8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Abacus

Volume 54 • Number 1 • September 2015

EQUIVALENCE AND RELATIONAL THINKING - ADDITION

TABLE OF CONTENTS

Abacus Editor Greetings 1
RESEARCH SUMMARY 2
- Equivalence and Relational Thinking
LINKS TO MANIPULATIVES . . 3
- Balance Scale
LET'S DO MATH: 4-5
- Open Number Sentences (bansho plan)
LINKS TO LITERATURE 6
Equal Shmequal
LET'S DO MATH: 6
- True/False Number Sentences
LET'S DO MATH: 7
- Open Number Sentences
NEXT STEPS FOR YOUR PROFESSIONAL LEARNING. 8
- Application to Your Classroom
- Suggested Readings

ABACUS CO-EDITORS



MARY LOU KESTELL works on K to 12 mathematics professional learning at the provincial level. She has spent 40+ years working in all aspects of Ontario mathematics education and is a past-president of both OAME and OMCA. **maryloukestell@gmail.com**



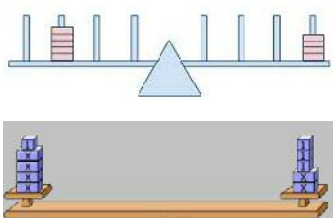
KATHY KUBOTA-ZARIVNIJ currently focuses on K to 6 mathematics professional learning at the provincial level. She is a long-time OAME board member and OAME past president. In relation to her work in mathematics education, she uses complexity thinking to interpret mathematics teaching and learning for students and teachers. **kathkubo@rogers.com**

The balance metaphor is a widely used conceptual image to represent the concept, equivalence. Where is the concept of equivalence meaningful in your kindergarten to grade 6 mathematics program? What are the ways that you develop your students' understanding of equivalence?









- Examine these different number/mass balance scales.
- How could students use these balance scales to construct models of equivalent addition number expressions and terms?
- How might you introduce the use of these balance scales to provoke students' thinking about equivalence of 1 digit addition expressions (e.g., Is it true that 5+4 = 2+7? How do you know that 3+2+4 ≠ 2+2+3)?

In this issue, the Research Summary and reasoning tasks (with multiple solutions) focus on equivalence and relational thinking, within addition contexts. The reasoning tasks in this issue are designed for use with a three-part problem-solving lesson or a Mental Math mini lesson structure. Consider including in your mathematics program planning daily learning and practice for solving addition true/false number sentences and open number sentences.

A Publication of the OAME/AOEM

RESEARCH SUMMARY – EQUIVALENCE AND RELATIONAL THINKING

Randall (2005) identifies equivalence as a big idea. He explains that "any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value (e.g., numerical expressions can be named in an infinite number of different but equivalent ways)" (p. 14). More specifically, a property of equality is that if the same real number is added or subtracted to both sides of an equation, equality is maintained; two quantities equal to the same third quantity are equal to each other.

What number would you put in the box to make this number sentence true? 8 + 4 = □+ 5?

According to Carpenter, Franke, and Levi (2003), 7, as the numeric value for □ in the equation above is not commonly selected (relational); two common responses (operational) are: 12 and 17.

12 → 8+4 = 12 and ignore the + 5.

17 → 8+4+5 = 17

Number in □ to Make Number Sentence True				
Grades	7	12	17	12 and 17
1 and 2	5	58	13	8
3 and 4	9	49	25	10
5 and 6	2	76	21	2

From this data, they concluded that most elementary students and many older students understand the equal sign from an operational view (here comes the answer) rather than a relational view (relating two equal quantities). Relational thinking incorporates the use of a big idea, comparison. Randall (2005) states that "numbers, expressions, and measures can be compared by their relative values; … one-to-one correspondence can be used to compare sets … numerical and algebraic expressions can be compared using greater than, less than or equal" (p. 15).

To improve students' understanding of equivalence, true/false and open number sentences are designed using an arithmetic property (i.e., Commutative, Associative and Identity Properties for Addition).

True/False Number Sentences Varying the Structure	**True/False Number Sentence Including and Varying Position of Zero**	**Open Number Sentences Varying the Position of the Unknown**
23 + 27= 50 (27 + 23 = 50)	134 = 109 + 25	23 + 109 + 53 = □ + 109 + 55
50 = 23 + 27; (50 = 27 + 23)		
50 = 50	109 + 25 = 134 + 0	23 + □ + 53 = 55 + 109 + 23
23 + 27 = 23 + 27		
23 + 27 = 27 + 23	134 + 0 = 0 + 109 + 25	23 + 109 + 53 = 55 + □ + 23
(commutative property for addition)		
23 + 27 = 25 + 25	0 + 109 + 25 = 0 + 0 + 134	□ + 109 + 53 = 23 + 109 + 55
23 + 27 = 2 + 23 +2 5		
23 + 27 = 2 + 25 + 23	25 + 0 + 109 = 0 + 134 + 0	23 + 109 + 53 = 55 + 109 + □
23 + 2 + 25 = 2 + 25 + 23		
(associative property for addition)	109 + 0 + 25 = 133 + 1 + 0	

Developing Relational Thinking Using True/False Number Sentences

Is this equation true or false? 7 + 5 = 6 + 6.

Students respond to this question in different ways:

- operational - add the numbers for each addition expression and compare the sums
- compensating - decomposing 7 (7=6+1) and associating the 1 with 5 (6=1+5), transforming the addition expression 7+5 to be the same as 6+6 , reducing the need to do the calculations; more specifically, 7+5 = (6+1)+5 = 6+(1+5) = 6+6
- relational - compare 7>6 (1 more than) and 5<6 (1 less than), so must be equivalent.

For students in grades two and three, equivalence can be developed to include 1-digit, 2-digit and 3-digit whole number true/false number sentences with more than 2 addends like 12+4+4 = 10+8+2. When thinking relationally, rather than operationally, students can see that equivalence is maintained no matter what operation is used, as long as the number operation is applied to both sides of the equation. 8 + 2 + 10 + □ = 12 + 4 + 4 + □

Developing Relational Thinking using Open Number Sentences

What's the value of in this open number sentence? 220+150+125 = + 215+120

Carpenter, Levi, Franke, and Zeringue (2005) and Molina, Castro, and Mason (2008) explain that when students solve open number sentences using relational thinking, they are looking at expressions and equations in their entirety. Students are noticing the fundamental number properties and structure of the number sentences in order to make distinctions and establish a relationship between the elements (i.e., sameness, difference), rather than paying attention only to the numbers involved and operations to perform on them. More specifically, Irwin and Britt (2005) argue that, to determine equivalence between the two related parts of number sentences, in order to find the value of a missing number, one has to know the direction in which compensation needs to occur.

- operational -> 220+150+125 = 495 -> 495-215-120=160 → □ = 160
- compensating → 220+150+125 = (220-5)+(150+10)+(125-5) = 215+160+120 -> □ = 160
- relational → 220 > 215 by 5; 125 > 120 by 5; so 150 < □ by 10 → □ = 160

LINKS TO MANIPULATIVES: BALANCE SCALE

Make a balance scale with a coat hanger (with notches), 240 cm string, single-hole punch; two identical clear plastic cups; scissors; measuring tape; masking or packing tape and same size cubes.



- Measure and cut 8 pieces of string exactly 30 cm long.
- Make a mark 2.5 cm below the rim on opposite sides of the cup on the outside. (Make 4 marks in the same place on both cups.)
- Tie off the end of each string with a knot to keep it from sliding out of the hole.
- Loop the 4 strings over the top of the hanger, knot them and rest in one of the notches.
- Steady the hanger and make sure the cups are hanging at the same level. Adjust and retie the string until they are even, then secure the string in the notch with tape.

To develop conceptual understanding of equivalence, ask questions like:

- One cup has 4 cubes and the other cup has 2 cubes. Why does one cup stay up while the other cups goes down?
- One cup has 5 cubes and the other cup has 5 cubes. The cups are balanced. Would these cups be balanced if the hanger is put somewhere else? Why or why not?
- One cup has 8 cubes. The other cup has 2 cubes. How many cubes are needed to balance the cups?

LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

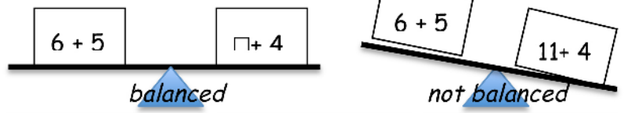
Grade 2	Grade 3	Grade 4
- determine the missing number in equations involving addition and subtraction to 18, using a variety of tools and strategies	- determine, the missing number in equations involving addition and subtraction of one- and two-digit numbers, using a variety of tools and strategies	- determine, the missing number in equations involving multiplication of one- and two-digit numbers, using a variety of tools and strategies

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

What is the Unknown Number?

What is □? Explain the method used.

6 + 5 = □+ 4



Solution 1
6+5 = 11
What number needs to be added to 4 to sum to 11?
6+4 = 10
6+1+4=11
7+4=11, □ = 7

Solution 2
6+5=11
What's the difference between 11 and 4?
11-4=7
□ = 7

Solution 3
6+5 = 7+4
What can be the same to see the difference?
6+1+4=□+4
4s the same, so to make same, □=7

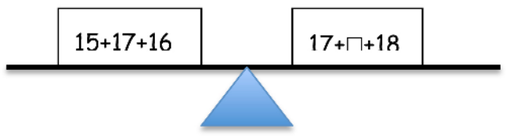
DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

What is the Unknown Number?

What is □? Explain the method used.

15 + 17 + 16 = 17 + □ + 18

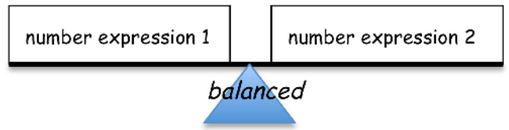
What information are we going to use to solve the problem?



□ *means the unknown number*

`AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

*Visualize the quantity for each number expression so there is balance (**equivalence**)*



Same Sum
*sum of number expression 1 (15+17+16) is **the same as (equivalence)***
sum of number expression 2 (17+□+18)

Solution 1
Add 3 numbers in the 1st expression to get sum
15+17+16 = 30+18=48

Add 2 numbers in the 2nd expression, then add on to get same sum as 1st expression
17+□+18 = 30+18
35+□ = 48
□=13

Solution 2
Add 3 numbers in the 1st expression to get sum
15+17+16 = 30+18=48

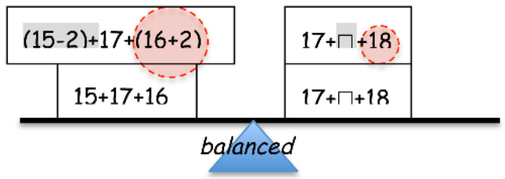
Want same sum for 2nd expression, so subtract the sum from the 1st expression the numbers in the 2nd expression
48-17-18
= 48-10-10-10-5
= 18-5 = 13 □=13

Operational
Add Terms for 1st and 2nd Expression Get Same Sum

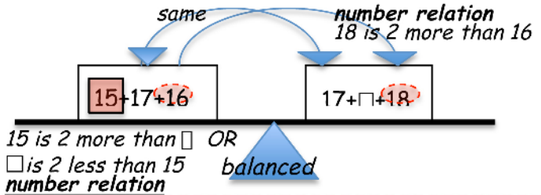
Operational
Add Terms (1st Expression) and Subtract Term (2nd)

Equivalence or Equality
An equation indicating that two quantities (or expressions) are in balance; two expressions that are equivalent (e.g., 6+5=7+4; 15+17+16=17+13+18).

Compensating Method for Equivalence



Relational Method for Equivalence
Comparing pairs of of numbers from expression 1 and 2



Solution 3
15+17+16 = 17+□+18
Making pairs of numbers the same to know unknown number

17s are the same
add 2 to 16, to make same as 18
subtract 15 from 2 to balance the (add 2) is 13

15-2 + 17 + 16+2

Solution 4
15+17+16 = 17+□+18
Comparing pairs of numbers from each expression

16 < 18 (2 more)
17 = 17 (0 more)
(2 more) 15>□= 15-2

balance →2 more (right side), 2 more (left side)
so □=13

Compensating
Make Terms Same To See the Unknown

Relational
Compare Pairs of Terms Across 2 Expressions

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed

Highlights/Summary

Determine unknown number in addition equations using different methods:
- ***operational** – calculating the operations for both number expressions (e.g., 15+17+16-17-18 = □)*
- ***compensating** – making pairs of numbers the same by adding or subtracting numbers (e.g., (15-2)+17+(16+2) = 13+17+18, □=13 (notice …15+17+16-2+2))*
- ***relational** – comparing pairs of numbers in expressions to know the number relations (e.g., 16<18 (2 more); 17=17; □<15 (2 less)*

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

What is the Unknown Number?

What is □? Explain the method used.

75 + 77 + 76 = 77 + □ + 78

Solution 1
□ = 73
because I know that
15+17+16 = 17+□+18
and □=13 so for
75+77+76 = 77+ □+78
equation is 6 tens or 60 more per number
so□= 13+60 = 73

Solution 2
76<78 2 more right side)
77=77 (same or equal)

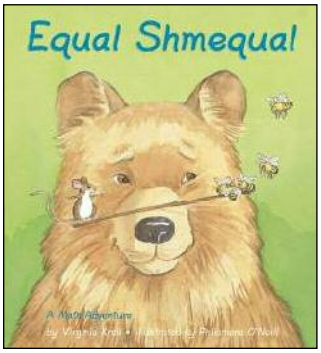
keep balance
□<75 (2 more left side)
(2 more left side, 2 more right side -> balanced)

so □=73

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO LITERATURE: EQUAL SHMEQUAL



What does it mean to be equal? *Equal Shmequal* introduces the concept of equal measures, by demonstrating what it means to have equal parts that are the same in weight, size, quantity and/or force. A group of animals see children playing a game of tug-of-war and try it. But they have to go through all sorts of adjustments to their two teams in order to come up with a fair game. They try dividing up by meat-eaters and plant-eaters; then those that have fur and don't have fur. These solutions do not work. Mouse starts thinking mathematically. They try three animals on each side, and the larger (heavier) animals win. Finally, they line up on the teeter-totter, shifting sides until the board balances.

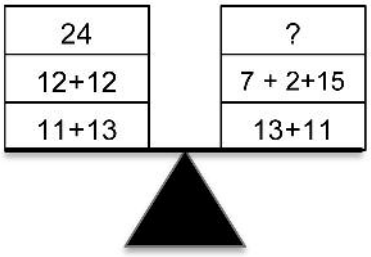
- Have students select different number of interlocking cubes to represent the relative sizes of the animals. Discuss choices like the one of the animals is 2X, 3X, 4X larger than one of the other animals. Identify which animal is represented as a unit of 1 interlocking cube.
- Model the animals on the teeter-totter from the story. Use interlocking cubes and a balance scale.
- Have the students write additional story pages and model the animals using interlocking cubes and the balance scale.

LET'S DO MATH – TRUE/FALSE NUMBER SENTENCES

Before (Getting Started)

What numbers could be added together to make a sum of 24?

- Record student responses on a drawn balance scale.
- Possible responses -> 11 + 13, 12 + 12, 13 + 11, and so on.
- Ask, "What addition expressions could be put on the other side to create a balance?"



During (Working On It)

Addition True/False Number Sentences:

True or False? 135 + 43 + 169 = 170 + 141 + 36

After (Consolidation)

Anticipating Student Responses:

Solution 1

135+43+169 = 170+41+136

135+43+169+1-2+1

= 135+43+169+0

= 135+1+43-2+169+1

= 136+41+170

136+**41**+170 = 170+**41**+136

True – compensated left expression to be the same as the right expressions

Solution 2

135+43+169

= 200+30+40+60+5+3+9

= 200+130+17 = **347**

170+141+36

= 200+70+40+30+1+6

= 200+140+7

= **347**

True –expressions sum to **347**

Solution 3

Left side	Right side
135	< 141 (+6)
(+7) 43	> 36
169	< 170 (+1)

both expressions have +7

7 = 7

True – differences between the terms balance (7=7)

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solution 1, and 3?

- *Solution 2* – operational -> calculated each expression and compared the sums
- *Solution 1* – compensating -> adds 1 twice and subtract 2 from terms in left (first) addition expression) which has a net result of 0, to make the terms the same as the right (2nd) addition expression
- *Solution 3* – relational – compares pairs of terms to determine how much the left side terms and right side terms have greater numbers to see which side has more (e.g., left side has 7 and right side has 6 + 1 = 7, so both have 7, so the same)

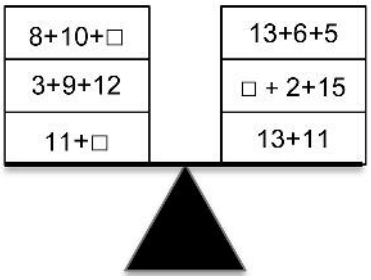
LET'S DO MATH – MENTAL MATH OPEN NUMBER SENTENCES

Before (Getting Started)

What is the value of □? Explain your method.

Record student responses on a drawn balance scale.

- Operational – added the terms in the 1st addition expression and subtract the terms from the 2nd addition expression to get value of □
- Compensating – adjust value of terms in the 1st addition expression to be similar to terms in the 2nd addition expression
- Relational – Compare pairs of terms from each addition expression to determine how much greater one term is from another term



During (Working On It)

Addition Open Number Sentences

What's the value of □? 135 + 43 + 169 = □ + 141 + 36

After (Consolidation)

Anticipating Student Responses:

Solution 1

135+43+169 = □+141+36

135+43+169 = □+41+136

135+1+43-2+169+1 = □+41+136

136+41+170 = □+41+136

136=136; 41=41; 170=□

Solution 2

135 < 141 (+6)

(+7) 43 > 36

7 > 6

169 + 1 = □

170 = □

Solution 3

135+43+169-141-36

= (100+100-100)+(30+40+60-40 - 30) + (5+3+9-1-6)

= 100+60+10

= 170

□ = 170

Coordinating Discussion for Student Learning:

Why might solution 3 be chosen first for student discussion, followed by solution 1 and 2?

- *Solution 3* - operational – added the terms in the 1st addition expression and subtract the terms from the 2nd addition expression to get value of □
- *Solution 1* - compensating – adjust value of terms in the 1st addition expression to be similar to terms in the 2nd addition expression
- *Solution 2* - relational – compare pairs of terms from each addition expression to determine how much greater one term is from another term

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how equivalence, reasoning tasks (e.g., True/False Equations, Open Number Sentences) develop students' understanding of number relationships and the use of different strategies for addition and relational thinking.
- To suit your students' learning needs, select and design reasoning tasks that are relative to grade specific curriculum expectations in the Patterning and Algebra strand (equivalence) and the Number Sense and Numeration strand (addition of 1 digit, 2 digit and 3 digit numbers; mental math addition), being mindful of the transition from calculating and comparing the results of addition expressions towards analyzing number relationships between addends when determining the equivalence of addition expressions.
- Incorporate daily practice of addition strategies and relational thinking practices through daily mental math mini lessons, two times a day. Design mental math equivalence reasoning tasks by using 1-digit, 2-digit and 3-digit whole numbers.
- Practise noticing the different methods and the breadth of the mathematics your students use to solve equivalence reasoning tasks, such as, carrying out the operations for each expression, comparing terms in the addition expressions, adjusting the terms to make them similar in order to see the difference or unknown (compensating).
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit addition and relational thinking strategies for solving equivalence reasoning tasks.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

- Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic & algebra in elementary school.* Portsmouth: Heinemann.
- Carpenter, T., Levi, L, Franke, M., and Zeringue, J. (2005). Algebra in elementary school: Developing relational thinking. *ZDM, 37*(1), pp. 53-59.
- Irwin, K. & Britt, M. (2005). The algebraic nature of students' numerical manipulation in the New Zealand Numeracy Project. *Educational Studies in Mathematics 58*(2), 169-188.
- Molina, M., Castro, E., and Mason, J. (2008). Elementary school students' approaches to solve true/false number sentences. *PNA 2*(2), pp. 75-86.
- Randall, C. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *NCSM Journal, 8*(1) 9-24.

CLOSING

Our next issue will focus on Equivalence and Relational Thinking within combinations of addition and subtraction contexts. As well, the ways that teachers can use these mathematical ideas and pedagogical strategies within learning and teaching mathematics within a three-part problem solving lesson framework are outlined.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or maryloukestell@gmail.com