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PROPOSED NEW CONSTITUTION OF
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E D I T O R I A L

It will not be new information to most readers of the Gazette that there have been a large number of new "experimental curricula" in North America during the past few years. These grew up in the United States after the Sputnik panic, and were characterized by somewhat extravagant claims which have become more moderate as time has gone on. After all, it did seem impossible to have half a dozen curricula in "new math", each one of which was the only perfect way of approaching new mathematics!

Now in Ontario we have reached the stage of curriculum change and have jumped into the swim with vigour. Most of the teachers reading the Gazette will have had a chance to teach the new Grade XI curriculum, and will have noted changes in the curricula in both Grades IX and X. Perhaps, now that curricular changes are under way, it would be appropriate to stand back and take a look at the changes which have been made and further changes which are proposed. Here we may receive some guidance from the United States.

The most striking feature of the American experience, and we must remember that the United States is usually somewhat ahead of Canada in its actions and in its reactions, is the fact that many eminent American mathematicians, including some leading enthusiasts for "new math", are sounding a note of alarm. The United States has reached the stage of curriculum evaluation, and many persons are not pleased with everything which they see in the new curriculum. One of the most prominent proponents of "new math" has been Professor Max Beberman of the University of Illinois and Professor Beberman is still a strong supporter of new-approach mathematics. However, a recent article in Time quotes Professor Beberman as stating that he never intended things to proceed as far as they have gone. Professor Beberman states that computation is the key feature of mathematics, and expresses real fear that we may be raising a generation of students, many of whom will be unable to calculate the amount of change they should receive in a restaurant. If Professor Beberman has this fear, I think that we should all have the same fear, and we should take a critical look at the way things are going.

This is not to indicate that there are not many points of value in the new curricula which are being introduced. Anyone would admit this. On the other hand, perhaps there has been too much of a tendency to eliminate material which was likewise of value. Experience with our Grade XI curriculum seems to indicate that there has been a tendency to cram too much material and too many topics into a limited space of time. We should probably be attempting to cover a smaller range of topics more thoroughly; dabbling is one of the worst tendencies in

Mathematics, and it is very easy to disguise dabbling under another heading such as "a spiral approach to learning". Dabbling does not become any better by being disguised, and we should make every effort to avoid dabbling. Dabbling may destroy a student's interest and may do him severe harm in his later mathematical work.

Since I have mentioned that the United States has reached the stage of evaluation, perhaps it would be appropriate if I drew the attention of readers of the Gazette to a publication which I have been receiving for the past year. This is "The A.C.C.E. Reporter" and is published by the American Council for Curricular Evaluation. Subscriptions may be obtained from Dr. Alexander Calandra, Executive Director, The American Council for Curricular Evaluation, 829 Woodruff Drive, Ballwin, Missouri. This publication is stimulating and provocative, gives one a large number of relevant quotations, and brings out points which you will see glossed over by many people in many places. I recommend it highly.

Perhaps here in Ontario we need an Ontario Council for Curricular Evaluation. After all, curricular change is not enough. Usually a curricular revision is made with great haste and under strong pressures of one sort or another. Textbooks are produced with haste and under pressure. Consequently, evaluation of curricular change becomes of great importance and frequently a revision is not nearly as significant as is the revision of the revision. I believe that we are now at this stage of evaluation, and that as much attention should be given to evaluation as have been given to curricular change, if not more.

And now farewell again for another year. I hope to see many readers of the Gazette at the annual Easter meeting of the O.E.A.

Ralph G. Stanton

R. G. Stanton,
Editor, Ontario Mathematics Gazette.

5. (a) Three functions f, g, and h of the same variable are said to be linearly dependent if there exist constants c_1 and c_2 such that $c_1f + c_2g + h = 0$ for all values of the variable. Show that the functions $k - 1$, $2k + 3$ and $3k + 2$ are linearly dependent functions of k in the sense of the above definition. As a result, what conslusion can be drawn about the following family of lines.

$$(k - 1)x + (2k + 3)y - (3k + 2) = 0 .$$

(b) The equations of two sides of a triangle are of the form $(k - 2)x - (2k + 1)y + (4k + 7) = 0$. One vertex is at the origin and another is on the positive y-axis. Find the ordinate of the vertex on the y-axis if the area of the triangle is 3 units.

Solution:

(a) The identity

$$c_1(k - 1) + c_2(2k + 3) + (3k + 2) = 0$$

implies that c_1 and c_2 satisfy the equations

$$c_1 + 2c_2 + 3 = 0 ,$$

$$- c_1 + 3c_2 + 2 = 0 .$$

These equations have the solution $c_1 = c_2 = - 1$.

It follows that the family of lines

$$(k - 1)x + (2k + 3)y - (3k + 2) = 0$$

all pass through the point $(1,1) = (-c_1, -c_2)$.

(b) Since two sides of the triangle are of the form

$$(k - 2)x - (2k + 1)y + 4k + 7 = 0 ,$$

it follows by (a) that one vertex of the triangle is $(2,3)$, the intersection of the lines

$$x - 2y + 4 = 0 ,$$

$$- 2x - y + 7 = 0 .$$

Since the area of the triangle with vertices $(2,3)$, $(0,0)$, and $(0,2)$ is 2, it follows that the required ordinate is 3.

6. For an ellipse and a hyperbola with the same foci, show that (a) the two curves intersect at right angles, (b) there is no common tangent.

Solution:

Any hyperbola or ellipse with foci $(\pm \sqrt{A - B}, 0)$ can be expressed in the form

$$\frac{x^2}{A+k} + \frac{y^2}{B+k} = 1$$

for a suitable choice of k. The tangent at the point (x_0, y_0) is

$$(1) \qquad \frac{x\, x_0}{A+k} + \frac{y\, y_0}{B+k} = 1 .$$

(a) Let the ellipse E and the hyperbola H correspond to the parameter values k_1, k_2, and let a point of intersection be (x_0, y_0). By (1), the slope of the tangent to the ellipse is

$$m_E = \frac{- x_0 (B + k_1)}{y_0 (A + k_1)} ,$$

and the slope of the tangent to the hyperbola is

$$m_H = \frac{- x_0 (B + k_2)}{y_0 (A + k_2)} .$$

Now, since (x_0, y_0) lies on the ellipse and on the hyperbola,

$$\frac{x_0^2}{A+k_1} + \frac{y_0^2}{B+k_1} = 1 , \quad \frac{x_0^2}{A+k_2} + \frac{y_0^2}{B+k_2} = 1 .$$

Substracting,

$$\frac{x_0^2}{(A+k_1)(A+k_2)} = \frac{- y_0^2}{(B+k_1)(B+k_2)} .$$

This can be written in the form

$$\frac{x_0^2 (B+k_1)(B+k_2)}{y_0^2 (A+k_1)(B+k_2)} = - 1 = m_E m_H .$$

(b) Suppose the ellipse and hyperbola have a common tangent. Let the points of tangency be $P_1(x_1,y_1)$

and $P_2(x_2,y_2)$ respectively. By (1), the common tangent has the equation

$$\frac{x\,x_1}{A+k_1} + \frac{y\,y_1}{B+k_1} = 1 \ , \tag{2}$$

and also the equation

$$\frac{x\,x_2}{A+k_2} + \frac{y\,y_2}{B+k_2} = 1 \ . \tag{3}$$

For these equations to be identical, it is necessary and sufficient that

$$\frac{x_1}{A+k_1} = \frac{x_2}{A+k_2} \quad \text{and} \quad \frac{y_1}{A+k_1} = \frac{y_2}{A+k_2} \ . \tag{4}$$

Further, since P_2 satisfies (2) and P_1 satisfies (3), we obtain by substitution and subtraction

$$\frac{x_1x_2}{(A+k_1)(A+k_2)} + \frac{y_1y_2}{(B+k_1)(B+k_2)} = 0 \ .$$

Finally, substituting from (4) in this result yields

$$\frac{x_1^2}{(A+k_1)^2} + \frac{y_1^2}{(B+k_1)^2} = 0$$

which can only occur for $x_1 = y_1 = 0$. But then P_1 does not lie on E. This contradicts the original assumption that a common tangent exists.

7. If P and Q are two points on and connected by a smooth curve, the tangent to the curve at P is defined to be the line through P with slope equal to the limit of the slope of the chord PQ as Q approaches P along the curve. Using this definition derive an equation of the tangent to the curve

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

at the point $P(x_1, y_1)$ on the curve.

Solution:

Denote the coördinates of Q by (x_2, y_2). Since P and Q are on the curve

$$ax_1^2 + by_1^2 + cx_1y_1 + dx_1 + ey_1 + f = 0 ,$$

$$ax_2^2 + by_2^2 + cx_2y_2 + dx_2 + ey_2 + f = 0 .$$

Substracting,

$$a(x_1^2 - x_2^2) + b(y_1^2 - y_2^2) + c(x_1y_1 - x_2y_2)$$

$$+ d(x_1 - x_2) + e(y_1 - y_2)$$

$$= a(x_1 - x_2)(x_1 + x_2) + b(y_1 - y_2)(y_1 + y_2)$$

$$+ c[(x_1 - x_2)y_1 + (y_1 - y_2)x_2]$$

$$+ d(x_1 - x_2) + e(y_1 - y_2) = 0 .$$

Thus the slope of the chord PQ is given by

$$\frac{y_1 - y_2}{x_1 - x_2} = -\frac{a(x_1 + x_2) + cy_1 + d}{b(y_1 + y_2) + cx_2 + e}$$

Setting

$$m = \lim_{Q \to P} \frac{y_1 - y_2}{x_1 - x_2} ,$$

It follows that

$$m = -\frac{2ax_1 + cy_1 + d}{2by_1 + cx_1 + e} .$$

By the given definition, the equation of the tangent to the curve at P is

$$y - y_1 = m(x - x_1)$$

where m has the value obtained.

8. The orbit of a satellite is an ellipse with one focus at the centre of the earth. The point of closest approach to the earth is called perigee and the point farthest from the earth is called apogee.

(a) Prove that perigee and apogee both lie on the major axis of the ellipse.

(b) If perigee and apogee are respectively 1000 and 3000 miles above the surface of the earth, and if the earth is a sphere of diameter 8000 miles, find the eccentricity of the ellipse.

(c) If the orbit is polar, with apogee directly over one pole, find the height of the orbit directly above the equator.

Solution:

Let the equation of the orbit of the satellite be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

and let the position of the earth (in a co-ordinate system moving with the earth) be $(c,0)$ where

$$c^2 = a^2 - b^2 .$$

(a) When the satellite is at the point (x,y), the square z of its distance from the earth is given by

$$z = (x - c)^2 + y^2 .$$

Substituting for y^2 yields

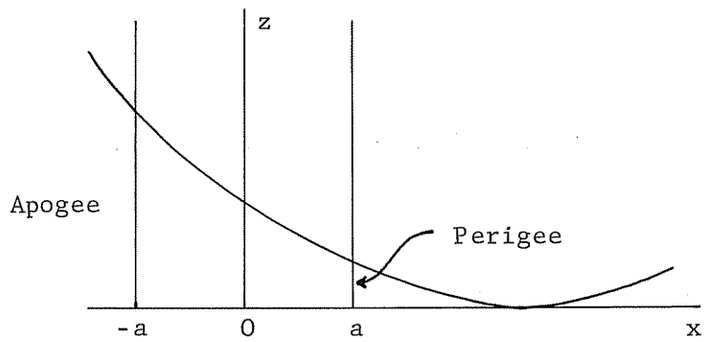
$$z = (x - c)^2 + b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= x^2\left(1 - \frac{b^2}{a^2}\right) - 2cx + b^2 + c^2$$

$$= x^2\,\frac{c^2}{a^2} - 2cx + a^2 \ .$$

Thus $z = \left(\frac{cx}{a} - a\right)^2$.

The graph of z is a parabola with vertex at $\left(\frac{a^2}{c},0\right)$ as shown in the figure. Since $\frac{a^2}{c} > a$, and $-a \leq x \leq a$, it follows that the maximum value of z, corresponding to the apogee occurs for $x = -a$ (and hence $y = 0$), and the minimum value of z corresponding to the perigee occurs for $x = a$ (and $y = 0$).

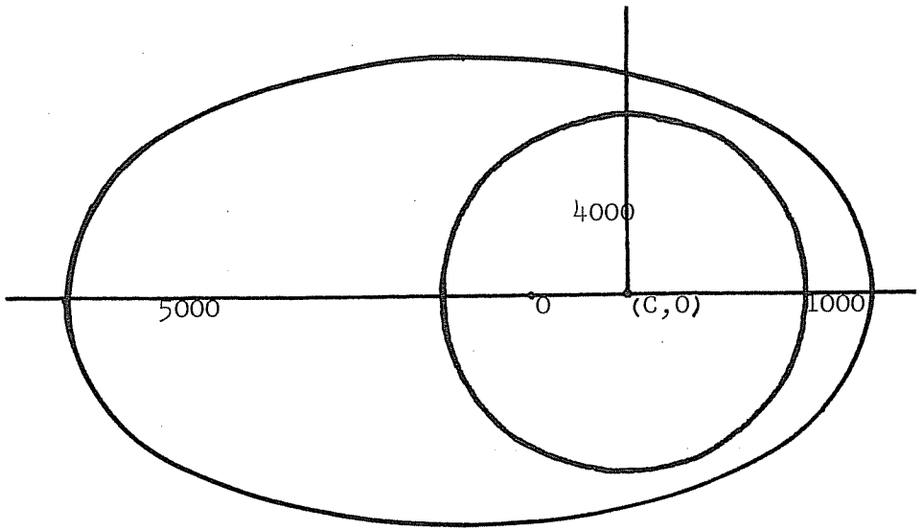


(b) Since the center of the earth is at $(c,0)$, we find

$$a - c = 5000, \quad a + c = 7000, \quad \text{as shown.}$$

Solving, $a = 6000$, $c = 1000$; thus $e = c/a = 1/6$.

(c) The height H_c of the satellite above the center of the earth at the equator is equal to the semi-



latus rectum (that is, the value of $y > 0$ when $x = c$).

Thus

$$H_c = \frac{b^2}{a} = \frac{35000}{6} = 5800 \text{ miles.}$$

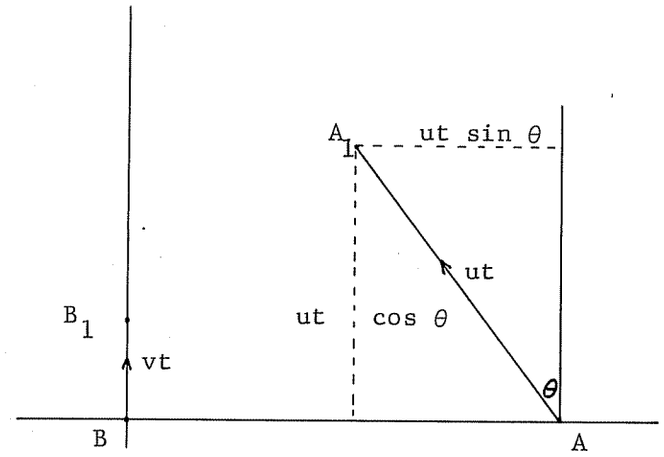
Thus the height of the satellite above the surface of the earth is 1800 miles.

9. A ship A is initially a miles due east of a second ship B. Ship B is sailing due north at a speed of v miles per hour in a direction N. θ^o W. Show that the shortest distance, in miles, between the ships is given by the numerical value of

$$\frac{a(v - u \cos \theta)}{\sqrt{u^2 + v^2 - 2uv \cos \theta}}$$

Solution:

Select rectangular co-ordinates so that at time $t = 0$ ship B is at the origin, A is at $(a,0)$, and the positive y axis is in the north direction. At time t the position of B is $B_1(0, vt)$ and the position of A is $A_1(a - ut \sin \theta, ut \cos \theta)$ as shown in the figure.



Then $(A_1B_1)^2 = (a - ut \sin\theta)^2 + (ut\cos\theta - vt)^2$

$= \alpha t^2 - 2\beta t + \gamma$

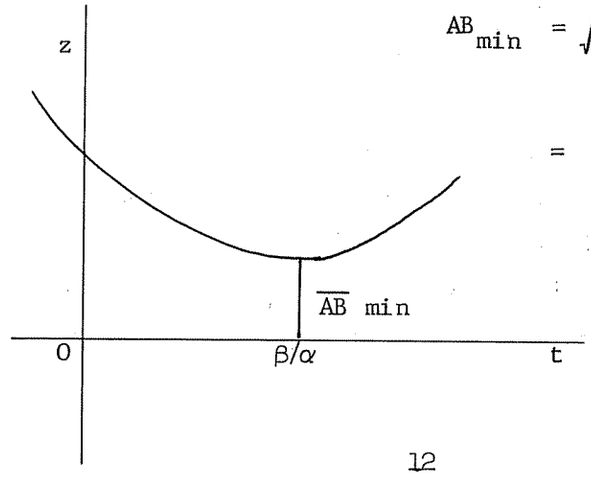
where

$\alpha = u^2 \sin^2\theta + (v - u\cos\theta)^2$,

$\beta = au \sin\theta$,

$\gamma = a^2$.

Since the lowest point (vertex) of the parabola $z = \alpha t^2 - 2\beta t + \gamma$ is $(\frac{\beta}{\alpha}, \frac{-\beta^2 + \alpha\gamma}{\alpha})$, it follows that the minimum value of AB is



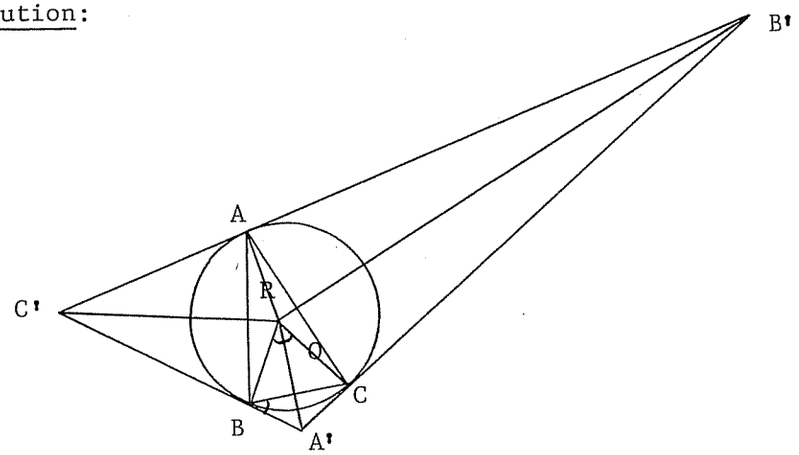
$$AB_{min} = \sqrt{\frac{-\beta^2 + \alpha}{\alpha}}$$

$$= \frac{a(v - u\cos\theta)}{\sqrt{u^2 + v^2 - 2uv\cos}}$$

10. The tangents at A, B, C to the circumcircle of a triangle ABC form a triangle A'B'C'. If R is the circumradius of the triangle ABC, prove that the area of the triangle A'B'C' is given by the product

$$R^2(\tan A + \tan B + \tan C).$$

Solution:



Consider triangles OBA', OA'C. Since BA' = CA' and BO = CO, $\angle$ OBA' = $\angle$ OCA' = 90°, it follows that the triangles are congruent so that $\angle$ BOA' = $\frac{1}{2}\angle$ BOC = $\angle$ A .

Thus $\quad \triangle$ OBA' + $\triangle$ OCA' = R(BA') = $R^2 \tan A$.

Similarly $\quad \triangle$ OCB' + $\triangle$ OAB' = $R^2 \tan B$

$\quad \triangle$ OAC' + $\triangle$ OBC' = $R^2 \tan C$

Adding $\quad \triangle$ A'B'C' = R^2 (tan A + tan B + tan C).

11. Let f be the function defined by

$$f(x) = \frac{\sin x}{\sin(x-a)\sin(x-b)}, \quad x \neq a, \; x \neq b .$$

If a - b is not an integral multiple of π, show that f(x) may be expressed in the form

$$\frac{A}{\sin(x-a)} + \frac{B}{\sin(x-b)}$$

by first finding A and B in terms of sines and cosines of a and b.

Solution:

If $a - b$ is not a multiple of π, the identity

$$\frac{\sin x}{\sin(x-a)\sin(x-b)} = \frac{A}{\sin(x-a)} + \frac{B}{\sin(x-b)}$$

is equivalent to the identity

$$\sin x = A \sin(x-b) + B \sin(x-a)$$

obtained by multiplying by $\sin(x-a)\sin(x-b)$. Expanding the right-hand side and equating the coefficients of $\sin x$ and $\cos x$ yields

$$1 = A \cos b + B \cos a ,$$
$$0 = A \sin b + B \sin a .$$

These equations have the solution

$$A = \sin a / (\sin a \cos b - \cos a \sin b),$$

$$B = - \sin b / (\sin a \cos b - \cos a \sin b).$$

Substitution yields the required identity.

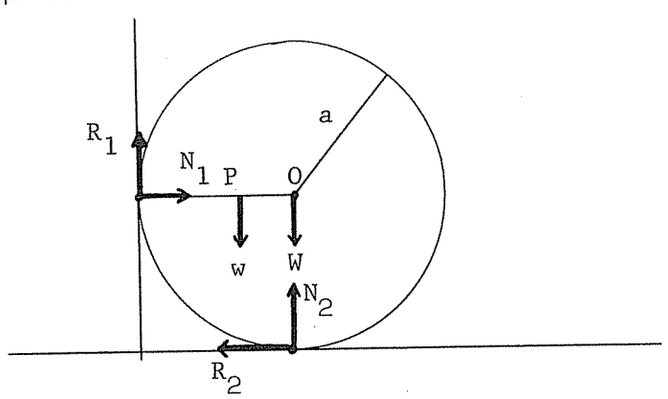
12. A uniform circular disc, of radius a and weight W, stands in a vertical plane upon a rough horizontal floor, with a point of its circumference in contact with an equally rough vertical wall. The plane of the disc is at right angles to the wall. A weight w is then attached to a point P of the disc at the same height as the centre O, and between O and the wall. If μ is

the coefficient of friction, prove that the disc is on the point of slipping when

$$OP = \frac{a\mu(1+\mu)}{1+\mu^2}\left(1 + \frac{W}{w}\right) .$$

Solution:

Let R_1, N_1 and R_2, N_2 denote the components of the reaction at the wall and floor respectively as shown in the diagram. The weight W acts through O and the weight w through P, both in a downward direction. Since μ is



the coefficient of friction,

(1) $$R_1 = N_1\mu, \quad R_2 = N_2\mu .$$

Equating forces in the horizontal and vertical directions, and taking moments about O yields three equations

(2) $$N_1 = R_2 ,$$

(3) $$R_1 + N_2 = w + W ,$$

(4) $$w\overline{OP} = R_1 a + R_2 a .$$

The required expression for $\overline{OP}$ may be obtained by eliminating R_1, R_2, N_1, and N_2 between the five equations.

This may be done as follows.

Substitute for N_1, N_2 from (1) into (2) and (3) to obtain

$$(5) \qquad \frac{R_1}{\mu} = R_2 , \qquad R_1 + \frac{R_2}{\mu} = w + W .$$

Eliminating R_2 between equations (5) yields

$$(6) \qquad R_1 = (w + W)\mu^2/(1 + \mu^2) ,$$

and again using (5),

$$(7) \qquad R_1 + R_2 = R_1 + \frac{R_1}{\mu} = \frac{R_1(1+\mu)}{\mu} .$$

Finally, substituting (7) and (6) in (4), and solving for $\overline{OP}$, we find

$$\overline{OP} = \frac{a\mu(1+\mu)}{1+\mu^2}\left(1 + \frac{W}{w}\right) ,$$

as required.

WHERE MUST EUCLID GO?

[Address given by Dr. M. Rosskopf, OEA, March 31, 1964; contributed by L. D. Aukland.]

It was a French mathematician, Dieudonné, I believe, who first coined the slogan which has become almost a battle cry in certain circles: Euclid must go! One group shouts the battle cry enthusiastically. Almost fanatically, they hack away at the body of Euclid, so dear to all geometry teachers. Another group fiercely defends Euclid as they know it. Changes may occur but only

after a last ditch stand has been made.

When one reads the summary of the address in which Dieudonné made the assertion (New Thinking in School Mathematics), one sees that he meant exactly what he said: Euclid must go. He meant that geometry as most of us know it and studied it really must go. He did not mean what he calls "patchwork"; that is, to make the geometric structure sound by introducing real number and order axioms, or to put more emphasis on co-ordinate geometry. He meant an almost complete replacement of geometry by an algebraic approach. He mentions the "silliness of the work on triangles" and circles. Furthermore, he bluntly says he is thinking about the average student and his experience in mathematics, not the training of mathematicians.

The address must have been quite persuasive. Even in summary, one cannot help but agree with what Dieudonné says. And yet...; aren't there values in the body of geometry that only a study of it can uncover? Are our students ready for a program that would depend so much on algebraic and vector space systems? Are our schools ready to prepare students in such a program and our colleges to receive them?

In the United States, geometry has traditionally been a year course, or a year and a half if you include solid geometry; and this course has been somewhat separated from the study of algebra. During the past ten years, there have been made several efforts to write a better geometry course. Some have tried a Hilbert axiom approach. Others have tried a carefully worked out logical, co-ordinate geometry. Then there have been the geometries that use the real number and order axioms to achieve a sort of merger of the synthetic and co-ordinate approaches. Of course, there is a group of programs, too, that Dieudonné would describe as "sloppy patchwork". All of these have had some success; perhaps the last the greatest success of all!

In view of what is taking place in the elementary and junior high school mathematics programs, what might we look forward to in the future? Let me

examine briefly the foundation for senior high school geometry. There has been enough work done on these two lower school levels, elementary and junior high school, so that some inferences can be drawn. If you study an elementary school series of textbooks, you will find that only a few more pages are devoted to what might loosely be called geometry. A similar assertion can be made about the junior high school textbooks. For example, the amount of geometry that appears today in junior high school textbooks is no more than I covered when I taught such students in the 1940's. The reason there is not more is available time. In order to do all the things that are believed necessary parts of the mathematical experiences of children of ages 6-14 does not leave much room for new material. Remember, that algebra also is in a process of migration down to earlier age levels.

But, and this it seems to me is the important difference, the "look" of the geometry for ages 6-14 is entirely different. The language used is much more mathematically correct. The spirit in which it is done provides an opportunity for children to develop that creativity so long thought reserved for the tenth grade. The emphasis is on relationships, the so-called non-metric geometry, rather than on computations involving perimeters and areas. Computations are subordinate to the geometry of relationships. These are not the only changes. Some of the axioms of Euclidean geometry are stated as "properties" of points and lines. There are proofs of a quality appropriate to the mathematical maturity level of the child.

A restriction of the discussion to geometry only does not tell the whole story. Both the arithmetic and the algebra included in Grades 1-9 are of a much higher quality from the point of view of both mathematics and the psychology of learning. Mathematically speaking, there is much emphasis on the structure of our number system, what principles permit us to do the things we do. Psychologically speaking, there is a unification of approaches to processes of arithmetic and algebra that should enable students to master just a few approaches instead of a large number of separate tricks.

So a student who reaches the tenth grade after nine years in one of these newer programs has already been exposed to many of the facts of geometry. For example, he knows about points, lines, planes and their inter-relationships; segments, separation, and rays; angles and angle measurement, as well as the different sets of angles; parallel and perpendicular lines, plus the properties of angles determined by parallel lines and a transversal; the quadrilateral family and its properties; polygons and regular polygons; the circle, and at least central angle properties; congruent and similar triangles; the Pythagorean Theorem. In addition, he knows many of the principles on which geometry rests. He has been led through some proofs. Some logical terms, like conjunction and disjunction, are familiar to him. In short, he has studied over a period of years a large body of geometric facts, principles, and language.

Perhaps, with such a background on which to build, our tenth grade student is ready for something different. Let us make some conjectures concerning the sort of tenth year course a student might face who starts now and works his way through such a strengthened elementary and junior high school mathematics sequence. In short, what might the course look like ten years from now?

One more comment, however, before we begin our climb out onto a limb. The way in which students must accumulate credits for college entrance, the Carnegie credit units, is going to militate against a carrying along during the same year of geometry and algebra. One might say the separation of mathematics into algebra for Grade 9, geometry for Grade 10, and algebra for Grade 11 will be the typical order of things for an overlong time. Some inroads on this separation have been made already. More will be made I am sure as we come closer and closer to some sort of national program in mathematics.

So first, I do not think Euclid will be banished, if by "Euclid" is meant synthetic geometry. Too many of us love synthetic geometry to give it up during the teaching life times of anyone in this audience. Perhaps, as the liberal arts colleges change their pro-

grams and send out mathematics majors to be teachers who are equipped with the "new mathematics" and have a different view of mathematics from ours, there will be more willingness to shift to an algebraic approach to all high school mathematics. But that remains to be seen. It is clear that mathematicians do not see eye to eye on these matters.

Second, it does not seem to me that the tenth year course will become a vector geometry course. This would be just substituting one sort of approach to geometry for our present one.

Third, it is quite unlikely that in the forseeable future the tenth year course will be taught from the point of view of finite vector spaces of two and three dimensions. The tenth graders, these fifteen year old students, would not have the mathematical background required for such an approach.

But what will the geometry course become, then? Well, I think there will be a gradual shift from a study of geometry as an example of an axiomatic system to a study of the properties of euclidean space of two and three dimensions. In such a study the various approaches -- synthetic, co-ordinate, vector -- become tools to use to uncover the properties. Sometimes one tool would prove to be more appropriate than another, and in this way the three approaches mentioned would be exploited.

My own choice would be to begin the year with a synthetic approach. I am not sure that a completely rigorous system should be used, but I am sure that complete honesty about gaps in proofs is necessary. Certainly transformations like reflections, translations, and homotheties would be discussed and used. In fact, since I assume students would be familiar with the structure of a group, the transformation groups of euclidean space might serve as a nice transition from the synthetic to the co-ordinate approach. This could be accomplished as it is in projective geometry by introducing a co-ordinate system and finding the algebraic equivalents of the synthetic approach, for example a linear equation for a line. To secure the benefit of the vector approach might require a bit

of backing up in order to develop some ideas concerning vectors. For I do not see yet how such ideas would be worked into the already crowded junior high school program. Great changes would have to occur there before it would be possible, it seems to me. But with a sufficient number of ideas concerning vectors and some familiarity with what can be done with them, a co-ordinatized vector geometry could be worked on to finish out a year's experience with the properties of a euclidean space and methods of attack on such a space.

Let me consider for a moment before closing an alternative possibility. But this possibility would require some bold planning, writing of materials, and experimentation in classrooms for its implementation. The possibility I have in mind is a linear algebra approach to geometry--an approach that would, of course, follow the path of vectors spaces of two and three dimensions. What such an approach requires is a firm grasp of the real number system. This means students would have to have such a mastery of the field and order axioms that proofs using them would not be beyond their grasp.

In order to achieve such mastery, I do not see how one could avoid following a strong ninth course with the material of one of the present strong eleventh grade algebra courses. Then the third year could, just possibly, be a linear algebra "geometry". Some group of experts, it seems to me, would have to evaluate carefully the pros and cons of a program of this sort. As mentioned before, bold thinking and bolder writing would be required. But it may be that this is the direction our planning should take.

And yet ... And yet ... I come back again to the former hesitancy. The earlier question: Aren't there values in geometry study as geometry that we do not want to lose? How can we be reasonably sure that some years from now there will not be a renewed interest in the synthetic approach. In spite of the implied boast in the title of my talk that I knew the answer to the question, you find me at the close of my allotted time as uncertain as at the beginning. Like every one else, I am making a guess; all I hope is that it is an intelligent one!

REPORT FROM THE O.M.C.

A.J. Coleman, Queen's University

In this particular issue we report briefly on the Fifth Annual Meeting of the Ontario Mathematics Commission which took place on September 12, 1964, in Trinity College, Toronto, and brought together 5 teachers from Elementary Schools, 10 from Secondary Schools, 12 from Universities, and 5 representatives of the Department of Education.

Prof. H. A. Elliott of the Royal Military College, having served as President for two years (the limit according to the constitution) automatically became Past President. Together with Prof. Elliott, the new Executive consists of President J. Norman C. Sharp, Vice-President Prof. Paul R. Beesack, Secretary-Treasurer Henry J. Campbell (to whom one writes at 1260 Bay St., Toronto, 5, for information about the O.M.C.), Members Wilfred Cunnington, Prof. George F. D. Duff, Rev. John C. Egsgard, Arnold W. Harris. Subsequent to the meeting, it has been announced that Arnold Harris is to be Professor of Mathematics at the New Ontario College of Education in London.

It was reported that:

(a) During the summer of 1964, with financial and moral support of the London Board of Education, the O.M.C. arranged for D. Boyle, a mathematics specialist, to experiment with a five weeks summer course (two hours a day), for 19 students from Grades 11, 12, 13. They studied Chapters 1-4 of Finite Mathematics by Kemeny, Smell, and Thompson. This was a "pilot project" in summer enrichment and a test for the idea of introducing Logic as a regular topic in the curriculum. Both teacher and students were enthusiastically satisfied. In the discussion at the Commission meeting, Prof. R. A. Staal, of the University of Waterloo, commented that he had used the above text and found it satisfactory.

(b) Brochures on Polar Coordinates, by Prof. Faught of the University of Windsor, and on Complex Numbers, by Profs. Hogarth and Kirby of Queen's University, have been printed and distributed to Schools.

Additional material on Logic and Probability should be ready for distribution before Christmas. On the recommendation of the O.M.C., the Department of Education has suggested to the Chief Examiner for the 1965 Problems paper that a few questions might be set on these topics in order to encourage teachers to test them for suitability for introduction into the regular curriculum.

(c) The Elementary Education Committee has been working on an In-Service Training Outline for teachers of Grade 7 mathematics; it will continue its study of the K-6 curriculum and prepare a submission to the Department of Education in the proper training of Elementary School teachers of mathematics.

(d) The Secondary Education Committe has begun to turn attention to the mathematics programmes for students in the Four Year Course, and at the same time to consider the pertinence of the visionary Report of the Cambridge Conference on School Mathematics, Goals for School Mathematics (Education Services Incorporated, 108 Water Street, Waterdown, Mass.), to the situation in Ontario. (We plan to discuss this Report more fully in the next issue of the Gazette.)

The Annual Meeting concluded with a supper meeting which, through the kindness of Dr. Alice Turner of York University, was held in the gracious atmosphere of the University Women's Club. The Guests of Honour were Mr. and Mrs. Frank Asbury, and James Cavanagh, a student of St. Michael's College School, who was the first recipient of the Frank C. Asbury Award. Mr. Asbury served as the first Secretary-Treasurer of the Ontario Mathematics Commission. Mr. Frank Kinlin, in a very moving speech, recalled the remarkable service Mr. Asbury had rendered in the cause of Education and of Mathematics over a period of fifty years. Prof. Coleman described Mr. Asbury's indespensable contribution to the development of the O.M.C. With his usual sparkle and acumen, Mr. Asbury modestly tried to turn the compliments aside. He was clearly touched and delighted by the establishment of the Asbury Award which is intended for the winner of the Junior Mathematics Contest.

COMMENTS ON TAKING EUCLID'S CONGRUENCY PROPOSITIONS AS AXIOMS

R. A. Honsberger

The purpose of this article is to register a criticism of taking Euclid's congruency propositions as axioms in grade ten and to discuss briefly a conceptual basis for congruency which would provide a sound pedagogical approach in a first course in deductive geometry.

A. I -

The motivation for placing the congruency propositions among the axioms is a desire to avoid the "method of superposition". In the past, incomplete coverage has been given superposition in the formal bases of the subject. From the point of view of full rigour, it needs to be put on a proper basis. Beyond this, however, superposition rubs some people the wrong way because it smacks so much of the physical world. The current trend is to drive home logical structure and consistency and it tends to soft-pedal the formally unnecessary conceptual notions which clutter up a purely logical presentation. Some authors foster the fond wish of grounding congruency in "number". Unfortunately, this is quite incompatible with the learning process.

With regard to a purely logical development, no meaning whatsoever need be attributed to the word-symbols employed. Rigour is what counts here and this depends solely upon consistency with the axioms. In an eagerness to impart to young minds the essence of a deductive system, we must be careful not to commit a serious pedagogical omission. In a first course, it is imperative that the elements of geometry be given a complete conceptual basis. It would be pedagogical suicide to attempt the course in terms of meaningless or partially understood symbols. In order to render a grade-ten conceptual basis for congruency complete, we have no choice but to admit many physical notions which are rooted in intuition. Filling congruency with content suitable for grade ten can almost be accomplished without the notion of superposition; but not quite! And in this case, nine-tenths of a solution to the problem is no solution at all. We must be psychologically complete as well as logically so. In fact, this requirement may be applied far beyond the level of grade ten.

II -

One gets oneself into difficulty in using superposition too freely. For example, it is too restrictive to define congruent figures just as those which can be made to coincide. This is all right when applied to plane figures because we conceive a third dimension to space in which oppositely orientated figures may be turned over to permit coincidence. In the illustration, one of the triangles needs to be turned over before coincidence can be effected. However, insurmountable orientation



difficulties are encountered with certain solid figures such as a right hand and its mirror image, which appears as a left hand. There is no way to bring them into coincidence in three-dimensional space. Yet their great degree of identification places them in the same category as figures which can be made to coincide. Accordingly, an inclusive general definition of congruence goes as follows:

Two figures are said to be congruent if they are related by an isometry; that is, if there is a one-to-one correspondence connecting the points of the two figures such that the distance between any two points of one of the figures is equal to the distance between the corresponding points of the other figure.

This contains two main aspects, one-to-one correspondence and equal distances. One-to-one correspondence presents no special conceptual problem and the latest textbooks pay it due attention. The breakdown in the "modern" meaning of congruence occurs in the notion of "equal distances". Here we can not avoid superposition in grade ten. If we say nothing about equal distances, then the student automatically assumes that superposition is intended by virtue of his earlier training in "measurement" or the common sense of plain coincidence. If we see the matter through, then there is no way out but to acknowledge superposition. Let us pursue the meaning in "equal distances" and see what we find.

III -

In searching many recent textbooks on this matter, one is likely to find that two distances are equal if the appropriate line-segments joining the pairs of points in question have the same "length" or "measure". That's fine. But what is meant by the "same length" or "same measurement"? This, they explain, means that the same "real number" has been assigned to each of the two segments. At this point, one can hardly suppress the question "By what method are the numbers assigned to segments?". This forces an admission of superposition in the guise of measurement or refuge is sought in the pedagogically untenable position that it doesn't matter to the logical structure how the numbers are assigned. The simple fact is that "equal distances" precedes measurement, not the other way around. In order to make a measurement, one must first make a measuring-stick. This, like laying out numbers on an axis, presumes the superposition of a constant unit at abutting intervals along the "stick". Here we have common ground that everybody understands.

With the futility of the attempt to banish superposition in grade ten, the case for taking the congruency propositions as axioms evaporates. In keeping with our obligation to admit superposition into the concept of congruency, the appropriate move is to incorporate superposition properly into the foundations of the subject. The essential characteristics of a deductive system will not in the least be subdued. After all, the subject is geometry, not pure logic. Let us turn briefly to certain aspects of such an undertaking.

B. I -

A conceptual basis for congruency, though containing considerable detail, need not be complicated and subtle. At the grade-ten level we should try to move into geometry as comfortably as possible. No difficulty with the meaning of geometric terms and procedures need be encountered if we adhere to the common notions of physical experience. Their statement accompanied by a minimum of description in the foundations is sufficient to legalize their appearance in the writings of the subject and to provide us with a pedagogically essential vehicle and guide for our thoughts. Deep and lengthly discussions of these notions are to be avoided in class for the simple reason that such common concepts do not need penetrating analysis in order to be understood. Their statement in the bases permits their use from the logical standpoint; they are primarily understood, not by explanation, but by experience. Understanding through explanation must give place to understanding

through experience. This is a necessity of language.

II -

The primary notion in "equal distances" is coincidence. This precedes measurement. It presumes the existence of "movable figures" (or images) and the validity of "undistorted motion" (the homogeneity of space), the act of superimposing line-segments and the state of coincidence. To say two line-segments are equal basically means that the one, or an image thereof, can be brought into coincidence with the other. The homogeneity of space asserts that the extent to which two segments coincide at one position of space is the same as at any other place. With the "transitivity" of coincidence we obtain measurement. This asserts the equality of two segments if each coincides with a third segment transported from the one to the other. "Equal angles" has a similar content. The basic idea involved is coincidence. With motion, homogeneity, etc., and transitivity we obtain the measurement of angles. As a result of measurement, we have two ways of thinking of equal segments and equal angles. We may go back to the basic "coincidence" or think in terms of equal measures, i.e., numbers.

To prove congruency, it is required that we establish a one-to-one correspondence which "preserves" distances. In the fundamental triangle theorems, by far the easiest way to set up such a correspondence is by superimposing the entire triangles. Not to proceed with the "traditional" method of superposition of the triangles is to make the problem exceedingly difficult, if it remains at all possible. Little wonder that it was decided, in the latest textbooks, to take these propositions as axioms upon the elimination of superpositions. However, with superposition, these theorems can be proved and they can be proved without difficulty.

In conclusion, the banishment of superposition robs congruence of its only sensible grade-ten meaning by blanking out "equal distances". Because numbers can not be laid out feasibly on an axis without superposition, its exile invalidates many illustrations in algebra (which have already been used in grade nine) and all the pertinent ones in analytic geometry. Finally, without motion and superposition, the topic of "constructions" in geometry is, to say the least, made unnatural and unwieldy.

INVERSE FUNCTIONS

R. G. Stanton

Suppose that a (single-valued) function f is given with the property that it possesses a (single-valued) inverse function f^{-1}. Students sometimes have trouble determining f^{-1} when given f. Perhaps a few examples may help (we use only real variables throughout).

Example 1 The function f is defined by the rule

$$y = 3x + 2$$

Thus any real number x is mapped by f on an "image" $3x + 2$.

(1) $$x \xrightarrow{f} 3x + 2.$$

The inverse function f^{-1} takes $3x + 2$ and maps it on x, that is,

$$3x + 2 \xrightarrow{f^{-1}} x.$$

We might write this also as

$$y \xrightarrow{f^{-1}} \frac{y-2}{3}.$$

Or, using x as variable,

(2) $$x \xrightarrow{f^{-1}} \frac{x-2}{3}$$

Probably (1) and (2) may not appear to be inverse functions; but if we check we see that

$$x \xrightarrow{f} 3x + 2 \xrightarrow{f^{-1}} \frac{(3x + 2) - 2}{3} = x.$$

In short, the result of applying operators f and f^{-1} in that order is just the identity function. Similarly,

$$x \xrightarrow{f^{-1}} \frac{x-2}{3} \xrightarrow{f} 3\left(\frac{x-2}{3}\right) + 2 = x;$$

the result of apply f^{-1} and f in that order is likewise thus the identity function.

Example 2 f is the exponential operator (function) defined by $y = 10^x$.

Thus $1 \xrightarrow{f} 10$

$-2 \xrightarrow{f} .01$

$x \xrightarrow{f} 10^x$.

By the inverse mapping,

$10^x \xrightarrow{f^{-1}} x$;

this may be written as

$y \xrightarrow{f^{-1}} \log_{10} y$

or alternatively

$x \xrightarrow{f^{-1}} \log_{10} x$.

Again we see that $ff^{-1} = f^{-1}f = i$, where i denotes the identity operator:

$x \xrightarrow{i} x$.

For $x \xrightarrow{f} 10^x \xrightarrow{f^{-1}} \log_{10}(10^x) = x$

and $x \xrightarrow{f^{-1}} \log_{10} x \xrightarrow{f} 10^{\log_{10} x} = x.$

Naturally $ff^{-1} = f^{-1}f = i$, always; but it is interesting to check in these particular cases.

Example 3 Suppose $y = f(x)$ and $y = f^{-1}(x)$ are inverse functions. Then

$$y = f^{-1}(x)$$

can also be written

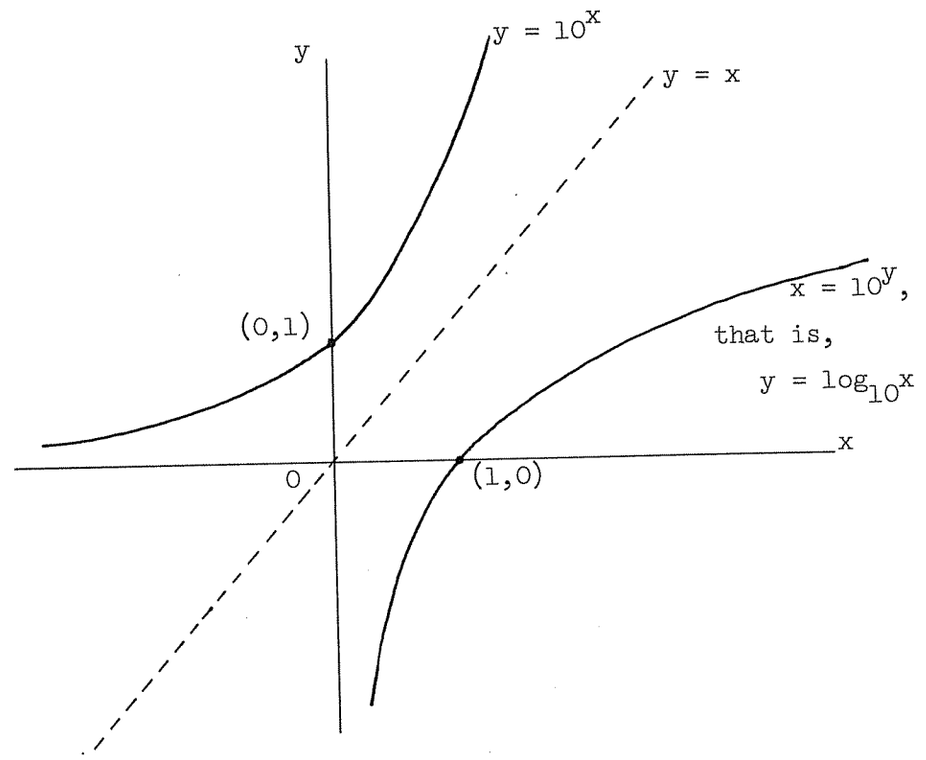
$$f(y) = f[f^{-1}(x)] = x.$$

Thus the inverse functions take the form

$$y = f(x) \text{ and } x = f(y).$$

Using the article on symmetry in an earlier Gazette, we see that: Inverse functions have graphs which are reflections of one another in the line $y = x$.

Thus the graphs in Example 2 are as shown:



MATHEMATICAL DAFFYNITIONS

Factors: maybe it's this way, and maybe it's that

Abscissa: what's behind a sore tooth

Square Root: vegetable sold in an honest market

Complex Fraction: when a piece of the broken bone sticks through the skin

Sum: a little, but not too much

THE ABSTRACT IN THE CONCRETE

A.J. Coleman, Queen's University

1. In the famous book Science and the Modern World, Alfred North Whitehead wrote:

> " The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact." (Chap. II, First Edition, P. 41)

The truth of this statement which explains the extraordinary influence of mathematics today will be illustrated by means of the concept "$\sqrt{-1}$" . Certainly this is an abstract idea, so much so that before 1850 many orthodox mathematicians reacted to it with cold shivers and denounced $\sqrt{-1}$ as "imaginary".

> " Actually, there is nothing imaginary about $\sqrt{-1}$, but unfortunately the description has been carried into engineering literature with stultifying effect upon the young student". (S.A. Stigant, Modern Electrical Engineering Mathematics, P. 13).

This abstraction, $\sqrt{-1}$, is an essential weapon for analyzing such diverse concrete phenomena as the vibration of a locomotive, the current or voltage in a radio or television set, the stability of an aeroplane wing, the energy states of an atom or any cyclic aspect of an economic system, in fact any phenomenon where oscillation is actually or potentially present.

The number $\sqrt{-1}$, like irrational numbers, the Heaviside calculus, and Dirac's delta function, was initially rejected by orthodox mathematicians. All those ideas ultimately proved themselves as very useful and important mathematical concepts.

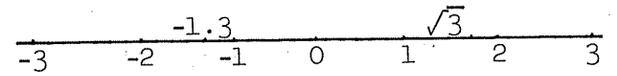
2. $\sqrt{-1}$ as a Number, i .

The symbol $\sqrt{-1}$ divides the world into two groups: Electrical Engineers and the rest of mankind! The former

denote the $\sqrt{-1}$ by the letter j, while the rest use i. We bow to social pressure and, at first, set $\sqrt{-1} = i$ so that

$$i^2 = -1 \qquad (1)$$

Thus, i is a solution of $x^2 + 1 = 0$. We assume that we can multiply i by the so-called "real" numbers which are representable as decimal fractions, and suffice to name all the points on the R-line



If x and y are real we can form the "complex number" $z = x + iy$. We agree that these new numbers will satisfy the usual C A D properties, and in particular $yi = iy$ for $y \in R$.

Theorem. If x, y, u, v are real, then

$$x + iy = u + iv \;:\Rightarrow\; x = u \text{ and } y = v.$$

Proof.

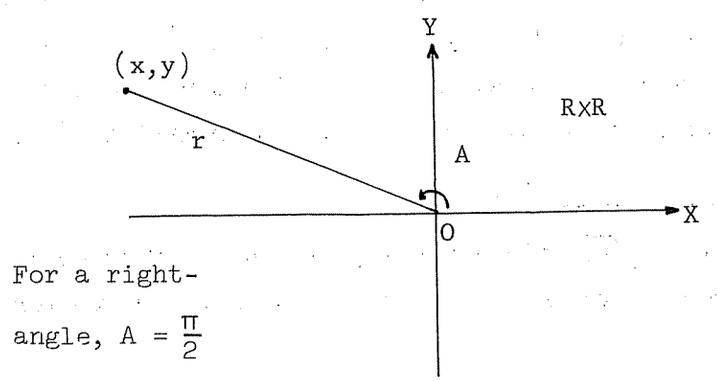
$$(x-u) = i(v-y)$$

therefore

$$(x-u)^2 = -(v-y)^2$$

But the square of a real number is positive or zero; therefore $x = u$ and $v = y$. q.e.d.

Since the complex number $z = x + iy$ uniquely determines the pair of real numbers (x,y), we can represent complex numbers as points in the Cartesian plane $R \times R$. We use radian measure for angles.



For a right-angle, $A = \frac{\pi}{2}$

The sine and cosine functions are defined so that

$$x = r \cos A$$

$$y = r \sin A$$

where $r^2 = x^2 + y^2$

Thus $z = x + iy$

$$z = r(\cos A + i \sin A) \qquad (2)$$

The number r is called the <u>norm</u> of the complex number z and is often written as $r = |z|$. To any complex $z = x + iy$ we associate its <u>conjugate</u>, $\bar{z}$, obtained simply by replacing i by $-i$. Thus

$$\bar{z} = x - iy \qquad (3)$$

Note that $z\bar{z} = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2$; therefore

$$|z|^2 = z\bar{z} = r^2 .$$

If $r = 1$, the point (x,y) is on a circle with unit radius and centre at the origin, O. And in this case

$$z = \cos A + i \sin A \qquad (4)$$

Supposing e and B are real numbers, can any meaning be given to e^{iB} ? If this is a complex number, say

$$e^{iB} = x + iy,$$

then, replacing i by $-i$,

$$e^{-iB} = x - iy.$$

Thus if the ordinary index laws are to hold for complex exponents,

$$x^2 + y^2 = (x+iy)(x-iy) = e^{iB} e^{-iB} = e^0 = 1 ;$$

so e^{iB} would be a complex number of the form (4).

It is a remarkable fact, used by Leonard Euler in 1747,

but known to the Englishman R. Cotes in 1722, that <u>there is a real number</u> (which, by a fabulous coincidence, is usually <u>denoted by e</u> !) such that

$$e^{iA} = \cos A + i \sin A \qquad (5)$$

It can be shown that for any complex z

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} \ldots\ldots \qquad (6)$$

where the r.h.s. is an infinite series. Thus, for $z = 1$,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = 2.7182818284\ldots \qquad (7)$$

For $z = iA$, (6) gives

$$e^{iA} = 1 + iA - \frac{A^2}{2!} - i\frac{A^3}{3!} + \frac{A^4}{4!} + i\frac{A^5}{5!} \ldots$$

$$= (1 - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \ldots) + i(A - \frac{A^3}{3!} + \frac{A^5}{5!} \ldots)$$

Comparison with (5) and use of our theorem, shows that

$$\cos A = 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \ldots \qquad (8)$$

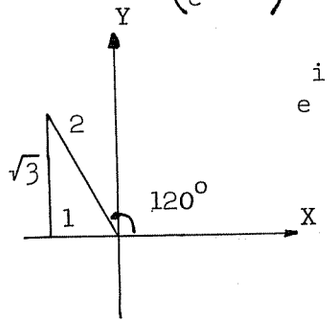
$$\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \ldots \qquad (9)$$

Remember that A is radian measure of the corresponding angle. If in (5) we set $A = \pi$, we find $e^{i\pi} = -1$ or

$$e^{i\pi} + 1 = 0$$

This fascinating equation relates the five most important numbers of mathematics. Notice that

$$\left(e^{i\frac{2\pi}{3}}\right)^3 = e^{i2\pi} = \cos 2\pi + i \sin 2\pi = 1 .$$



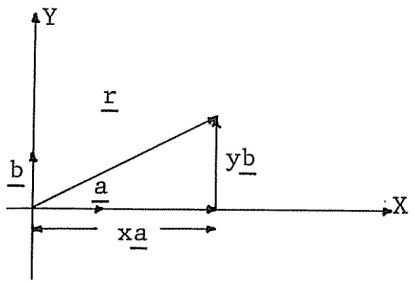
$$e^{i\frac{2\pi}{3}} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

<u>is a cube root of 1</u>.

Problem Show that $e^{i\frac{4\pi}{3}}$ is another. There are three. What is the third? Show that the same results are obtained by solving $x^3 - 1 = 0$.

3. $\sqrt{-1}$ as an Operator, j .



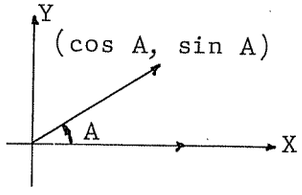
We underline vectors. Let $\underline{a}$ denote a vector of unit length in the X-direction, so $x\underline{a}$ is a vector in the same direction of length $|x|$.

Similarly, $\underline{b}$ is a unit vector in the Y-direction, and $y\underline{b}$ has length $|y|$.
In the figure, the vector

$$\underline{r} = x\underline{a} + y\underline{b} ,$$

and it has length $r = \sqrt{x^2 + y^2}$.

What happens to $\underline{a}$ if it is rotated through an angle of A radians? Denote the act of rotating through angle A by $S(A)$, so that $\underline{a} \rightarrow S(A)\underline{a}$. From the diagram, we see that



$$S(A)\underline{a} = \cos A\, \underline{a} + \sin A\, \underline{b} \qquad (10)$$

Similarly,

$$S(A)\underline{b} = -\sin A\, \underline{a} + \cos A\, \underline{b} \qquad (11)$$

Notice that these formulae check for $A = 0$ and $A = \pi/2$.

If we now rotate $S(A)\underline{a}$ through the angle B, from (10) we will get $S(B)\,(S(A)\underline{a}) = \cos A\, S(B)\underline{a} + \sin A\, S(B)\underline{b}$ and using (10) and (11) we easily show that

$$S(B)(S(A)\underline{a}) = (\cos A \cos B - \sin A \sin B)\underline{a} + (\sin A \cos B + \cos A \sin B)\underline{b} \qquad (12)$$

However, the net effect on $\underline{a}$ is to rotate it through $A + B$ radians, thus by (10),

$$S(B)S(A)\underline{a} = S(A+B)\underline{a} = \cos(A+B)\underline{a} + \sin(A+B)\underline{b} \ . \qquad (13)$$

Comparing (12) and (13) gives the addition formulae for trigonometrical functions:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

from which all the other addition formulae are easily obtained

Let us denote the <u>rotation through a right-angle</u> by j.

$$J = S(\tfrac{\pi}{2}) \qquad (14)$$

Then $j^2 \underline{a} = S(\frac{\pi}{2})\, S(\frac{\pi}{2})\, \underline{a} = S(\pi)\, \underline{a} = -\underline{a}$ and $j^2 \underline{b} = S(\pi)\, \underline{b} = -\underline{b}$.

Therefore, j^2 has the same effect on both $\underline{a}$ and $\underline{b}$ as multiplying by -1, so that it seems that $j^2 = -1$, and we have given the abstract $\sqrt{-1}$ a "concrete" representation as a <u>rotation through a right angle</u>.

Since a knowledge of the coefficients of $\underline{a}$ and $\underline{b}$ in (10) and (11) tell us what $S(A)$ does, it is natural to identify the rotation $S(A)$ with the <u>matrix</u>

$$S(A) = \begin{pmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{pmatrix} \qquad (15)$$

In particular,

$$S(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \ , \text{ say}$$

and

$$S(\tfrac{\pi}{2}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = j$$

so that $S(A) = (\cos A)I + (\sin A)j = e^{jA}$

A <u>matrix</u> is a rectangular array of numbers. Here it arises in providing us with a <u>representation of the group of rotations.</u>

Such representations of groups are an important tool in quantum mechanics, characterizing the possible state of atoms, molecules and nuclei.

Matrices arise also from systems of linear equations and linear inequalities. Large matrices (many rows and columns) arise frequently in economic problems and in Operations Research. For example, in designing optimal power grids, mathematicians in the Ontario Hydro Electric frequently handle matrices with 100 or more rows and columns.

It is clear the $j\underline{a} = \underline{b}$; therefore $\underline{r} = x\underline{a} + y\,\underline{b}$

$$= (x + yj)\underline{a}$$

Thus we may now think of $x + yj$ as an operator which transforms $\underline{a}$ into $\underline{r}$, whereas in Section (2) we thought of $x + yi$ as the point with coordinates (x,y).

3. An Isomorphism of an Additive Group in a Multiplicative Group.

The technical term group is used by mathematicians to denote a set, G, of "objects" which is closed under one binary operation, say $\perp$, which satisfies three simple properties. For $a, b, c \in G$.

(i) $a \perp b \in G$. Closure

(ii) $(a \perp b) \perp c = a \perp (b \perp c)$ Associative

(iii) $a \perp x = b$ has a unique solution for any given a and b.

You are familiar with many examples of groups. Here are three.

G_1 : the real numbers with addition as the binary operation.

G_2 : the positive reals with multiplication as the binary operation.

G_3 : the rotations of a wheel about a fixed axis with successive application of two rotations as the operation.

In G_3 we speak of the operation as the <u>multiplication</u> of rotations just as we denoted $S(A+B)$ by $S(A)S(B)$ in Section (2).

A slide rule changes multiplication into addition. Can we change addition into multiplication?

In fact, we have already done it. Thus for A, B ϵ G_1

$$S: A \rightarrow S(A) = A'$$
$$S: B \rightarrow S(B) = B'$$
$$S: A + B \rightarrow S(A+B) = S(A)S(B) = A'B'$$

Thus S effects a correspondence of the elements of G_1 to the elements of G_3 such that the <u>sum</u> of two elements of G_1 corresponds to the <u>product</u> of the two corresponding elements of G_3. Such a correspondence which preserves the group <u>structure</u> is called a <u>homomorphism</u>. If it is also 1:1 it is called an <u>isomorphism</u>. G_3 is homomorphic to G_1 but not isomorphic.

However, G_2 is isomorphic to G_1. Consider, for example, the correspondence

$$A \longrightarrow e^{A} = A'$$
$$B \longrightarrow e^{B} = B'$$
$$A+B \longrightarrow e^{A+B} = e^{A}\, e^{B}$$

therefore $(A+B)' = A'.B'$

This last equation means that the correspondence preserves the group structure. It can also be shown to be 1:1, and is therefore an isomorphism of G_1 onto G_2.

The fact that of the two homomorphisms

$$A \longrightarrow e^{A} \quad \text{and} \quad A \longrightarrow e^{jA}$$

only the first is an isomorphism, is intimately related to the association of $\sqrt{-1}$ with vibratory phonomena. A topologist would make the same point by saying that G_3 is a compact topological space and G_2 is not.

4. Conclusion

It was an obvious, apparently innocent, trivial, even inconsequential, question to ask

"Is it possible to solve the equation $x^2 + 1 = 0$?"

Any school-boy who liked solving equations could have asked this question, but in his wildest dreaming he would not have glimpsed the manifold consequences of his question in atomic physics, in mechanical and electrical engineering, in areas of pure mathematics such as the theory of complex functions and the theory of Hilbert space; and in many other realms.

One of the most striking aspects of contemporary civilization is the rapidity with which mathematics is penetrating into every area. The truth of Whitehead's remark is proved over and over again.

> " It is with the help of intricate mathematical calculations that space ships are guided in their orbit. Mathematics is applied in the design of radar and similar electrical equipment, in the day-to-day operations of insurance companies, in the control of the inventories of large companies, in discussions of the behaviour of atoms and nuclei, and in attempts to explain the constitution of the stars and the properties of light emanating from the nebulae millions of light years away.
>
> Perhaps the most startling mathematical development since the War has been the enormous increase in speed and capacity of electronic computers, which has opened up extraordinary new possibilities. It has been claimed, without exaggeration, that computers are in the process of bringing about a revolution in society second in importance only to the Industrial Revolution.

High-speed, large-memory computers are used in medical diagnosis, for composing music, for translating from one language to another, for comparing the style of the various epistles attributed to St. Paul, and for solving problems in mathematics, chemistry, physics, and engineering which could not have been tackled twenty years ago.

Whole new branches of mathematics, such as homological algebra, normed rings, and the theory of games, have come into being in the last three decades, each of which is more important than many older branches of mathematics such as trigonometry or Euclidean geometry. And, illustrating Whitehead's dictum, these new branches of mathematics (invented originally with no practical end in view but simply to satisfy the curiosity or the aesthetic sense of the pure mathematician) have frequently found practical application, with astonishing rapidity, in physics and engineering. Topological groups, a subject which has fascinated me for over twenty years, is one of the most abstract areas of mathematics; yet now it has applications to the study of the structure of molecules and crystals and to the classification of the myriads of new fundamental particles which the physicists have discovered in the past fifteen years.

It is impossible to foresee what new mathematics will appear even one year ahead, but we can be certain that its effects on our civilization will be even more startling than at present, and that an understanding of mathematics will aid us all to be home in the society of the future."

5. Problems

The following problems may interest you.

(1) Find five distinct fifth roots of unity.

(2) Show that i^i could be any member of a certain infinite set of positive real numbers which contains no element between 1,000 and 10,000. How many of the set are less than 0.0001 ?

6. Further Reading

(a) A. N. Whitehead, An Introduction to Mathematics, Oxford University Press (paperback). Chaps. VII and VIII discuss complex numbers.

(b) A. J. Coleman, J. J. Del Grande, H. A. Mulligan, and H. E. Totton, Mathematics 9, published by W. J. Gage Company, has a detailed treatment of the fundamental properties of numbers. Chaps. 6, 7, 8 and 9 are especially useful.

(c) E. W. Hobson, Plane Trigonometry. This rather old book (1897) has recently appeared in a Dover edition. The author was a great mathematician and writes very clearly. Chaps. 13-18 contain huge quantities of fascinating and useful material including careful proofs of everything in my lecture.

(d) N. Miller, Limits, Blaisdell and Company. This book is written for High School students and teachers to provide a firm basis for the study of calculus.

(e) G. H. Hardy, Pure Mathematics, Cambridge University Press. This exists in paperback. Chapter III gives a thorough discussion of complex numbers. Euler's equation (5) occurs in Chapter X.

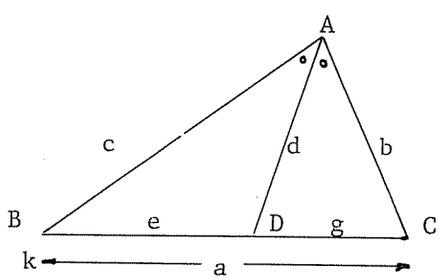
(f) W. Lederman, Complex Numbers, Free Press, Glencoe, Ill.; a very useful small paperback. i^i appears in his final set of exercises!

All of the above books, or any book, can be ordered by writing to the Book Store of any Ontario University.

LENGTH OF THE ANGLE BISECTORS IN A TRIANGLE

Riko Winterle, Hamilton

Problem: Given the sides a, b, c of a triangle ABC, it is required to find the length of the bisector of $\angle A$ to side BC = d in terms of a, b, c.



Solution: Let D be the intersection of bisector of $\angle A$ and BC of $\triangle ABC$. Let the lengths of AD, BD and DC be d, e, and g units repectively.

Since AD bisects $\angle A$

$$\frac{b}{c} = \frac{g}{e} = k \quad \text{(a constant);} \tag{1}$$

also $$e + g = a. \tag{2}$$

From (1), $$g = ek \tag{3}$$

and from (2) $$g = a - e \ ; \tag{4}$$

therefore, $ek = a - e$ or

$$e(k + 1) = a.$$

But $$k = b/c.$$

Therefore $$e = \frac{a}{k+1} = \frac{ac}{b+c} \tag{5}$$

From (4), $$g = a - \frac{ac}{b+c} = \frac{ab}{b+c} \tag{6}$$

A theorem of geometry says that if D is a point in the side BC of $\triangle ABC$ such that

$$\frac{BD}{DC} = \frac{m}{n}$$

then

$$nAB^2 + mAC^2 = (m+n)AD^2 + nBD^2 + mDC^2$$

(from "A Modern Geometry for High Schools" pg. 248.)

Since BD:DC is equal to c:b ,

$$bAB^2 + cAC^2 = (c+b)AD^2 + bBD^2 + cDC^2$$

or $$bc^2 + cb^2 = (b+c)d^2 + \frac{ba^2c^2}{(b+c)^2} + \frac{ca^2b^2}{(b+c)^2}$$

hence $$d^2 = (bc^2 + cb^2 - \frac{ba^2c^2}{(b+c)^2} - \frac{ca^2b^2}{(b+c)^2}) \ /(b+c)$$

$$= \frac{bc(b+c)}{b+c} - \frac{ba^2c^2 + ca^2b^2}{(b+c)^3}$$

$$= bc - \frac{a^2bc}{(b+c)^2}$$

$$d^2 = \frac{bc(b+c)^2 - a^2bc}{(b+c)^2}$$

$$= bc[(b+c)^2 - a^2] \div (b+c)^2$$

$$= \frac{bc(b+c-a)(b+c+a)}{(b+c)^2} \quad .$$

If $s = \frac{a+b+c}{2}$,

then $d^2 = \frac{bc(2s - 2a)(2s)}{(b+c)^2}$

$$d^2 = \frac{4bcs(s-a)}{(b+c)^2}$$

hence $d = \frac{2\sqrt{bcs(s-a)}}{b+c}$.

This formula can also be established by the theorem that says "If the bisector of the vertical angle of a triangle is produced to meet the base, the rectangle contained by the two sides is equal to the rectangle contained by the segments of the base, increased by the square on the bisecting line."

$$d^2 = cb - ef$$

$$= cb - \frac{a^2bc}{(b+c)^2}$$

$$= cb \left(1 - \frac{a^2}{(b+c)^2}\right)$$

$$= cb \left(\frac{b+c-a}{b+c}\right)\left(\frac{b+c+a}{b+c}\right)$$

The rest is as in the previous proof.

SECOND THOUGHTS ON THE NEW APPROACH

J. Bellamy, Georgetown.

There are so many excellent ideas in the 'NEW APPROACH' maths texts and courses that it may be ungracious to criticize, particularly when the criticism is not of detail, but of principle. I believe that New Approach contravenes basic principles of mathematics, pedagogy, and administration, and that courses based on texts now circulating could do as much harm as the Spelling Reform movement has done in its own field in the past few years.

Mathematics. New Approach stresses the logical structure of mathematics: precise definitions, formal laws, and new symbolism and vocabulary are used to this end. But creative mathematics is not a logical process - mathematical discovery is tested by logical analysis, not produced by it. High school maths students are engaged in creative maths at their own level, and it is misleading as well as repellent to over-stress the logical side of the subject. It is dishonest to pretend that the new treatments are morally superior to the old because of their more formal logical treatment: there are no degrees of rigour, any more than there are of virginity. Full rigour is impossible in high school, and it is a matter of taste and expediency, not conscience, where to stop.

The obsession with the structure of mathematics has a missionary zeal which characteristically belongs to the newly converted. It is no doubt true that the structure the missionaries perceive was not taught them at school, nor perhaps in their undergraduate studies. Their perception of structure is the result of their progressive exposure to mathematical knowledge and thought over the course of years. It is a sophisticated response, just as the original investigations of structure were sophisticated. It is psychological nonsense to suppose that one can appreciate the structure of mathematics without considerable mathematical experience.

Pedagogy. It is a platitude of pedagogy to approach

the unknown from the known, the general from the particular, the abstract from the concrete. New Approach, emphasizing the abstract, defies these principles. Euclid has served for 2,000 years to introduce deductive logic, with the advantage of having concrete illustrations to hand. Now he is out and deductive logic is to be taught through formal algebra, without any appeal to the concrete. The years of Euclid were not conspicuously successful in teaching, but it is hard to see why a more abstract presentation should be more successful. A sounder deduction would be that most of the time spent on Euclid was wasted - without any disastrous results - and that any similar massive teaching of deductive logic will be equally fruitless.

There is a degree of fanaticism in the doctrine of 'nothing but the truth' which reminds one of Brock Chisholm on Santa Claus. In so far as anyone arrives at the truth, it is by progressive revelation and discovery, and each stage of development is related to the age and inclinations of the aspirant. The Bobbsey twins and Hardy boys are not great literature, but they do more for the ten year old than Shakespeare and Tolstoy. Similarly the congruence propositions are for most students the only satisfying piece of deductive logic they encounter: to deny them this seems the height of puritanical virtue - or hypocrisy.

Administration. New Approach has come to us from the U.S. where as a rule high school maths is a real option, taken by the more interested students. If we suppose that in Ontario the Technical and Commercial courses remain outside its orbit, there still will be the whole of the four and five year arts and science classes to deal with. Such a program will require a large number of teachers who are not now, nor ever will be, qualified to teach mathematical semantics on this scale. The worst of the teacher shortage may be over, but maths teachers will never be over abundant. It is poor administration to make excessive demands on an exiguous supply. If New Approach is to be taught effectively in grades 9 and 10, there will be no properly qualified maths teachers for 11, 12, and 13.

Reading the more apocalyptic New Approach protagonists, nearly every student seems destined to be a

space scientist or a computer designer. Certainly there will be a better market for mathematical skills than in the past, but not 1% will ever graduate from an honour math course, not 5% will ever use math above high school level, and not 20% will need more than public school arithmetic in after life. We cannot mass produce Socrates, Russells or Wieners, and if we try we shall fail to do what can and should be done.

An Alternative. We hear a great deal about the Revolution in Mathematics, but there has been no revolution - only evolution. Our courses need review, but from the point of view of the students' needs, not from the standpoint of mathematical purity. For most students maths has no vocational value - it occupies its classic place in the curriculum, and may continue to hold that place, as a mental discipline. For the minority of students who do make use of their maths later, the main requirement is high competence in certain skills. It is a great weakness of our courses new and old that they ignore the fact that maths is the best loved and worst hated of all high school subjects. The current concern over the grade 13 cram, and the introduction of the four year arts and science courses afford some hope that a reasonable degree of specialization from grades 11 and 12 may soon start, and different levels of achievement set for major and minor subjects. In maths the minority should achieve higher levels of executant skill over a wider range than they do now: European maths students have two years of calculus at school for instance - surely we can manage one. For the majority we should expect a much lower degree of executant skill, and use a more descriptive approach, with something of the history of the subject and its correlation with developments in civilization, science and technology. Teachers of English and Music know that appreciation of structure and real executant skill are vouchsafed to few, but they do their best to make their subjects meaningful to the rest. We could and should do the same in maths.

To sum up, New Approach with its excessive emphasis on symbolism and formal structure ignores the historical and creative processes of mathematics: it ignores the psychology of the high school student, and

the educational needs of the majority, and it ignores the teacher resources available.

CORRESPONDENCE

The Editor-in-Chief,
The Ontario Mathematics Gazette,
University of Waterloo,
Waterloo, Ontario.

Dear Sir:

I have decided to enter the battle of the function.

Professor Wittenberg's comments on the subject are the most misleading yet, and should not be left unanswered.

I disagree with Professor Coleman's ordered-pair definition of function on pedagogical grounds and on these grounds alone. Mathematically, it is unassailable. For teaching purposes at the Secondary School level, the approach to functions through the idea of a "rule of correspondence" is probably the easiest to grasp, and it captures one of the basic ideas involved. But at a price! It is conceptually misleading in a way that the ordered-pair definition is not. I am afraid that Professor Wittenberg's comments have added to the misconception.

To finish my argument, I must become technical. We have at our disposal in any given context a finite alphabet. There exists, therefore, a countably infinite ($\aleph_0$) set of linguistic expressions (meaningful or otherwise). If we consider a function whose domain (and hence range) is finite, then the function can be completely described by a finite table. This table is the function, and it is a set of ordered pairs. The rule of correspondence is categorically given by the table as well. If we consider a number-theoretic function (from the natural numbers to the natural numbers), then we cannot in general write down the "table" which defines (describes or is) the function. One immediate reply to this is: Then give the rule of correspondence! But how do we do this in every case? There are uncountably

many ($2^{\aleph_0}$) number-theoretic functions, but only countably many ($\aleph_0$) linguistic expressions available to us. There just do not exist enough names or descriptions to go around. However, the notion of a set of ordered pairs, with the appropriate uniqueness condition, is an obvious direct generalization into the transfinite of the "table" referred to above. But, to repeat, there are uncountably many of these in the number-theoretic case, and we cannot give them all names.

This situation is worse for functions from the real numbers to the real numbers. There are uncountably many ($2^{\aleph_0}$) real numbers to begin with, and so even these cannot all be named. (But each one is a number-theoretic function, and as such it is a set of ordered pairs.) The set of functions in question has even higher cardinality

$$(2^{2^{\aleph_0}}).$$

Now Professor Wittenberg's letter gives the impression that for every function (from the real numbers to the real numbers, say) one can find a "formula" for the rule of correspondence. Furthermore, he appears to confuse the rule with the function itself. There do not exist "formulas" for all of these $2^{2^{\aleph_0}}$ functions among the $\aleph_0$ available linguistic expressions. Hence, at a suitably sophisticated level, we are forced to concede that a function is a set of ordered pairs with the appropriate uniqueness properties.

Now personally, I feel that the notion of function should be introduced via the concepts of domain, range and rule of correspondence. It should be made clear, however, that the rule is not always expressible by means of a formula or any finite expression. Then the graph of a function is defined as a set of ordered pairs. Finally, it can be argued that by moving to a higher level of abstraction, the function and its graph can be identified as the same thing. When we do this, we regain what we lost when we left the finite tables behind.

Yours very truly,

D.A. Clarke,
Assistant Professor of Mathematics,
University of Toronto.

ONTARIO ASSOCIATION OF TEACHERS OF MATHEMATICS AND PHYSICS

During the last two years, a committee of the Council of the O.A.T.M.P. has been preparing a revision of the Constitution. At the last meeting of the Executive and Council, the following Constitution was approved for presentation to the members at the Annual Meeting on April 21, 1965.

Members of the Association are asked to consider this constitution and to be prepared to vote on a motion to adopt the constitution at the Easter Meeting.

REVISED CONSTITUTION OF THE MATHEMATICAL AND PHYSICAL ASSOCIATION OF ONTARIO (1891)

Article I Name

The name of this organization shall be "The Ontario Association of Teachers of Mathematics" (O.A.T.M.)

Article II Purpose

The purpose of this association shall be to promote the study and teaching of mathematics in Ontario.

Article III Membership

All persons who subscribe to the purpose of the association are eligible to become members.

Article IV Executive Committee

A. The executive committee shall consist of six officers and a body of Councillors.

B. Officers:
The officers shall be:
1. Honorary President
2. Past President
3. President
4. Vice-President
5. Secretary-Treasurer
6. Publicity Representative

C. Councillors:

1. The number of councillors shall be decided by the Executive Committee.

2. The councillors shall be members of the association and among them must be representatives from the following areas of education in Ontario: The Department of Education, The Universities, The Colleges of Education, The Elementary Schools, The Secondary Schools.

Article V Procedure for Amendments

A. Amendments to the constitution or to the by-laws shall be agreed upon at an annual meeting.

B. Details of amendments to the constitution or to the by-laws must be presented to members at least two weeks before the annual meeting.

C. A motion of amendment to the constitution or to the by-laws must receive a favourable vote of not less than two-thirds of the members in attendance at an annual meeting.

BY-LAWS

By-Law I A. Duties of the Officers

(1) President - The President shall preside at all meetings of the Association, shall serve as Chairman of the Executive Committee and shall carry on all duties pertaining to the office and shall act as a representative of the Association on the O.E.A. Senate.

(2) Past President - The Past President shall be the chairman of the Nominating Committee.

(3) Vice-President - The Vice-President shall take over the duties of the President whenever he is required to do so, and shall act as a second representative of the Association on the Senate of the O.E.A. when necessary.

(4) Secretary-Treasurer - The Secretary-Treasurer shall keep a record of all meetings of the Association and of the Executive Committee.
He shall be custodian of the income of the Association and shall be responsible for the payment of all duly authorized bills and expenses of the Association.

Immediately before the annual meeting he shall prepare a financial report, covering his year in office. He shall receive an honorarium.

(5) Publicity Representative - The Publicity Representative shall be responsible for publicizing the activities of the Association. At the annual meeting he shall be responsible for the registration of members.

B. Duties of Councillors

It shall be the duty of a councillor:

1. To attend meetings of the Council as called, to submit suggestions, resolutions or requests to the Executive Committee.
2. To act as a liaison between the Executive Committee and teachers in his area and to keep these teachers informed of the work of the association.

By-Law II Admission to Membership

Membership in the Association shall be obtained by joining the Mathematics section of the Ontario Educational Association.

By-Law III Election of the Executive Committee

(1) Election of the Executive Committee shall be held by open ballot at the annual meeting.

(2) The president shall appoint a nominating committee comprising the past president as chairman and two other members of the Executive Committee for the purpose of preparing a complete slate for presentation to the Annual Meeting.

(3) Additional nominations from the floor shall be called for at the annual meeting.

(4) The consent of each nominee shall be obtained before an election is held.

(5) The councillors shall have a term of three years with approximately one-third entering and retiring each year.

By-Law IV Meetings

A. Association Meetings.

(1) An annual meeting shall be held in Easter week at the time of the annual convention of the Ontario Educational Association.

(2) Other meetings may be held at the discretion of the Executive Committee.

B. Executive Meetings.

(1) The Executive Committee shall meet in the spring, in the fall, and at such other times as are deemed necessary for the conduct of business.

(2) The president shall call other meetings upon the written request of at least one-third of the executive members or with the consent of the majority of officers.

(3) A quorum of the Executive Committee shall consist of not less than one-fourth of its members.

By-Law V Affiliation with the National Council of Teachers of Mathematics (N.C.T.M.)

(1) The Secretary-Treasurer shall pay to the N.C.T.M. an annual fee for renewal of affiliation of the O.A.T.M. with the N.C.T.M.

(2) The Publicity Representative shall be the representative to the National Council of Teachers of Mathematics.

(3) The President shall appoint a delegate to the Annual Meeting of the Delegates of The National Council of Teachers of Mathematics.

A MESSAGE FROM THE SCHOLARSHIP COMMITTEE OF THE ONTARIO MATHEMATICS COMMISSION

[The following has been received from Mr. J. McKnight for publication.]

Calling All Mathematics Clubs of Ontario!

Let us hear that you exist!

Let us know (1) the names of your executive
(2) the number of members
(3) the name of your staff advisor

Send this information, together with any requests or further items of information that arise in the paragraphs which follow, to

Mr. John I. R. McKnight,
Master Teacher - Mathematics and Science,
Scarborough Board of Education,
2472 Eglinton Avenue East,
Scarborough, Ontario.

The Scholarship Committee can put you in touch with a professor at almost any university of your choice. He in turn can give you invaluable help in extracurricular reading and projects in Mathematics.

If your Mathematics Club is operating successfully at present, and if you have completed some interesting project, we urge you to send a description of this project to the editor of the Gazette. Your Club may get into print.

Remember the Mathematics Fair at the O.E.A. during the Easter holidays. For detailed information write to -

Mr. J. W. Fencott,
c/o R. H. King Collegiate Institute,
3800 St. Clair Avenue East,
Scarborough, Ontario.

If you are seeking projects for your Club, here are some suggestions:

Grades 9, 10: Fibonacci Numbers. Reference - "Fibonacci Numbers", N.N. Vorob'ev. A Blaisdell Scientific Paperback (95 cents) Ginn and Co., 35 Mobile Drive, Toronto 16.

Grades 11, 12: Infinite sequences and series - convergence. Reference - "Sequences and Series", J.A. Green, Library of Mathematics Series, Routledge and Kegan Paul (London).

Grade 13: Probability. References - (i) "A First Look at Probability". This booklet, published for the

Ontario Mathematics Commission may be obtained upon request from the Ontario Mathematics Commission,
1260 Bay Street,
Toronto, Ontario.

(ii) "Probability, an Introduction" by Greenberg, published by Prentice Hall.

Have you tried Seminars?

A Seminar provides a splendid opportunity to unite several small clubs, or groups of interested students, in some central location. At a Seminar, a programme can be attempted that might be beyond each individual group. A well planned Seminar provides a genuine "Mathematical lift", and there are many university professors who are more than willing to assist in a major capacity. Seminars presently being held are proving to be very successful.

Solutions to Problems

All of us have at some time encountered mathematical problems whose solutions elude us. The Scholarship Committee invites you to send any such problems to Mr. John I. R. McKnight (address as above). The Committee will assign these problems to someone who is qualified to answer them.

BOOK REVIEW

"The New Mathematics Dictionary and Handbook", by R.W. Marks; Grosset and Dunlap (180 pages).

This book is intended as a classroom companion, giving definitions of many mathematical concepts. Basically a dictionary, it contains interesting items from the history of mathematics and many short biographies; for example, the method used by Eratosthenes (200 B.C.) for finding the circumference of the earth is explained, and the notes on Gauss take up half a page. Quotations, word derivations, and tables are other features of the book.

The book is up-to-date, liberally sprinkled with diagrams, and of broad appeal; for example, there are 8 pages, extensively illustrated, explaining how to use a slide-rule.

The main drawback of the book is a reasonably frequent incorrectness of expression. Here are a few examples. Under the heading "Commutative Law", we find this statement "the operations of subtraction and division are not satisfied by the Commutative Law". In the example under "Equation", the surd sign is incorrectly employed. The section "Exponent" could lead a student into serious difficulties if he tried to employ it; the laws are given for "any number a", and there is no mention whatever of the necessity for a to be positive. The definition of the verb "to factor" makes no reference to the type of expression allowable, and thus is almost meaningless; the book says that "to factor an expression is to divide it into smaller terms (or expressions) which, on multiplication, will yield the original expression." It is hardly clear to say that "the sum of an infinite series is defined as the limiting value of the first n terms as the number of terms increases indefinitely". Nor is it desirable to say that a sequence is "a set of ordered elements, each related to one or more of the preceding elements in some specifiable way".

In conclusion it would appear that the book could be very useful to a teacher whose background in mathematics was sufficient to enable him to spot errors and imprecisions such as those mentioned in the preceding paragraph; it would seem better not to leave the book in the hands of students without guidance. It is to be hoped that, if there should be a later edition of the book, careful revision could be made. The usefulness of the book would thereby be much increased.

R. A. Honsberger
R. G. Stanton

Geometry and the Imagination - David Hilbert and S. Cohn-Vossen, translated from the German by P. Nemenyi (New York, Chelsea Publishing Company, 1952).

This work, though not new, deserves being brought to the attention of all geometry teachers. One can do no better than quote from the preface, written by the great mathematician, David Hilbert.

"In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations.

"As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that intuitive understanding plays a major role in geometry. And such concrete intuition is of great value not only for the research worker, but also for anyone who wishes to study and appreciate the results of research in geometry.

"In this book, it is our purpose to give a presentation of geometry, as it stands today, in its visual intuitive aspects. With the aid of visual imagination we can illuminate the manifold facts and problems of geometry ... . We want to take the reader on a leisurely walk, as it were, in the big garden that is geometry, so that each may pick for himself a bouquet to his liking."

The treatment is masterful, being Cohn-Vossen's reworking of lectures given by Hilbert, called Intuitive Geometry; the content is all-encompassing, as the chapter headings will indicate: The simplest curves and surfaces, Regular systems of points, Projective configurations, Differential geometry, Kinematics, Topology.

Every teacher of geometry owes it to himself to read this book. If he has a copy, he is sure to see that his best students become acquainted with it.

R.S.D. Thomas