8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Abacus

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FRACTIONS AS OPERATOR

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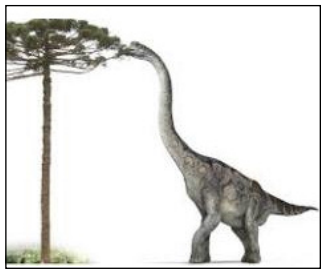
MARY LOU KESTELL works on K to 12 mathematics professional learning at the provincial level. She has spent 40+ years working in all aspects of Ontario mathematics education and is a past-president of both OAME and OMCA. **maryloukestell@gmail.com**



KATHY KUBOTA-ZARIVNIJ currently focuses on K to 6 mathematics professional learning at the provincial level. She is a long-time OAME board member and OAME past president. In relation to her work in mathematics education, she uses complexity thinking to interpret mathematics teaching and learning for students and teachers. **kathkubo@rogers.com**

How are fractions represented in these 2 examples? How are these examples of fractions different from fractions as part of a whole, as division, or as a quotient?

A model of a Brachiosaurus could be 1/100 the size of the actual dinosaur, which is 26.35 m feet long from its snout to its tail. How long is the model of the Brachiosaurus?



If 1/100 = 0.01, then 26.35 m x 1/100

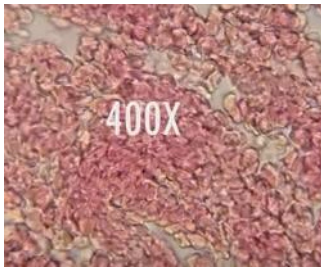
= 26.35 m x 0.01 = (2635 x 0.01) m x 0.01

= 2635 x (0.01 x 0.01) m = 2635 x (0.0001) m

= 2635 ten thousandths = 0.2635 m

In another example, a red blood cell might be magnified under a microscope from 100 times, to 400 times and then to 1000 times its actual size.

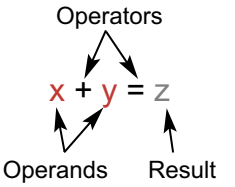






RESEARCH SUMMARY – ABOUT FRACTIONS AS AN OPERATOR

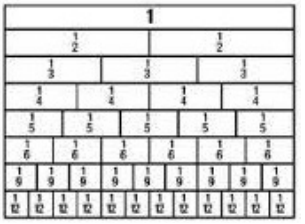
About Operand and Operator



- An operand is the object of a mathematical operation; that is, a quantity on which an operation is performed.
- An operation is a mathematical function performed on one or more operands.
- An operator is a symbol representing a mathematical operation.

About Unit Fractions as Operators

Empson and Levi (2011) explained unit fractions as any fraction that has the form, 1/*n* (where *n* is any positive whole number). A unit fraction is defined by its relationship to the whole; that is, 1 unit can be divided into any number of equal parts (1 ÷ *n* = 1/*n*) and those parts can be recombined to make 1 whole (*n* x 1/*n* = 1; *n*/*n* = 1).

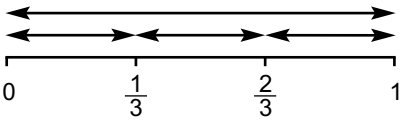


According to Watanabe (2007), multiplication is a way to quantify something that is composed of several copies (iterations) of identical size. Multiplication is the basis of understanding of fraction m/n, as it means m times 1/*n* (m x 1/*n* = *m*/*n*) or m times the unit fraction.

1 ÷ 4 = 1/4, 4 x 1/4 = 1

1 ÷ 12 = 1/12, 12 x 1/12 = 1

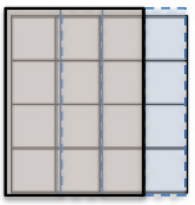
In a measurement context, when 1 whole (1 m) is divided into 3 equally partitioned parts (3 equal size lengths), then a length equivalent to two of the three equally partitioned parts of 1 metre, it is referred to as 2/3 of 1 metre (2/3 m). So, the part-whole meaning of fraction is based on unit fractions, such as 1/3 (or 1/3 m), whereby non-unit fractions (2/3 m) are considered a collection of unit fractions (two one-thirds).

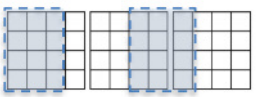


Non-Unit Fractions as Operators

A fraction can be used to operate on a quantity (operand). In other words, a single fraction acts as a function that describes a composite operation, when two multiplicative operations are performed, the one on the result of the other.

- 3/4 of a quantity (16) has several combinations of operations: (a) divide by 4, multiply by 3 **or** (b) multiply by 3, divide by 4. When the multiplier is a quantity less than 1, then the result is a smaller quantity than the original quantity (16 x ¾ = 12). This is the same result as if we multiplied this quantity by ¾.





16 ÷ 4 = 4

4 x 3 = 12

16 x 3 = 48

48 ÷ 4 = 12

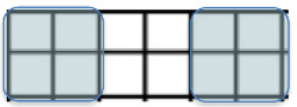
- 1-3/4 or 7/4 of a quantity (16) has several combinations of operations: (a) divide by 4, multiply by 7 or (b) multiply by 7 then divide by 4. In this case, the result is a larger quantity than the original (16 x ¼ = 28). This is the same result as if we multiplied this quantity by 7/4.

Lamon (2012) also defines operators as transformers that lengthen or shorten line segments, increase or decrease the number of discrete objects in a set, or a geometric figure that is mapped onto a larger or smaller figure of the same shape in a geometric plane. A fraction can be an operator that enlarges or reduces a quantity. More specifically, Charalambous and Pitta-Pantazi (2006) explained that the quantity upon which the operator acts is exchanged for a conceptually new quantity in such a way that the ratio of the input to the output quantity is equal to the numerator-to-denominator ratio of the operator.

- stretching/shrinking transformation results in the same number of units of different size (e.g., ¾ as a fraction multiplier means 3 x (1/4 of a unit))
- duplicating/partition-reducing transformation creates a different number of units of the same size (e.g., ¾ as a fraction multiplier means ¼ x 3 units)

LINKS TO MANIPULATIVES: SQUARE TILES AND NUMBER LINES

a) How much is 2/3 of 12 tiles? (12 x 2/3)





Divide 12 into 3 groups then multiply by 2

12 ÷ 3 = 4; 4 x 2 = 8

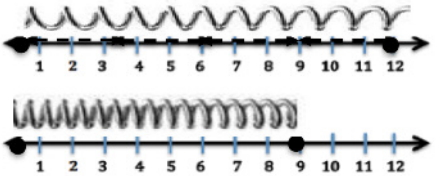
12 x 2/3 = 12 ÷ 3 x 2 = 4 x 2 = 8

Multiply 12 x 2 groups then divided by 3

12 x 2 = 24; 24 ÷ 3 = 8

12 x 2/3 = 12 x 2 ÷ 3 = 24 ÷ 3 = 8

b) A plastic coil that was 12 cm in length is compressed to ¾ its original length. What is the new length?



Think of length 12 cm the whole

12 cm x 1/4 = 12 ÷ 4 = 3 cm

3 lengths out of 4 lengths is ¾ of whole

3 cm x 3 lengths = 9 cm

12 cm coil is compressed by ¾ resulting in a 9 cm length coil.

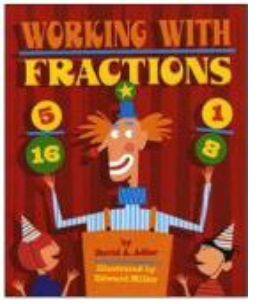
c) How much is 5/12 of 2 m? (2 m x 5/12)



5/12 of 1 m = 5/12 m because 5/12 = 1/12 m x 5

5/12 of 2 m = 5/12 m x 2 = 5/12 m + 5/12 m = 10/12 m

LINKS TO LITERATURE: WORKING WITH FRACTIONS



Working with Fractions by David Adler and Edward Miller offers different contexts for representing fractions as part of a whole. For example, the story says, "You can see fractions at Birthday parties. If there are 3 red balloons at the party and 10 balloons in all, then a part, a fraction, of the balloons are red. 3/10 of the balloons are red."

Use the contexts from this picture book to pose questions about fractions as an operator:

- How many balloons are red if 3/10 of the balloons are red?
- 3/10 of 20 balloons are red. How many balloons are red?
- 6/10 of 20 balloons are red. How many balloons are red?
- Jan says that 4/10 of the balloons is 12 red balloons. How many balloons does Jan have?

LESSON LEARNING GOALS
Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

Grade 7	Grade 8	Grade 9
-demonstrate using concrete materials the relationship between the repeated addition of fractions and the multiplication of that fraction by a whole number (1/2+1/2+1/2 = 3 x 1/2)	-represent multiplication and division of fractions using a variety of tools and strategies (e.g., use an area model to represent ¼ multiplied by 1/3)	-simplify numerical expressions involving integers and rational numbers with and without the use of technology

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Marbles Problem

How much is 3/4 of 200 marbles?

Solution 1
I know that three-fourths of 100 marbles is 75 marbles.
Double 100 marbles
(Three-fourths of (100x2)
= (3/4x100) x 2 = 150 marbles

Solution 2
200 x .75
= (2x100) x (75x 0.01)
= (2x75) x (100x 0.01)
= [(2x70) + (2x5)] x 1
= (140+10) = 150 marbles

three fourths of 1 whole ➔ 3/4 of 1 = 0.75
3/4 x100 = 0.75 x 100 = (75x 0.01) x 100
= 75x (0.01 x 100)
(fourths of 100 means 4 parts of 100 -> 25 per 1/4)

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Beading Problem



Jesse and Jaime shared 600 beads: 3/4 for Jesse and the rest for Jaime. Jaime made a necklace with 2/5 of her beads. How many beads were used to make Jaimie's necklace?

What information are we going to use to solve the problem?
- 600 beads altogether
- 2/3 of beads for Jesse
- 1/3 of beads for Jamie
- Jamie used 2/5 of her beads to make a necklace
- How many beads did Jamie use?

` AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Fraction as an Operator *– fraction quantity used to operate (multiplication of quantity less than 1) on a quantity (operand). Multiplying a whole number quantity by a fraction is the same as:*
- *dividing by total parts (denominator in the unit fraction) and duplicating parts*
- *using composite operation of 2 multiplicative operations (multiply by 2 and divided by 5 for the fraction operator 2/5).*

operand *operator*

Solution 1
Jesse 600 x 0.75
= (6x100) x (75x 0.01)
= (6x75) x (100x 0.01)
= (6x (70+5)) x 1
= (6x70) + (6x5)
= 420 + 30
= 450
Jaime beads
600-450=150
2/5 = 4/10 = .4
150x .4 =(15x10) x (4x 0.1)
= (15x4) x (10x 0.1) = 60
Jaime used 60 beads to make a necklace.

Solution 2
600 x ¼ = 600÷4
= 600÷2÷2 =
= 300÷2=150

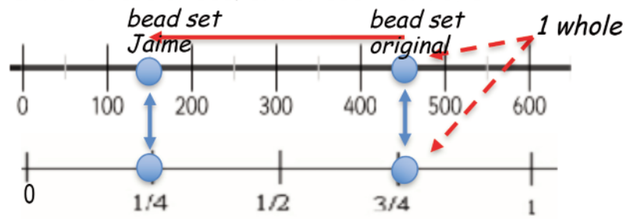
150÷5 = 15x10÷5
= (15÷5) x 10 = 3x10
1/5 of 150 = 30
2/5 of 150 =30x2 = 60

Jaime used 60 beads to make a necklace.

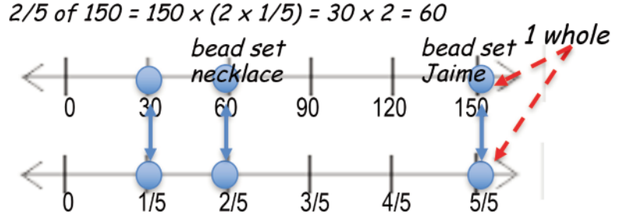
Decimal Multiplication

Duplicating Unit Fraction (1/4, 1/5).

Whole is the full set of beads
Jaime's set of beads (subset of set of all beads)
1/4 of 1 = 150 of 600; 3/4 of 600 = 3/4 of 1



New Whole is Jaime's set of beads
Jaime's necklace beads (subset of Jaime's set of beads)
2/5 of 150 = 150 x (2 x 1/5) = 30 x 2 = 60



Fraction as an Operator (less than 1) reduces the quantity of beads from the original set, to Jaime's set, to necklace set.

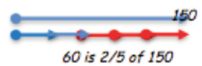
Solution 3
600 ÷4 = 150
150 ÷ 5 x 2
= 30 x 2
= 60

or

600÷4 = 150
150 x 2 ÷5 = 300÷5 = 60

Jaime used 60 beads to make a necklace.

Solution 4
Original bead set is 600. Imagine a number line showing 600 beads long then it is compressed to ¼ of its original length.,
Original bead set -> 600
Jaime's bead set 600x1/4 = 600÷4 = 150 beads



Jaime's necklace is 2/5 of 150 = (150÷5) x2=60

Composite Multiplicative Operations

Shrinking Transformation

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria

Highlights/Summary

Fraction as an Operator -multiplication of a fractional quantity (operator, < 1) on a quantity (operand) by:
- *dividing by total parts (denominator in the unit fraction) and duplicating parts (150x 2/5 =(150x1/5) x 2 = (150 ÷5) x2)*
- *using composite operation of 2 multiplicative operations (multiply 150 by 2 and divided by 5 for the fraction operator 2/5).*

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Juice Problem

600 juice boxes: 2/3 are orange, the rest is apple. 3/5 of apple is empty. How much apple juice is left.

Solution 1
600 ÷3 = 200
1/3 of 600 = 200
200÷5 =40 (1/5x2)
40x2= 800
80 apple juice boxes are left

Solution 2
600 ÷3 = 200
2/5 of apple juice is left
200 x 2 ÷5
= 400 ÷5 = 80

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

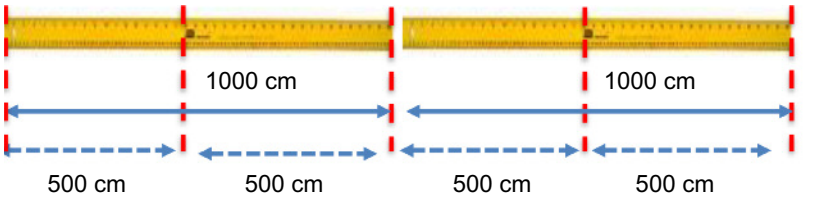
LET'S DO MATH – HOW MUCH IN A SET, OF AN AREA, OF A LENGTH?

Before (Getting Started)

How much is ¼ of 2 m?

Solution 1

I will divide 2 m into 4 equal parts and each part will show ¼ of 2 m.



Solution 1

¼ of 2 m

= 2 ÷ 4

= ½ m = 500 m

¼ of 2000 cm

= 2000 ÷ 4

= 500 cm

During (Working On It)

Use the fraction as an operator in a problem. Solve your problem.

After (Consolidation)

Anticipating Student Responses:

Solution 1

There are 10 stamps in this set. How many are in 4/5 of the set.

What is 4/5 of 10

Think of 1/5 of 10 which is 2. 4 parts of 1/5 is 4/5

So 4 x 2 = 8. So, 4/5 of the set is 8 stamps



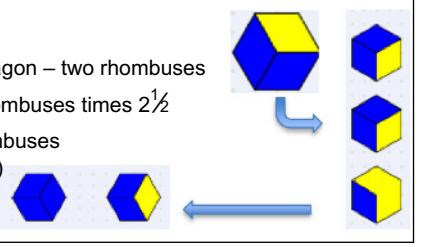
Solution 2

2½ x ⅔ = ?

Think of ⅔ of a hexagon – two rhombuses

Now multiply the rhombuses times 2½

The result is (5 rhombuses on top of a hexagon) Or 1⅔



Solution 3

What is ¾ of 5?

3 lengths of 5 looks like this:

The new line is 15 units and I want them divided into 4 equal pieces for ¼ of 15

15/4 – 3 ¾. So ¾ of 5 = 3 ¾

Coordinating Discussion for Student Learning: Solutions 1, 2, and 3

Why might solution 1 be chosen first for student discussion followed by solution 3 and then 2?

- *Solution 1* – shows a fraction of a set calculation asking how many stamps are in ¾ of the collection. The student identified a unit fraction, ⅛ of the collection and took 4 of these to arrive at an answer of 8 stamps. This solution show a unit fraction calculation followed by a whole number multiplication.
- *Solution 3* – For ¾ of 5, the student tripled the length of the line and then found ¼ of the new length 15. This solution shows the whole number multiplication followed by a unit fraction calculation.
- *Solution 2* – shows a mixed number (2 ½) multiplied by a fraction (⅔). The student created an image of ⅔ of a hexagon with pattern blocks. Then s/he created a copy of 2 ½ of these. When the parts are reconfigured, you can see that the result is 1 ⅔ of a hexagon.

LET'S DO MATH – HOW MUCH LENGTH AND AREA?

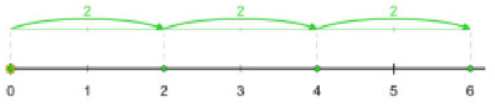
Before (Getting Started)

Carol is cycling to her Grandma's house that is 6 km away. She has travelled ⅓ of the way. What distance does she have to complete?

Solution 1

Divide the 6 km into 3 parts. Each part is 2 km long. She has travelled ⅓ if the way or 2 km.

She has 4 km to go.



Solution 2



6 km x ⅓ =
6 km ÷ 3 = 2 km

2 km = _____ + 6km → 4 km to go

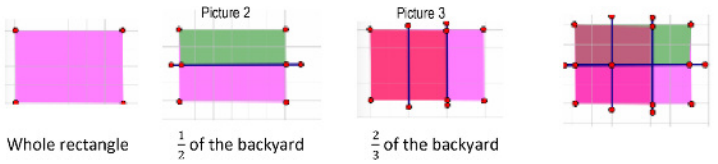
During (Working On It) – Paving for Basketball Court

Jeremy agreed to pave ⅔ of half of the backyard so the children could play basketball. What would of ⅔ of ⅓ be? What fraction of the yard is left?

After (Consolidation)

Anticipating Student Responses:

Solution 1



Overlap pictures 2 and 3.

⅔ of ½ of the backyard. Shows 2 squares

in top left corner or ²⁄₆ or ⅓ of the whole.

⁴⁄₆ = ⅔ of the yard is left unpaved.

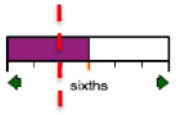
Solution 2



½ is shown here.

To take ⅔ of this strip I need to break the dark part into thirds and therefore the whole strip into sixths.

⅔ of ½ = ²⁄₆ the whole.



Notice when you multiply fraction x fraction the units change.

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solution 2?

- *Solution 1* – The rectangle makes you think of a yard so the solution feels like breaking up a yard. It shows taking half the yard and then ⅔ of that half to pave for the basketball court.
- *Solution 2* – The strip solution is similar in that there are two steps – finding the first part of the whole and then the second fraction of that fraction.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how fraction reasoning tasks (e.g., as operator) develop students' understanding and use of methods of representing and comparing fractions with like and unlike denominators
- To suit your students' learning needs, select and design reasoning tasks that are relative to grade specific curriculum expectations in the Number Sense and Numeration (e.g., fractions as operator)
- Practise noticing the different methods and the breadth of the mathematics your students use to solve fraction as an operator reasoning tasks, such as, "In which case will the result be greater than the operand: ⅞ of 63 or ⅝ of 63?"
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit aspects of different methods for representing fractions as an operator using a set, area or measurement model.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

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