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Ontario Mathematics Gazette

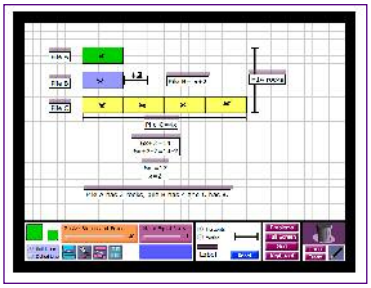
OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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Vol. 53 #4
June 2015

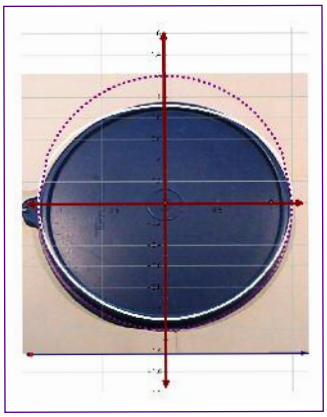
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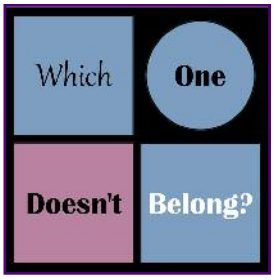
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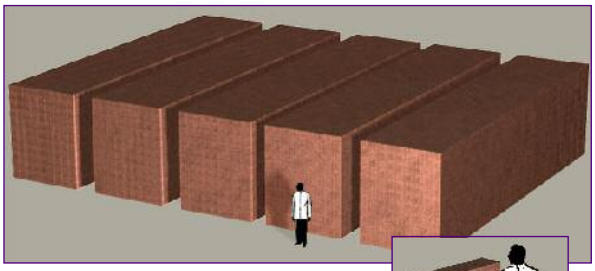
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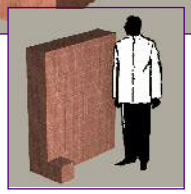
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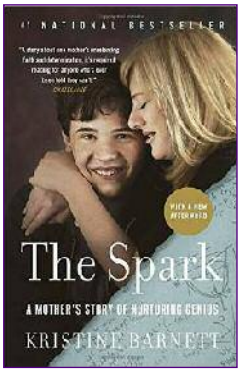


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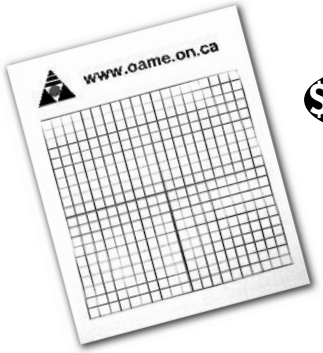


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The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Dan Jarvis, along with **separate files for each of the text, graphics, and/or photographs**. Please e-mail all files to Dan Jarvis at **dan.jarvis@oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and e-mail address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME.

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Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

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▲ Editor's Message



DANIEL H. JARVIS, O.C.T., Ph.D.
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Dr. Daniel Jarvis is Professor of Graduate and Mathematics Education in the Schulich School of Education at Nipissing University, North Bay, Ontario. His research interests include instructional technology, integrated curricula, and mathematics of the workplace.

Warm summer greetings to all math teachers, coordinators, researchers, and general math enthusiasts!

Many of you will be aware of, and no doubt quite familiar with, the Ministry of Education's EduGAINS website (**www.edugains.ca**), including the rich series of web pages and associated multimedia resources that are available by clicking through the Resources > Mathematics links. If not, I'd like to draw your attention to this well-organized Ministry of Education resource, and perhaps just highlight a few items therein.

Under *Curriculum & Teaching* you will find *Ontario Curriculum: Mathematics* documents, as well as grade-specific support resources (e.g., *Guides to Effective Instruction*, classroom posters, sample lessons, *Mathematical Processes Package*). Under *Classroom Practices*, you'll find sub-links to *Differentiated Instruction, Learning Environment, Lesson Design,* and *Technology & Teaching Tools*. Under the *Assessment for/as/of* tab you'll find information on *Communication, ELL, Effective Questioning, EQAO, Gap Closing, Math Processes,* and *Student Voice*. The *Home Support* area provides links to *CLIPS, Gap-Closing ePractice, Mathies Interactive Electronic Supports,* and *Doing Math with Your Child* and *Partnering with Your Teen* documents for parents/guardians. Under *Paying Attention to Mathematics Education* you'll find a series of helpful monographs focusing on challenging topics such as fractions and algebraic/proportional/spatial reasoning. It is here that you'll also be introduced to OME's "Seven Foundational Principles for Improvement in Mathematics, K–12" which are as follows:

- Focus on mathematics;
- Coordinate and strengthen mathematics leadership;
- Build understanding of effective mathematics instruction;

- Support collaborative professional learning in mathematics;
- Design a responsive mathematics learning environment;
- Provide assessment and evaluation in mathematics that supports student learning; and,
- Facilitate access to mathematics learning experiences.

Under *Professional Learning* you can access a resource created by Shawn Perry from York Region District School Board and which focuses on *Teaching through Problem Solving: Three-Part Lesson*. Finally, under the *Resource Collection* there are helpful links to *Manipulative Use* summaries, C*apacity Building Monographs*, downloadable classroom math posters, software supports, and to more than 20 mathematics education video webcasts. Truly this represents an incredible collection of teaching tools, strategies, and related literature for parents/guardians, for those teachers wishing to further their own knowledge base and teaching effectiveness, and for those in positions of teacher support and professional development. Further, by using the "Contact Us" feature on the EduGAINS website, one can request a provincial facilitator to speak to their resources.

The June 2015 issue of the *Gazette* includes three new articles and ten regular columns. In their co-authored work, *Revealing Subjective Probability in the Middle and High School Mathematics Classroom*, Egan Chernoff and Jack Chernoff contrast subjective probability with classical and frequentist probability foci. Using a deck of standard playing cards, they further share an innovative approach to the teaching and learning of a central tenet of subjective probability, for recommended use in the middle or high school mathematics classroom.

In his article entitled *The Mathematics of Photographs*, Tim Sibbald builds upon his previous work (*Mathematics of a Hill*, 2014) and makes the case that sometimes modelling a physical object, seen in terms of the photographic image, can be useful for identifying patterns. Through the use of a number of intriguing geometric diagrams and photographs with grid/intercept overlays, Sibbald models his stated goal of providing a workable sense of how the taking of a photograph transforms the geometry of a physical object.

Roger Herz-Fischler, in his article entitled *Doing Maths Algorithmically in the Age of the Computer*, introduces and advocates for the use of a comprehensive, open-source software known as Octave. He also makes the case for introducing high school students to an algorithmic approach to solving certain complex problems, and points out the perceived dichotomy existing in the Ontario Mathematics and Computer Science curricula. Through the analysis of a familiar "birthday problem," Herz-Fischler describes the utility of the software Octave, and further offers a link to a free introductory resource.

Regular columns include the following highlights: OAME President, Paul Alves (President's Message) reflects fondly on his term as President; Assessment Abby (eponymous) answers a submitted question regarding the evaluation of curriculum expectations; Mary Bourassa (Technology Corner) highlights a new website entitled *Which One Doesn't Belong*, created by Christopher Danielson; Mirela Ciobanu (In the Middle) looks at Singapore's model-drawing approach for solving word problems using text, pictorial, and symbolic representations; Greg Clarke, Agnes Grafton, Ross Isenegger, and Markus Wolski (Provincial Digital Learning Resources) present and extol their two new web/mobile learning tool apps, Money and Notepad; Lynda Colgan (Hey, It's Elementary) tackles large numbers such as billions, trillions, quadrillions, and… googolplexes [see also Richard Schwartz's fantastic new book/website on a similar theme, *Really Big Numbers* (AMS, 2014)]; Stewart Craven (Fields Institute MathEd Forum Report) provides speaker highlights from the Saturday, March 27 meeting; Shawn Godin (What's the Problem?) poses a problem regarding telling time with two given hourglasses; Ann Kajander (MB4T) discusses student explorations of pattern rules, using virtual algebra tiles; Todd Romiens (OAME/NCTM Report) discusses the teaching of mathematics to preschoolers; and, as usual, this issue also features the rich contribution of ideas for elementary math teaching as found in the Abacus insert, co-edited by Mary Lou Kestell and Kathy Kubota-Zarivnij, and focusing here on *Spatial Reasoning via Patterning and Algebra and Data Management and Probability* strand problems.

Volume 53 Issue 4 also includes several special features: (i) a review by Tim Sibbald of Barnett's inspiring documentary tale, *The Spark: A Mother's Story of Nurturing Genius*, in which she recounts her autistic son's incredible and ongoing journey through life and learning; (ii) several further entries of Ontario mathematics education researchers and their respective

projects; and (iii) selected excerpts of Canadian researcher contributions to book editor Lerman's new *Encyclopedia of Mathematics Education* (2014).

A number of math conferences should be noted: OAME 2015 "Building Mathematical Mindsets" (see **http://oame2015.ca/**) was successfully hosted at Humber College, Toronto (May 7–9); The Canadian Mathematics Education Study Group's (CMESG) annual conference (see **www.cmesg.org/**) took place at the Université de Moncton, Moncton, New Brunswick (June 5–9); and a Fields Institute-sponsored *Math + Coding Symposium* (June 19-21) will be hosted later this month by the Faculty of Education at Western University, London, Ontario (see **www.researchideas.ca/coding/**).

Erratum, mea culpa: In the March issue editorial, in discussing Abbott's classic *Flatland* tale, while I correctly noted the narrator's pseudonym, A. Square, I mistakenly listed "*A. Sphere*" as the other main character, whereas it should have read simply, all in lower case, "… a sphere."

I trust you will find the content of the final issue of the 53rd Volume of the *Gazette* to be stimulating and informative (and the math joke on p. 40 to be quite funny), and I look forward to receiving your new submission(s) in the weeks to come!

References

Lerman, S. (Ed.). (2014). *Encyclopedia of mathematics education.* London, UK: Springer.

Schwartz, R.E. (2014). *Really big numbers.* Providence, RI: American Mathematical Society. ▲

Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

"Similarly, studies in workplace mathematics show that although the people working in a fish hatchery may make extensive use of mathematical processes and objects, they do not understand themselves as doing mathematics—they will describe themselves as raising fish. In their hands, mathematical entities are radically different than these are in the hands of mathematicians or mathematics teachers." (p. 318)

Roth, W.-M. (2014). Interdisciplinary approaches in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 317–320). London, UK: Springer.

▲ President's Message



PAUL ALVES
E-MAIL: paul.alves@oame.on.ca

In preparing to write this final President's Message, I reviewed the final messages of my predecessors. The theme was consistent. This is a time for pausing, to reflect on the year that has passed, and to express gratitude to those who have helped make the year a success or just helped you make it through the year. I am sure that many of you are thinking back on this past year, despite the challenges that may have surfaced toward the end of the school year. What goals did you have? Which ones did you feel were achieved and which ones were left unattained? Why?

I have also been reflecting on the year that has passed in my classroom and the goals I set for my own professional development, as well as those for my department. I am very fortunate to work with a group of teachers at Fletcher's Meadow Secondary School in Brampton, who are continually challenging themselves to examine their practice. It is a challenge at times to try to harness so much talent in a single group, but I always find that I am along for a fun and enriching ride.

As I reflect back on my term as President and consider what has happened, I am reminded of what Cathy Hall mentioned in her parting message—the months have indeed passed all too quickly. The Leadership Conference was a highlight for me. Registration climbed to the point where we had to shut it down and create a wait list. This was no doubt due to the exceptional slate of speakers (Amy Lin, Jo Boaler, Dr. Cathy Bruce, and Shelley Yearley), who provided a memorable learning experience for all attendees. I want to thank them again for making it a remarkable Leadership Conference. Don't miss out on next year's Leadership Conference taking place November 5–7.

One of my goals upon starting my term was to look at other ways that OAME can serve its membership. As co-chair of OAME 2014, I was pleased to support a new initiative which we hope will become an annual tradition–the OAME eConference.. The eConference has created new opportunities for mathematics educators in this province and beyond—from the participation of Nova Scotia educators, to receiving professional development and sharing in the learning. Webinars were also

provided through the leadership of President-Elect Tim Sibbald. As well, OAME has reached out into the world of social media. The OAME Twitter feed (@oamecounts) has over 800 followers and is a great way to connect with the organization and other educators.

The provincial conference that took place May 7–9 at Humber College was, as always, a highlight for OAME. This year, there was the added significance in that the OAME board was in charge of planning and running the conference. Usually, one of the 15 OAME chapters hosts the conference. I want to take this opportunity to thank all of the organizers who volunteered their time to make the conference such a resounding success. For the second year in a row, registration had to be closed early. This speaks to the strength of the program that is assembled annually for this conference, and how well it runs, given the large attendance numbers and everything that happens over a jam-packed three days. Special thanks go to our conference chair, Wayne Erdman, for his dedication and composure in guiding all aspects of the conference. Its success is in large part due to his efforts. I hope that all of you have already marked May 5–7, 2016 on your calendars so you can "Leap into Math!" at next year's conference in Barrie, hosted by MAC2.

Hopefully many of you have also taken advantage of the great professional-development opportunities offered by your local chapter. These provide a way for many educators to network and share their learning. Perhaps you have even reached out to your local chapter to inquire about getting involved in its work. Chapters, and by default OAME, exist because of the dedication of volunteers who make each of the chapters a vital link for teachers to share their passion for teaching mathematics. If you haven't considered volunteering with your local chapter, it is never too late to get involved. Check the OAME website for the e-mail addresses of chapter representatives.

In closing, I would like to offer a special thank you to those who have made my term and OAME run smoothly this year:

- Executive Directors – Fred and Lynda Ferneyhough, the brains and brawn behind the operation. For those of you who know them, you know who is the brains and the brawn. It is impossible to convey how much easier they made my job. Thank you!
- The members of the OAME Executive – Past President Sonia Ellison, President-Elect Tim Sibbald, Vice-Presidents Bill Otto and Judy Mendaglio. With so much brain power around one table, it was a piece of cake making it through this term. The wise counsel you provided to me and OAME ensured that the best interests of the organization and its members were front and centre at all times. Thank you for the light-hearted moments as well, which helped as we navigated some challenging times. See you in the bowling lanes.
- The members of the OAME Board – OAME directors and chapter reps. Thank you on behalf of all OAME members for giving of your time and your expertise on the various OAME committees and especially for pulling double duty with planning the provincial conference.
- OAME webmaster extraordinaire and Barbu wizard – Greg Clarke. Thank you for your constant support and incredible good nature, despite the 17 MCIS registrations I created for myself. OAME's web presence and digital archive are an exceptional resource, and it is thanks to your work and dedication to the organization.
- The Executive and/or Council of our 15 regional chapters – CHAMP, COMA, GOLDEN, GVMA, ISOMA, MAC2, NOMA, NWOAME, PRMA, QSLMA, SAME, SWOAME, TEAMS, WOMA, Y4MA. The reorganization of the OAME board two years ago reflected a belief that the lifeblood of OAME is in its chapters. Your time and effort in planning events and working for chapter members makes the entire organization vibrant and relevant.
- Editor of the *Gazette* – Dan Jarvis. Thank you and your team of contributors, including Abacus Editors, Kathy Kubota-Zarivnij and Mary Lou Kestell, for continuing the tradition of producing a respected resource for all math educators. Despite my best efforts to derail the process with the lateness of my submissions, Dan guided each issue to its successful and rewarding conclusion.
- The many other hands that chipped in when we needed a representative to attend forums, workshops, and other initiatives. We are fortunate to have so many distinguished educators to support our work and partnerships.

- All of our OAME members – Thank you for making OAME such a respected and relevant professional organization. Please continue to be involved and to share your love of teaching math. It has been an honour serving you as President.

I take my leave now and reflect on a wonderful year. I am humbled by the experience and the honour of serving as President, especially when I consider who served before me. I wish all of you the best for a restful summer and look forward to intersecting paths in the future. ▲

▲ Call for Manuscripts

The *Ontario Mathematics Gazette* is inviting manuscripts for all grade levels.

Instructions for submission of manuscripts are found on page 1 of any *OMG*.

Contact the Editor for further details.



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▲ Mathematics Education Researcher Highlights

Math Education Researcher: Dr. Donna Kotsopoulos, Wilfrid Laurier University

Projects: Donna is an Associate Professor of mathematics education, and is the director of the Mathematical Brains Laboratory (**www.wlu.ca/mathbrains**). Her research explores how young children learn about mathematics before the start of formal schooling and during their early years of schooling. Through her research, and in collaboration Dr. Joanne Lee, she has developed the early mathematics learning program called *LittleCounters*®. This program developed for parents, caregivers, early childhood educators, and teachers focuses on how play can be made mathematical in order to support young children's understanding of numbers. The program is highlighted in her recently published book called *Let's Talk About Math: The LittleCounters® Approach to Building Early Math Skills* (**http://products.brookespublishing.com/Lets-Talk-About-Math-P763.aspx**). The program is being offered at Ontario Early Years Centres and libraries throughout Ontario.

Mathematics Education Researcher: Dr. Walter Whiteley, York University

Projects: One of Walter's many interests is that of spatial reasoning, at all levels of education. Related projects would include the following: (i) of special interest for elementary teachers is the Spatial Reasoning Group out of the University of Calgary which has just published a book on *Spatial Reasoning in the Early Years: Principles, Assertions, and Speculations* (Routledge, 2015); (ii) ongoing research with Dr. Ami Mamalo and Robyn Ruttenberg on optimization with spatial reasoning, including development of cognitive networks for scaffolding geometric learning through explorations with specially developed manipulatives; (iii) development of spatial reasoning courses to support students entering universities in engineering and science, and to support pre-service elementary teachers; and (iv) Geometry education focusing on symmetries, transformations, and invariance.

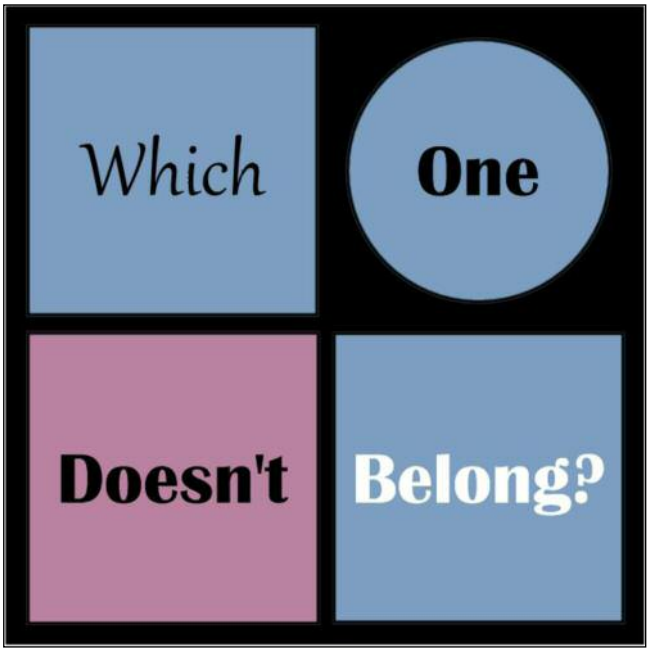
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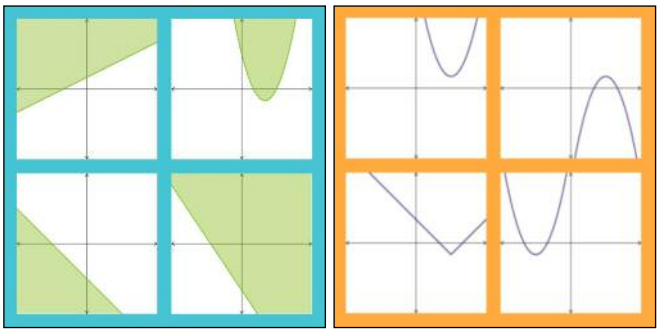
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Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

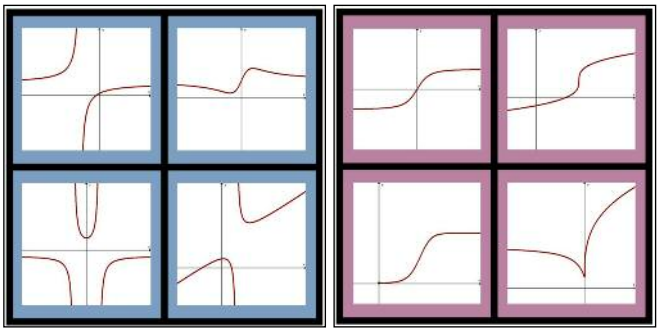
In March, a new math website called *Which One Doesn't Belong?* (*WODB?*) was launched at **www.wodb.ca**. Take a moment to look at the *WODB?* logo and decide which doesn't belong.



Did you choose one? Could there be more than one? The top left doesn't belong because the font is different. The top right has a circle. The bottom left has a different-coloured background (pink, if you could see it in colour). The bottom right has white font. Each set on the website shares this property—there is a reason for each object not belonging. In some cases, there are multiple reasons. The website was inspired by the *Which One Doesn't Belong?* shapes book, written by Christopher Danielson (@trianglemancsd), and by tweets posted by Chris Hunter (@chrishunter36), including these:



This led to the idea of making some sets specifically for my calculus classes. We were working on curve sketching at the time and this seemed like a perfect fit. I created the following two *WODB?* sets:

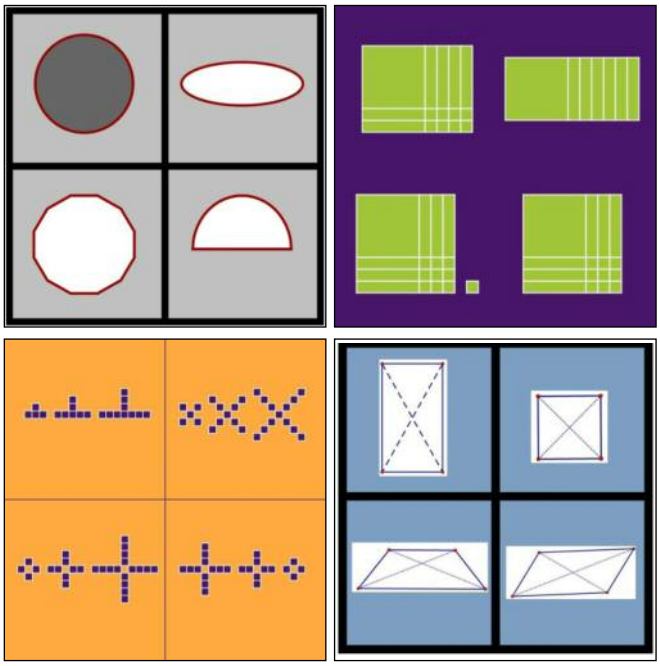


In class, I showed my students the logo and asked them which one didn't belong. This ensured that they understood the concept. Then I asked them all to be quiet for two minutes as I showed them the first calculus set. Once the two minutes were up, they discussed why each one didn't belong in their groups, and then we discussed reasons as a class. I loved that they came up with reasons I had not thought of when I created the sets. We repeated the process for the second set. Then I gave each group 12 graphs and asked them to choose 4 that made a good *WODB?* set. They did this really well, although the struggles they went through to choose the fourth one in a set were very interesting to listen to. I heard conversations about asymptotes, non-differentiable points, even functions, maximum and minimum points, etc. It was great! The third part of this activity involved them creating their own set. They used Desmos to graph their equations and soon discovered just how difficult it is to create a set.

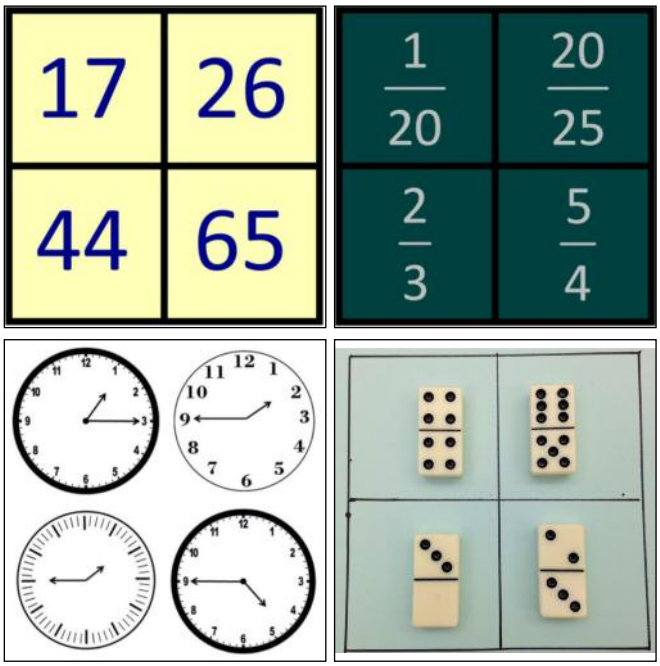
I had blogged about my plans to do this with my calculus classes (**http://marybourassa.blogspot.ca**), received a lot of good feedback, and someone suggested that there needed to be a website to house all of these *WODB?* sets. I took on that challenge and the response has been phenomenal.

There are currently four categories on the website: shapes, numbers, graphs & equations, and incomplete sets. Here are some examples.

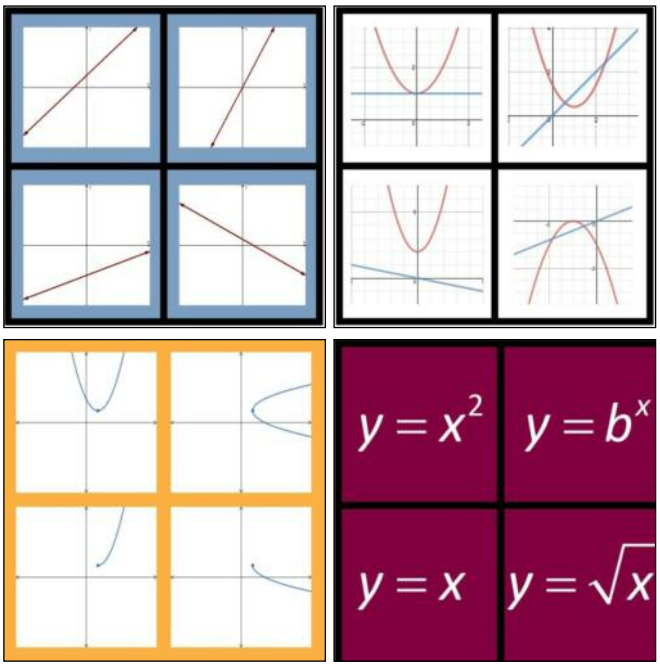
Shapes:



Numbers:



Graphs & Equations:



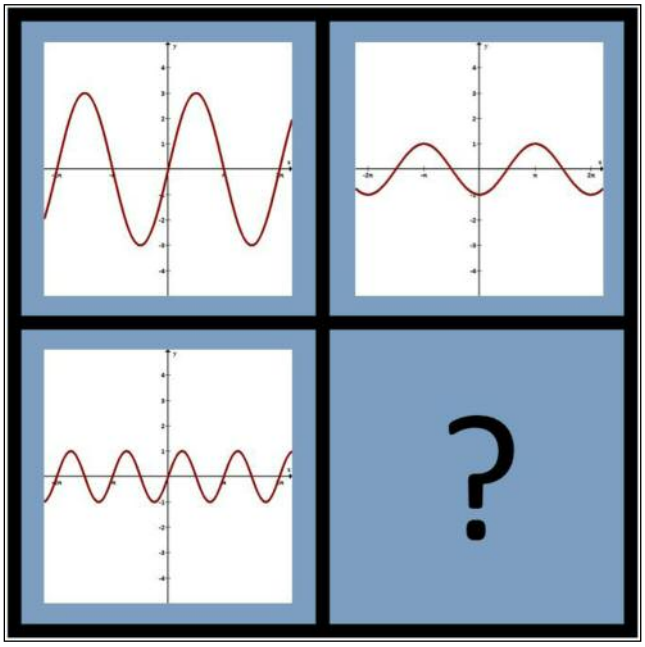
These are accessible to students from JK to Grade 12. My younger children enjoy the shapes, and I love the graphs & equations. I have had submissions from my own children and from teachers and students alike. There are also links to blog posts written by teachers who have used *WODB?* in their classes.

One of the most powerful features of *WODB?* is that it gives every student a voice. The student who often struggles in class is able to find a reason why at least one object does not belong, and it may be a reason that the high flyer in the group did not see. The importance of having all students see multiple solutions is huge in promoting a growth mindset. These puzzles allow great conversations that promote correct vocabulary and broaden problems-solving strategies. The focus is on the "why?"—students have to be able to explain their reasoning and make their thinking visible to others. They must all be able to critique their own reasoning and the reasoning of others. They can work quietly or discuss everything they are doing. When they are creating, they will think that they have a set that works, but then discover that it doesn't and now need to fix it. The process of taking apart what they have, deciding what to keep, and what they need to add makes them think deeply and demonstrate a solid understanding of the mathematical concepts at play.

There are more simple benefits, too. How do we refer to each object? You could use top left, top right, and so on. But you can also have students refer to the objects by quadrant (thanks to Sam Shah for that idea). These

sets can also be used to motivate new vocabulary. If you showed a set involving quadratic graphs before students have learned about them, they might struggle with ways of describing the axis of symmetry or the vertex. This could provide the perfect opportunity to introduce those important terms.

The latest addition is the page of incomplete sets. Here is an example:



The idea is that students would have to logically determine what is present or absent in each of the objects shown, and find a way of completing the set. This can act as a stepping stone between students finding which one doesn't belong and creating their own set. Chris Hunter showed a great way of organizing characteristics that would look something like this for the above set:

I hope you find a few *WODB?* sets that you can use with your students. If you don't find what you are looking for, submit one! The site continues to grow as teachers and students submit sets and inspire each other with new ideas. Enjoy! ▲

▲ Ask Assessment Abby: Help Me with Evaluating

ASSESSMENT ABBY
E-MAIL: assessmentabby@oame.on.ca

Ask Assessment Abby A^3 is a regular column in the OAME *Gazette*, where teachers can share concerns and best practices about assessment, evaluation, and reporting of mathematics. Please send your questions to Ask Abby at **assessmentabby@oame.on.ca**.

Dear Assessment Abby,

The school year is coming to an end and I am running out of time. What I want to know is, do all curriculum expectations need to be evaluated equally?

Help Me with Evaluating in June

Running Out of Time

The good news is that there is some flexibility in evaluating curriculum expectations. Even though all expectations are accounted for in instruction and assessment, *Growing Success* states that "evaluation focuses on students' achievement of the overall expectations" (2010, p. 38). This means that all of the overall expectations will be evaluated, but not necessarily all of the specific expectations. In other words, when evaluating the overall expectations, teachers use professional judgment in selecting the specific expectations they will include in that evaluation.

Keep Math Rich with Students in Mind,

Assessment Abby

If you have any questions or comments for Assessment Abby: **assessmentabby@oame.on.ca**

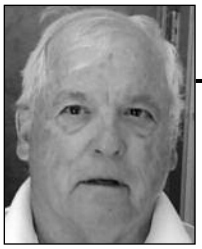
Growing Success: **www.edu.gov.on.ca/eng/policyfunding/growSuccess.pdf**

Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

"Teachers' self-efficacy sits on the boundary between beliefs and attitudes as it also incorporates emotional factors, i.e., confidence and anxiety. ... There has also been considerable interest in the factors that influence self-efficacy ..., particularly in preservice teacher education, as it has been suggested that self-efficacy is most malleable early in teachers' careers." (pp. 584–585)

Liljedahl, P., & Oesterle, S. (2014). Teacher beliefs, attitudes, and self-efficacy in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 583–586). London, UK: Springer.

▲ OAME/NCTM Report



TODD ROMIENS
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Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a past president of OAME and a life member, as well as a previous editor of the Gazette.

Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a past president of OAME and a life member, as well as a previous editor of the *Gazette*.

An interesting article came out of National Public Radio a few months ago, discussing the effect of emphasizing teaching mathematics to young preschoolers and whether or not this results in improving later academic performance. Researcher Pamela Morris stated, "We think math might be sort of a lever to improve outcomes longer term." She also indicated that there is plenty of evidence on the long-term importance of preschool, and she cites a 2013 study by Greg Duncan at the University of California, Irvine's School of Education, that showed math knowledge at the beginning of elementary school was the single most powerful predictor determining whether a student would graduate from high school or attend college.

One of the problems, however, is the lack of teachers who feel comfortable spending time teaching mathematics to preschoolers. In one study, only 58 seconds out of a 5-hour preschool day was spent on mathematics activities. University of Denver professor, Doug Clements, stated, "Most teachers, of course, have been through our (USA) mathematics education, so they tend to think of mathematics as just skills. They tend to think of it as a quiet activity." This seems to be a rather harsh commentary on mathematics teachers, as well as on the programs with which they have been involved. I don't believe this is the case with Ontario teachers.

Clements is the creator of "Building Blocks," which is designed to be engaging, exciting, and loud for preschoolers. Google "Building Blocks" for more information.

On a related note, did you know that Kindergarten Entry Assessments (KEAs) are on the horizon for schools in the United States? Apparently there has been a lot of discussion regarding the role of KEAs. "This year (2015), education systems will invest in the needed resources to develop and administer developmentally appropriate KEAs and thus improve instruction for young children," stated Rhian Evans Allvin, Executive Director, National Association for the Education of Young Children. Does this mean a child can fail pre-Kindergarten? Can we ever have enough testing?

President Obama's announcement in 2010, urging all states to embrace the Common Core State Standards (Core Standards) for mathematics, created quite a stir in the United States. The Common Core concentrates on a clear set of math skills and concepts. Students will learn concepts in a more organized way, both during the school year and across grades. The standards encourage students to solve real-world problems, and define what students should understand and be able to do in their study of mathematics from Kindergarten to Grade 12.

One of the goals of the Common Core State Standards initiative would be to leave behind a patchwork of 50 different sets of standards measured by 50 different tests, which would facilitate direct comparison by parents and policy makers of student performance in one state versus nationally. It would also be much more difficult for lagging states to manipulate the system in order to hide weak performance.

That goal seemed within easy reach by the end of 2011, as 44 states and the District of Columbia had adopted the new Standards. The Obama administration spent hundreds of millions of dollars to help states develop two new online tests, known as PARCC (Partnership for Assessment of Readiness for College and Careers) and Smarter Balanced, which would measure student progress on the Common Core. Most states signed on to administer these tests in the Spring of 2015.

There was, however, was a great deal of criticism of the Common Core from conservatives, who view it as an attempt to nationalize education. As a result, only a few states have signed on to administer either of the two tests mentioned above, which represents only about 40 percent of students nationwide. Many states have decided to administer their own tests, making state-to-state comparison very difficult.

Opposition to this whole idea appears to stem from the age-old arguments about standardized testing. Advocates claim that the benefits of implementing the Common Core far outweigh the value of the tests because the standards push teachers to better prepare students for life after high school.

If interested, sample questions from both of these tests may be obtained by Googling "PARCC" or "Smarter Balanced."

Enjoy your summer! ▲

▲ Fields Institute MathEd Forum Report



STEWART CRAVEN
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Stewart Craven presently teaches courses for the Continuing Education Department and the Transition Years Program at York University. He is an active participant in the Fields Mathematics Education Forum.

Our meeting on Saturday, March 27 was particularly interesting. Our speakers included mathematics education professors, classroom teachers, board consultants, and even a boat builder. The general theme was about Aboriginal ways of knowing, with a focus on learning and teaching mathematics.

Lisa Lunney Borden (St. Francis Xavier University) joined us by Skype to present her topic entitled *Learning Together to Transform Mathematics for Aboriginal Students*. Lisa has been supporting Mi'kmaw youth in learning mathematics for over 20 years, first as a teacher and currently as a researcher and teacher educator. Her presentation highlighted a model—developed in conjunction with teachers and community Elders—that identifies potential tensions and areas in need of transformation when considering mathematics education for Mi'kmaw students. In addition to discussing this model, Lisa described the Show Me Your Math program, which has generated new inquiry-based projects in Mi'kmaw schools that honour community ways and knowledge as a context for learning mathematics.

Ruth Beatty (Lakehead University) and **Jody Alexander (Renfrew County District School Board)** discussed *Exploring Connections Between Indigenous and Western Ways of Knowing*. For the past three years, Ruth Beatty (Lakehead University) and Danielle Blair (Ontario Ministry of Education) have been working with four research teams to explore connections between indigenous ways of knowing and the Western mathematics that is represented in current mathematics curricula. They have worked with Elders and community members from Indigenous communities across Ontario to co-design units of instruction that are culturally responsive and mathematically rigorous. Ruth Beatty and Jody Alexander, a member of the Algonquins of Pikwàkanagàn First Nation and an educator with the Renfrew County District School Board, highlighted the mathematical potential of Algonquin loom beading and described how it supports students' algebraic, proportional, and spatial reasoning.

Experiences and Challenges of Infusing Aboriginal Perspectives in Mathematics (Education in an Urban Context) was the title of **Marilyn Maychak's (Toronto District School Board)** presentation. In 2007, the Ontario Ministry of Education launched an Aboriginal Education Strategy to support the learning and achievement of First Nations, Métis, and Inuit students. The First Nations, Métis, and Inuit Education Policy Framework outlines strategies for integrating First Nations, Métis, and Inuit ways of knowing throughout the curriculum. Marilyn invited two guests, who described their project, in which Aboriginal students investigate mathematics while building a boat. This provided an excellent example of the authentic infusion of Aboriginal perspectives in mathematics classrooms.

The audience size at our monthly meetings continues to grow, and many newcomers are actively participating, so please don't hesitate to join us! (Agendas and information can be found at **www.fields.utoronto.ca/programs/mathed/forum/**.) I also invite you to make suggestions regarding themes you think would be of interest to the mathematics education community. You can drop me an email at **numeratecitizen@mac.com**.

Our last meeting for this school year will take place on Saturday, April 25, but we will start up again near the end of September. ▲

Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

"The role of teacher questioning in developing students' algebraic reasoning has been found to be no less important than that of good task design. . . . Skillful teacher guidance is needed in order to help students engage in the algebraic reasoning that is intended by the tasks." (p. 31)

Kieran, C. (2014). Algebra teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 27–32). London, UK: Springer.

▲ What's the Problem? Telling Time



SHAWN GODIN
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Shawn is the head of Mathematics, Business Studies, and Library at Cairine Wilson Secondary School in Orleans, Ontario. He is a past member of the OAME Board of Directors and is currently Editor-in-Chief of the Canadian Mathematical Society's problem-solving journal, Crux Mathematicorum*.*

Welcome back, problem solvers. Last time, I left you with the following problem:

> You have two hourglasses: one that measures 9 minutes and one that measures 13 minutes. Determine how to measure 30 minutes, using these two hourglasses.

This problem is the "easy version" of question P6 from the book *The Inquisitive Problem Solver* by Paul Vaderlind, Richard Guy, and Loren Larson.

This problem may look familiar, and we will explore some of the related problems later on. First, we'll use a useful problem-solving technique and consider an easier problem. Instead of looking at 9-minute and 13-minute hourglasses, we will work with 3-minute and 5-minute hourglasses and look for some patterns.

With small numbers, we can attack the problem with pencil and paper, or we could use some manipulatives to help us out. For example, if you used linking cubes, you could make groups of threes and fives to see which numbers can be created. After some playing around, you should get: 3, 5, 6, 8, 9, 10, 11, 12, 13,… and every number after that. At this point, I am wondering which numbers can and cannot be represented by adding groups of 3's and 5's, or any two numbers.

We can start to see some patterns if we write all possible numbers down in the right way. For our simplified version, since we are using 3- and 5-minute hourglasses, I will record all the numbers in a table with three columns. In Figure 1, we can see that once we reach a number such as 3, then all other numbers following it in the same column are reachable just by adding another 3. Thus, if we are searching for numbers that we can represent, we just need to fill all the columns. Since the 3 takes care of the last column (multiples of three), the 5 just needs to take care of the other (3 – 1) = 2 columns. Thus (3 – 1) x 5 = 10 is the last number to be filled, and (3 – 1) x 5 – 3 = 7 is the last number that we cannot represent.

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15

Figure 1

What we have actually solved is a version of the coin problem, which states: *Given unlimited coins of denominations d1, d2,…, dn, what is the largest amount of money that* ***cannot*** *be represented using these coins?* In our case, we discovered that the largest amount of money that cannot be represented with 3¢ and 5¢ coins is 7¢. The solution to the two-coin version has been known for a long time, while for three or more coins, no explicit formula is known. If you have unlimited coins of denominations a¢ and b¢, where a and b have no common factors (why is this important?), the problem is solved in exactly the same way as we did with the 3¢ and 5¢ coins. If $a < b$ and we put all numbers into a table with a columns, then a takes care of the last column (the multiples of a), and each multiple of b takes care of a new column. The last column will be taken care of by $(a – 1) \times b$; therefore, the smallest number that cannot be represented is $(a – 1) \times b – a = ab – a – b = (a – 1)(b – 1) – 1$.

If our problem were dealing with 9¢ and 13¢ coins, the smallest amount not representable would be 95¢.

Unfortunately, this problem isn't equivalent to our problem. At this point, we will look at another related problem. You have two unmarked jugs that hold 5 litres and 9 litres, respectively. You have a vat and you want to put exactly 7 litres into the vat. How do you do it?

Unlike the coin problem, with the jug problem, we can dump out previously poured water, so this problem involves addition and subtraction. You can obtain a solution by adding 9's and subtracting 5's, or vice versa. For example, I can add a 9, then continue subtracting 5's until I don't have a full 5 litres in the vat, at which point, I

add another 9. The contents of the vat would be 9, 4, 13, 8, 3, 12, 7—and the problem is solved. You may want to try adding 5's and subtracting 9's, to show that the problem can be solved this way as well. It is also interesting to note that when this problem appears, there usually isn't a vat. Try your hand at measuring 7 litres if all you have are two unmarked jugs that hold 5 litres and 9 litres. It is possible.

Both the coin problem and the jug problem are examples of problems that can be modelled with *Diophantine equations*. Diophantine equations are those where we are interested in integer or whole-number solutions. The coin problem could have been modelled by the equation $3x + 5y = 30$, where the variables are non-negative integers. The jug problem can be modelled by the equation $5x + 9y = 7$, where the variables are integers. The method used in Figure 1 is closely related to *modular arithmetic*, sometimes called clock arithmetic. All numbers in the same column are equivalent *modulo 3* (in this case). A surprising amount of interesting mathematics occurs when we consider not the numbers themselves, but the *residue* classes (columns) to which they belong.

Our problem lies somewhere in between the coin and the jug problem. All "solutions" to the coin problem work for the hourglass problem. On the other hand, only some of the new "solutions" to the jug problem work for the hourglass problem (the "hard" version at least).

In our problem, we can start both timers simultaneously and turn over the 9-minute timer when it is done. When the 13-minute timer is done, we know that 4 minutes have passed for the 9-minute timer, so we could turn the timer back over and measure 4 minutes. We could also have "started timing" when the 13-minute timer finished and the 9-minute timer would be a 5-minute timer (or a 4-minute timer, if we turned it over). Similarly, if we start both timers simultaneously, turn them both over when they are done. Then when the 9-minute timer is done for the second time, 5 minutes will have passed on the 13-minute timer. This means that we use this as our "start" point, and we can use the 13-minute timer to time 5 or 8 minutes.

If you don't mind waiting, you can use this process to turn the 9-minute timer into a 1/8-, 2/7-, 3/6-, or a 4/5-minute timer, while the 13-minute timer can be turned into a 1/12-, 2/11-, 3/10-, 4/9-, 5/8-, or 6/7-minute timer. Thus, we can solve the problem, with a head start, by doing 3 + 9 + 9 + 9 = 30 or 4 + 13 + 13 = 30 (or 8 + 9 + 13, or several other ways).

The original problem from *The Inquisitive Problem Solver* asked you to time 30 minutes without a head start, which makes it a little trickier, but possible. There are a couple of possible solutions. One possible solution follows. Turn both hourglasses over at the same time, and turn the 9-minute hourglass over as soon as it finishes. When the 13-minute hourglass is finished, 4 minutes will have passed in the 9-minute glass, turn it over and let it run out. When it runs out, 17 minutes will have passed. Now turn over the 13-minute hourglass, and when it is finished, you will have passed 30 minutes.

This problem goes to show the power of considering an easier problem, and the fact that sometimes the questions you ask yourself yield more interesting mathematics than the question you were asked. Now for your homework:

In Figure 2, a rectangular ceiling PQRS measures 6 m by 4 m and is to be completely covered using 12 rectangular tiles, each measuring 1 m by 2 m. There is a beam, TU, that is positioned so that PT = SU = 2 m and that cannot be crossed by any tile. Determine the number of possible arrangements of tiles.

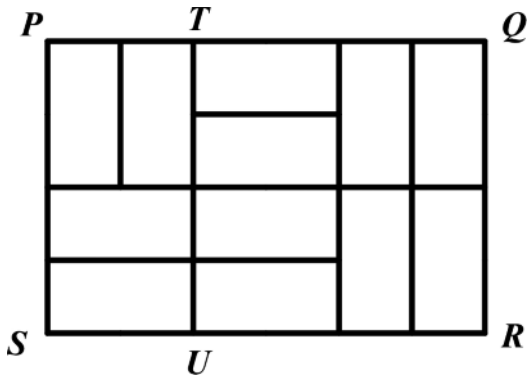


Figure 2

Have a great summer, and happy problem solving! ▲

▲ Call for Manuscripts

The *Ontario Mathematics Gazette* is inviting manuscripts for all grade levels.

Instructions for submission of manuscripts are found on page 1 of any *OMG*.

Contact the Editor for further details.

▲ Hey, It's Elementary: If I Had a Million Dollars...



LYNDA COLGAN
E-MAIL: colganl@educ.queensu.ca

Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.

As I was walking along Kingston's lakefront path one hot and humid morning last summer with my friend and neighbour, Bonnie, two equally sweat-soaked walkers approached us from the opposite direction. As they passed, one person made the comment: "It must be a gazillion degrees out here already and it's only 7:00 a.m." Bonnie and I both nodded our heads to communicate our absolute agreement with her amazingly accurate approximation of the temperature.

It wasn't until much later in the day, when I was sitting in a graduate seminar, listening to one of my students presenting a talk on a particular theory and stating categorically that "there are about a *bazillion* different interpretations of this concept," that I got thinking about how fictitious numbers are used in a *kajillion* different ways in colloquial contexts.

Clearly, in the first case, one *gazillion* degrees was understood by the speaker, Bonnie, and me to mean convey that it was very hot. In the second case, by claiming that there were at least one *bazillion* different ways of thinking about a theory, the speaker was, though exaggerating, successfully communicating her synthesis of the research, which demonstrated that the concept was enormously complex and ambiguous.

In addition to *gazillion*, *bazillion*, and *kajillion*, you may have heard or spoken words like *zillion*, *jillion*, *bajillion*, *kabillion*, *gajillion*, or *umptillion* to convey a number so large that only a special word can make it fathomable to our mind's eye—for example, the number of grains of sand on a beach. The amazing thing is that most people understand the "definition" of such words in spite of the fact that they are entirely fictitious names, and represent entirely *faux* numbers and quantities; i.e., these numbers, though regularly used in everyday conversation, have neither mathematical definition nor mathematical validity. That means that no one knows how many zeroes a *jillion* has or whether a *kajillion* is bigger or smaller than a *kabillion*.

However, we do have many *illions* numbers with official names, such as *billion*, *trillion*, *quadrillion*, *octillion*, *centillion*. When there are so many zeroes that *illions* just won't do, there are even special words to represent the quantity, as you will see.

We regularly hear or read about *illions* numbers in the news: Italian soccer player Lionel Messie earned $41 million in 2013; Bill Gates' wealth in 2015 is estimated to be $79.3 billion; there are 35 million people living with HIV/AIDS (23 million of whom live in Sub-Saharan Africa); 6 million new HIV infections develop each year; and 1.1 million deaths are attributable to the disease annually. And on its website, the U.S. Energy Information Administration reports that in 2012, Canadians consumed 13.5 quadrillion BTUs (in this context, a unit that measures the energy consumed in heating and air conditioning applications).

Though we read and hear about *illions* of dollars or people, do we really understand what those *illions* mean? Have we become desensitized to mere *millions* and *billions* because they are small in comparison to the *kabillions* and *bajillions* that we use in conversation daily?

To understand big numbers, we need to understand how they are constructed, how they are named, and what they represent.

To write large numbers, we use exponents. Exponents are a shorthand way to show how many times a number is multiplied times itself. A number with an exponent is said to be "raised to the power" of that exponent. For example, 100 is written as 10^2 (10 to the power of 2), which means that we multiply 10 times 10. Notice that 100 has two zeroes and an exponent of 2. To write 1000, or 10 x 10 x 10, we use 10^3. 10^6 is written as the number 1 followed by six zeroes—1 000 000—and 10^9 can be written as 1 000 000 000, and so on.

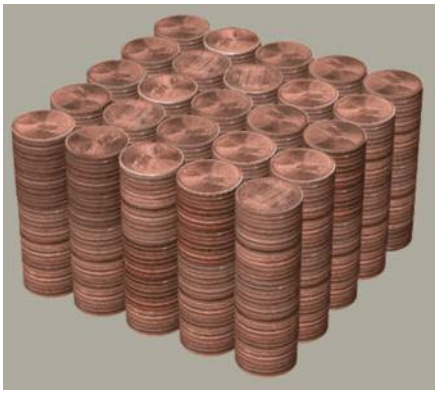
Do you know the everyday name for the number 1 followed by six zeroes or 1 000 000 000? The table below lists many of the official terms for the big numbers that are officially recognized in the world of mathematics. Although these are official names, most are rarely used. In fact, it would be rare to talk of any number beyond a quadrillion by using its "name." If a number such as 1045

needs to be referred to in words, it is simply read out "ten to the power of forty-five" because it is easier to say, easier to understand, and less ambiguous than "quattuordecillion."

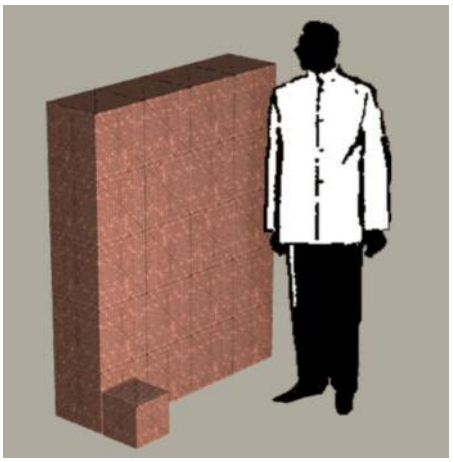
Name	Exponential Form
million	10^{6}
billion	10^{9}
trillion	10^{12}
quadrillion	10^{15}
quintillion	10^{18}
sextillion	10^{21}
septillion	10^{24}
octillion	10^{27}
nonillion	10^{30}
decillion	10^{33}
undecillion	10^{36}
duodecillion	10^{39}
tredecillion	10^{42}
quattuordecillion	10^{45}
quindecillion (quinquadecillion)	10^{48}
sexdecillion (sedecillion)	10^{51}
septendecillion	10^{54}
octodecillion	10^{57}
novemdecillion (novendecillion)	10^{60}
vigintillion	10^{63}
googol	10^{100}
centillion	10^{303}
googolplex	$10^{10^{100}}$

But knowing the mathematical names of big numbers is just the first step in understanding the quantity they represent. Without concrete examples, the difference between 100 000 and 1 000 000 can look like just another zero.

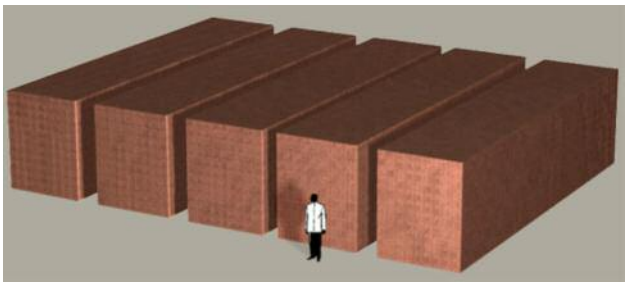
Even though the penny has been discontinued by The Canadian Mint (a disastrous decision in terms of teaching quantity and base 10 systems because it removes the unit upon which all others are based), most adults can picture 100 pennies—getting a quick mental image of ten stacks of ten pennies or a roll of 100 pennies. The fact that we can connect the word *hundred* to that mental image helps us to understand the quantity of one hundred. Understanding a million, like anything else, comes from experience. A million is a difficult concept for many of us to grasp, because one rarely sees that many of anything collected in one place.



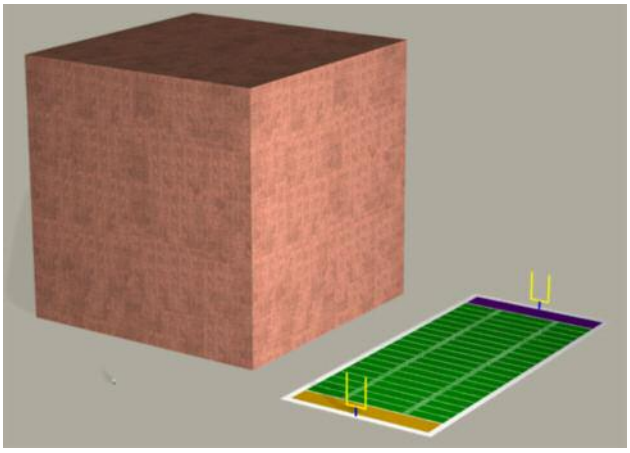
One thousand pennies



One million pennies



One billion pennies



One trillion pennies

Now use the pictures above to help you contextualize the number one million in other ways. Imagine counting aloud to one million by ones. Picture yourself starting out by counting "one, two, three, four…" and not stopping until you say "one million." How long do you think it will take? Assuming you could say one number per second, it would take you 11.57 days—non-stop.

Do you know anyone who is one million days old? Not likely, since 1 000 000 days is approximately equal to 2738 years.

Imagine you won one billion dollars in a lottery. How long would it take you to count your winnings, loonie by loonie? Assume you can count one loonie per second.

You could count 60 loonies in one minute. Since there are 60 minutes in one hour, you could count 60 x 60, or 3600, loonies in one hour. There are 24 hours in one day, so you could count $86 400 in one day. There are 365 days in a year, so there are 365 x 86 400 loonies, or 31 536 000 loonies in one year. But remember that all that counting leaves no time to sleep, eat, shower, use the phone, pay your bills, or go to the bathroom!

So how many years does it take to count 1 000 000 000 loonies?

$$31 \times 31\,536\,000 = 977\,616\,000 \text{ and}$$

$$32 \times 31\,536\,000 = 1\,009\,152\,000$$

So 31 years is not enough, and 32 years is too much. Looks like it would take you 31 years plus a little bit to count your winnings.

Young children are enthralled by the names of large numbers—they are reminiscent of dinosaur names—fun to say and act out (imagine a Kindergartner with arms outstretched to approximate her conceptualization of magnitude). Building mathematics vocabulary is an important foundation, and early experiences such as reading and watching David Schwartz's brilliant and visually potent *How much is a million?* By piquing students' motivation, emotion, interest, and attention, we set the stage for further learning: initial curiosity in mathematics has the potential to evolve into genuine interest. In fact, the "wow" factor that first captures a child's attention is likely to contribute to the development of greater understanding of, and positive attitudes toward, mathematics.

That initial curiosity about large numbers can take a more concrete turn for Junior Division students, who are beginning to have more responsibility for weekly allowances, birthday and holiday monetary gifts and personal expenses. At their stage of development, Junior Division students are ripe for early and evocative lessons about financial literacy. A classic problem comes to mind: *For your allowance, would you rather have $100 at the end of one month (30 days) or would you prefer to begin with one penny on the first day of the month and have the amount double each day for 30 days?* Of course, this rich problem garners surprising results: if you chose the latter, you would have $5 368 709.12 at the end of the month! Exponential growth with a personal context and a *WOW!* factor that concomitantly resonates and illuminates.

As anyone who has taught secondary students knows there is no doubt that one must appeal to their passions and interests and make learning profoundly personal. When students tire of real-world examples like cellphone plans, credit card interest, and mortgages, it is likely time for less practical activities—ones with surprising or unexpected results. Fermi problems bring together many skills: research, estimation, calculation, interpretation, and synthesis. Imagine asking students to consider the quantity of fast-food containers disposed of daily, weekly, monthly, and annually in their community. Researching and answering such questions may have long-term impact in terms of environmental awareness and sensitivity. For example, people in the United States throw away 2.5 million plastic bottles every hour and less than 3 percent are recycled. In the United Kingdom, of the 15 million plastic bottles used every day, fewer than 3 percent are recycled. And, less than 1 percent of the billions of plastic bags used each year are recycled, and the majority are used only once. Complementing such exercises with powerful visuals (such as videos of *The Great Pacific Garbage Patch*) continue the *WOW!* in age-appropriate ways, meaning that mathematics exercises cannot only contribute to the development of general adult numeracy skills, they may improve lifestyle and our world.

We need to build in opportunities to understand numbers like *million*, *billion*, and *trillion*—throughout a student's mathematics journey in school. Large numbers have definitions, but more importantly, they have *meaning*. Maybe the next time we read that Canada's total debt is $1.2 trillion dollars or hear a politician say, "We gave $125 million dollars in foreign aid in 2014," we won't just hear "we have 'a large' deficit" or "we spent 'a jillion' dollars." Without the proper knowledge to put these large numbers into a context we can understand, the numbers are utterly meaningless and productive debate is impossible.

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▲ In the Middle: Using Singapore's Model-Drawing Approach for Solving Word Problems



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Model Drawing, or "Bar Modelling," is the most distinctive and unique feature of Singapore's highly acclaimed math curriculum. In 1983, Singapore's Ministry of Education (MOE) introduced the diagram or model drawing in the primary school curriculum to help students solve both arithmetic and algebraic word problems involving fractions, whole numbers, ratios, and percents (Ng & Lee, 2009).

When using the model, students work with three models of representation: text, pictorial, and symbolic. Students create a diagram, using rectangles that represent both known and unknown quantities in the problem context. When the rectangles are labelled with a letter to represent unknown quantities, the model is used to create "pictorial equations" (Ng & Lee, 2009) and looks similar to modelling equations, using algebra tiles.

Rather than adopting a "shortcut" approach (Duval, 2006) focusing on the use of algorithms to solve the problem or translating key phrases of the problem directly into symbolic representations, model drawing forces students to focus on understanding the problem first and then identify the appropriate computation that will translate the diagram to solve the problem. In other words, aside from being a model *of* the problem, the diagram becomes a model *for* solving the problem, the base of the solution plan.

Common Model Structures

There are three main types of problem structures (models) taught to primary school children in Singapore: part-whole, comparison, and multiplication and division (Ng & Lee, 2009).

a. The Part–Whole Model

This model can be used for arithmetic situations translated as $a + b = x$ (arithmetic model) or $x + a = b$ (the algebraic model) (Figure 1).

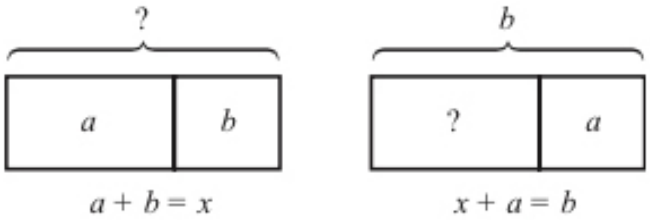


Figure 1: *(Ng & Lee, 2009, p. 286) Part-whole models: Arithmetic model (left) and algebraic model (right)*

Here's an example of a Patterning and Algebra curriculum expectation for Grade 5 (MOE, 2005, p. 83) that can be addressed through the use of the part-whole model.

> Demonstrate, through investigation, an understanding of variables as unknown quantities represented by a letter or other symbol (e.g., 12 = 5 + □ or 12 = 5 + s) can be used to represent the following situation: "I have 12 stamps altogether and 5 of them are from Canada. How many are from other countries?)

This is the way the model would be used to solve this problem (Figure 2):

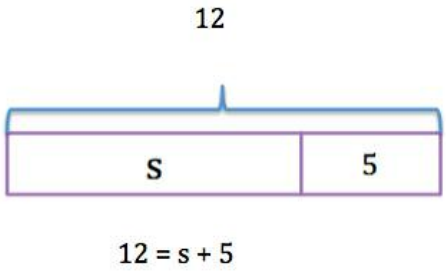


Figure 2: *Part-whole algebraic model of representation*

b. The Comparison Model

This model involves working out the relationship between two or more quantities represented by rectangles that vary in length.

Figure 3 (by Ng & Lee, p. 287) below illustrates this structure for both the arithmetic model ($a + b = x$) and the algebraic one ($x + b = c$).

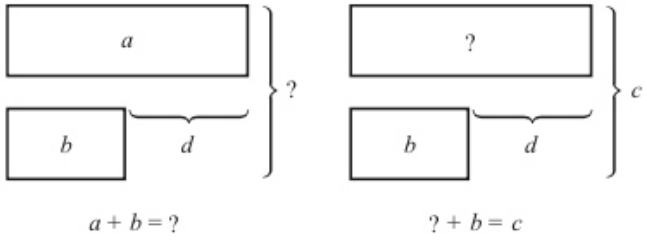


Figure 3: *(Ng & Lee, 2009, p. 287) Comparison models: Arithmetic model (at left) and algebraic model (at right)*

Let's consider the following problem:

The sum of four consecutive 3-digit numbers is 866. What are the four numbers?

(Addison Wesley, *Math Makes Sense 6*, 2006, p. 63)

This is the way model drawing could be used to solve this problem (Figure 4)

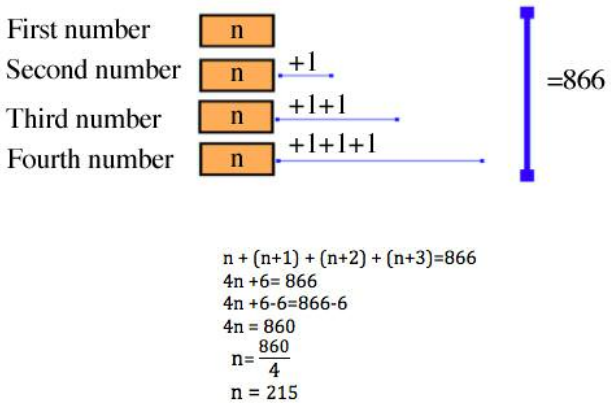


Figure 4: *Model drawing for the comparison model*

c. The Multiplication and Division Model

This is one of the models that helps students see relationships between the quantities presented in the problem's context. In this model, one of the quantities in the problem can be represented as a multiple of the other (i.e., *a* = *b* x *n*). Let's consider the following problem, given by Bell, Malone, and Taylor (1987), to explore how students solve the same problem in different ways by writing various equations for *x*.

> "There are 3 piles of rocks, A, B, and C, where B has 2 more than A, and C has 4 times as many rocks as A. The total number of rocks is 14. Find the number of rocks in each pile, using *x*."

Here's how the model could be drawn for this problem, using Thinking Blocks, a free online modelling tool, which can be accessed at **www.mathplayground.com/ThinkingBlocks/thinking_blocks_modeling%20_tool.html** (Figure 5).

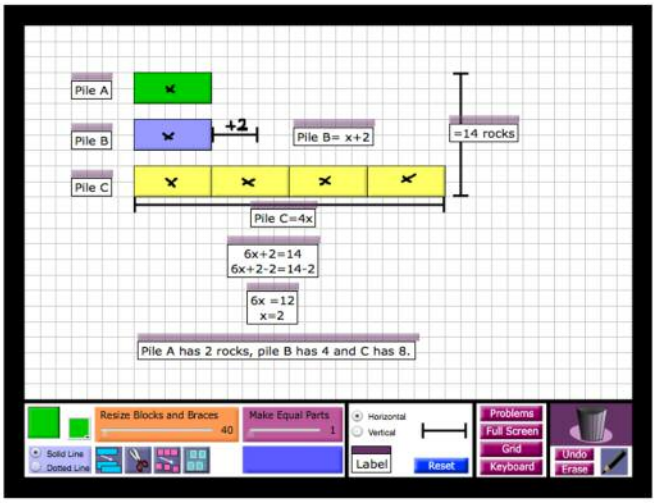


Figure 5: *Modelling the problem, using Thinking Blocks*

Alternatively, any quantity in each pile can represent *x* and function as the generator of the model. A model like this can enable students to see the relationship between bars expressed as equations involving fractions. In fact, another way of solving the problem that students in Bell, Malone, and Taylor's research (1987) reported is for pile C = x. Therefore, $A = \frac{x}{4}$ and $B = \frac{x}{4} + 2$. The equation students could write is $\frac{x}{4} + \left(\frac{x}{4} + 2\right) + x = 14$.

Considerations for Classroom Implementation

Model drawing represents a "macrostructure capturing the inputs, the relationships between the inputs, and the output of the problem" (Ng & Lee, 2009, p. 291). The model involves three stages through which students alternate between the text phase, the structural phase, and then the procedural-symbolic phase. It is not easy to develop mastery in drawing the model, and an erroneous model would lead more likely to a wrong solution. In Singapore, the model is introduced as early as Grade 2 and in Grade 5, teachers demonstrate the use of the model to solve algebraic problems.

When introduced to our students, the model needs a lot of guided practice. It can be as simple as demonstrating arithmetic equations, using the rectangular bars to make students understand the approach. One of the best online resources that I have used in my classes is Singapore Math Teacher (**www.thesingaporemaths.com**), an interactive site that provides lots of examples and independent practice opportunities. The students can see how the model for each problem develops step by step, starting with Primary 3 and ending with Primary 6 of the Singapore curriculum.

Additionally, teachers can use two great print resources to practise model drawing:

a. Forsten, C. (2010). *Step by step model drawing: Solving word problems the Singapore way.* Peterborough, NH: Chrystal Spring Books. (Best for Grades 4–6, or to introduce the model)

b. Walker, L. (2010). *Model drawing for challenging word problems: Finding solutions the Singapore way.* Peterborough, NH: Chrystal Spring Books. (Best for Grades 6–9)

For teachers who wish to see how to model the approach to their students, this is a video of Dr. Yeap Ban Har. A renowned researcher from the National Institute of Education (NIE) in Singapore, he is the current principal of the Marshall Cavendish Institute, Singapore, and in 2013, was part of the team that reviewed the Singapore Math Syllabus. This video shows Dr. Yeap modelling this process to teachers in United Kingdom, and introducing the progression of the approach in Singapore. (Online access: YouTube title: "Model method (The Singapore Bar Model)"; link: **www.youtube.com/watch?v=Em2yERb3Kfs**)

An Illustration of the Model-Drawing Approach Step by Step

Let's consider one of the "Problems of the Week," proposed by the University of Waterloo, Centre for Mathematics and Computing, to further demonstrate the progression of the model-drawing approach in detail at each stage:

> "A fast-food restaurant sells three items: pitas, baked potato chips, and bottled water. The price of an item does not change, regardless of the quantity purchased.
>
> Purchasing a pita, a bag of baked potato chips, and a bottled water costs $6.00. For $9.50 you can buy two pitas, a bag of baked potato chips, and a bottled water. A pita and a bottled water can be purchased for $4.70.
>
> Cyril Dorder, known by his friends as "Cy," purchases three pitas, two bags of baked potato chips, and a bottled water. What will lunch cost Cy Dorder?"

(Available online at **www.cemc.uwaterloo.ca/resources/potw/2014-15/POTWC-14-NN-PA-05-P.pdf**)

Phase 1: Understanding the Problem

a. Identify the known and unknown quantities. (Several readings of the text are needed.)

b. Chunk the information to represent visually (both the givens/known and the unknown).

Phase 2: Structural Phase – Drawing the Model

a. Students draw rectangles to represent the known and the unknown information. Attention to the proportions is needed, and use of colours can enhance the relationships between the quantities (Figure 6).

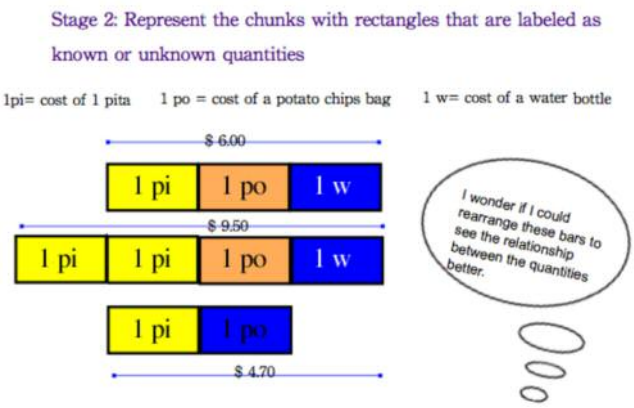


Figure 6: *Illustration of Stage 2 of the approach*

The model is revised. As you can see, it is easier to see the role of colour coding. The part-whole relationships are more obvious when the students align the rectangles representing the same quantities in each visual statement (Figure 7).

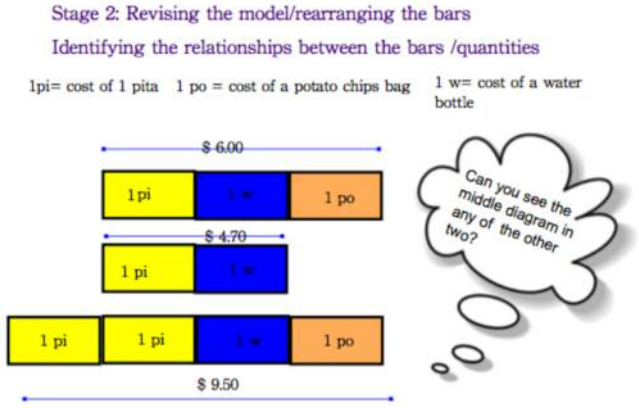


Figure 7: *Students can manipulate the model by rearranging the bars to better highlight the relationships among the quantities in the problem*

Phase 3: Procedural-Symbolic Stage

a. Students translate the diagrams into math sentences, in this case, algebraic equations.

b. Students further work out the relationships between equations. They can see, for instance,

how the middle diagram can be found in the other two, or how the first can also be found in the last one. By subtracting the second diagram from the first, they can find the price for a bag of potato chips.

c. They continue to revise the model by adding the found information to the model and solving all equations. This stage works closely with the other two and, in fact, students will alternate between them several times to solve the problem (Figure 8).

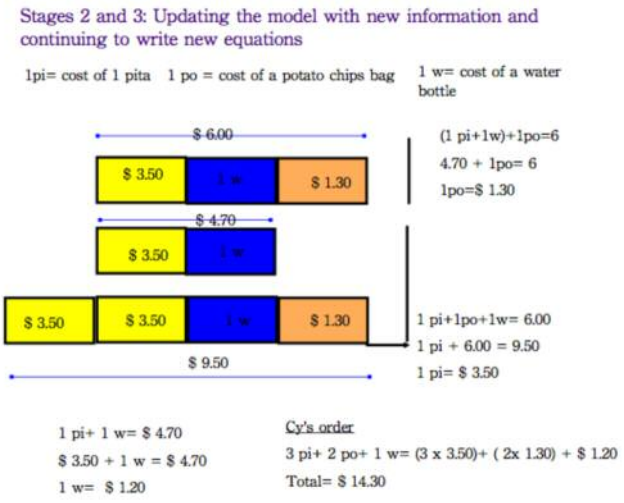


Figure 8: *The diagram also helps students keep their communication clear and concise*

As you can see, the model is not only an efficient visual representation of the problem, but it is the base of the solution plan, and helps students to solve the problem by manipulating the model and communicate clearly, concisely, and effectively. Most importantly, it forces the students to move beyond "short-cut" approaches and truly work on understanding the structure and the relationships in the problem.

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▲ MB4T (MATHEMATICS BY AND FOR TEACHERS): PATTERN RULES THAT "LOOK" DIFFERENT – THE PERFECT MOTIVATION FOR ALGEBRA TILES



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Ann Kajander is an experienced classroom teacher currently teaching mathematics and mathematics education at Lakehead University. Her research interests relate to teachers' enhanced learning of mathematical concepts. She and her classroom-teacher colleague, Tom Boland, have recently co-authored a book for teachers called Math Without Memorization: Big Ideas in Modelling and Reasoning for Elementary Teachers and Other Learners.

The previous column discussed the importance of encouraging students' multiple ways of seeing and connecting geometric patterns to algebraic reasoning, as encouraged both by our curriculum (Ontario Ministry of Education, 2005) and researchers such as Boaler (2014). A flexible understanding of building pattern rules provides a solid foundation for the development of algebraic thinking, and the emphasis on creating and connecting different rules based on different ways of visualizing patterns explicitly attends to the processes of problem solving, reasoning, communicating, connecting, and reflecting in our classroom discourse. In the previous column, an example from Kajander and Boland (2014) was used to illustrate the wealth of ways pattern rules might be constructed from a suitable geometric pattern. Here, the use of such multiple rules for the further development of algebraic reasoning will be explored.

When facilitating a lesson in which students are asked to come up with different rules for a pattern, a natural question one might expect is for students to want to know if the rules are the "same." How can all these different-looking rules all "work" to describe the pattern? This is a wonderful "teachable moment," which now motivates an exploration of algebraic methods of simplification. Once again, this reminds us that we must work consciously to resist our own (preprogrammed)

desire to encourage or insist on simplifying all rules to a given format. Rather, leaving the rules in the original form allows the initial geometric ways the students first "saw" the rules to remain visible. It is interesting to explore how the rules can "look" different algebraically, but still be equivalent. Questions such as these, which emerge from classroom explorations, provide the perfect introduction to the need for algebra tiles.

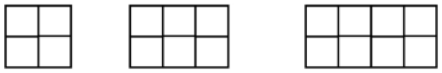
Simplistically, algebra tiles are a manipulative for both numbers as well as variable amounts. They have a resemblance to the concept of base-10 blocks, but include representations of lengths and areas that *vary*. Thus, the key additional feature of algebra tiles is that they contain *variable*-length pieces. (This may require some imagination... a very useful mathematical ability!) While the number one is still represented by a unit tile, some additional pieces are included in algebra tiles.

The best way to visualize algebra tiles is to see them dynamically at work, such as on The National Library of Virtual Manipulatives site (**http://nlvm.usu.edu/en/nav/category_g_3_t_2.html**). Choosing a variable piece (which we might call *n*, *x*, or *y*, and so on) and dragging it onto the working area, then using the mouse to change its size, gives the idea that the variable pieces are indeed designed to represent amounts that *vary*. The variable pieces represent a general value—any value—rather than representing a specific number.

In my own classroom teaching practice, I often found that the number one issue that held back some students from success in Grade 9 was the lack of a deep understanding of the role of a variable. In particular, students with an incomplete understanding often confuse *equations* and *expressions* in the same way they might confuse *variables* and *unknowns*. Foreshadowing this important distinction is important from the start. This will be further illustrated via an example.

An Initial Example

Starting with a simple example, particularly one for which the rules tend to use positive constants, can be a good way to begin. For example, consider the first three terms of the following growing pattern:

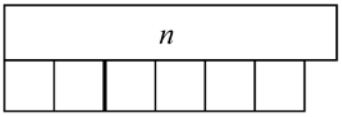


n: 1 2 3

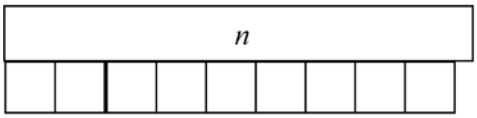
Some students might "see" the general rule as $2 \times n + 2$, while others might find $2 \times (n + 1)$ as a rule. It is important that we model the correct terminology here. These are *expressions*. It's easy to accidentally use the word "equation," but a key idea is that an equation contains an *equals* sign. Hence, $2 \times n + 2 = 8$ is an *equation*. In this case, there is enough information in the equation to define only one possible value for *n* which makes the equation a true statement. So in this case, the variable *n* has only *one* possible value; thus, it is often called an "unknown." Specifically in this case, the equation $2 \times n + 2 = 8$ for the above pattern can be thought of as saying, "If we have 8 tiles, what picture number can we build?"

In linear relations, we may go on to explore all the possible relationships. So for the equation $2 \times n + 2 = y$, there are many choices of combinations of *n* and *y* that will satisfy the equation; hence, in this case, both *n* and *y* represent *variables*. I have found that the confusion between unknowns (only one correct answer such as in the case $2 \times n + 2 = 8$), and variables (representing a relationship between values, often with many possible solutions, such as in the case of $2 \times n + 2 = y$) can really affect students' success in early secondary school. For example, I have seen students in Grade 9 or 10 Applied who struggled with this distinction, and would literally try to "solve" an equation such as $2x + 3 = y$ by saying that it means that "*y* is 3, so *x* has to be 2" (Kajander, 2002).

In the case of the geometric pattern presented here, the pattern rules are in the form of expressions. Let's return to the original question of exploring the different forms of these expressions to see how they relate. To explore this using algebra tiles, we will designate a variable piece as *n*. We note that when using algebra tiles, this amount is represented by a plastic rod whose width is one, but must be understood as representing a variable length, here denoted by *n*. It can be thought of as the sum of *n* unit pieces, lined up in a rod. So its width is 1, and the length is *n*. A key feature of this variable piece is that since it can take on any value (it varies), we can't randomly give it a particular value by lining up unit tiles next to it. Again, the virtual versions of these tiles make this clear. So either

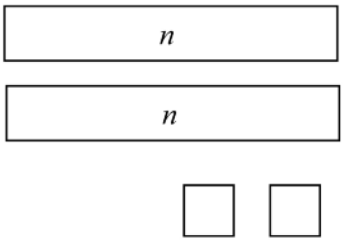


or

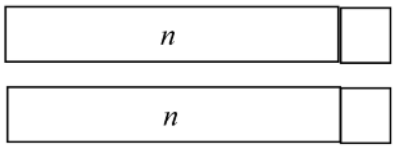


or any other length for the n piece is possible.

Returning to the introductory task, to model the rule $2 \times n + 2$, we can use two n pieces and two unit tiles:



Next we model $2 \times (n + 1)$ so that we can compare to the previous version of the rule. To do this, we must start with the bracketed quantity, so we add one unit tile on the end of an *n* tile to represent the expression in the brackets. Then we take two groups of this amount.



Students can see immediately how these two models are equivalent. Further, the tiles make explicitly teaching the algebraic method of "distributing" the 2 "over the bracket"


$2 \times (n + 1)$

unnecessary. In fact, the idea of an area model has been created here—we note that if we slide the pieces together, they form a rectangle whose side lengths are 2 and $(n + 1)$. Thus, we know the area of this rectangle will be the product, which we can thus also see (by counting the n's first and then the units) as $2 \times n + 2$. A visual understanding of the algebra tile model makes the simplification method obvious, rather than requiring a "rule" to be remembered.

Other models using algebra tiles can be used to develop other algebraic concepts. In this column next year, I anticipate sharing examples of the usefulness of such algebraic models in constructing more complex algebraic concepts.

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Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

To recap, complexity has emerged in education as a set of mathematical tools for analyzing phenomena, as a theoretical frame for interpreting activity of adaptive and emergent systems, as a new sensibility for orienting oneself to the world, and for considering the conditions for emergent possibilities leading to more productive 'intelligent' classrooms. ... As complexity becomes more prominent in educational discourses and entrenches in the infrastructure of 'classrooms,' mathematics education can move from a culture of cooperation to one of collaboration, and that has entailments for the outcomes of schooling—articulated in, e.g., movements from generalist preparation to specialist expertise, from independent workers to team-based workplaces, and from individual knowing to social action." (p. 91)

Davis, B., & Simmt, E. (2014). Complexity in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 87–91). London, UK: Springer.

Provincial Digital Learning Resources: What's New? Recently Released Learning Tool Apps!









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Last year, we highlighted a Rekenrek app that the team had developed for the desktop and mobile devices (see Vol. 52 #3 – March 2014). As mentioned in the last issue, that app was finally approved and is now available in both the Google Play store and in the Apple App Store for iOS devices. Having finally broken into that scene, we are continuing to develop learning tools for three platforms simultaneously (Desktop, Android, and iOS) and have just published two more Apps by mathies: *Money* by mathies and *Notepad* by mathies.

Money by mathies







Icons as found in mobile app stores, in the CLIPS tool container, and on **mathies.ca**

The *Money* by mathies app can be used to represent various money amounts, using realistic images of Canadian coins and bills used with the permission of the Royal Canadian Mint and the Bank of Canada. It is used to:

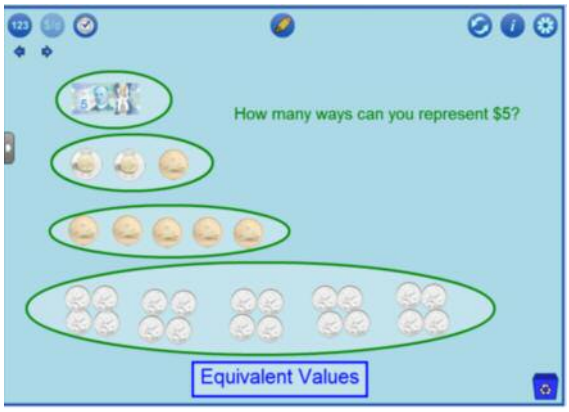
- estimate and count a collection of money
- skip count using the value of one of the coins or bills
- represent relationships between coins and bills
- add and subtract money amounts by making simulated purchases and providing change
- explore place value and unit conversions

Like the Rekenrek app, the Money by mathies app has the Annotation tool (accessible via the Pencil icon at the top middle of the screen). We've used the annotation tool to highlight examples of how a student might use the Money by mathies app.





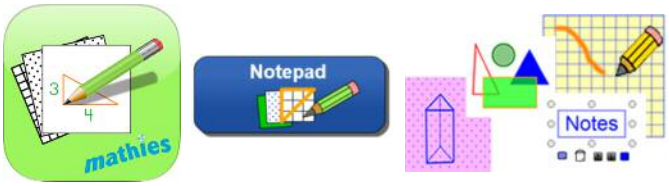






As the above screen suggests, there are a number of tweaks that can be made to how the Money by mathies app looks (customizing the background colour) and operates (choosing which coins and/or bills are available in the cash drawer).

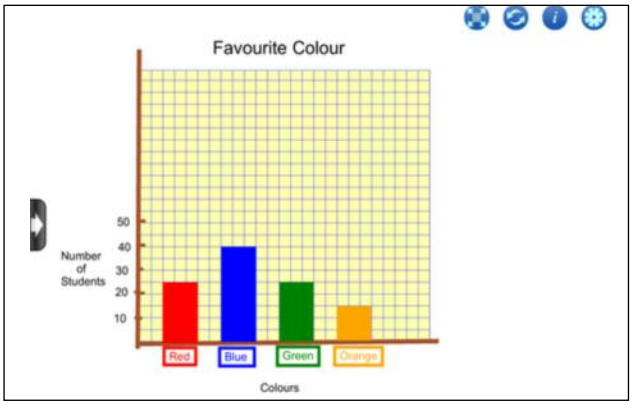
Notepad by mathies

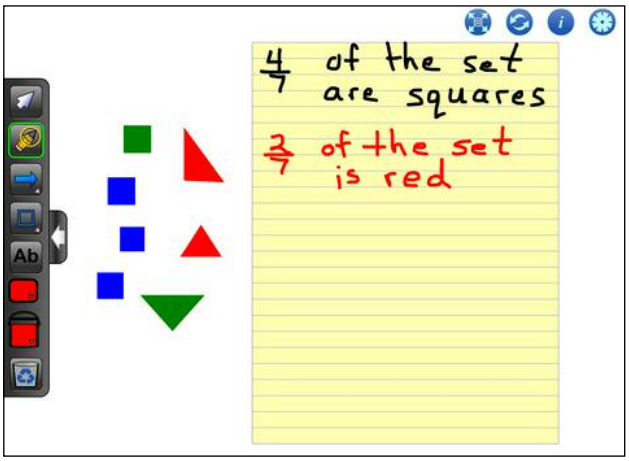


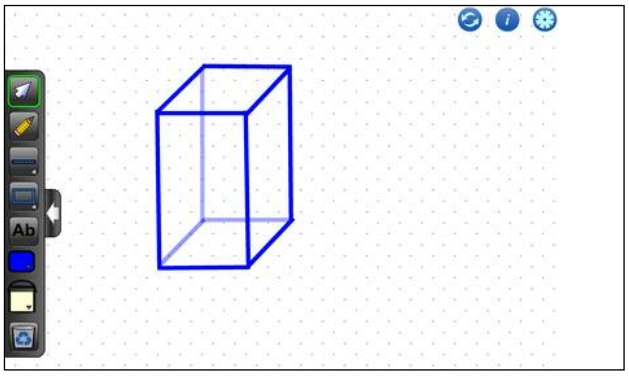
Icons as found in mobile app stores, in the CLIPS tool container, and on ***mathies.ca***

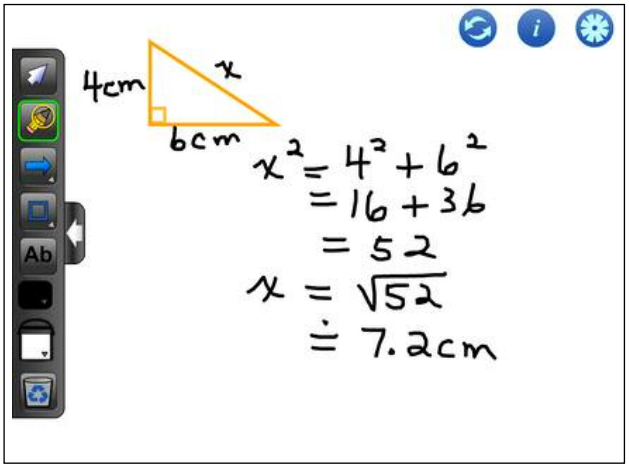
The *Notepad* tool provides a vehicle for students to write solutions, sketch diagrams, create graphs, and record their mathematical thinking. It combines the Annotation tool with useful backgrounds. Four backgrounds are available: blank paper, grid paper, isometric dot paper, and lined paper. Students can take a screenshot of their work to share with their teacher or add to their portfolio.

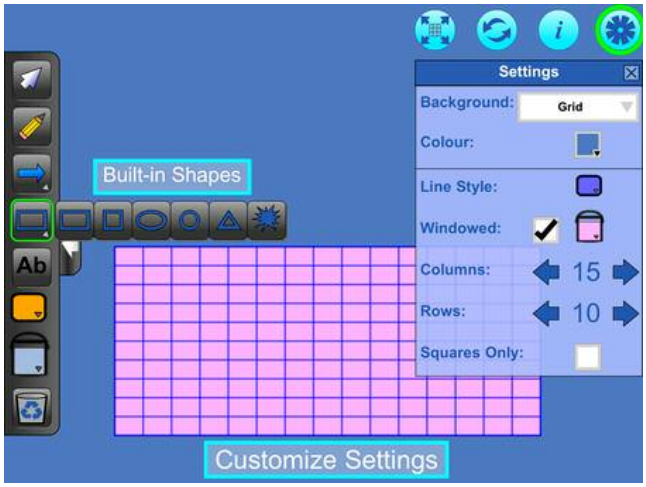
Here are a sampling of screenshots created with the *Notepad* by mathies app:











Like the other apps by mathies, there are a number of customizations available with this app— background type as well as colour, and then specific customizations about the look and setup of that background, including whether it should be contained within a window on the screen, or take up the full screen.



The screenshot above was created by a Grade 5 student, who used a beta version of the Notepad by mathies app to test out its features during development. Thank you Felicity!

Desktop versions of these mathematics learning tools, as well as many others, are available from the Learning Tools page at mathies.ca. Use the i button to access **mathies.ca**, as well as helpful videos and wiki page supports.

Additional learning tool apps are in development as this article goes to press, and ideally will be published before the end of the school year. To search for all our apps on the Google Play store or the App Store for iOS, search for "mathies" or "OAME," then download them for free and tell all your friends. ▲

▲ The Mathematics of Photographs



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The use of photographs in math classrooms has drastically increased with the availability of digital projectors and SMART Boards®. The usage has highlighted shapes, interpretations, and recognizing patterns (Lancaster, 2000–2008). There are exceptions (Sibbald, 2011), and sometimes modelling the physical object, seen in terms of the photographic image, can be useful for identifying patterns. In some cases, it is the shape of the feature on the ground, as opposed to shape as it appears in the photograph, that is of interest, and this leads to the question of how the analysis of photographs should handle the geometry of the photographic process.

In a previous article that examined the mathematics of a hill (Sibbald, 2014), there was a mention of two-dimensional images, of three-dimensional features, leading to effects such as foreshortening of the third dimension. This article takes a step back and looks at the underlying mechanics of basic photography, with an eye to understanding how the deeper mathematics may be used for instruction in secondary classrooms. It begins by looking at how a coordinate system on a photograph implies a coordinate system on the ground. Then the process is reversed to examine how a function based on ground coordinates becomes a corresponding function based on photographic coordinates. This is not intended to be classroom ready; rather, it is intended for teachers to be sufficiently informed that they feel they can begin to explore these ideas.

A camera can be modelled as a regular rectangular-based pyramid. The apex of the pyramid is the focal point of the camera—the lens is drawing light into a point and that point is the peak of the pyramid. The base of the pyramid is the photograph. It is the plane where the rectangular digital sensor (or in days gone by, the frame of film) sits. The key here is that a camera is designed so that the pyramid is regular, with the focal point centred over the middle of the rectangular area.

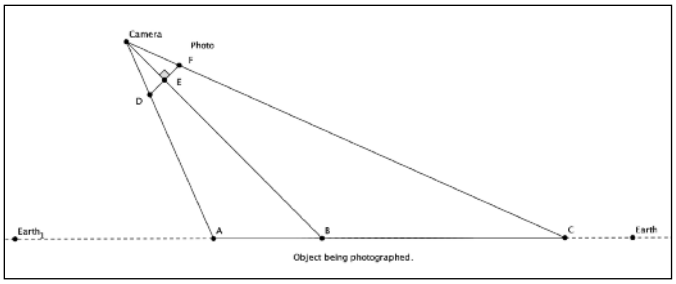


Figure 1: *Side view of basic geometry of photography*

The ground that is being photographed is, in the simplest model, a section of a plane that is at an arbitrary angle to the camera (see Figure 1). Typically a camera is held at an angle to a flat surface to avoid glare and that leads to the arbitrary angle. If one imagines holding a camera and taking a picture of a floor, then the photograph and associated coordinate system is at an angle to the floor and its associated coordinate system. The focal point of the camera along with the photograph forms a rectangular-based pyramid. As the four sides of that pyramid are extended toward the floor, the projection of the photograph makes the shape of the ground area that is being photographed. Note that the term "ground" is used, rather than an object, because it is the overall relationship between the coordinate system on the ground and the coordinate system in the photograph that is important.

Figure 2 expands on Figure 1 by showing a three-dimensional portrayal of what is going on. In this figure, it can be assumed that the "vertical" line in the middle of the photograph is the midline of the ground area (assuming the ground is flat). In effect, if one drew *x*- and *y*-axes on the photograph, with the origin at the centre of the photograph, then it can be assumed that Figure 1 is showing a side view of the *y*-axes and their geometry; in particular, that the *y*-axis in the photograph corresponds to a straight line on a flat earth. It is also clear that the triangle, composed of the camera and *y*-axis in the photograph (DF), is not similar to the triangle composed of the camera and the *y*-axis on the ground (AC) unless the line of sight of the camera is perpendicular to the ground. A similar effect takes place for the *x*-axis of the photograph; that is, the triangle formed by the camera and photographic *x*-axis will not, generally, be similar to the triangle formed by the camera and ground *x*-axis.

The good news is that the axes on the photograph correspond to two lines on the ground. Perpendicular axes in the photograph correspond to perpendicular axes on the ground, provided the camera is held so that the bottom of the photograph is not tilted relative to the ground (i.e., the line at the bottom of the photograph will never intersect the plane of the ground), which will be assumed throughout this article. This assumption does not always apply; shapes do not always align with the coordinate system that is imposed by the angle of the camera. For example, if one is in an airplane and takes a photo of two roads crossing at right angles, the perpendicular axes of the photograph correspond to perpendicular axes on the ground, but these are unlikely to correspond to the roads. Therefore, the roads, as they appear in the photograph, are unlikely to actually appear to be at right angles until mathematically transformed to correct for the photographic process.

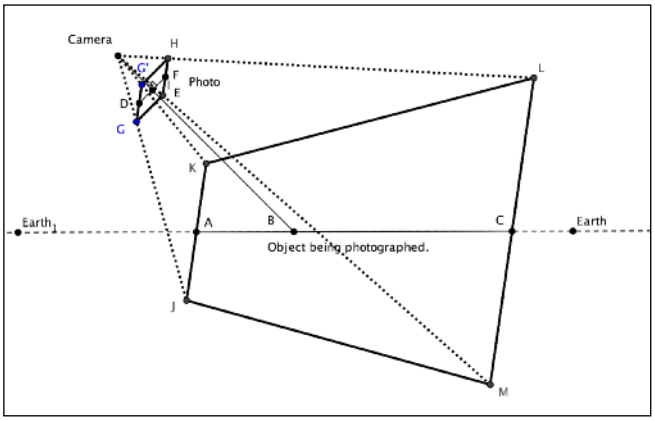


Figure 2: *Geometry of a whole photograph*

In terms of a photograph, as a whole, the rectangular area of the photograph projects onto the ground as a trapezoid. In Figure 2, if the bottom edge of the photograph (G,G') is parallel to the bottom edge of the corresponding ground (J,K), then along with the camera vertex, they form a pair of similar triangles. In addition, every line in the photo that is parallel to that bottom edge of the photograph will form a distinct similar triangle with the corresponding line on the ground that is parallel to the bottom line of the ground area. Each similar triangle is distinct because the distance between the line in the photograph and the corresponding line on the ground is increasing as one moves from the bottom of the photograph to the top.

> *"Mathematics curricula worldwide aim at teaching students to understand and produce proofs, both to reflect proof's central position in mathematics and to reap its many educational benefits. . . . Mathematics educators and students have already benefited greatly from educational software . . . such as Dynamic Geometric Systems (DGS) and Computer Algebra Systems (CAS), and researchers are working on advanced proof software specifically for mathematics education (e.g., eduTPS)." (pp. 404, 408)*
>
> Hanna, G. (2014). Mathematical proof, argumentation, and reasoning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 404–408). London, UK: Springer.

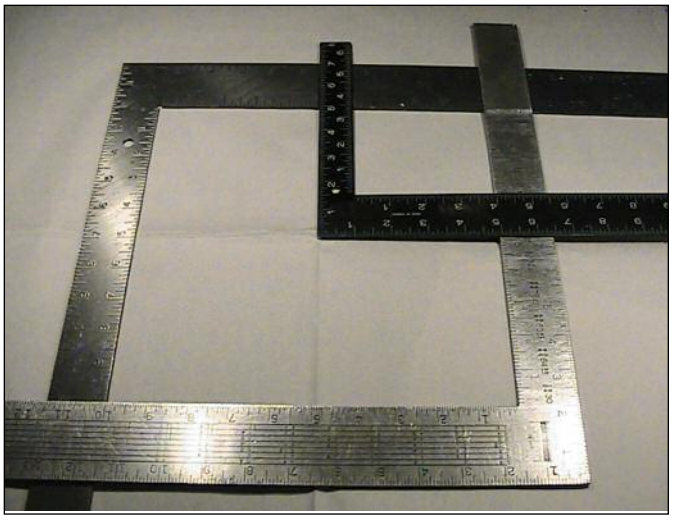


Figure 3: *Photograph of carpenter squares*

You can, of course, test out these ideas and, if you are like me, might find a practical exercise helpful. Figure 3 shows a photograph that was taken with the line of sight in the plane of the *y*-axis at an angle of approximately 45° above the plane, and with the centre of the photo approximately lined up with the centre of the ground. In the figure, there is a creased sheet of newsprint, with creases corresponding to the *x*- and *y*-axis, that has two large carpenter squares and a smaller carpenter square. The smaller square has a right angle at the origin. The inside measure of the two large squares is ten inches, and lined up to be five inches either side of each axis. They form a square on the ground, but the photograph has a trapezoidal appearance because of the effects described. The square appears to have been shrunk at the top of the figure, relative to the bottom because the top is farther away from the camera than the bottom.

Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

"One of the challenges of teacher educators is to create genuine learning situations for teachers, often through carefully designed tasks, in which teachers experience as learners the kind of learning that the mathematics teacher educator wishes to convey." (p. 431)

Krainer, K., Chapman, O., & Zaslavsky, O. (2014). Mathematics teacher educators: Definition. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 431–434). London, UK: Springer. London, UK: Springer.

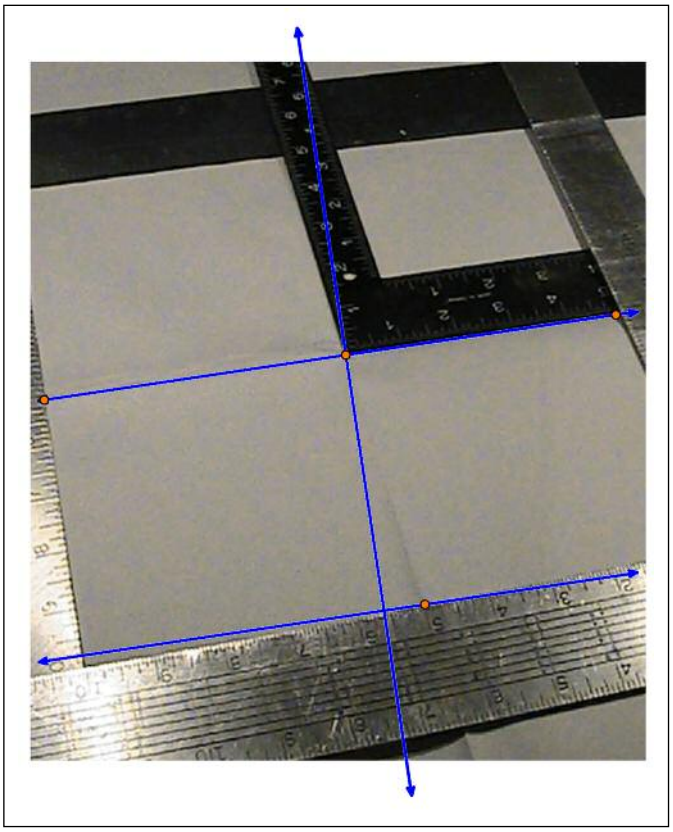


Figure 4: *Photograph taken at an angle to the plane of the* Y*-axis*

Since the photograph in Figure 3 was taken with the camera in the plane corresponding to the *y*-axis, and centred on the ground centre, the two axes appear to be perpendicular. If the photograph is taken at a different angle, relative to the plane of the *y*-axis, as shown in Figure 4, then the axes are not perpendicular (the constructed lines are perpendicular). Also note that a line parallel to the *x*-axis has been added and is slightly skew with the square at the bottom of the photograph. In spite of these aberrations, the axes in the photograph, when projected onto the ground, correspond to perpendicular lines (because that was the set-up of the squares). They just don't happen to be the natural coordinates we see on the ground. This naturally leads to the question of how to remove the effects of the transformations caused by the photographic process.

Defining the Photographic Transformation

Begin by considering a photograph of an item of interest. Assume the camera is in the plane of the *y*-axis, of the ground, with photographic centre perfectly aligned with the centre of the item of interest on the ground. Assume an *x*-axis on the ground that is centred on the item of interest. Further assume that points on the

photograph can be thought of in terms of a photographic coordinate system with origin in the photographic centre and using horizontal (u) and vertical (v) axes. These may be thought of as pixels, provided they are counted relative to the centre of the photograph.

In this geometry, the rectangular photograph projects onto a trapezoidal area of ground (as shown in Figure 2). Horizontal lines in the photograph correspond to horizontal lines on the ground and, along with the camera, form similar triangles. So there is a constant of proportionality, which we generally do not know, and $u = kx$ for some value of k. In other words, line segments along the x-axis on the ground have a length proportional to the u-axis in the photograph. The trapezoid shape on the ground indicates that k is a function of y and should be described as $u = k(y)x$. The dependence on y is due to the ground corresponding to the bottom of the photograph being closer than the ground corresponding to the top of the photograph. It may be feasible to ignore this detail if the near and far aspects of the item of interest are small, relative to the distance from the camera. The crucial detail is how the v coordinate depends on the y coordinate.

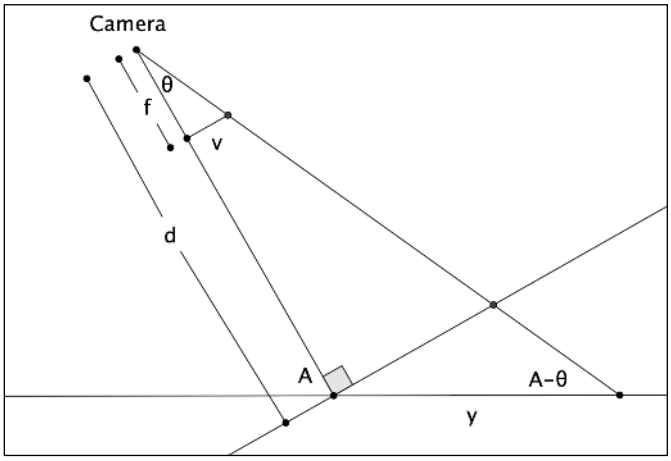


Figure 5: *Diagram for determining the relationship between v and y*

The determination of the relationship between v and y is found using Figure 5. In the diagram, f is the (exaggerated) focal length of the camera, v is the vertical position on the photograph, d is the distance to the centre of the item of interest on the ground, A is the angle of the line of sight to the ground, and θ is the angle inside the camera from photographic centre to the point that is "v" away from the photo centre and y is the ground distance from the ground-origin to the point corresponding to v. Finding the value of y can be done using the sine law:

$$\frac{\sin(\theta)}{y} = \frac{\sin(A-\theta)}{d}$$

This was purposely written with y in the denominator so that when the sine of the difference is expanded, cancellations are clearer.

$$\frac{1}{y} = \frac{\sin(A)\cos(\theta) - \cos(A)\sin(\theta)}{d\sin(\theta)}$$

$$\frac{1}{y} = \frac{\sin(A)}{d\tan(\theta)} - \frac{\cos(A)}{d}$$

This form is useful because $\tan(\theta) = \frac{v}{f}$, which effectively eliminates θ. Making the substitution and inverting the fraction leads to:

$$y = \frac{dv}{f\sin(a) - v\cos(A)}$$

Since d, f, and A are not usually known, it is the general form that is important, and the determination of the parameters based on the photograph will circumvent the problem. The algebraic form of this transformation is:

$$y = \frac{\alpha v}{\beta - v} \text{ and } y = \frac{\beta y}{\alpha + y}$$

It is, in principle, feasible to determine the parameters based on characteristics of the camera and knowing the distance from the camera to the ground-centre, but realistically this does not generally happen. This makes it important to recognize that the specifics of the parameters can be determined experimentally from fitting in software and through calculations based on the domain, range, or intercept properties.

In practice, one wants to model a real-phenomenon $y = f(x)$ that is on the ground, but the photograph uses the u, v coordinate system. If the distance to the object is relatively large, so that the trapezoidal edges have relatively steep slopes, then the general transformation is:

$$y = f(x) \leftrightarrow \frac{\alpha v}{\beta - v} = f\left(\frac{u}{k}\right)$$

While photo analysis can be tricky, this approximation, along with the use of technology, allows us to play with this. To examine the accuracy of the theory, photos were taken of a circular lid from a variety

of angles. The photos were brought into Geometer's Sketchpad® and an arbitrary coordinate system imposed. In this case, the equation on the ground was known, and it was the relationship between *u* and *v* that was needed to fit an appropriate curve on the photograph. The algebraic equation is:

$$\left(\frac{u}{k}\right)^2+\left(\frac{\alpha v}{\beta-v}\right)^2=r^2$$

where α, β, *k* are scale factors. The scale factors can be extracted heuristically from a photograph, and that will be demonstrated. Consider the photo of a circular lid that is shown in Figure 6. In this photo, there is an off-white background with a circular lid sitting on it. There is a dotted line that reaches the top of the square background, and that is the equation of a circle that has been placed on the photograph that used the two *x*-intercepts and matched the bottom of the photo as well as possible. Clearly there is a mismatch at the top of the photograph. The white curve is based on the theory and corrects for the geometry of the photographic process as outlined below.

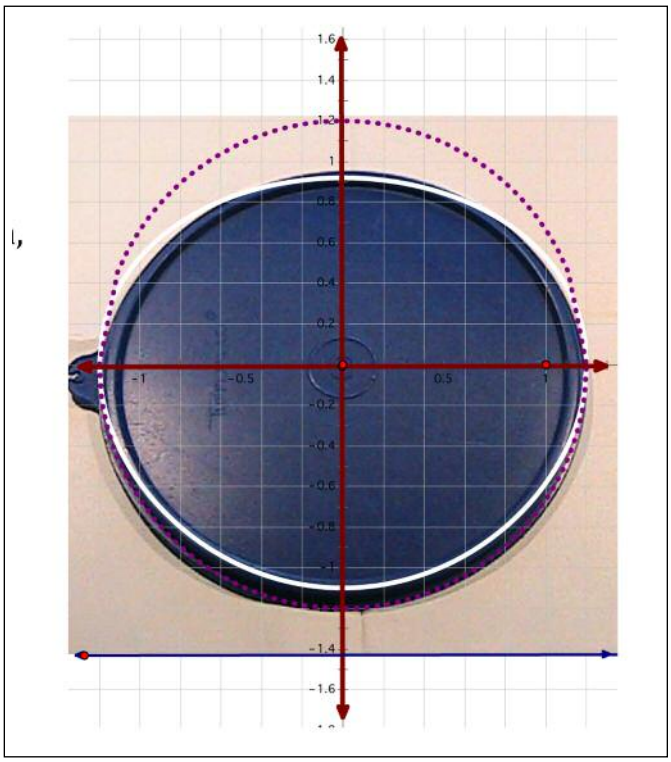


Figure 6: *Photo of a circle*

Fitting the curve to the circle requires resolution of the parameters. To do this, the y-intercepts, when $u = 0$, were used to obtain two equations to solve for two parameters. These equations were:

$$\frac{\alpha v_{min}}{\beta-v_{min}}=r \text{ and } \frac{\alpha v_{max}}{\beta-v_{max}}=r$$

The minimum and maximum values of *v* were read off the imposed grid as -1.10 and 0.92, respectively. The value of $r = 1.20$ was read off the *x*-axis because it is the basis for the analysis and interpreted as undistorted. This provided two equations with two unknowns and led to estimates α=11.24 and β=13.47. The approximation was also made that the scale of in the *x*-direction changes negligibly across the photograph so that $k = 1$ (see the "Going Further" box for more details). The assumption that $k = 1$ has led to some distortion away from the *x*-axis, and the white curve is not a perfect fit to the circular lid; however, it is clearly much improved over the dotted curve that is a pure circle. A second example is shown in Figure 7, which followed the same process, but for a smaller line-of-sight angle.

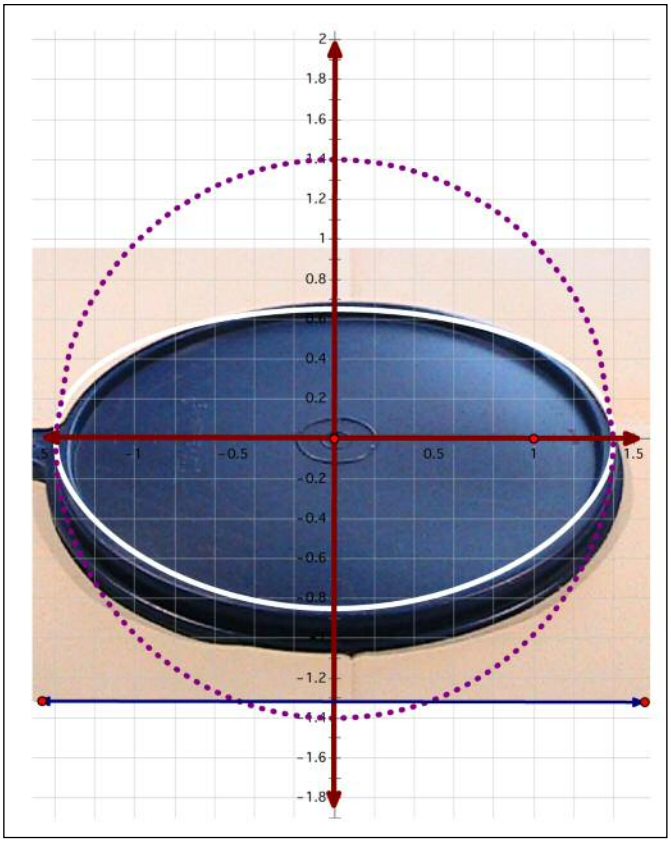


Figure 7: *A second example*

The goal of this article is to provide a workable sense of how the taking of a photograph transforms the geometry of a physical object. The approximation, ignoring variations in the *x*-axis scaling, creates a realistic opportunity. Further, the extraction of scale factors from the photograph simplifies the process and

allows pragmatic curve fitting for teachers. Would I use this content with students? Perhaps I would broach the subject in a highly scaffolded manner with an advanced functions class. The transformation fits with rational functions and provides an opportunity to explore a modelling application. For the most part, the intention was to provide a resource to teachers so that they can explore this and use fitted curves to enhance photographs they may use in their classrooms.

Going Further

A challenging issue in analyzing photographs with high accuracy is that the focal point of the camera implies that the photograph and ground both have a trapezoidal grid. Unlike the Cartesian grid we learned in our own education, a trapezoidal grid implies dependence between the variables. This is revealed where the *k* parameter depends on *y* and therefore *v*; however, it is scaling the *u* coordinate.

What makes the process more complex is that on a Cartesian grid, intervals on both axes create a square on the plane. In the geometry of photographs, intervals on both axes create trapezoids on the ground. While scaling the gridlines parallel to the *x*-axis is one issue. A further complication is that the grid lines in the *y*-direction are not parallel and, in fact, are increasingly tilted as one moves away from the *y*-axis. This means that the range of *v* changes when *u* changes.

These issues go beyond this article and do not help teachers to tinker with the availability of digital photographs. I would invite readers to play with the content of this article and take the thinking beyond what has been presented.

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Sibbald, T.M. (2011). Helicopters and defence spending. *OAME Gazette, 49*(3), 30–32. ▲

▲ Revealing Subjective Probability in the Middle and High School Mathematics Classroom



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Prelude

Imagine, if you will, a standard deck of 52 playing cards. Consider, now, the following two scenarios. *Scenario 1:* A card is drawn from the deck; the card is replaced; then a second card is drawn from the deck. What is the probability that the second card drawn is a king? The probability, in this particular scenario, *Scenario 1,* to no one's surprise, is 4/52 or, if you like, 1/13. *Scenario 2:* A card is drawn from the deck; the card is not replaced, but, rather, is placed face down beside the deck; then a second card is drawn from the deck. What is the probability that the second card drawn is a king?

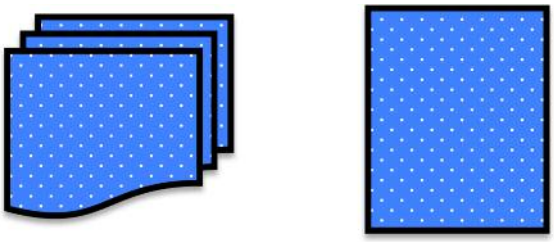


Figure 1: *Scenario 2: A card is drawn from the deck and not replaced, but placed beside the deck.*

Introduction

Jones, Langrall, and Mooney (2007), in their efforts "to inform the teaching and learning of probability by synthesizing research in the field" (p. 909), declared the "focus on both the classical and frequentist approaches to probability" (p. 913) as the strongest feature of the middle and high school mathematics curricula of Australia (AU), the United Kingdom (UK), and the United States of America (USA). Subjective probability, a third interpretation of probability, briefly mentioned in the mathematics curricula of AU and the UK, was not considered one of the big ideas in probability for the middle and high school grades in the USA. At the end of their synthesis, the authors, speaking in more general terms, indicated that the research they reviewed calls for "a more unified development of the classical, frequentist and subjective approaches to probability" (p. 949). This particular call is not new.

Calls for a more unified approach to teaching and learning the varying probabilistic interpretations can be found in books (Burrill & Elliot, 2006; Chernoff & Sriraman, 2014; Jones, 2005; Kapadia & Borovcnik, 1991; Shulte & Smart, 1981), special journal issues (Biehler & Pratt, 2012; Borovcnik & Kapadia, 2009), and research syntheses (Borovcnik & Peard, 1996; Garfield & Ahlgren, 1988; Hawkins & Kapadia, 1984; Shaughnessy, 1992, 2003), dedicated to research on the teaching and learning of probability. Despite repeated efforts—for more than three decades—the calls continue, which is due, in part, to the hot mess that is subjective probability (Chernoff, 2014).

Subjective Probability

It appears difficult to pin down what is actually meant when one uses the term *subjective probability*. Chernoff (2008) concluded that use of this term was subjective; and, he found inconsistent use of a variety of terms, such as *Bayesian*, *intuitive*, *personal*, *individual*, *epistemic*, *belief-type*, *epistemological*, and others, which were all used as synonyms for subjective probability. Further complicating matters, Chernoff also determined that the term *subjective probability* in mathematics education has dual meaning. The term is concurrently used as a general classifier and as a specific probabilistic theory. In sum, the term *subjective probability*, with respect to the teaching and learning of mathematics, means different things to different people.

Despite the above, we believe that subjective probability can, in fact, be brought into the middle and high school mathematics class. In order to do so, we propose an exercise, which will introduce and illustrate to students a central notion of subjective probability: "Probabilities quantify not events but *our information* about events" (Devlin, 2014, p. xii). As you will see in what follows, the task requires an understanding of the probabilistic notions of "and" and "or" and, further, the ability to work with fractions. To elaborate on the former point, we now return to the problems discussed in the Prelude.

The "Replaced Equals Not Replaced" (R=NR) Problem and Extensions

In our Prelude, we provided the answer to *Scenario 1* $\left(\frac{4}{52}\right)$, but we did not provide the answer to *Scenario 2*, that is, until now. We hope, at this point, that you have already embraced our cliffhanger approach to *Scenario 2*, and have worked out the probability for yourself. If not, this is your last chance; in other words, in what follows we detail how we determined the answer (i.e., spoiler alert). In terms of notation, we adopt $P(K1)$ to denote drawing a king for the first card drawn, and $P\left(\overline{K1}\right)$ to denote not drawing a king for the first card drawn. Similar notation will be used for $P(K2)$, etc.

$$P(K2) = P\left(K1\,K2 \cup \overline{K1}\,K2\right) = P(K1\,K2) + P\left(\overline{K1}\,K2\right)$$

$$= \left(\frac{4}{52}\times\frac{3}{51}\right) + \left(\frac{48}{52}\times\frac{4}{51}\right) = \left(\frac{4}{52}\times\frac{3}{51}\right) + \left(\frac{4}{52}\times\frac{48}{51}\right)$$

$$= \frac{4}{52}\left(\frac{3}{51}\times\frac{48}{51}\right) = \frac{4}{52}\left(\frac{51}{51}\right) = \frac{4}{52}(1) = \frac{4}{52}$$

Recalling *Scenario 2* from the Prelude, the probability that the second card drawn is a king, denoted $P(K2)$, is $\left(\frac{4}{52}\right)$.

We see, then, that $P(K2)$ is $\left(\frac{4}{52}\right)$ in *Scenario 1*, and $P(K2)$ is also $\left(\frac{4}{52}\right)$ in *Scenario 2*. In other words, for this particular problem, the probability that the second card drawn is a king is the same, whether the first card is replaced, as in *Scenario 1*, or not replaced, as in *Scenario 2*. We denote this *The Replaced Equals Not Replaced Problem* or, more succinctly, the *R=NR Problem* (which, yes, is also play on the word "problem").

Based on our experiences with the R=NR Problem, to some, the answer is intuitive; but, to others, also known as the majority, the answer is counterintuitive. Primarily, the counterintuitive nature of the R=NR Problem stems

from the probability being the same, whether the card is replaced or not, which is anathema to the secondary intuition (Fischbein, 1975) that is developed when the teaching and learning of probability hinges largely upon the overarching bifurcation of "with replacement" or "without replacement" leading to different probabilities. Cementing this counterintuitive nature, the answer to the R=NR Problem remains the same as the problem is extended.

We ask that you consider, now, a third scenario to the R=NR Problem. In *Scenario 3*, a card is drawn from a standard deck of cards; the card is not replaced, but, rather, is placed face down beside the deck; a second card is drawn from the deck, which is also not replaced, but, rather, placed face down beside the first card that was not replaced; then a third card is drawn from the deck. What is the probability that the third card drawn is a king?

$$P(K3) = P\left(K1\,K2\,K3 \cup K1\,\overline{K2}\,K3 \cup \overline{K1}\,K2\,K3 \cup \overline{K1}\,\overline{K2}\,K3\right)$$

$$= P(K1\,K2\,K3) + \left(K1\,\overline{K2}\,K3\right) + \left(\overline{K1}\,K2\,K3\right) + P\left(\overline{K1}\,\overline{K2}\,K3\right)$$

$$= \left(\frac{4}{52}\times\frac{3}{51}\times\frac{2}{50}\right) + \left(\frac{4}{52}\times\frac{48}{51}\times\frac{3}{50}\right) + \left(\frac{48}{52}\times\frac{4}{51}\times\frac{3}{50}\right) + \left(\frac{48}{52}\times\frac{47}{51}\times\frac{4}{50}\right)$$

$$= \frac{4}{52}\left[\left(\frac{3}{51}\times\frac{2}{50}\right) + \left(\frac{48}{51}\times\frac{3}{50}\right) + \left(\frac{48}{51}\times\frac{3}{50}\right) + \left(\frac{48}{52}\times\frac{47}{51}\right)\right]$$

$$= \frac{4}{52}\left[\frac{6+144+144+2256}{2550}\right] = \frac{4}{52}\left[\frac{2550}{2550}\right] = \frac{4}{52}$$

That's right, $P(K3) = \frac{4}{52}$.

Our final extension to the R=NR Problem, which we call *Scenario 4*, extends the problem in a similar fashion to the extension from *Scenario 2* to *Scenario 3*. In *Scenario 4*, a card is drawn from a standard deck of cards; the card is not replaced, but, rather, is placed face down beside the deck; a second card is drawn from the deck, which is also not replaced, but, rather, placed face down beside the first card that was not replaced; a third card is drawn from the deck, which, again, is not replaced, but placed face down beside the two cards that were not replaced; then a fourth card is drawn from the deck. What is the probability that the fourth card drawn is a king?

$$P(K4) = P\left(K1\,K2\,K3\,K4\right) + P\left(K1\,K2\,\overline{K3}\,K4\right)$$

$$+P\left(K1\,\overline{K2}\,K3\,K4\right) + P\left(K1\,\overline{K2}\,\overline{K3}\,K4\right)$$

$$+P\left(\overline{K1}\,K2\,K3\,K4\right) + P\left(\overline{K1}\,K2\,\overline{K3}\,K4\right)$$

$$+P\left(\overline{K1}\,\overline{K2}\,K3\,K4\right) + P\left(\overline{K1}\,\overline{K2}\,\overline{K3}\,K4\right) =$$

$$\left(\frac{4}{52}\times\frac{3}{51}\times\frac{2}{50}\times\frac{1}{49}\right) + \left(\frac{4}{52}\times\frac{3}{51}\times\frac{48}{50}\times\frac{2}{49}\right)$$

$$+\left(\frac{4}{52}\times\frac{48}{51}\times\frac{3}{50}\times\frac{2}{49}\right) + \left(\frac{4}{52}\times\frac{48}{51}\times\frac{47}{50}\times\frac{3}{49}\right)$$

$$+\left(\frac{48}{52}\times\frac{4}{51}\times\frac{3}{50}\times\frac{2}{49}\right) + \left(\frac{48}{52}\times\frac{4}{51}\times\frac{47}{50}\times\frac{3}{49}\right)$$

$$+\left(\frac{48}{52}\times\frac{47}{51}\times\frac{4}{50}\times\frac{3}{49}\right) + \left(\frac{48}{52}\times\frac{47}{51}\times\frac{46}{50}\times\frac{4}{49}\right)$$

$$= \frac{4}{52}\left[\frac{6+288+288+6768+288+6768+6768+103776}{124950}\right]$$

$$= \frac{4}{52}\left(\frac{124950}{124950}\right) = \frac{4}{52}$$

Once again, the answer is $\frac{4}{52}$, that is, $P(K4) = \frac{4}{52}$.

As demonstrated, the solution to all four of our scenarios to the R=NR Problem is the same: $\left(\frac{4}{52}\right)$. As such, our following general discussion of the solution, while specific to *Scenario 3*, applies to both *Scenario 2* and *Scenario 4*.

The probability that the third card drawn is a king, in *Scenario 3*, is predicated on the first and second card, the ones that are not replaced, being placed face down. Worthy of note, whether the card was actually placed "beside the deck" is, for all intents and purposes, irrelevant. In other words, the cards that are placed face down could, if one so chooses, be placed on the floor, on an adjacent table, at the back of the room, in another room, on a wall, or wherever one sees fit. Wherever the first and second cards are placed, however, it is crucial that the cards are placed face down. In addition to the card remaining face down, it is also of vital, vital importance that, in the act of placing the cards face down beside the deck, one does not get a peek at the

card. As long as the cards remain face down, no information is gathered and, as such, the card that is placed beside the deck (or wherever) must be taken into consideration both as a king and not as a king.

After all, peeking at any of the cards, however brief, changes our information about the event(s) and, thus, changes, for the person that got the peek, the probability that the third card drawn is a king. To illustrate our point, we look at four particular cases for *Scenario 3*.

Exposing a Central Tenet of Subjective Probability

For those of you well versed in the *Men in Black* film franchise, you know what a neuralyzer is and what it does; but on the off chance you are not familiar with a neuralyzer, we will provide a brief explanation of its key points. A neuralyzer is a cigar-sized metal tube that, when a particular button is pressed, emits a bright flash—like that of a camera. Unlike a camera, though, the flash erases the memories for any person or persons who see the flash. The period to be erased, that is, how many seconds, minutes, hours, days, weeks, months, and years, is entered into the neuralyzer, via buttons and dials, before the flash takes place. Presently, the neuralyzer is not real. However, we wish, in what follows, as you would in a movie, suspend your reality (even if just for a few moments) and embrace the notion that seeing a flash results in a memory wipe.

As claimed earlier and throughout, the R=NR Problem is an opportunity to bring an accessible version of a central tenet of subjective probability to the middle and high school mathematics class: probabilities are subjective, that is, they quantify our information about events. In an attempt to support our claim, we now discuss a few different cases of *Scenario 3* of the R=NR Problem.



Figure 2: *Case 1: All cards remain face down.*

Case 1 for *Scenario 3* is one that we are already familiar with—where all of the cards in the deck remain face down, which is denoted by the blue face. As shown in Figure 2, the deck is on the left, the middle card represents the first card that was not replaced, and the card on the far right represents the second card that was not replaced. In *Case 1*, the probability that the third card drawn is a king is $\left(\frac{4}{52}\right)$. However, as we will now see in cases two through four, information we obtain changes the probability that the third card drawn is a king. But, before we get to that… neuralyzer flash!



Figure 3: *Case 2: Information about the first card not replaced has been exposed.*

There is but one subtle change from *Case 1* to *Case 2*. In *Case 2*, the first card that was not replaced, represented by the middle card in Figure 3, has been exposed. We have gained information that we previously did not have and we now know that the card is not a king. Perhaps the card dropped on the floor face up while transitioning it to beside the deck, perhaps one could just not help oneself and ever so discretely and quickly picked up the bottom right corner of the card to get a peek, or some other scenario (which you as the teacher will have to cleverly conjure up to help with the task when you implement it in your classroom). Either way, the cat is now out of the bag.

$$P(K3) = P\left(\overline{K1}\,K2\,K3\right) + P\left(\overline{K1}\,\overline{K2}\,K3\right) =$$

$$\left(\frac{48}{52}\times\frac{4}{51}\times\frac{3}{50}\right) + \left(\frac{48}{52}\times\frac{47}{51}\times\frac{4}{50}\right) = \frac{48}{52}\times\frac{4}{51}\left(\frac{3}{50}+\frac{47}{50}\right) =$$

$$\frac{48}{52}\times\frac{4}{51}\left(\frac{3}{50}+\frac{47}{50}\right) = \frac{48}{52}\times\frac{4}{51} = \frac{192}{2652}$$

At this point, once you know the card is not a king, the information that we have garnered has changed, from *Case 1*, the probability that the third card drawn is a king (which you will remember you don't remember). We must, in our calculation, take into consideration that the first card drawn was not a king. In doing so, for *Case 2*, the probability that the third card drawn is a king is no longer $\left(\frac{4}{52}\right)$, but, rather $\frac{192}{2652}$. Neuralyzer flash!

There is but one subtle change from *Case 1* to *Case 3* and, for that matter, from *Case 2* to *Case 3*. In *Case 3*, information about the second card that was not replaced has, somehow, been exposed. Remember, though, because of the neuralyzer flash that you were exposed to a few moments ago, you do not remember the

information about the first card not replaced. Essentially, we are not privy to the information associated with *Case 2* at this point.



Figure 4: *Case 3: Information about the second card not replaced has been exposed.*

Nevertheless, we are, for *Case 3*, privy to the fact that the second card not replaced is, for this example, a king.

$$P(K3) = P(K1\,K2\,K3) + P(\overline{K1}\,K2\,K3) =$$

$$\left(\frac{4}{52}\times\frac{3}{51}\times\frac{2}{50}\right)+\left(\frac{48}{52}\times\frac{4}{51}\times\frac{3}{50}\right)=\frac{4}{52}\times\frac{3}{51}\left(\frac{2+48}{50}\right)=$$

$$\frac{4}{52}\times\frac{3}{51}\left(\frac{50}{50}\right)=\frac{4}{52}\times\frac{3}{51}=\frac{12}{2652}$$

Now, a new cat is out of the bag and, as a result, the information that we have garnered impacts the probability that the third card drawn is a king. $P(K3)$ for *Case 3* is not $\frac{4}{52}$, as in *Case 1*, and not $\frac{192}{2652}$, as in *Case 2*, but, rather $\frac{12}{2652}$. Neuralyzer flash!

Not that you will remember (because of the flash), but *Case 4* is unlike *Cases 1, 2*, and *3* because, in *Case 4*, information about both the first and the second cards that were not replaced, somehow, has been exposed, as seen in Figure 5. For *Case 4*, yet another cat is out of the bag. The first card that was not replaced was not a king and the second card that was not replaced was a king. The probability that the third card drawn is a king depends on information about the cards, plural this time, that have been exposed.



Figure 5: *Case 4: Information about the first two cards not replaced has been exposed.*

The information that we have garnered in this case results in the probability that the third card drawn is a king is not $\frac{4}{52}$, as in *Case 1*, not $\frac{192}{2652}$, as in *Case 2*, not $\frac{12}{2652}$, as in *Case 3*, but, rather, is $\left(\frac{48}{52}\right)\left(\frac{4}{51}\right)\left(\frac{3}{50}\right)=\frac{576}{132600}$. Neuralyzer flash!

One Scenario, Four Cases, Four Probabilities

As has been demonstrated, the four different cases presented for *Scenario 3* of the R=NR Problem have resulted in four different probabilities that the third card drawn is a king. Specifically, $P(K3)$ was $\frac{4}{52}$ in the case where none of the cards were exposed; $\frac{192}{2652}$ in the case when it was found out that the first card not replaced was not a king; $\frac{12}{2652}$ in the case when it was determined that the second card not replaced was a king; and $\frac{576}{132600}$ in the case when it was established thatthe first card not replaced was not a king and the second card not replaced was a king. Generally speaking, the different cases, which each revealed different, but similar information, all resulted in different probabilities that the third card drawn is a king. In other words, probabilities are subject to our particular information about an event, that is, they are subjective.

Classroom Activity

While the neuralyzer approach to explaining the R=NR Problem does work in certain venues (e.g., one-on-one, traditional lecture format, dinner parties, and elsewhere), it is not the most effective way to bring the activity to your mathematics class. To work the N=NR Problem into your math class and, thus, bring a central tenet of subjective probability to students in Grades 5 to 9, we advocate the following teaching strategy.

This approach utilizes a fundamental resource in your classroom: the students. Of course, the role of teacher in this scenario, as is always the case, is vital to the task. As you probably suspect, you, as the teacher, set up the question while at the front of the room. However, instead of utilizing the flashes of a neuralyzer (remember, they don't exist), you divulge certain information to different students or groups of students in your room. For example, if you, as the teacher, have one group not peek at any of the cards, tell one group of students about the first card not replaced, a second group of students about the second card that was not replaced, and a third group of students about both cards not replaced, there will be four different probabilities that the third card drawn is a king floating around in your math class. (For bigger classes of students, *Scenario 4* provides the opportunity

for eight different possible probabilities, which corresponds to revealing information about none, one, two, or three cards that are not replaced.) In addition, the point where there are different groups of individuals with different probabilities for an event can result in further teaching and learning opportunities (e.g., guess the card that was covered based on a group's probability and others). Most importantly, though, when the exercise is being wrapped up, there is an opportunity to discuss subjective probability with your middle and high school mathematics class.

Discussion and Conclusion

The R=NR Problem, and the expanded versions, provide an opportunity to address a number of the General and Specific Outcomes for the Statistics and Probability (Data Analysis) Strands found in the Western and Northern Canadian Protocol's *Common Curriculum Framework for K–9 Mathematics and 10–12 Mathematics*. Further, the R=NR Problem also addresses a number of the General and Specific Outcomes for the Number Strands. Similarly, overall and specific expectations from *The Ontario Curriculum: Mathematics*, Grade 8 (Data Management and Probability strand) and Grade 12 (Probability Distributions strand in Math of Data Management MDM4U) courses can also be addressed. More generally speaking, the R=NR Problem provides an excellent opportunity to demonstrate that probabilities quantify our information about events, which will become one of the key enduring understandings of probability for well-informed citizens as we, as a society, move toward a world where more and more information becomes readily available.

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"A mathematician is a device for turning coffee into theorems"
(P. Erdős, 1913-96)



▲ Doing Maths Algorithmically in the Age of the Computer: Introducing Octave, A Comprehensive, Open-Source Software Resource



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This article has several goals:

1. To put forward my view that we should be introducing high school students to an algorithmic approach to solving certain problems where either there is no nice "formula" (in the classic sense, e.g., the Pythagorean Theorem), or where a complicated situation is better resolved by working through a sequence of more manageable steps
2. To point out the virtually complete dichotomy between the Ontario secondary school computing and mathematics curricula, and how this affects students, not only at the secondary level, but also during their further studies
3. To encourage teachers to introduce students to Octave, a relatively easy-to-learn, open-source programming language, with which students can do most, or perhaps even *all* of their assignments involving statistics, data analysis, equations, matrices, plotting, etc., and which they can *continue to use* in their post-secondary studies

To illustrate, I am going to discuss a variation of the well-known birthday problem:

When we ask k students to state their birthday, we are essentially asking them to choose a number at random from 1, 2, ..., 365. Instead of 365, we can ask them to pick an integer from 1, 2, ..., n. We want to find p, the probability that at *least two* students pick the *same* number:

$$p = 1 - \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-k+1}{n}$$

Before you read any further, you are invited to try and calculate, using a calculator, spreadsheet, etc., the value of p when k = 100 people, and n = 3000. Now let us see how we would do this using Octave. Being a very high-level language, the commands correspond to mathematical statements, and this makes programming quite simple. Suppose that we want to create the row vector [1, 2, ..., 100], we simply write: a = [1 : 1 : 100], where the 1 in the middle indicates that we are increasing by steps of 1. To go in descending order, as in the formula for p, we simply use a negative step, e.g., b = [100 : -1 : 1]. To find the product of the numbers 100, 99, ..., 1, we need only write **prod**(b).

Multiplication and division are indicated by "*" and "/," and when applied to a row vector, the operation will be performed on each element of the vector, e.g., we could write c = (1/100)*b. Then we could write prod(c) to obtain the product of the elements of c.

So, to do the calculations for k = 100 people and n = 3000, we simply type in the following at the Octave prompt (the semi-colon prevents displaying long vectors):

```
a = [3000 : -1 : (3000 – 100 + 1)];
b = (1/3000)*a;
c = prod(b);
p = 1 – c
```

Octave quickly responds with the answer p = 0.81148. Here we have used Octave as a powerful calculator. If we wanted to perform the same calculation for many combinations of k and n, we would create a short program with two input statements:

```
k = input('how many people?');
n = input('what is n?');
a = [n : -1 : (n – k + 1)];
b = (1/n)*a;
c = prod(b);
p = 1 – c;
disp(p)   % disp = display; % is used for comments
```

The program is simply an algorithmic transcription of the mathematics. Instead of input statements, we could just as easily have created a function of k and n. If we wanted to calculate the birthday probabilities for k = 10, 20, ..., 60, we could add a loop. This is the way computing is done in the real world: use a text editor to create a program, run your program, debug it, expand its

capabilities, store the values, and do a plot.

Above, we were given *k* and *n* and obtained the value of *p*. We can consider this as a *direct* problem and solution. A more complicated situation occurs because of the nature of the birthday problem. If we run the last program with $k = 30$ and $n = 365$, we find that $p = .71$. This means that if you try the birthday experiment in a class of 30, you have the large probability of .29 of looking rather silly. This leads to an *indirect* problem and solution. We now start off by fixing a level of tolerance ahead of time, for example .95, which corresponds to the "19 times out of 20" used by pollsters. Our task now is to find as *large a value of* n *as possible* so that if 30 people pick a number from 1, 2, ..., *n*, then the probability that at least two people pick the same number is *at least* .95. In view of the fact that the quantity *n* appears *k* times in our formula for *p*, finding a closed-form formula for *n* is probably an impossible task. However, by thinking algorithmically and then using an extension of the above Octave program, we can easily find the largest value of *n* for given values of *k* and *p*.

The algorithm for finding *n*:

1. Start with the value for *k* and the desired level of tolerance.
2. Begin with $n = k$ (when *p* is virtually equal to 1).
3. For each value of *n*, find *p* in terms of *k* and *n*.
4. Is *p* still greater than the fixed level?

   If yes, increase *n* by 1 and redo the calculation for *p*.

   If no, *n* is now too large; subtract 1 and stop.

To translate this algorithm into an Octave program, we take the above program and put it inside a "while-loop." To simplify, we will take $k = 30$ and set the tolerance at .95. The program would then read:

```
n = 30;   % start with n = k = 30
p = 1;    % we start above the tolerance of .95
while p >= .95
   n = n + 1;    % increment n by 1
   a = [n: – 1 : (n – k + 1)];
   b = (1/n)*a;
   c = prod(b);
   p = 1 – c
endwhile    % loop again while p >= .95
disp('maximum n is'), disp(n – 1)   % we went too far,
%    so use n – 1
```

If we run this program, we find that if we want to be certain 19 times out of 20, then the maximum allowable value of *n* is only 155, which is well below 365. For 40 students, we can go up to 273, and in an auditorium with 100 people, up to 1685!

Here is another set of direct/indirect problems. The direct problem is taken—with a modification of the wording—from a Grade 12 textbook: "At a manufacturing plant, 35% of the employees call in sick on any given day. If the plant has 20 employees on the payroll, what is the probability that no more than 7 call in sick on any given day?" The corresponding indirect problem would be: "What is the *minimum* number of employees required so that the probability that *no more than* 7 employees are sick on a given day is *at least* .90?" To solve this, we would proceed in the same algorithmic manner as with the pick-a-number problem above.

Some Thoughts on Mathematics and Computer Science Curricula

If you download the 72-page Ontario Computer Science curriculum and search for "mathematics," you will note that there are zero occurrences. All that we find are vague references to "mathematical literacy." We also learn that in course ICS4U (A3.5), students will learn algorithms to perform matrix operations. In other words, there is apparently no place for applied computing—mathematics, as such, lives in another, separate world. In my opinion, Ontario mathematics education reflects the 1980s era, as far as computing is concerned. When many of our students go on to university or college, they will not likely be using spreadsheets or programmable calculators, but rather, working with real computers. Perhaps what is needed is a course entitled something like, "Applied Computing for Mathematics and Science," so that students will be prepared for the practical side of the next stage of their studies.

Why am I advocating that Octave be introduced into the Ontario secondary school curriculum? To begin with, every engineering program in Canada uses Octave (or a comparable, commercial product such as MATLAB) in their courses. Students in higher-level science courses often need to do applied programming, complicated graphing, etc. In first- and higher-year courses in linear algebra, students learn about row reduction (another example of an algorithmic process). Octave easily handles this, as well as every other linear algebra calculation.

Further, at the high school level, Octave would be ideal for the course "Mathematics of Data Management." With this one tool, students would be able to do all the

computations, and do them in a *systematic* manner. As an example, suppose we take 11 measurements on each member of a group of people. For each person, we create a 1 x 11 row vector of data. We label these row vectors a_1, a_2,.... We then create a matrix DATA1 by simply listing the vectors and separating them by semi-colons: DATA1 = [a_1 ; a_2 ; ...]. Then all we have to do is apply the Octave functions—"mean," "median," "mode," "std," "quantile," etc.—to DATA1. If later on we have another data set, DATA2, we can work with the two sets separately and then see what happens when we combine them via DATA3 = [DATA1 ; DATA2]. Simple commands also allow us to obtain bar graphs, to generate large amounts of random data, or to toss coins thousands or millions of times.

Octave

Octave originated in 1988, when early versions were used in chemical engineering courses at the University of Wisconsin. Unlike the commercial MATLAB, Octave is open-source and thus may be freely downloaded for Linux and BSD (the two systems used on most mainframes), Windows, and Mac OSX [**www.gnu.org/software/octave/download.html**].

For the benefit of teachers and students, I have put together a series of support resources that can be freely downloaded from the website: **web.ncf.ca/en493/STUDENT_LINUX/STUDENT_LINUX.html**. The resources include, in addition to software, my instructional booklets entitled *An Introduction to Octave for High School and University Students* and *An Introduction to Student Linux*. Linux, because of the possibility of having multiple "desktops," is ideal for systematic teaching and learning. The booklets can be freely copied and distributed to colleagues and students. All of the software, including the Linux distribution itself, is open source so that there are absolutely no user fees or restrictions associated with their distribution or use.

Acknowledgements

I read about the pick-a-number problem at least 40 years ago, but I have not seen it mentioned since. Unfortunately, I no longer have the reference, but as I recall, it was in a magazine for high school teachers of mathematics. A special thanks to Bruce McLaurin, the head of mathematics at Glebe Collegiate in Ottawa, for taking time to discuss the high school curriculum with me. I also much appreciated the comments of the various reviewers concerning the original version of this article.

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Haese, R., Haese, M., Humphries, M., Haese, S., & Owen, J. (2004). *Mathematics for the international student.* [The direct factory question above was adapted from p. 723, exercise 3.]

Herz-Fischler, R. (2014). *An introduction to Octave for high school and university students.* [The text can be freely downloaded (**web.ncf.ca/en493/STUDENT_LINUX/STUDENT_LINUX.html**), may be modified, if desired, and then reproduced, on a non-commercial basis, for distribution to students.] See section 11 on how to present the solution of simultaneous equations by the method of row reduction and also by using Octave. For data analysis and statistics, see sections 05, 10, 19. Matrix operations are discussed in section 20.

Herz-Fischler, R. (2000). *The shape of the great pyramid.* [See the notes relating to the computations, p. 197. All computations were done in a systematic, algorithmic fashion; for certain repeated calculations, e.g., Pythagorean theorem, laws of sine and cosine, functions were first defined in Octave files and then called when needed.]

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Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

"The politics of language are to some extent self-evident in the structure of the research community. In particular, English is the predominant language of this community; the leading international journals and conferences all prefer English. ...There are thousands more languages in the world that are entirely absent from mathematics education research discourse. The preference for English makes things easier for English-speaking researchers...and more challenging for everyone else. It also, however, privileges certain ways of thinking about mathematics, teaching and learning, while rendering invisible other alternatives." (p. 335)

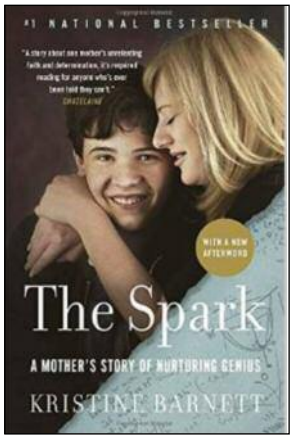
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▲ Autism and the Spark of Genius: A Mother's Perspective

TIM SIBBALD BOOK REVIEW AUTHOR

Book review of:

Barnett, K. (2013). *The spark: A mother's story of nurturing genius.* Toronto, ON: Random House Canada.



I confess to picking up this book when I was in a pinch for something to read on an airplane flight. It caught my eye primarily because I saw some math on the dust jacket and the word "genius." The fact that it is written by a mother about her prodigious son, who joined the *Perimeter Institute* in Waterloo at the age of 15 (Perimeter Institute, 2013; note the book predates this event), was all part of the fascinating discovery path to which the book and subsequent reading has led me.

The story highlights a bright youngster, who at a very young age, took a turn for the worst and began withdrawing to the extent that he was diagnosed with autism. Subsequently, when he was old enough to enroll in school, there were struggles, and the book gives a very insightful picture of the parental perspective of autism. The need for different therapies, in addition to school, and the overwhelming issues of handling day-to-day life are eye-opening. This alone would be beneficial for all teachers to read.

As the book progresses, I was struck by the need for teachers to be on the lookout for enrichment opportunities—not programmatic opportunities; rather, the moments to steer students toward learning that may not be for every student in the class. Programmatic options entail special education that addresses 17 percent of students (Gallagher-Mackay & Kidder, 2014), which includes 2 percent deemed to be gifted (Matthews, 2006). This discrepancy suggests looking for talent and exploring what students can do beyond the curricular expectations. In the case of the autistic prodigy in the book: "Jake had been pestering me about algebra for more than a year. Unfortunately, third-grade math meant multiplication tables and long division, not the algebra he was so desperate to learn" (p. 135).

As the book continues, the line between prodigious prowess and autism generally becomes less and less clear. The book focuses on the mother's experience raising an autistic prodigy. She also was involved with the larger community of autistic children and their parents. The message that it is important to have family downtime that does not entail therapy is clear. The need for parents to have some time set aside to engage with their children in a manner that is not promoting conformist behaviour is clearly stated. However, the book is intriguing because it is clear that the parents were underestimating their child, and there is an ongoing tension in the book about what will happen.

In terms of math education specifically, the mother describes the discovery that her son had developed a theory of some importance—the reader is given credible information substantiating what is said, but it is unclear how much depth of understanding the mother actually has. As she describes her understanding, we find some math education nuggets:

> Jake had started with a picture. Now he needed the mathematical notation to describe what he was seeing. I've come to understand that math is really a language, a way of describing what people like Jake see. He had all the rudiments already, but a concept of this complexity required a vocabulary he didn't yet have. (p. 206)

This is one simple example, but it is really a complex theme throughout the book—it is unclear or uncertain the extent of parental comprehension of the capacity of their own child. That is not to suggest any less of the parents who, as the story shows, advocated for their children. It is to say, however, that teachers may see and learn a considerable amount about the interplay of teacher and parent with regard to a single individual—the student.

The book also includes many anecdotes that may be familiar to math educators, such as the first time one memorized digits of pi. In my case, I was glad to see one of my personal pet peeves address: "Like many people with autism, Jake finds it difficult to read fiction. He says that for him, reading a made-up story is like converting a Microsoft Word file into an Excel spreadsheet" (p. 220). For me, my abhorring of fiction started in elementary school, and I have since found increasing evidence that it may have been a bias of the teachers, rather than the

individual affliction many teachers suggested.

Along similar lines are some perceptions that likely represent much more common views than we care to admit. Consider, for example, the following excerpt, which is ill informed, but will resonate with many math educators:

> Jake's story is an American story, too. Whenever you read about those terrible test scores, the article always ends in the same way. China, or India, or wherever is going to take over because their kids can do math and our kids are hopeless. Of course, that's not true, but many of the people who contacted us said that's why they were so heartened to know that Jake was out there. (p. 230)

As a whole, this book is terrific because it is readable and effective in communicating what was clearly a difficult experience for both the mother and child (and the rest of the family). It leaves the reader with questions because of the combination of autism and giftedness that confounds understanding either. The key message in the mother's words is, "I'm not suggesting every autistic child is a prodigy, or every typical child, for that matter. But if you fuel a child's innate spark, it will always point the way to far greater heights than you could ever have imagined" (p. 249). That, in a nutshell, is the strength of this book because the issues arose in a confounded manner that may reflect the common occurrence of simultaneous gifted qualities with learning disabilities. Ultimately, perhaps it is not disentangling the issues that matters, but how we go about providing meaningful education without being able to disentangle the details.

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"What's your favourite thing about mathematics?"

"Knot theory."

"Yeah, me neither."

▲ Mathematics Education Researcher Highlights

Mathematics Education Researcher: Dr. Richard Barwell, University of Ottawa

Projects: Much of Richard's research is about different aspects of language and mathematics, including extensive work on the mathematics learning of English Language Learners (ELLs) and students in bilingual, immersion, or multilingual classrooms. He is currently writing up findings from a project that compared second language learners' participation in mathematics in these different kinds of context. More generally, Richard has written about the role of language in mathematics classrooms, in mathematics, and in mathematics education research. A second area of research looks at mathematics education in relation to climate change, and environmentally sustainability, more generally. Finally, Richard is the current Editor of the international journal, *For the Learning of Mathematics*.

Mathematics Education Researcher: Dr. Daniel Jarvis, Nipissing University

Projects: Dan's research interests include instructional technology, curriculum integration, and mathematics of the workplace. He invites teachers of mathematics to visit his web/mobile resource, RE4MUL8 (**www.RE4MUL8.ca**), to view videos of rich problem-based learning lessons and reflective teacher/coordinator interviews. He is currently involved in the following collaborative research projects: co-teaching in graduate studies; mathematics and technology use in nursing education and practice (with School of Nursing); *GeoVisage*, a spatial decision support tool for northern Ontario farmers (with Geography & Computer Science); and, instructional CAS technology in undergraduate mathematics teaching (with colleagues from Brock U, U Alberta, and U Auckland. Dan is the co-author (with Dr. Irene Naested) of the book, *Exploring the Math and Art Connection: Teaching and Learning between the Lines* (Brush Education, 2012), and currently serves as the Editor of the *Ontario Mathematics Gazette*.

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