8 PAGE INSERT ELEMENTARY MATH

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SPATIAL REASONING WITHIN MEASUREMENT CONTEXTS

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We use spatial reasoning to position and orientate ourselves in everyday environments. Spatial reasoning consists of two key elements: (a) spatial visualization ability, which is your capacity to retrieve images in your mind and reason with these images and (b) spatial orientation, which is the capacity to look at a fixed figure from several different points of view (Bahr and de Garcia, 2010). Spatial reasoning leverages information that is already present in our memory due to earlier observations or learning processes, as well as information, which is learned within the spatial characteristics of a location.



- Try to visualize a room or a location when you are not actually physically there. The better your spatial reasoning skills are, the more detail you will be able to visualize, thus the better is your spatial visualization. However, your spatial visualization also influences your capacity to mentally manipulate these images in your mind. Imagine how the room or location would look from another viewpoint, like top view, or from a window.



- As another example, how do you decide if a box is large enough for the objects you want to put into it? Imagine placing the objects into the box, shaped as a rectangular prism and noticing how much of the length, width and depth of the box is in relation to the lengths of the objects.

In this issue, the Research Summary and problems (with multiple solutions) focus on the use of spatial reasoning within different measurement contexts. The problems we provide are designed for use with a three-part problem-solving lesson structure.

RESEARCH SUMMARY – SPATIAL REASONING WITHIN MEASUREMENT CONTEXTS

According to Small (2009) "Measurement is about assigning a numerical value to an attribute or an object, relative to another object called a unit. A greater measurement implies that one object has "more" of a particular attribute than another… ." Measurements are used to describe attributes of an object in one dimension (length), two dimensions (area), or three dimensions (volume, mass or capacity). Other measurements studied in elementary school mathematics include angles, money, temperature and time.

Clements and Sarama (2014) emphasizes that measurement is a key real-world application of mathematics. Counting, as a type of measurement determines how many items are in a collection. As children develop their understanding of measurement, they begin to use personal referents as units of measure and then tools to measure and understand the significance of standard units of measure. Clements and Sarama describe this developmental path as a part of learning trajectory for measurement.

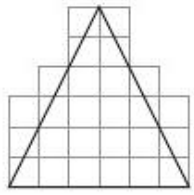
Personal Referents for linear units are familiar benchmarks of particular lengths used to estimate the lengths of other objects. Referents for standard units of length include:

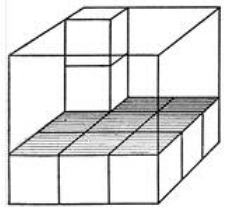
- centimetre – width of the little finger or width of a standard paper clip
- metre – length of a stride or the width of a classroom door; height from door knob to the floor
- kilometre – distance you could walk comfortably in 12 minutes

If spatial reasoning involves thinking and reasoning by comparing, manipulating and transforming mental images, then how is spatial reasoning integral to the development of measurement sense and understanding? As one example, let's examine a part of Clements and Sarama's (2014) linear measurement learning trajectory, which is as follows:

- Length quantity recognizer – identifies length as an attribute
- Length direct comparer – physically aligns 2 objects to determine if same or if one is longer
- Indirect length comparer – compares length of 2 objects by representing them with a 3rd object, like string
- Serial ordered to 6+ – orders lengths marked in one to six units
- End-to-end length measurer – lays units end-to-end, may not use equal-length units
- Length unit iterater – uses a ruler and sees need for equal-length units
- Length unit relater – relates size and number of units (need more units if units are smaller)
- Length measurer – begins to measure, knowing the need for identical units, relationship between different units, partitions of one unit, zero point on a ruler
- Conceptual ruler measurer – has an "internal" measurement tool – mentally moves along an object, segmenting it and counting the segments.

Spatial reasoning is implicated explicitly in length direct comparer, indirect length comparer, serial ordered to 6+, length unit relater, length measurer. Similarly, Battista (2003) explains that to measure area and volume in standard measurement systems, the number of unit squares or cubes in the region being measured requires the use of four mental processes:





- "Forming and Using Mental Models process – individuals create and use imagistic or recall-of-experience-like mental representations to visualize, comprehend and reason about situations
- Spatial Structuring process – individuals abstract an object's composition and form by identifying and interrelating and organizing its components
- Units Locating process – locates squares and cubes and composites of squares and cubes by coordinating their locations along the dimensions of an array
- Organizing-by-Composites process – combines array's basic units (squares or cubes) into more complicated, composite units that can be repeated or iterated to generate the whole array." (pp. 122-123)

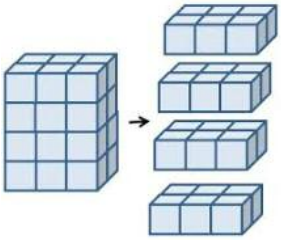
Often, students make explicit their use of spatial reasoning when measuring through gestures. According to Goksun, Goldin-Meadow, Newcombe and Shipley (2013), a gesture is a natural medium for reflecting mental representations involving action. They reported that in mental rotation tasks "gestures are common when people convey



spatial information, for example, when they give directions or describe motion in space … low-spatial individuals (as assessed by a standard paper-and-pencil measure) gestured more to explain their solutions than high-spatial individuals. While this finding may seem surprising, finer-grained analyses showed that low-spatial participants used gestures more often than high-spatial participants to convey "static only" information, but less often than high-spatial participants to convey dynamic information" (p. 122). Consider the gestures that students use to communicate their use of different measurements.



- For length measurement, do students extend their arms to show comparative lengths or trace a length in the air with their finger?
- For area measurement, are students visualizing the number of square units in a row and column and counting them out (grade 3) or gesturing the number of linear units for 2 dimensions of a rectangle?
- For volume measurement could students see how many objects fit within a rectangular prism using a referent like the number of objects in 1 layer, then building up layers of the object?
- For time measurement, might students gesture a full, half, or quarter of a circle to show passage of time or fold a length of paper to represent 12 hours, 24 half hours and 48 quarter hours as particular lengths (scale units) on a time line?



LINKS TO MANIPULATIVES – USING SQUARE TILES AND CUBES

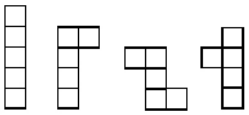
Use square tiles and cubes to develop your students' spatial reasoning within measurement contexts.



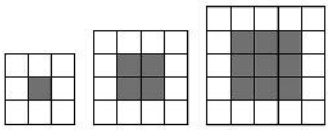


Square Tiles:

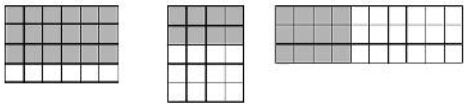
- What are the different 2D shapes that have an area of 3, 4, 5, 6, 7 … square units? Square tiles must connect side to side only.



- What is the area of the 5th pool and its border? The 10th pool and its border?

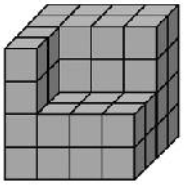


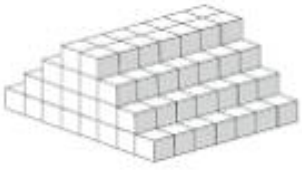
- What is the area of double, triple or half of these rectangles?

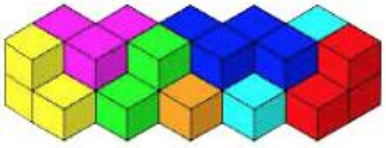


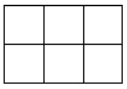
Cubes:

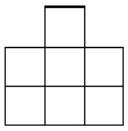
- What are the different 3D shapes that have a volume of 3, 4, 5, 6, 7 … cubic units? Cubes must connect face to face.
- What is the volume of these 3D shapes in cubic units, if they are doubled, tripled or halved?

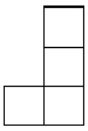







Top view


Front view


Right View

LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

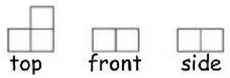
Grade 4	Grade 4	Grade 5
-estimate, measure and record the capacity of containers using the standard unit of the litre	-estimate, measure using concrete materials and record volume and relate volume to the space taken up by an object (use centimetre cubes to demonstrate how much space a rectangular prism takes up	-determine through investigation using stacked congruent rectangular layers of concrete materials, the relationship between the height, the area of the base and the volume of a rectangular prism

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

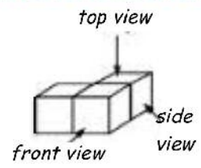
Composite Cube Problem

What is the volume of this 3D composite shape? Explain.



Solution 1.
I imagine that the top view has under it 1 row of cubes; so if there are 3 cubes in a "v" shape; the volume of the 3D shape is 3 cube units

Solution 2
I look at the front side and imagine 2 cubes and then rotate the shape to the side view and I see 2 cubes. But I count 3 cubes not 4 because I see one cube on both views; volume of the 3D shape is 3 cubic units because 1 cube is 1 cubic unit.

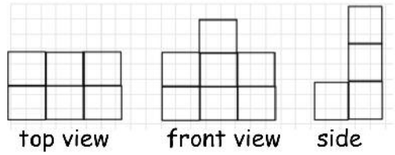


3D Composite shape is made up of smaller 3D shapes, like cubes. Volume of a 3D composite shape is the number of cubes or cubic units.

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

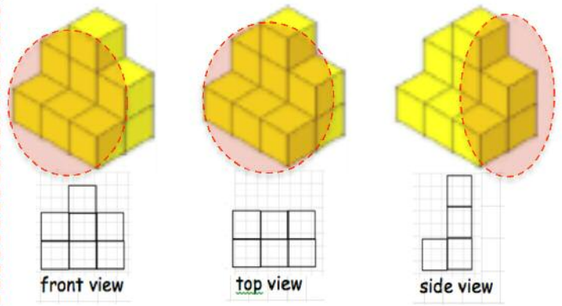
Composite Cube Problem 2:
What is the volume of this 3D composite shape? Explain.



What information are we going to use to solve the problem?
- *number of cube faces shown on top, front and side views*
- *shape of the cubes on the top, front and side views*

AFTER (Consolidation) – 20 to 25 minutes

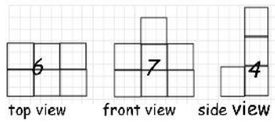
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.



For each view, 1 face is 1 cube, but sometimes 2 or 3 faces is the same 1 cube, not 3 cubes

Used "spatial visualization" to imagine the number of cubes for each face and imagine what the 3D shape looks like when the 3 views are look at when the 3D shape is rotated in our mind.

Counted 7 cubes two times

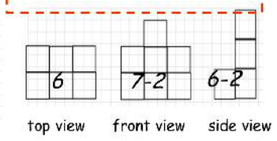


Solution 1
Top view means 6 cubes
Front view means 7 cubes
Side view means 4 cubes

At the most the 3D composite shape has 6 cubes + 7 cubes + 4 cubes = 17 cubes, so volume of the 3D shape is 17 cubic units

Counts Each Face on Each View as 1 Cube

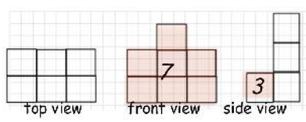
Counted 3 cubes two times



Solution 2
Top view means 6 cubes
Front view means 7 cubes,
Side view means 4 cubes
6 + 7 + 4 = 17 cubes
BUT has to be 4 less cubes because the side view and the front view each show 2 less cubes from a rectangle,
So volume is 17 – 4 = 13 cube units

Counts Each Face for Each View as 1 Cube and Adjusts

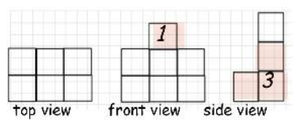
Counted 10 cubes once



Solution 3
Front view has 7 cubes
Top view says there are 2 rows x 3 columns on at least the 1st row.
Side view says that there is 1 row like the front view (7 cubes) and 1 row that is only 3. So 7 + 3 = 10 cubes
Volume of the 3D shape is 10 cubic units

Selects view that accurately represents # of cubes and adds on cubes from other views

Counted 10 cubes once



Solution 4
If put the front view with the side view, there are 3 sets of 3 cubes in a L shape, with 1 cube on top in the middle.
3 cubes x 3 groups = 9 cubes
9 cubes + 1 cube = 10 cubes
Volume of the 3D shape is 10 cube units.

Identifies 1 group of cubes (shape of cubes and number) that repeats and adds on

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed

Highlights/Summary

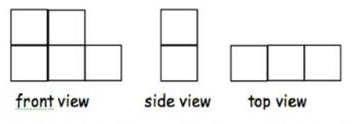
- A 3D composite shape is made up of smaller 3D shapes, like cubes.
- Volume of a 3D composite shape is the amount of space taken up by an object

Measure the volume of a 3D composite shape, made up of cubes by:
- use spatial visualization to see parts of the composite shape and # of cubes
- counting the number of cubes or cubic units of the 3D shape

AFTER (Practice) – 5 to 10 minutes

- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Composite Cube Problem 3 :
What is the volume of this 3D composite shape? Explain.



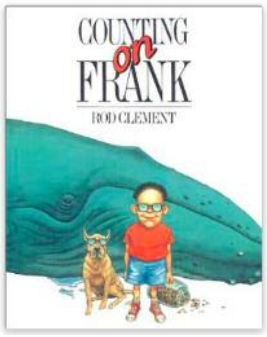
Solution 1
Top view means there is only 1 column of cubes and the side view says there are at the most 2 cubes high, but the front view shows the 2nd row has 2 cubes not 3, so ...
3 cubes x 2 rows – 1 cube
= 6 cubes – 1 cube = 5 cubes
Volume is 5 cubic units.

Solution 2
Front view shows 5 faces meaning 5 cubes; top view shows 3 columns of cubes, so ....
5 faces means 5 cubes and for volume, the 3D shape is 5 cubic units.

Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO LITERATURE



In *Counting on Frank*, by Rod Clement, questions are asked in the context of 'what if' scenarios. Linear, capacity and volume measurements are threaded throughout the story. Each of the scenarios could easily be the context for open-ended or open routed problems.

- How long could a pen write before it runs out of ink? (time or linear measure)
- How many dogs would fit into this bathtub? (capacity, comparative volume)

How many whales would fit in his house? (capacity, comparative volume)

Because the problems do not require exact answers, as long as the answers are within a reasonable magnitude, theses kinds of problems are often referred to as Fermi problems. A Fermi problem or question seeks a rough estimate of quantity, which is either difficult or impossible to measure directly.

- How long a pen writes before it runs out of ink is measured?
- How many drops of water fill your bathtub?
- How many cubes (1 cubic centimetres each) fit in your house?
- How many whales would fit in his house?

One caution … the measurements are in imperial units, so they would have to be adjusted.

LET'S DO MATH – FEET LENGTHS LINE PROBLEMS



Before (Getting Started)

Feet Problem 1:

The children are measuring the length of the line using their bare feet. How many "feet lengths" do you think Manuel, at the front of the line, has counted? How long is the line in "feet lengths"?

Solution 1

8 "feet lengths" because there are 4 children on the line and each child has 2 "feet lengths" covered.

Solution 2

14 because to measure you put the toe to heal and cover all the space. Manuel counted 8 spaces where feet are and 6 spaces in between. Looks like 2 more "feet lengths" to go. The line would be about 16 "feet lengths" long.

Solution 3

When I look at my friend's feet, I notice they are different lengths. So, for the girl behind Manuel, the line might be 18 "feet lengths" long because her feet are smaller and it will take more of her feet to cover the same distance. Looks like the boy third in line has bigger feet. He might measure the line as only 15 "feet lengths" long.

During (Working On It)

Feet Problem 2: Which is the Longest?

How long is each?

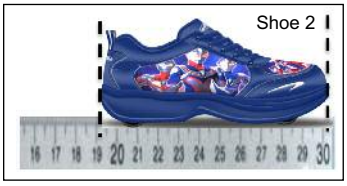
Read the centimetre ruler under the shoe.

Note: Some pictures were stretched and some were shrunk.



After (Consolidation)

Anticipating Student Responses:







Solution 1

Shoe 1 – 11 cm (11 – 0)
Shoe 2 – 11 cm (30 – 19)
Shoe 3 – 8 cm (100 – 92)
Shoe 4 – 8 cm (9 – 1)
Shoe 1 and 2 are longer than shoes 3 and 4.

Solution 2

It looks like shoe 1 is the longest because it starts at zero and ends at 11. It is 11 cm long. But shoe 3 ends at 99 so I'm not sure.

Solution 3

Shoe 3 because it is 100 cm long. But it looks like the shortest.

Solution 4

I saw the rulers and saw that they don't all start at 0. Then I realized 1 and 2 were the same. For 2, I counted 11 spaces between 19 and 30.
Shoe 2 = shoe 1 = 11 cm.

Coordinating Discussion for Student Learning:

Why might solution 3 be chosen first for student discussion followed by solution 2, 4, and 3?

- *Solution 3* – invites discussion about how to read the centimetre ruler measure; represents a mis-matched understanding that the last number on the right gives the measure for length; focus discussion on the idea that measuring starts at the left and counts the spaces between left and right
- *Solution 2* – show some understanding that measuring must start at the left side and count the spaces between where the measuring starts to where it ends; second part of the solution shows how fragile the understanding of measuring is when student says shoe 3 is 99
- *Solution 4* – shows a simple description of how to read the rule — count the spaces between the endpoints on the left and right
- *Solution 3* –shows measures for all the shoes and concludes that shoes 1 and 2 are both 11 cm long, greater than the length of shoes 3 and 4 at 8 cm

LET'S DO MATH – PROBLEMS

Before (Getting Started)

Problem 1: Time Passes

What time is it?







Solution 1	*Solution 2*	*Solution 3*
I see 13:50 on clock 1 but if I think of the 12 h clock, this means 1:50 or 10 min to 2:00.	Clock 2 rotates and each line shows ¼ h so this one shows 10:45 or ¼ h to 11:00.	Clock 3 shows 4 min to 4:00. This is the same as 3:56.

During (Working On It)

Problem 2: Duration of Time

Georgio arrived to visit his Grandmother at 1:50 and left at 7:30. How long did he visit?





After (Consolidation)

Anticipating Student Responses:

Solution 1	*Solution 2*	*Solution 3*
I am looking at the face of an analog clock. 10 min to 2:00 5 h to 7:00 30 min to 7:30 That means he visited for 5 h and 40 min.	7 : 3 0 → 6 : 9 0 regrouping 60 min + 30 min 1 : 5 0 → 1 : 5 0 subtract 5 : 4 0 He visited for 5 h and 40 min.	10 min to 2:00 Count 2 to 3 to 4 to 5 to 6 to 7 – that's 5 h And 30 more min He visited 5 h and 40 min.

Coordinating Discussion for Student Learning:

Why might solution 3 be chosen first for student discussion, followed by solution 1 and 2?

- *Solution 3* - shows the count from 2:00 to 7:00.
- *Solution 1* - uses the jumps between 2 and 7 on the face of the clock.
- *Solution 2* - shows a regrouping of 7:30 to 6:00 + 60 min + 30 min to 6:90 and then the starting time is subtracted from 6:90.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how these measurement problems include visual-spatial reasoning with a emphasis on proportional reasoning or multiplicative comparisons of measurements.
- To suit your students' learning needs, begin designing reasoning tasks or problems that are relative to grade specific curriculum expectations in the Measurement strand, being mindful of the different models and units measurement, such as linear, area, volume, time, mass.
- Select and design measurement problems, which provoke the use of proportional reasoning; that is, multiplicative comparisons of measurements.
- Incorporate daily practice of spatial reasoning through visual-spatial Quick Images. Quick images materials could include a specific linear unit to measure the length of images of objects; a specific square unit to measure the area of a given polygon; a specific cubic unit to measure the volume of a 3D object composed of rectangular prisms. Quick images are presented to the learner for a few seconds, and then the learner is prompted to identify the number of units needed to measure the length, area or volume of an object.
- Practise noticing the different strategies and the breadth of the mathematics your students use to solve measurement reasoning tasks or problems, such as, decomposing or composing numbers using iterations of same size measurement units.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit spatial reasoning strategies for measuring different attributes objects using visual inspection.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

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CLOSING

Our next issue will focus on spatial reasoning within patterning and algebra and data management contexts. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within learning and teaching mathematics.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email

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