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INSERT
ELEMENTARY
MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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DIVISION OF WHOLE NUMBERS

TABLE OF CONTENTS

Abacus Editor Greetings 1
RESEARCH SUMMARY 2
- About Division of Whole Numbers
LINKS TO MANIPULATIVES . . 3
- Base Ten Blocks and Square Tiles
LET'S DO MATH: 4-5
- Bansho Plan
Flags for Schools / basketball
LINKS TO LITERATURE 6
- *One Hundred Angry Ants* by Elinor J. Princzes
LET'S DO MATH: 6
- Red Delicious Apples from Farm to Home
LET'S DO MATH: 7
- Visiting the Parks
NEXT STEPS FOR YOUR PROFESSIONAL LEARNING. 8
- Application to Your Classroom
- Suggested Readings

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July 1, 2017 was the sesquicentennial anniversary of Canada's birth. Canada became a country on July 1, 1867 when the British North America Act was passed by the British Parliament. Some fun facts about Canada are used as contexts for math problems. Though the contexts are different, what is similar about these problems?



- Apples are the most significant fruit produced in Canada (in terms of tonnage, volume and value). Pat eats 6 Red Delicious apples. Sofie eats $\frac{1}{3}$ as many apples as Pat. How many apples does Pat eat?
- The Maple Leaf was made the official flag of Canada in 1965. For a school Canada 150 celebration, Stephanie needs 48 Canada flags. If 1 box has 3 flags, how many boxes should she buy?
- There are over 36 million people living in Canada. About one in five Canadians were born in another country. About how many Canadians were born outside of Canada?
- In Ontario, there are 334 provincial parks. For an Algonquin Park camping trip, Joel packed 6 outfits for camping. He has 2 pairs of shorts. How many shirts did he pack?
- The Canadian province, P.E.I. only has two cities (Charlottetown and Summerside), but 33 golf courses. If I visit P.E.I. for 1 month, is there enough time to play 1 round of golf (4 hours) at each golf course?

In this issue, the Research Summary and reasoning tasks (with multiple solutions) focus on division of whole numbers using different methods and modes of representation, within real life contexts. Reasoning tasks in this issue are designed for use with a three-part problem-solving lesson.

RESEARCH SUMMARY – ABOUT DIVISION OF WHOLE NUMBERS

Division is generally defined as the act of separating something into equal parts or groups. Another explanation, is that it is an arithmetic operation of determining how many times one quantity is contained in another. Division is the inverse of multiplication. A division equation is comprised of a dividend (the whole or the number being divided), the divisor (number by which the dividend is divided) and the quotient (the result of division).

20	÷	4	=	5
dividend		divisor		quotient

How is it possible that these two problems represent the same division equation, 20 ÷ 4 = 5?

- A class of 20 students is divided into 4 equal size groups. How many students are in each group?
- A class of 20 students is divided into groups of 4 students each. How many groups are there?

Each of these problems is based on a different conceptual representation of division:

- *Partitive* (equal sharing) – It is a process of division in which a given number or whole (dividend) is separated into a given number of groups, so that each of the groups has the same number of objects.

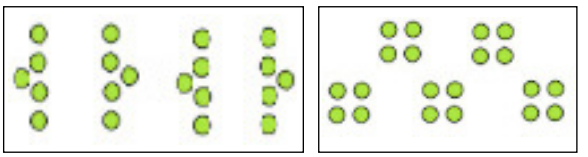
20	÷	4	=	5
dividend (whole)		divisor (# groups)		quotient (same number in each group)

- *Quotative* (equal grouping) – It is a process of division in which a given number or whole (dividend) is separated by the same number of objects in each of the groups.

20	÷	4	=	5
dividend (whole)		divisor (same number in each group)		quotient (# groups)

Visual Representations of 20 ÷ 4 = 5

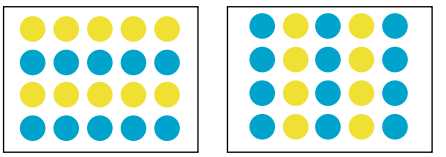
Set Model (Discrete Objects)



paritive
4 groups
5 in each group

quotative
4 in each group,
5 groups

Set Model (Array)



paritive
4 groups,
5 in each group

quotative
4 in each group,
5 groups

Linear Model



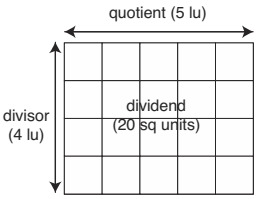
paritive
4 groups
5 in each group

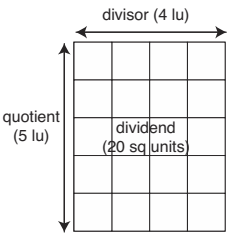


quotative
4 in each group
5 groups

Area (Rectangle) Model

This area model emphasizes the inverse relationship between the operations, multiplication and division. For multiplication, the known side lengths (factors) are multiplied to determine the area of rectangle (product).





- side length 1 (rectangle) x side length 2 (rectangle) = area (rectangle)
- 4 x 5 = 20 and 5 x 4 = 20

For division, the area of the rectangle (dividend, sq units) and one side length (linear units, lu) are known, while the other side length (linear units, lu) is unknown.

If 20 ÷ 4 = 5, then 20 ÷ 4 = quotient and 20 ÷ divisor = 5

Symbolic Notation

Symbolic representations of a division equation includes:

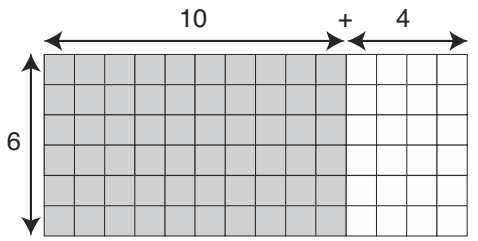
- division equation as a binary operation → **20** ÷ 4 = 5
- division algorithm → $4\overline{)20}$ with 5 above
- division as a fraction → $\frac{20}{4} = 5$

False Generalizations Made About Division

Because multiplication and division are inverse operations of one another, teachers often mistakenly identify repeated addition as a representation of multiplication, when it is simply a bridge from additive to multiplicative reasoning. Similarly, Fosnot and Dolk (2001) noted that "American teachers often tell children that division is nothing more than repeated subtraction. This idea not only is insufficient but also hinders children's ability to construct an understanding of the part/whole relationships in multiplication and division." (p.53)

LINKS TO MANIPULATIVES: BASE TEN BLOCKS AND SQUARE TILES

Square tiles and base ten blocks both can represent quantity as a set of objects in arrays (rows x columns). Square tiles are more suited for dividends no greater than 100. Base ten blocks model units of quantity flexibly; for example, whole numbers can be modelled in units of ones, tens, hundreds, and thousands. Such units compel the use of and make explicit the units of tens and ones in the quotient. In addition, base ten blocks can represent quantity as lengths in centimetre units (cm) and area in square centimetre units (sq cm or cm2), which emphasizes the relationship between quantity and measurement units. Also, the modelling of division equations should be accompanied with symbolic notation and explanatory labels. For example, 84 ÷ 6 = 14 can be modelled.



84 ÷ 6
= (60 + 24) ÷ 6
= (60 ÷ 6) + (24 ÷ 6)
= (10 + 4) = 14

partial dividend (60 + 24) divided by divisor results in partial quotients (10 + 4).

LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

Grade 5	Grade 6	Grade 7
divide 3-digit whole numbers by 1-digit whole numbers, using concrete materials, estimation, student-generated algorithms, and standard algorithms;	solve problems involving the multiplication and division of whole numbers (4-digit by 2-digit), using a variety of tools (e.g., concrete materials, drawings, calculators) and strategies	solve problems involving the multiplication and division of decimal numbers to thousandths by 1-digit whole numbers

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Canada Flag Problem

162 ÷ 27 = ? Write a partitive and quotative problem for this division expression. Use a Canada 150 fun fact as a context for the prolem.

Solution 1

162 Canada flags were given to a school. 27 classes will be given flags. How many Canada flags will each class get?

Partitive Division
Whole (162 flags) is divided by # groups (27 classes)

Solution 2

It takes 4 hours to play 1 round of golf. If 162 hours of golf were played, how many rounds of golf were played?

Quotative Division
Whole (162 hours golf) is divided by size of 1 group (4 hours for 1 round of golf)

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Basketball Team Problem 1

To celebrate the Canadian invention of the game, basketball, 2112 players signed up to play in a cross-Canada basketball tournament.
How many teams of 11 players each could 32 sponsors support?

What information are we going to use to solve this problem?
- *2112 basketball players*
- *1 team has 11 players*
- *32 sponsors*
- *How many basketball teams for each sponsor?*

AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

divisibility *– dividing a dividend (whole) by a divisor evenly (no remainders)*

Division *is the action of separating a whole into equal parts (partitive) or equal groups (quotative)*

2112 ÷ 11 ÷ 32 = 6
dividend divisor quotient

Solution 1

If 22÷11=2, then
22x100÷11=2x100
2200 players÷11 players for 1 team
= 200 teams
2200 players-2112 players =88 players
88 players÷11 players for 1 team = 8 teams

Too many teams, so adjust by subtracting 8 teams:
200 teams – 8 teams = 192 teams

192 teams ÷32 sponsors = 6 teams for each sponsor

Solution 2

2112 players ÷ 32 sponsors
=2112(÷2÷2÷2÷2÷2)
=(2112÷2)÷2÷2÷2÷2
=(1056÷2)÷2÷2÷2
=(528÷2)÷2÷2
=(264÷2)÷2
=(132)÷2
= 66 players for each sponsor

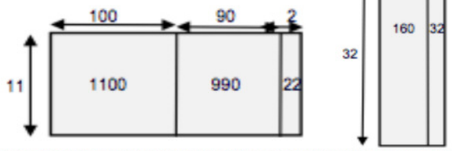
teams for 1 sponsor:
66 players÷11 players on 1 team
= 6 teams for 1 sponsor

Compensation for Divisibility

Decomposing Divisor

Division and Multiplication are inverse operations
2112 ÷ 11 ÷ 32 = 6 and 6 x 32 x 11 = 2112
2112 ÷ 11 = 192 and 192 x 11 = 2112
192 ÷ 32 = 6 and 32 x 6 = 192

Area model shows multiplication and division as inverse operations.



Dividend Decomposed
Decompose whole (dividend) into parts that are divisible by the divisor to get ***partial quotients***
2112 ÷11 = (1100+990+22) ÷11 = (1100÷11) + (990÷11) + (22÷11) = 100 + 90 + 2 = 192

dividend *(2112 (# players), 192 (# teams))*
divisor *(11 # players per team, 32 # teams per sponsor))*
partial quotient *(result of division of partial dividends)*

Solution 3

2112 ÷11 = # of teams

100+90+2 = 192

What factor do I multiply by the divisor to get close to the dividend?

```
11 ) 2 1 1 2
   -1 1 0 0   11 x 10 tens
     1 0 1 2
    -9 9 0    11 x 9 tens
       2 2
     - 2 2    11 x 2 ones
       0
```

192 ÷32 = # of teams for each sponsor

5 + 1 = 6

What factor do I multiply by the divisor to get close to the dividend?

```
32 ) 1 9 2
   -1 6 0    32 x 5 ones
     3 2     (30+2)x5=(30x5)+(2x5)
    -3 2     32 x 1 ones
       0
```

Solution 4

Players per Sponsors
2112÷32÷11
= (2112÷32) ÷11
= [(1600+512) ÷32]÷11
= [(1600÷32) + (512÷32)] ÷11
= (50+16) ÷11
(# players per team per sponsors)

66 ÷11 = 6 teams per sponsors

Expanded Long Division Algorithm

Decomposing Dividend

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria

Highlights/Summary

Division is about separating a whole into equal parts (partitive) or equal groups (quotative)

2112 ÷ 11 ÷ 32 = 6
dividend divisors quotient

Divide 4 digit whole numbers by 2 digit whole numbers by: <include math examples>
- ***compensation for divisibility*** *(adjust dividend to divide divisors evenly then adjust back to original dividend)*
- ***decomposing divisor*** *(divide by sequence of factors of divisor (÷32 -> ÷2÷2÷2÷2÷2)*
- ***decomposing dividend*** *(separate whole into parts for divisibility of divisor)*

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Basketball Team Problem 2

4224 players are playing in a Canada-US tournament. How many teams of 11 players could 64 sponsors support?

Solution 1

4224÷ 11
= (4180+44) ÷ 11
= (4180÷11)+ (44÷11)
= 380+4=384
3844÷64
3844÷8÷8 =48÷8 =6
6 teams for each sponsor

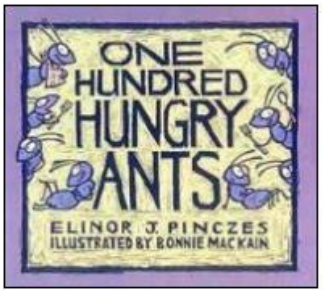
Solution 2

2112 x 2 = 4224
32 x 2 = 64 … so
(2112x2)÷11÷(32x2)
= (2112÷11÷32) x (2÷2)
= 192÷32 x 1
6 teams for each sponsor

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO LITERATURE: *ONE HUNDRED HUNGRY ANTS*



The book, *One Hundred Angry Ants* is written by Elinor J. Princzes. This picture book is a funny read-aloud about 100 hungry ants, desperately trying to get to a picnic as fast as possible. Initially, the 100 ants are hurrying, in one long row to go to the picnic. They deem that it will take too much time to go single file, so, the littlest ant suggests that they reorganize themselves into two rows with fifty ants in each. The ants scurry into the new formation. Again, the littlest ant stops them to reorganize the 100 ants into four lines with twenty-five in each. Still dissatisfied, the littlest ant has them reorganize twice more, once into five rows with twenty ants in each, and finally into ten rows with ten ants in each. After all the rearranging, will they make it to the picnic in time?

This whimsical context helps learners visualize rectangular arrays and think about pairs of factors that have a product of 100. As well, organization of the ants demonstrates division of 100 into 1, 2, 4, 5, and 10 equal groups structured using arrays. Have the students:

- model the division of the 100 ants into different size groups using different visual models (square tiles, number line) labelled by the division equation (i.e., dividend, divisor, quotient);
- explain how each visual model shows division;
- explain the relationship between the divisor and quotient from each division equation (e.g., as the divisor increases, the quotient decreases).

LET'S DO MATH – RED DELICIOUS APPLES FROM FARM TO HOME



Before (Getting Started)

The farm has 65 bushel baskets of Red Delicious apples to distribute to 5 fruit stands. How many should go to each?

Solution 1

Fruit Stand	Number of bushels		
	Try	Try	Try
FS 1	10	5	3
FS 2	10	5	3
FS 3	10	5	3
FS 4	10	5	3
FS 5	10	5	3
Total	50→ 15 left to share	25→ 50 + 25 too many	15→ 50 + 15 just right

If each stand gets 13 bushels, they will be evenly distributed.

Solution 2

I can think of 65 as 30 + 35

30 ÷ 5 = 6

35 ÷ 5 = 7

and 6 + 7 = 13

Each stand gets 13 bushels.

During (Working On It)

The Smiths have a large family of 7. They get a bushel basket with 84 Red Delicious apples. If the apples are given evenly to each family member, how many for each person?

After (Consolidation)

Anticipating Student Responses

Solution 1

$$\frac{84}{7} = \frac{70+14}{7}$$

$$= \frac{70}{7} + \frac{14}{7} = 10 + 2 = 12$$

Each person gets 12 apples.

Solution 2

84 ÷ 7 = ?

Try

If 84 ÷ 7 = 9 then 9 x 7 = 84 Not true as 9 x 7 = 63

If 84 ÷ 7 = 11 then 11 x 7 = 84 Not true as 11 x 7 = 77

If 84 ÷ 7 = 12 then 12 x 7 = 84 TRUE

Each person gets 12 apples.

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solution 1 in this partttive division problem? We know the number of groups and want to determine how many in each group.

- *Solution 2* – The solution shows the student working with division as the inverse of multiplication. Through guess and test, the student determines the quotient 12 gives the correct result when used as multiplier with 7 because 7 x 12 = 84.
- *Solution 1* – This solution shows partial division. The student has decomposed the dividend to numbers s/he knows are divisible by 7. Then by breaking up the dividend into those multiples, dividing each by 7 and recombining the quotients, the answer is determined.

LET'S DO MATH – VISITING THE PARKS

Before (Getting Started)

In Ontario, there are 334 provincial parks. Teams of 8 workers were sent out to clean up 248 parks in southern Ontario. How many parks can each team clean?

Solution 1

248 = 200 + 48

$$\frac{248}{8} = \frac{200}{8} + \frac{48}{8} = 25 + 6 = 31$$

Each team could clean up 31 parks.

Solution 2

To divide by 8, I can take half three times.

$$\frac{248}{2} = 124$$

$$\frac{124}{2} = 62$$

$$\frac{62}{2} = 31$$

Each team can clean up 31 parks.

During (Working On It)

The Parks Commission asked the individual park managers which camps wanted to run summer camps for kids. The Commission interviewed and was willing to pay 243 camp counselors. If they decided each camp would be allotted 9 staffers, in how many parks could they run summer camps for kids?

After (Consolidation)

Anticipating Student Responses:

Solution 1

243 ÷ 9
270 ÷ 9 = 30
But 270 – 243 = 27
And 27 ÷ 9 = 3

So, they could staff 30 – 3 or 27 summer camps.

Solution 2

$$\frac{243}{9} = \frac{180}{9} + \frac{36}{9} + \frac{27}{9}$$

= 20 + 4 + 3
= 27

The Commission was willing to staff 27 camps.

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solution 1in this quotative division problem? We know the size of the groups and we need to determine how many groups.

- *Solution 2* – This solution shows partial division. The student has decomposed the dividend to numbers s/he knows are divisible by 9. (Other combinations are possible.) By breaking up the dividend into multiples, dividing each by 9 and recombining the quotients, the answer is determined.
- *Solution 1* – This solution shows a compensation method.
  The student knows 9 x 30 = 270 and starts with that. But the 270 is 27 more than the dividend so the students subtracts $\frac{27}{9}$ to adjust the quotient so the answer shows $\frac{243}{9}$. (It is like partial division $\frac{243}{9} = \frac{270-27}{9} = \frac{270}{9} - \frac{27}{9}$).

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how whole number problems and/or reasoning tasks develop students' understanding (dividend (whole number), divisor (whole number), quotient (whole or decimal number)) and use of methods of representing (set and set array model, linear model, area model) and calculating whole number (1, 2, and 3 digit) division using different methods (i.e., partial dividend). Unlike multiplication, the distributive, associative and commutative number properties do not apply to division. Unlike multiplication, for multiple divisions in a row, the order of operation goes from left to right.
- To suit your students' learning needs, select and design whole number division problems and/or reasoning tasks that are relative to grade specific curriculum expectations in the Number Sense and Numeration
- Practise noticing the different methods and the breadth of the mathematics your students use to solve ratio reasoning tasks using equivalence.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit aspects of different methods for representing ratios using different models, including fraction notation.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

Boerst, T. (2004). Division discussions: Briding student and teaching thinking. *Teaching Children Mathematics*, 233-236.

Carpenter, T., Fennema, E., Franke, M.L., Levi, L., & Epson, S. (2015). *Children's mathematics: Cognitively guided instruction.* (2nd edition). Portsmouth, NH: Heinemann.

Fosnot, C., & Dolk, M. (2000). *Young mathematicians at work: Multiplication and division.* Portsmouth, NH: Heinemann.

Hedges, M. Huinker, D., & Steinmyer, M. (2005). Unpacking division to build teachers' mathematical knowledge. *Teaching Children Mathematics, 11*(9), 478-483

Humphreys, C., & Parker, R. (2015). *Making number talks matter: Developing mathematical practices and deepening understanding, grades 4-10*. Portland, ME: Steinhouse Publishers.

Jong, C., & Magruder, R. (2014) Beyond cookies: Understanding various division models. *Teaching Children Mathematics, 20*(6), 367-373.

Kamii, C. (2000). *Young children reinvent arithmetic: Implications of Piaget's theory* (2nd edition). New York, NY: Teachers College Press.

Martin, J. (2009). The goal of long division. *Teaching Children Mathematics*, 482-487.

Small, M. (2009). *Big ideas from Dr. Small, Grades 4-8*. Toronto, ON: Nelson

Small, M. (2009). *Big ideas from Dr. Small, Grades K-3.* Toronto, ON: Nelson

CLOSING

Our next issue will focus on division of whole numbers (with decimal number quotients) and rates, within local community applications. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a three-part problem-solving lesson design framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or maryloukestell@gmail.com