8 PAGE INSERT ELEMENTARY MATH

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FRACTIONS AND RATIOS

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We regularly use mathematics in our everyday life: from measuring distances, comparing prices, estimating how many children and adults are on a class trip to interpreting percentages in newspapers. What do these statements about everyday life have in common? Are any of these statements equivalent by quantity?

- A recipe for orange juice calls for 3 cans of cold water for 1 can of frozen juice concentrate. How many cans of cold water for 7 cans of frozen juice concentrate?
- A certain shade of green paint requires 3 parts of blue paint for every 2 parts of yellow paint. How could the shade of green paint be made darker?
- You need 1 cup of rice for every 2 cups of water. How much water would you need if you wanted to use 4 cups of rice?
- If 1 nickel makes 5 pennies, 2 nickels make 10 pennies, and 1 quarter makes 5 nickels, then what different ways can 1 loonie or 100 cents be made?
- If 4/4 time, 1 half note is for 2 beats, 2 half notes for 4 beats, how many beats are 8 quarter notes?
- If 1 cup of flour makes 4 servings, then how many servings are made with 8 cups of flour?
- For one person, there is 1 mouth, 2 arms, and 10 fingers. Is it true that 5 people have 40 fingers?

In this issue, the Research Summary and reasoning tasks (with multiple solutions) focus on ratios with different representations, including fraction notation, within real life contexts. Reasoning tasks in this issue are designed for use with a three-part problem-solving lesson.

RESEARCH SUMMARY – ABOUT FRACTIONS AND RATIOS

A ratio is a comparison of the size of one number to the size of another number. The numbers compared in a ratio can be any quantities of a comparable kind, such as objects, persons, lengths, or even cupfuls. A ratio is written as an ordered pair (a particular order) of numbers that expresses a comparison between the numbers in the following forms: “a to b”, “a : b”, or sometimes expressed as a quotient of the two quantities in a fractional form.



- ratio of oranges to apples is 2 : 3 or 2 to 3
- ratio of oranges to the total number of fruit is 2 : 5 or 2 to 5.
- $\frac{2}{3}$ as many oranges as apples
- $\frac{2}{5}$ of the pieces of fruit are oranges

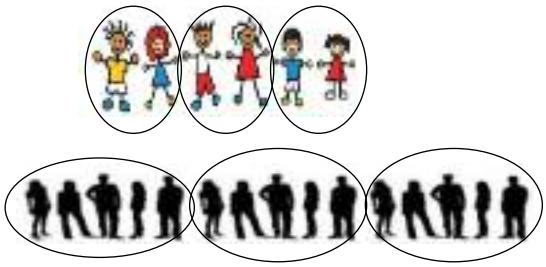
Part-to-Whole Ratios

A ratio can compare a part to a whole, then the part-to-whole interpretation of fraction is being used. For example, if there are 15 adults at a family party compared with a total of 21 people, we can write this comparison using a ratio:

- 15 to 21 or 15 : 21 which is equivalent to 5 to 7 or 5 : 7
- $\frac{15}{21} = \frac{5}{7}$

Part-to-Part Ratios

A ratio can compare two or more parts of a set to other parts of a set; for example, 6 children compared to 15 adults at the family party, part-to-part ratio is 6 to 15 or 6 : 15 which is equivalent to 2 : 5.



Ratio as a Quotient

More specifically, a ratio is a relationship between two numbers indicating how many times the first number contains the second. Another way to think about a ratio is how many times bigger one object is than another. For example, in a ratio of oranges to apples of 2 : 3, there are $\frac{2}{3}$ as many oranges as apples. In a ratio of children to adults of 6 : 15, there are $\frac{6}{15}$ or $\frac{2}{5}$ as many children as adults.

Therefore, ratios and fractions do not have identical meanings. Fractions are ratios, but not all ratios are fractions. If the two quantities have the same units, their ratio is a dimensionless number. If the two quantities have different units, then the ratio is referred to as a rate (e.g., distance per time or km per hour).

Equivalent Ratios and Percents as a Special Ratio

Lemons	Oranges	Lemons to Oranges	Oranges to Lemons	Lemons to Fruit
8	6	8 : 6	6 : 8	8 : 14
16	12	16 : 12 (8 : 6) x 2	12 : 16 (6 : 8) x 2	16 : 28 (8 : 14) x 2
4	3	4 : 3 (8 : 6) ÷ 2	3 : 4 (6 : 8) ÷ 2	4 : 7 (8 : 14) ÷ 2
		$\frac{4}{3}$ as many lemons as oranges	$\frac{3}{4}$ as many oranges as lemon	$\frac{4}{7}$ as many lemons as fruit

Multiply all quantities in a ratio by the same number, so that the ratio maintains its proportionality. The ratio of two quantities remains constant even as the corresponding values of the quantities change. Equivalent ratios can be created by iterating and/or partitioning a composed unit. If one

quantity in a ratio is multiplied or divided by a factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship. Furthermore, it is important to be clear what is being compared to what. Consider the ratio table that describes comparisons lemons, oranges and fruit.

Part A	Part B	Part C	Part D	A + B + C + D
5	9	4	2	5 + 9 + 4 + 2 = 20
10 (5 x 2)	18 (9 x 2)	8 (4 x 2)	4 (2 x 2)	40 (20 x 2)
25 (5 x 5)	45 (9 x 5)	20 (4 x5)	10 (2 x 5)	100 (20 x 5)
$\frac{25}{100}$	$\frac{45}{100}$	$\frac{20}{100}$	$\frac{10}{100}$	$\frac{100}{100}$
25%	45%	20%	10%	100%

In another example, for a mixture that contains substances A, B, C and D in the ratio 5 : 9 : 4 : 2, there are 5 parts of A for every 9 parts of B, 4 parts of C and 2 parts of D. It is equivalent to the ratio of 25 : 45 : 20 : 10. Using equivalent ratios, the special ratio, percent (quantity per hundred) can be determined. See the ratio table above.

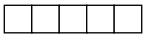
LINKS TO MANIPULATIVES: SQUARE TILES AND BAR MODEL

The bar model, represents the relationship between known and unknown numerical quantities, as rectangular “bars” to solve problems. Rectangles are used because they are easy to draw, divide, represent larger numbers and show proportional relationships, and the rectangles enable students to see number relationships.

A. There are 3/5 as many girls as boys in the grade 6 classes. If there are 60 boys, how many girls are there?’

3/5 means a ratio of 3 : 5 (girls : boys)

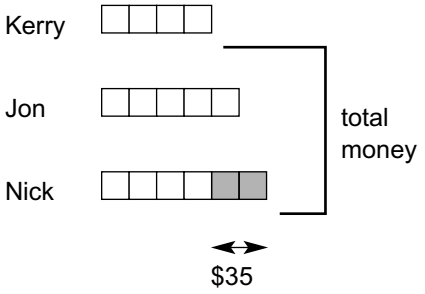
girls 

boys  60

5 square tiles (rectangle units) is 60 boys, so 1 square tile (unit) is 12 boys.

Girls are 3 units x 12 girls = 36 girls

B. Kerry, Jon and Nick shared a sum of money in the ratio of 4 : 5 : 6. If Nick has $35.00 more than Kerry, how much total money is shared by the three friends?



Nick has $35 more than Kerry which is 2 square tiles (2 units).

1 unit is $35 ÷ 2 = $17.50.

Total units = 4 + 5 + 6 = 15 units

15 units x $17.50 = (14 + 1) x $17.50

(14 x $17.50) + $17.50 = 7x (2 x $17.50) + $17.50

= 7 x $35 + $17.50 = (6 x $30 + 1 x $35) + $17.50
= $210 + $35 + $17.50 = $262.50

The friends share $262.50.

LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

Grade 5	Grade 6	Grade 7
-describe multiplicative relationships between quantities by using simple fractions and decimals	-represent ratios found in real-life contexts, using concrete materials, drawings, and standard fractional notation	-Solve problems involving proportions using concrete materials, drawings and variables

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Contest Prizes Problem

Three context prizes has a ratio of winnings for for first:second:third is 9:3:2. If third prize is 4 tickets to Jump Dome, then how many tickets in total are given out for all three prizes?

Solution 1

2x2=4 (3rd), 3x2=6 (2nd) and 9x2=18 (1st) → 4+6+18 = 28; 28 tickets for Jump Dome are given out altogether.

Solution 2

(9+3+2) x 2 = 14 x 2 = 28
28 tickets for Jump Dome are given out for all three prizes

Ratio 9:3:2 says the number of tickets, then if the 3rd prize is 4 or double 2, then it is double 3 and double 9

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Hockey Card Problem 1

Lisa and Luch agreed that they would share any hockey cards they get at a 3:5 ratio. Pat gave them 32 hockey cards. How many hockey cards did Luch and Lisa each get?

What information are we going to use to solve the problem?
- *Lisa : Luch is a 3:5 ratio*
- *32 hockey cards altogether*
- *Numbers of cards Lisa and Luch each get is unknown*

AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Ratio *-comparison of 2 or more quantities*

Part to Part Ratio *-comparison of 2 or more parts of a set to other parts of a set (3:5 ratio is comparing 3 cards to Lisa and 5 cards to Luch in a set of cards)*

Solution 1

From a pile of 32 hockey cards give 3 to Lisa and 5 to Luch again and again until there are no more cards left.

(3+5) + (3+5) + (3+5) + (3+5) = (3+3+3+3) + (5+5+5+5) = 12 + 20 = 32

Lisa gets 12 cards and Luch gets 20 cards, which totals 32 hockey

Grouping and Distributing

Solution 2

In every unit of size 8, 3 cards belong to Lisa and 5 cards belong to Luch.

4 units of 8 hockey cards makes 32 total hockey cards.

(3+5) x 4 = 12+20 = 32 hockey cards.

Lisa gets 12 cards and Luch gets 20 cards, which totals 32 cards.

Composing the Ratio Unit

Ratio as Quotient *- is about how many times bigger one object is to another. For ratio 3:5, (Lisa:Luch)*
-Lisa gets 3/5 as many cards as Luch.
-Luch gets 5/3 as many cards as Lisa

Ratio as Fraction (Part-Whole Ratio) *- compares a part to a whole, Ratio 3:5 is 3 cards for Lisa and 5 cards for Luch; total cards is 8*
-Lisa cards -> 3/8 of 32 total cards
-Luch cards -> 5/8 of 32 total cards

Equivalent Ratios
-multiply all quantities in a ratio by the same number
ratio maintains its proportionality. -
ratio of 2 quantities remains constant even as the corresponding values of the quantities change
- 3:5 is equivalent to 6:10 (3x2:5x2)
- 3:5 is equivalent to 12:20 (3x4:5x4)

Solution 3

The ratio 3:5 means 3 hockey cards for Lisa and 5 hockey cards for Luch.

Lisa	Luch	Total Cards
3	5	8
6	10	16
9	15	24
12	20	32

3:5 = 6:10 (3x2):(5x2)
3:5 = 12:20 (3x4) : (5x4)

Lisa gets 12 cards and Luch gets 20 cards, which totals 32 hockey cards.

Equivalent Ratios

Solution 4

The ratio 3:5 means 3 cards for Lisa, 5 cards for Luch and 8 total cards, between them.
Lisa:total cards is 3:8 → 3/8 of 32 cards = (32 ÷ 8) x 3 = 4x3 = 12 cards
Luch:total cards is 5:8 -> 5/8 of 32 cards = (32 ÷ 8) x 5 = 4x5 = 20

Lisa gets 12 hockey cards and Luch gets 20 hockey cards, which totals 32 cards.

Part-Whole Ratio Fraction of a Whole

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria

Highlights/Summary

A RATIO is the comparison between two numbers. It can be part-part (Lisa:Luch) *and part-whole* (Lisa:total cards)

Determine quantities of whole number ratios by:
-grouping and distributing
(3+5) +(3+5) + (3+5) +(3+5)
- composing the ratio unit (3+5) x 4
- equivalent ratios (3:5) = (3x4) : (5x4)
- part-whole ratios (fraction of a whole)
3/8 x 32 = (32 ÷ 8) x 3 = 4x3 = 12

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Hockey Card Problem 2

Lisa and Luch share hockey cards at a 5:7 ratio. With 60 cards, how many did Luch and Lisa each get?

Solution 1

Unit as 12 cards (5+7) so 12 x 5 = 60
Lisa -> 5x5= 25 cards
Luch-> 7x5=35 cards
Lisa gets 25 cards and Luch gets 35 cards which sums to 60 hockey cards.

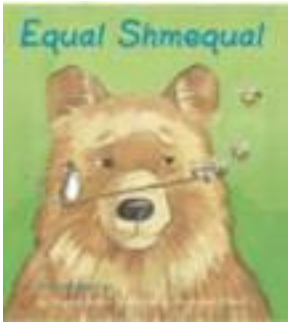
Solution 2

Lisa gets 5/12 of 60 > (60 ÷12) x5=5x5
Luch gets 7/12 of 60 > (60 ÷12) x7=5x7
Lisa gets 25 cards and Luch gets 35 cards and sums to 60 cards.

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO LITERATURE:



The book, *Equal Shmequal* is written by Virginia L. Kroll.

What does it mean to be equal? Mouse and her friends want to play tug-of-war but they can't figure out how to make teams that are equal. After multiple attempts are made, they finally arrive at a solution that makes both sides equal with the same number of animals, but there were other attributes to consider, such as mass and size. They try dividing up by meat-eaters and plant-eaters; then they try those that have fur versus those without fur. Neither of these solutions works. Nothing works until Mouse starts thinking mathematically. Then they try three animals on each side.

To make the balance equal, a set of ratios can be used to represent the number of animals. For example, bear : wolves → 1 : 5; wolves : turtles → 2 : 4;

turtles : mice → 1 : 3

Given these ratios, what are the ratios for the following animals?

- wolves : mice
- bear : turtles
- mice : turtles
- bear : mice

LET'S DO MATH – JEREMIAH'S FAMILY: CHILDREN AND PETS

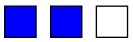
Before (Getting Started)

Represent a ratio in three or more ways. Explain how your models are the same and different.

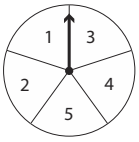
Solution 1



cats : dogs = 2 : 3



blue : total squares = 2 : 3



even numbers : odd numbers = 2 : 3

1 and 3 are part : part ratios.
2 is a part to whole ratio.ratios.

Solution 2

I picked the ratio 4 : 1



red apples : green = 4 1 or $\frac{4}{1}$

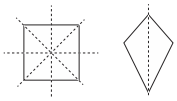
There are 4 times as many red apples green.

almonds : cashews = $\frac{4}{1}$



There are $\frac{1}{4}$ as many cashews as almonds

Number of lines symmetry



square : kite = $\frac{1}{4}$

A square has 4 times as many lines of symmetry as a kite.

During (Working On It)

In Jeremiah's family, there are two children for every one pet. If there are 4 pets, how many children are there?

After (Consolidation)

Anticipating Student Responses:

Solution 1



children : pet = 2 : 1



children : pets = 4 : 2



children : pets = 6 : 3



children : pets = 8 : 4

There are 8 children and 4 pets.

Solution 2

I can show the children with C and the pets with P.

CC : P = $\frac{2}{1}$

CCCC : PP = $\frac{4}{2}$

CCCCCC : PPP = $\frac{6}{3}$

CCCCCCCC : PPPP = $\frac{8}{4}$

Solution 3

Children	Pets
2	1
4	2
6	3
8	4

children : pets = 8 : 4

If there are 4 pets, there must be 8 children.

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solution 2?

Solution 1 – The solution shows images of the numbers of children and pets and the corresponding ratios. Every time you add 2 children, the number of pets increases by 1. This continues until there are 4 pets.

Solution 2 – This solution shows letters corresponding to the number of children and the number of pets. When the first term is scaled up by 2, the second term is scaled up by 1. This can be seen in the number of Cs and the number of Ps and in the fraction-representations of ratios.

Solution 3 – In the table of values, the step-by-step increases of children and pets can be clearly seen row by row.

LET'S DO MATH – THE SOCCER TEAM

Before (Getting Started)

How can this illustration be said to show the ratio 3 : 5 and the fractions $\frac{3}{8}$ and $\frac{5}{8}$?



Solution 1

The ratio 3 : 5 shows part-to-part ratio of white balls : black balls.

There are 8 balls altogether so

$\frac{3}{8}$ *represents 3 of the 8 balls are black*

$\frac{5}{8}$ *represents 5 of the 8 balls are white*

The fractions represent part-to-whole ratios.

Solution 2

Solution 2

black : white = 3 : 5

black : total number of balls = 3 : 8

white : total number of balls = 5 : 8

We say $\frac{3}{8}$ of the balls are black and $\frac{5}{8}$ of the balls are white.

During (Working On It) –

The City Soccer Team has a ratio of new to experienced players of 3 : 8. If the ratio for all the teams in the league is consistent and there are 55 players altogether, how many would you expect to be new players?

After (Consolidation)

Anticipating Student Responses:

Solution 1

new : experienced = 3 : 8 with a total of 11

If the new total is 55, each term in the ratio needs to be scaled up by a factor of 5 because $\frac{55}{11} = 5$.

new : experienced = 3 x 5 : 8 x 5 = 15 : 40

Of the 55 players, 15 would be new to the league.

Solution 2

new	3	6	9	12	15	18
experienced	8	16	24	32	40	48
Total	11	22	33	44	55	66

To preserve the ratio, I scaled up each term and the total by 2, 3, 4, and 5 until I got 55 as the total.

Then I can tell that in 55 players 15 would be new and 40 experienced players.

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solution 1?

- *Solution 1* – This solution shows that by comparing the total for one team and the total for all teams in the league you can conclude the factor of the ratio is 5. So, each term in the ratio is multiplied by 5 to determine the results. The initial ratio of 3 : 8 is preserved in 15 : 40 and the total number of players is 55.
- *Solution 2* – In the ratio table, the part-to-part ratios and the parts-to-whole ratios are recorded so the reaction of the numbers is clear. Each term is scaled up by the same factor each time.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how ratio reasoning tasks (e.g., part-part, part-whole) develop students' understanding and use of methods of representing and comparing whole number and fractional ratios.
- To suit your students' learning needs, select and design reasoning tasks that are relative to grade specific curriculum expectations in the Number Sense and Numeration (e.g., whole number and fractional ratios)
- Practise noticing the different methods and the breadth of the mathematics your students use to solve ratio reasoning tasks using equivalence.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit aspects of different methods for representing ratios using different models, including fraction notation.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

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CLOSING

Our next issue will focus on division of whole numbers (with whole number and decimal number quotients), decimal numbers and fractions with real world applications. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a problem solving-based teaching and learning lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or maryloukestell@gmail.com