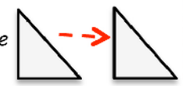
Diagonal Reflection *about line of reflection*

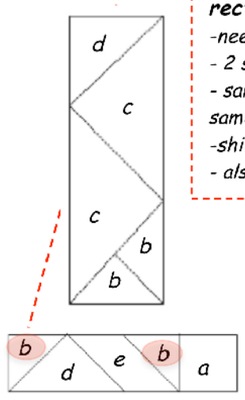
Composition of Transformations *– result of two or more transformations combined, forming a new transformation:*
- → *1st transformation produces an image upon which the second transformation is then performed.*
- → ***Glide Reflection*** *- composition of a translation and reflection (e.g., triangle b to triangle b)*

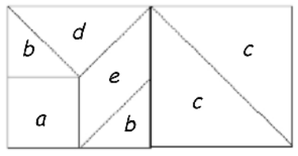


Or reflect then translate

Why do none of the tangram rectangles show a translation?
-need 2 same size 2-D shapes
- 2 sets of same size triangle tans
- same size triangles need to be in the same orientation
-shift or slide a shape
*- also called a **glide***







5 tans
8 to 12 small triangular units

7 tans
16 small triangular units

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed

Highlights/Summary:

Translations are slides, rotations are turns and reflections are flips of two-dimensional (2-D) shapes, which are congruent.

Analyse tangram designs using transformations:
- ***Reflections*** *– flip across a line of reflection a 2-D shape where its image is reversed from its pre-image*
- ***Rotations*** *– turn of 2-D shape around a centre of rotation*
- ***Translations*** *– slide a 2-D shape horizontally, vertically or diagonally*

AFTER (Practice) – 5 to 10 minutes

- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Tangram Shapes Problem C:

What ways can a trapezoid be made using 7 tans only?

Solution 1

Keep the 2 large triangles on the outer edges of the trapezoid.



Solution 2

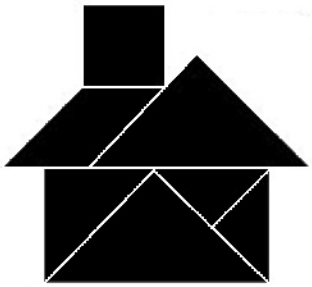
Think of a rectangle and right triangle.



Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.

LET'S DO MATH – PUZZLE COMPARING PROBLEMS



Before (Getting Started)

Puzzle Comparing Problem 1:

What shapes do you see? How do their dimensions compare?

Solution 1

I see:

- 2 big triangles
- 1 medium triangle
- 2 small triangles
- a rhombus
- a small square

Solution 2

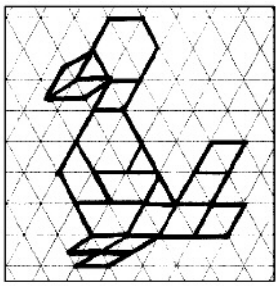
I see:

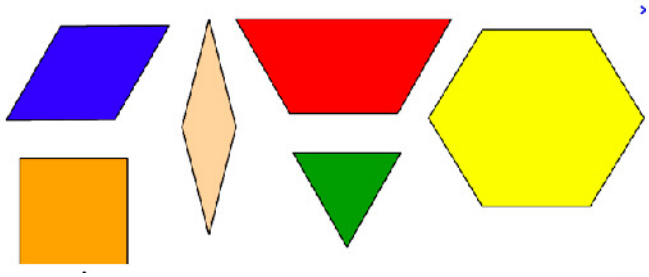
- 2 congruent big triangles
- one side of the large triangle is equal length of one side of the medium triangle
- one side of each of the small triangles is half the length of one side of the large triangle
- the square's side length is equal to the side length of the rhombus.

During (Working On It)

Puzzle Comparing Problem 2:

Pick a section of the pelican and match pattern blocks to it. Explain your thinking.



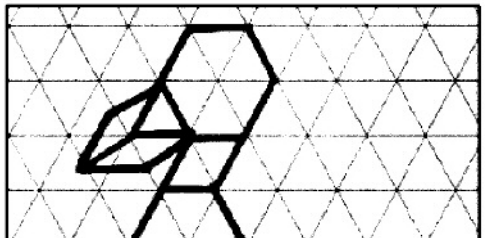


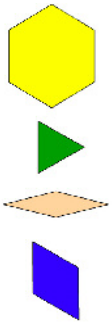
After (Consolidation)

Anticipating Student Responses:

Coordinating Discussion for Student Learning:

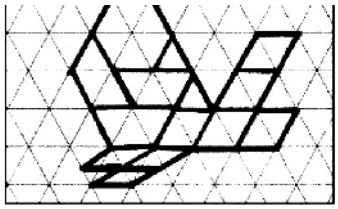
Solution 1

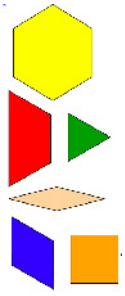




- A hexagon with 6 sides makes the head.
- The bigger rhombus with 4 sides and bigger acute angles makes the neck.
- The beak is made from 1 triangle (3 sides) and two small, light rhombus (4 sides) with the smaller acute angles.

Solution 1



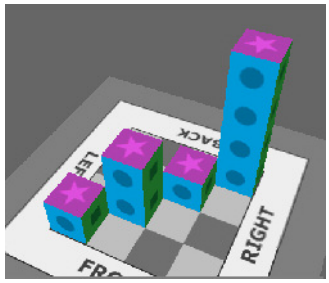


- I see three trapezoids (red). These are 2 units on bottom, 1 unit on top and two sides.
- I see three small rhombus (browny) with opposite sides equal and parallel. All sides are equal to 1 unit.
- I see one larger rhombus (blue) with bigger acute angle and smaller obtuse angle than the small rhombus. All sides are 1 unit.
- I see two little equilateral triangles 1 unit by 1 unit by 1 unit.

Why might solution 1 be chosen first for student discussion followed by solution 2?

- *solution 1* – shows comparing by direct matching of side lengths and angle measure, listing pieces that fit into the shape. The student reasoned using the number of sides and angles.
- *solution 2* – shows a more complicated part of the bird and names the properties that make pattern blocks match the puzzle pieces. Students manipulated and transformed the shapes to make them fit the puzzle pieces.

LET'S DO MATH – DIFFERENT PERSPECTIVES PROBLEMS

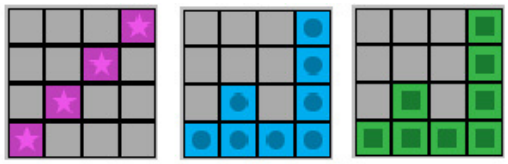


The images in this problem are taken from an applet from Explore Learning – Gizmos
http://www.explorelearning.com/index.cfm?method=cResource.dspView&ResourceID=170

Before (Getting Started)

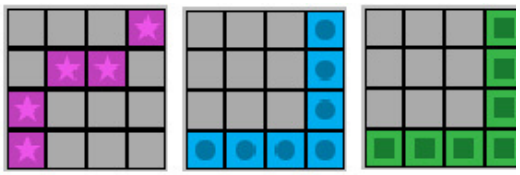
What are the top, front, and right views of this structure?

Solution 1



Top Front Right

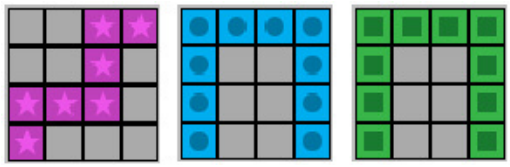
Solution 2



Top Front Right

During (Working On It)

Build a structure on the 4x4 grid so that these are the top, front, and right views of the structure.

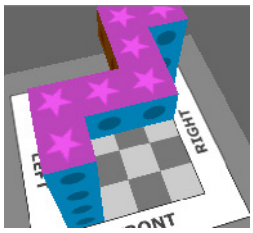


Top Front Right

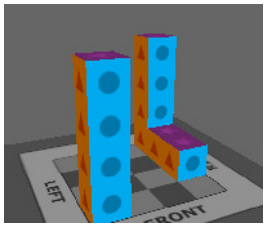
After (Consolidation)

Anticipating Student Responses:

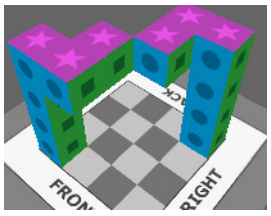
Solution 1



Solution 2



Solution 3



Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion, followed by solution 1 and 3?

- *solution 2* – this solution invite the discussion about the continuity or the lack of continuity from the front view. There are no cubes in column 2 or 3 (counting from left). The view from the Right is difficult to see and because on paper you cannot move around and "see" the structure from the Right. You may need to build the model and compare the right view.
- *solution 1* – this solution looks like it will work from the Front and the Right side. The 2 outside edges are 4 cubes high and there is a band across the top from left to right. Imagine viewing this from above and you see that the legs that extend across the structure are only 2 cubes in length rather than three.
- *solution 3* – this is an accurate solution. This perspective shows the Top and Front views clearly and you have to imagine the Right view.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Try these spatial reasoning tasks or problems yourself first and then offer them to your students.
- To suit your students' learning needs, begin designing spatial reasoning tasks or problems that are relative to the grade specific curriculum expectations in the Geometry and Spatial Sense strand. However, grade specific curriculum should not limit the kinds of spatial reasoning experiences we make available to students. Results from the Math For Young Children (M4YC) project remind us that early primary students have the capacity to engage in spatial reasoning tasks that are thought to be well beyond their years.
- Provide problems that have students work with 2-D plans, sketches and designs; read and draw maps of different scales, complete and create visual puzzles, draw and model three-dimensional objects given different views (side, top, front) and angles.
- Incorporate spatial reasoning daily, as Quick Images for practise and exploration.
- Concrete or virtual learning materials to use include tessellations, pentominoes, nets, isometric dot paper, blocks for building, tangrams, pattern blocks, PolydronsTM, Geometer's SketchpadTM and everyday classroom objects.
- Practise noticing the different strategies and the breadth of the mathematics your students use to solve spatial reasoning tasks or problems, such as, decomposing or composing shapes to make different shapes, moving one's body in space, perspective-taking, shifting dimensions between three-dimensional figures and two-dimensional shapes, comparing object attributes, imagining moving objects in space.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit spatial reasoning strategies and concepts as the students discuss and analyze each other's solutions to co-construct success criteria for the lesson learning goal.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below:

Bahr, D.L., & de Garcia, L.A. (2010). *Elementary mathematics is anything but elementary.* Belmont, CA: Wadsworth, Cengage Learning.

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Jolicoeur, P., Regehr, S., Smith, L., & Smith, G. N. (1985). Mental rotation of representations of two-dimensional and three-dimensional objects. *Canadian Journal of Psychology, 39*(1), 100-129.

Linn, M. C., & Peterson, A. C. (1986). A meta-analysis of gender differences in spatial ability: Implications for mathematics and science achievement. In J. S. Hyde & M. C. Linn (Eds.), *The Psychology of Gender: Advances Through Meta-Analysis* (pp. 67-101). Baltimore: The John Hopkins University Press.

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Ontario Ministry of Education. (2014). *Paying attention to spatial reasoning, K - 12: Spatial reasoning.* Toronto, ON: Queen's Printer for Ontario.

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CLOSING

Our next issue will focus on spatial reasoning within number sense contexts. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within learning and teaching mathematics within a three-part problem solving lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email

kathkubo@rogers.com or mkestell@sympatico.ca