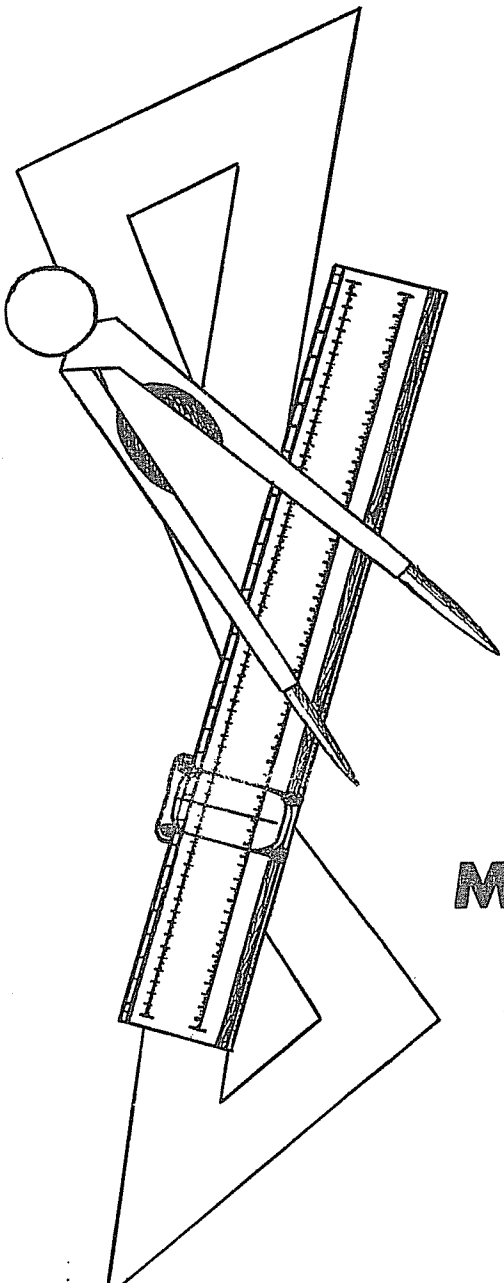


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E D I T O R I A L

It is the editorial policy of this Gazette to present a variety of views and disparate opinions on all matters more or less related to mathematics education in Ontario. For this reason, our editorials are written by different persons.

Furthermore, we encourage the submission of articles and letters defending any tenable position with respect to current, relevant issues of curriculum reform, teaching methods, etc. Our only criteria are that these submissions be well informed, well written, and in good taste. We especially encourage letters to the Editor; this is everyone's opportunity to participate even if he does not have the time or the inclination to write an article for us. This issue contains at least one article of the type referred to; we would be delighted to receive hundreds of letters of agreement or protest.

It is definitely time that the recent curriculum changes in Ontario received detailed criticism of a constructive nature. This should come from a variety of sources; in particular, it should come from those who are teaching the new courses.

For a number of years now, there has been considerable discussion among secondary school teachers concerning the status of teaching as a profession. Never before have the mathematics teachers of this province had a better opportunity to demonstrate the professional nature of their vocation. Those who have been attending refresher

courses, advanced courses, or serious seminars to prepare themselves for the many recent changes have been making the best possible demonstration of professional status.

As we write this, there is considerable controversy concerning the Metropolitan Toronto teachers' demands for salary increases of 23-25%. Might we recommend that now is the time for those who have been making the efforts and sacrifices referred to above to demand salary scales based on merit. It seems quite reasonable to suggest that beginning salaries should be a function of knowledge of subject matter, that subsequent increments should be based at least partly on increased knowledge, and that in any case increments should be contingent upon the retention of the initial level of knowledge.

* * * *

It is also the policy of this Gazette to provide all the important factual information that we can. In this connection, we have in this issue a number of "news" articles, informing you of the latest developments concerning various projects, organizations and institutions; we plan more of these for future issues. In a slightly different category, Professor Bush's excellent article will go a long way toward clarifying certain misconceptions that may have arisen from Curriculum S12.

We have some innovations in this issue.

First, we are proud to publish a paper of one of the world's foremost geometers, Prof. H.S.M. Coxeter. It is not often that an advanced research paper is suitable for publication in a journal like the Gazette, but this is one that we think you will be able to read and enjoy, and

with a bit of work you should be able to get right to the heart of it. Note how Prof. Coxeter begins with a few simple ideas and proceeds to do a number of quite fascinating things.

We have introduced two new departments in this issue: Notes from Students and Problems and Solutions. Please read them; let us know what you think of them, and help us to maintain them.

* * * *

After this issue, your editorial board will change. Those of us who are leaving extend our best wishes for success to the new editorial staff.

D.A.C.

MID-CIRCLES AND LOXODROMES

by

H.S.M. Coxeter
University of Toronto

1. INTRODUCTION

Any two distinct circles a and b (in the real inversive plane) have one or two mid-circles which may be described as the locus of the point of contact of a variable pair of tangent circles that touch a and b [7, p.31; 4, p.74]. We easily see by inversion that two non-intersecting or tangent circles have a unique mid-circle, but two intersecting circles have two orthogonal mid-circles, bisecting the angles of intersection. The circles a and b are interchanged by inversion in their mid-circle (or in either of their mid-circles). This subject has been surprisingly neglected, possibly because of the terrifying name "circles of antisimilitude" [1, p.28].

A construction for a mid-circle is described in Section 3. The next section deals with loxodromes, which are inverses of equiangular spirals. These plane curves (related to the better known spherical loxodromes by stereographic projection) give their name to the most general kind of Möbius transformation: the "loxodromic" homography. Certain positions for the centre of inversion yield centrally symmetric loxodromes. In Section 5, mid-circles are found helpful in constructing the invariant

points of the Möbius involution $(A_1B_1)(A_2B_2)$ that interchanges two given pairs of points.

2. A SUMMARY OF KNOWN RESULTS.

The ideas of orthogonality and inversion are closely related. The inverse of a point A in a circle b (not through A) is the second intersection of any two circles through A orthogonal to b ; and the circle through A orthogonal to two circles b_1 and b_2 is the circle AA_1A_2 where A_1 and A_2 are the inverses of A in these circles [3, p.80].

Any two distinct circles a and b determine a pencil of coaxal circles, consisting of all the circles orthogonal to a and b ; they determine also another pencil of coaxal circles consisting of all the circles orthogonal to two (and therefore all) members of the first pencil. Since the second pencil includes a and b themselves, we call it the pencil ab . The circles a and b may be non-intersecting, tangent, or intersecting, and then the orthogonal circles (such as c and d in Figure 1) must be intersecting, tangent, or non-intersecting, respectively. If a and b are non-intersecting, the common points of the orthogonal circles are called the limiting points of ab (or of a and b); they are interchanged by inversion in a or b .

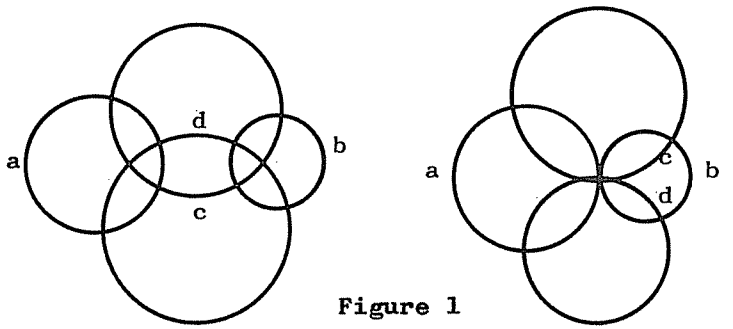


Figure 1

3. EUCLIDEAN CONSTRUCTIONS

The theory of orthogonal pencils, of inversion, and of mid-circles, belongs properly to inversive geometry, but Euclidean ideas often provide useful short-cuts. The pencil ab includes one straight line, called the radical axis of a and b , which is the line of centres of the orthogonal pencil cd ; similarly, the radical axis of c and d joins the centres of a and b . If three circles have non-collinear centres, their three radical axes have a common point, called their radical centre. This provides an easy construction (Figure 2) for the radical axis ℓ of two non-intersecting circles a and b , and thence for their limiting points, L and L' (Figure 3).

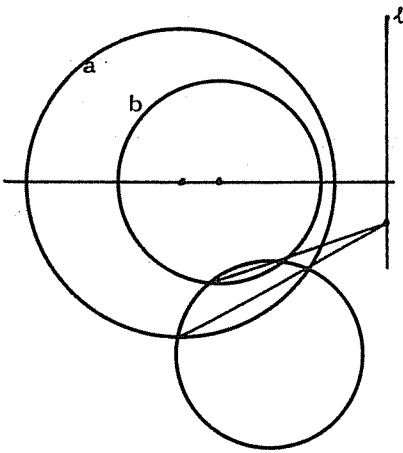


Figure 2

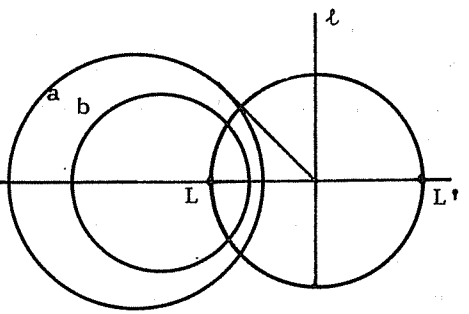


Figure 3

The mid-circle of two non-intersecting circles a and b (of different sizes) can be constructed as follows. Let their diameters on their line of centres be AA' and BB', so named that A and B' separate B and A' (as in Figure 4). Let L and L' be the limiting points of the two non-intersecting circles whose diameters are AB and $A'B'$. Then the circle m on LL' as diameter is the (unique) mid-circle of a and b. For, this circle, being orthogonal to the circles on AB and $A'B'$, inverts A into B, and A' into B'.

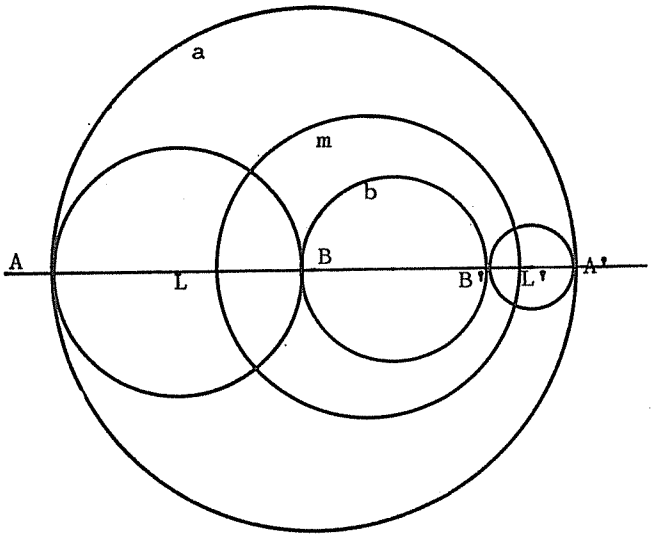


Figure 4

A construction for the mid-circle of two tangent circles can be derived by making A' and B' coincide with the point of contact, C . The circles now have diameters AC and BC , L is the inverse of C in the circle with diameter AB , and m is the circle on LC as diameter.

The problem of constructing the two mid-circles of two intersecting circles is trivial.

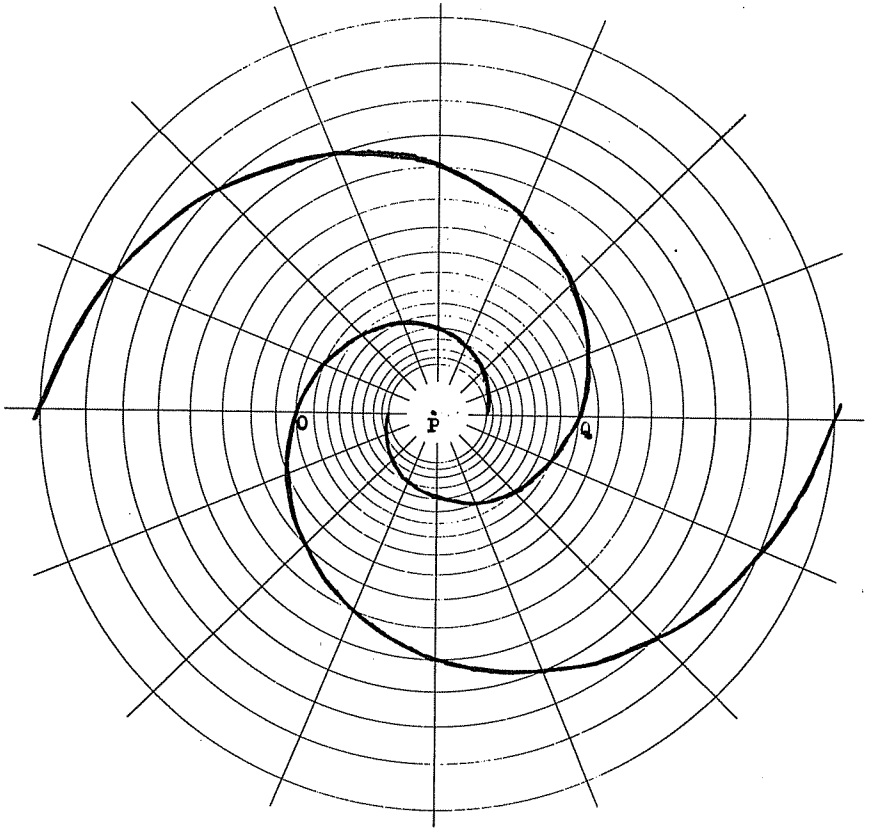


Figure 5[1] .

[1] Figures 5 and 6 were drawn by Mr. John Wilker, University of Toronto.

4. LOXODROMES

Figure 5 shows part of a sequence of concentric circles whose radii are in geometric progression, and a sequence of diameters, evenly spaced. Each circle, interchanging its two neighbours by inversion, is the mid-circle of those neighbours. Similarly, each diameter, interchanging its two neighbours by reflection, is the internal bisector of the acute angle between them. The points of intersection evidently lie on a sequence of equiangular spirals, two of which have been sketched in the figure. These two are related by the half-turn about their common pole P .

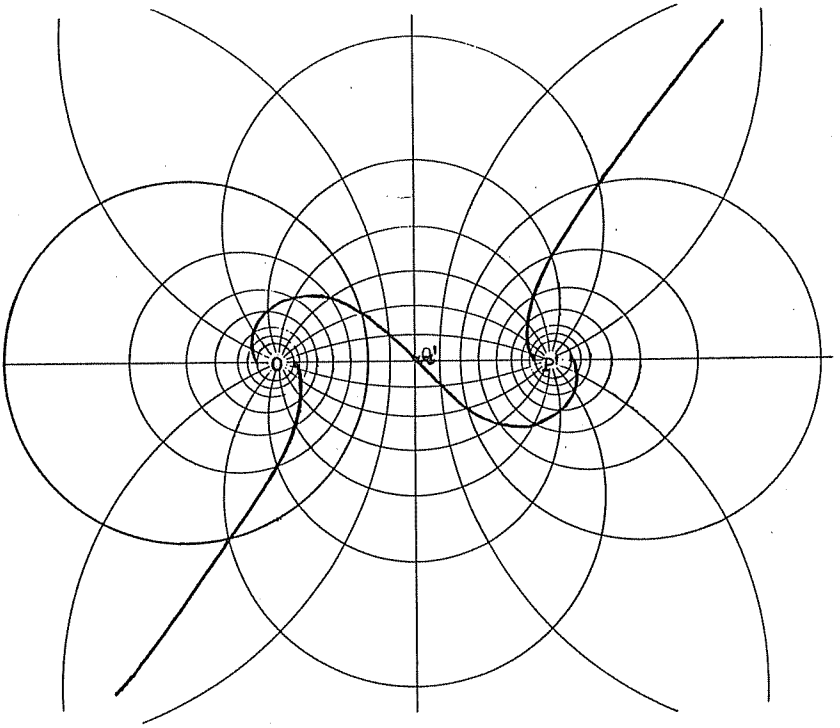


Figure 6[1] .

An arbitrary inversion [3, p.85] transforms the "spider's web" into two sequences of circles (each a mid-circle between its two neighbours) belonging to two orthogonal pencils of coaxal circles (see Figure 6). It therefore transforms the sequence of equiangular spirals into a sequence of loxodromes, two of which have been sketched in the figure. A loxodrome, being the inverse of a logarithmic spiral, cuts the circles at a constant angle. It is a "double spiral" having two poles, one of which is P', the inverse of the pole P of the equiangular spiral, while the other is the centre O of the inverting circle. In fact, the equiangular spiral itself may be regarded as a loxodrome having the point at infinity of the inversive plane as its second pole.

The two loxodromes shown in Figure 6 were selected because they are symmetrical by the half-turn that interchanges their two poles. They are the inverses of the two "opposite" equiangular spirals shown in Figure 5. The spiral that passes through the centre of inversion, O, yields the loxodrome that has an asymptote (and thus passes through the point at infinity, which is the inverse of O). The spiral that passes through Q, diametrically opposite to O (so that the pole P is the mid-point of OQ) yields the loxodrome that passes through Q', the mid-point of the segment OP' formed by the two poles of the loxodrome. (Since P' and Q' are the inverses of P and Q,

$$\frac{OP'}{OQ'} = \frac{OQ}{OP} = 2.\ )$$

The equiangular spirals (Figure 5) slide along themselves by a continuously applied dilative-rotation (or "spiral similarity"): the product of a dilatation

from P and a rotation about P [3, pp.91, 125; 8, p.67]. Analogously, the loxodromes (Figure 6) slide along themselves by a continuously applied loxodromic transformation: the product of hyperbolic and elliptic homographies with invariant points 0 and P' . Since any inversion transforms a loxodrome into another loxodrome that winds in the opposite sense, any homography transforms a loxodrome into one that winds in the same sense.

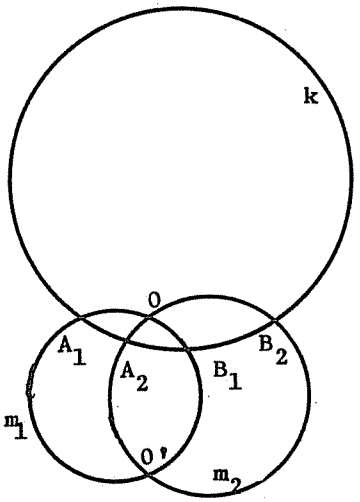


Figure 7

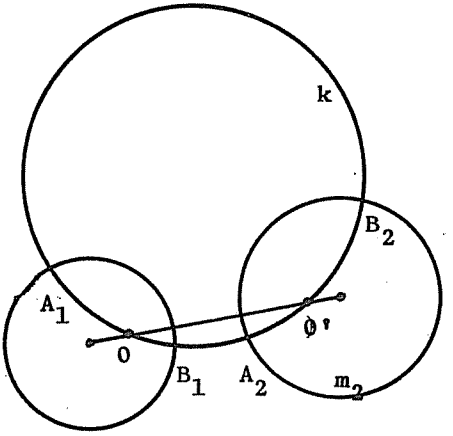


Figure 8

5. THE JACOBIAN OF TWO POINT PAIRS

S.L. Greitzer has suggested the problem of constructing a point O which is the centre of a circle inverting four given points A_1, A_2, B_1, B_2 into the four vertices of a parallelogram. Leon Bankoff has proposed an elementary solution involving angles, but the following different approach has some interesting features. (Compare [6a, p. 99].)

Suppose first that the point pairs A_1B_1 and A_2B_2 separate each other, so that every circle through A_1 and B_1 intersects every circle through A_2 and B_2 (as in Figure 7). Let m_1 and m_2 be such a pair of circles, orthogonal to the circle k on which the four points all lie [6, Section 5.1]; let O and O' be their points of intersection. Any circle with centre O (or O') will invert m_1 and m_2 into two diameters of the inverse of k, and thus will invert $A_1A_2B_1B_2$ into a rectangle whose centre is the inverse of O' (or of O).

Suppose next that the point pairs do not separate each other, but still lie on a circle k (as in Figure 8). The circles m_1 and m_2, orthogonal to k, are now non-intersecting. Let the line joining their centres meet k in O and O'. Any circle with centre O (or O') will invert m_1 and m_2 into concentric circles, k into a diameter, and $A_1A_2B_1B_2$ into a degenerate parallelogram.

The only remaining possibility is that A_1, A_2, B_1, B_2 are non-concyclic (as in Figure 9). Let a and b denote the circles $A_1B_1A_2$ and $A_1B_1B_2$, and let m be one of their two mid-circles, namely the one that separates A_2 from B_2. Let a' and b' be the circles through A_2 and B_2 (respectively) orthogonal to a, m, and b. Let n be the mid-circle of a' and b' (or, if they coincide, let n coincide with

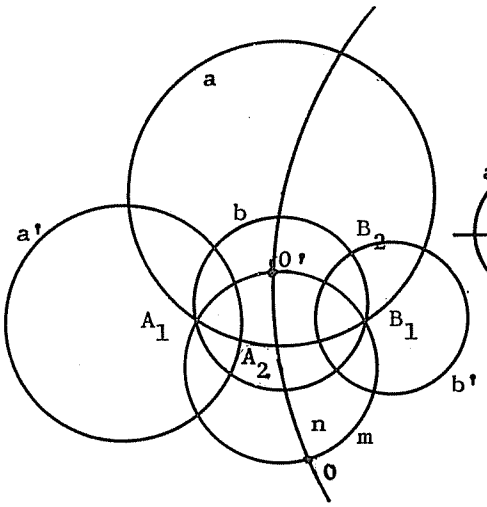


Figure 9

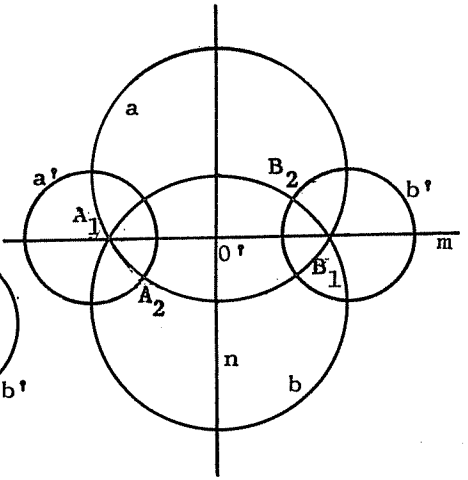


Figure 10

them). Finally, let 0 and 0' be the common points of the two orthogonal circles m and n .

Inversion in a circle with centre 0 transforms Figure 9 into Figure 10, where, for simplicity, the symbols for the points and circles (or lines) have been repeated without embellishment. Now m and n are perpendicular straight lines through 0' ; a and b are congruent by reflection in m ; a' and b' are congruent by reflection in n ; A_2 is one of the common points of a and a' ; B_2 is one of the common points of b and b' , namely the one on the other side of m (so that A_2 and B_2 lie in opposite "quadrants"). The

whole figure is symmetrical by the half-turn about $0'$. Therefore $A_1A_2B_1B_2$ is a parallelogram.

Returning to Figure 9, we see that the product of inversions in the two orthogonal circles m and n is the Möbius involution $(A_1B_1)(A_2B_2)$ which interchanges these pairs of points [5, p.233]. If we represent the points of the inversive plane by complex numbers z, we can describe the point-pair $A_\lambda B_\lambda$ ($\lambda = 1$ or 2) as the pair of roots of a quadratic equation

$$z^2 - 2p_\lambda z + q_\lambda = 0 .$$

Since the condition for three such paint pairs to belong an involution is

$$\begin{vmatrix} q_1 & p_1 & 1 \\ q_2 & p_2 & 1 \\ q_3 & p_3 & 1 \end{vmatrix} = 0$$

[2, p.181], the invariant points $00'$ of the involution $(A_1B_1)(A_2B_2)$ are the roots of the equation

$$\begin{vmatrix} q_1 & p_1 & 1 \\ q_2 & p_2 & 1 \\ z^2 & z & 1 \end{vmatrix} = 0 \qquad \text{or} \qquad \begin{vmatrix} z-p_1 & p_1z-q_1 \\ z-p_2 & p_2z-q_2 \end{vmatrix} = 0 .$$

Writing u/v for z, we recognize

$$-4 \begin{vmatrix} u-p_1v & p_1u - q_1v \\ u-p_2v & p_2u - q_2v \end{vmatrix}$$

as the Jacobian of the two quadratic forms

$u^2-2p_1uv + q_1v^2$ and $u^2-2p_2uv + q_2v^2$.

Accordingly, the point pair $00'$ can reasonably be called the Jacobian of the two point pairs A_1B_1 and A_2B_2 .

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ON THE TERMINOLOGY OF FUNCTIONS

by

George C. Bush

Queen's University, Kingston

The study of functions in Mathematics A of the Grade 13 course has led to some confusion for teachers who detect inconsistencies between the text they are using and other books they have read. This confusion seems to be concentrated in three areas:

(a) The meaning of the words "domain" and "range" (and "codomain").

(b) The conditions under which a relation is called a function.

(c) The composition of functions.

It is impossible to remove this confusion by simply stating what is the correct view or terminology, because leading mathematicians are not agreed on these points. The concept of function appears in all branches of mathematics and a variety of terinology has arisen to emphasize the particular aspect that is important in each application. We should not even claim that one approach is better than the others. Each has its advantages and the choice is largely a matter of preference and style. The best we can do is to make a summary of what various authors have done and attempt to emphasize the essential points that are being communicated in various ways.

In a note such as this we cannot undertake a very extensive survey of the mathematical literature. Since we are especially concerned with the Grade 13 course we shall consider three texts in use at this level in Ontario.

Since each of these have rather lengthy lists of authors we shall refer to them by the publishers; Gage (5), Holt (6), and Ryerson (3). From the many introductory university texts we have chosen three; Allendoerfer and Oakley (1) because it is widely used, Fine (7) which we chose more or less at random, Bush and Obreanu (4), which we chose out of personal prejudice. As authoritative treatises we accept Kelley (9) and Halmos (8). Finally we shall look at the French work that is published under the pseudonym of Bourbaki (2). This work has already played an important role in standardizing terminology in many branches of mathematics and may point the way to an eventual solution of the present problem.

1. Domain and Range

In most of the books referred to above the study of functions is preceeded by a study of relations. We begin with two sets A and B and their Cartesian product

$$A \times B = \{(x,y) | x \in A;\ y \in B\}$$

Any subset of $A \times B$ is called a relation. If we denote a relation by G we can also define sets F and S such that

$$F = \{x | (\exists y)(x,y) \in G\}$$

$$S = \{y | (\exists x)(x,y) \in G\}$$

That is, F is the set of all first components of the ordered pairs in G , and S is the set of all second components. The following subset relations among these sets are obvious:

$$G \subseteq A \times B$$
$$F \subseteq A$$
$$S \subseteq B$$

We can summarize the terminology used by the various authors in the following table

Text	G	A	B	F	S
Gage	Relation	Domain of x	Domain of y	Domain of G	Range of G
Holt	Relation	_____	_____	Domain of G	Range of G
Bush and Obreanu	Graph of relation	Domain	Codomain	_____	_____
Fine	Relation	_____	_____	Domain of G	Range of G
Halmos	Relation	_____	_____	Domain of G	Range of G
Kelley	Relation	_____	_____	Domain of G	Range of G
Bourbaki	Graphe d'un correspondence	Ensemble de déport	Ensemble d'arrivée	Ensemble de Définition	Ensemble des valeurs

Ryerson, and Allendoerfer and Oakley do not base their definition of function on the concept of relation and so they do not appear in this table.

The important thing to observe here is the role played by the various sets. This is far more important than the particular names attached to them. When we turn to a consideration of functions rather than relations we shall see that some of the discrepancies of terminology are less acute.

II. What is a Function?

All of the authors we have cited as defining a function as a relation with certain special properties are agreed that in order for a relation to be a function it is necessary that distinct ordered pairs in the relation must have distinct first components. In terms of

our previous notation this means that for each $x \in F$ there is one and only one $y \in S$ such that (x,y) is in G.

But two of the references, Bush and Obreanu and Bourbaki require in addition that each element of A appear as a first component of some ordered pair in G, that is, that $F = A$. This leads to the rather unsatisfactory situation in which the answer to the question of whether a certain relation is a function will depend on the text in which the question appears.

We must also consider the two references in which functions are introduced directly without making use of relations. Both of these, Allendoerfer and Oakley, and Ryerson, give two definitions of functions, one in terms of correspondence and the other in terms of ordered pairs. If we attempt to reconcile these definitions we find that a function can be thought of as a collection of ordered pairs such that each first component appears with exactly one second component. The set of first components is called the domain and the set of second components the range.

In the face of this variety of terminology what is meant by the statement "f is a function from U to V"? In each of our references (but not in all mathematics books) it is correct to think of f or its graph as a collection of ordered pairs, a subset of $U \times V$ such that for each $x \in U$ there is exactly one $y \in V$ such that (x,y) is in the subset f. In each of our English language references the set U is called the domain of f and the set of second components $\{x \mid (\exists x)(x,y) \in f\}$ is called the range of f. We can also say that the function f associates with each element of the domain U one and only one element of V.

Out of a variety of approaches we have found a rather surprising unity of the concept of a function

in the form in which it is usually encountered.

We still have no unanimity, however, in the name to be used for the set V. It is called the counter-domain (Ryerson), codomain (Bush and Obreanu), ensemble d'arrivée (Bourbaki) and the domain of y (Gage). The other authors do not find it necessary to have a name for this set.

There is one other piece of terminology that is worth mentioning. The idea of a mapping and the idea of a function as a collection of ordered pairs may suggest different things intuitively, but the vast majority of mathematicians use the words "function" and "map" or "mapping" interchangeably.

III. Composition of Functions

The outline of the Mathematics A course does not mention the composition of functions but the concept appears in the unit on groups in Mathematics B. The Gage and Ryerson texts have included the material in the Mathematics A text and have handled it in different ways. In view of this it seems appropriate to include composition among the areas of confusion that we are attempting to clarify.

Let us consider a function f that maps A into B and a function g that maps C into D. Under what conditions can we form a composite function $g \circ f$ such that $(g \circ f)(x) = g(f(x))$? The Holt text does not deal with composition. All our other references agree that if $B = C$ then the composite $g \circ f$ is defined and that it is a function from A into D. This case is illustrated in Figure 1. Kelley and the Gage text also consider a more general case, but for the rest the composition is considered only in this case where $B = C$.

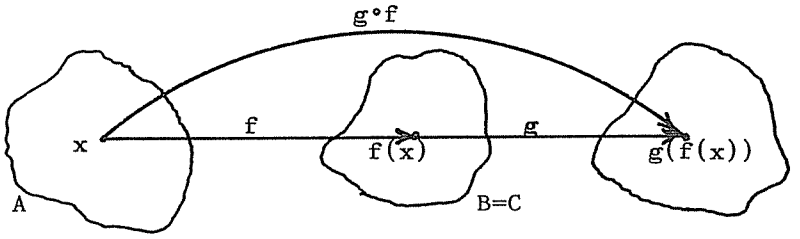


Figure 1

Kelley and Gage put no restrictions on the sets A,B,C,D . They define $g \circ f$ as the set of ordered pairs $(x, g(f(x)))$ where $x \in A$ and $f(x) \in C$. It is not difficult to show that for each such x we have a unique value $g(f(x))$ and so $g \circ f$ satisfies the only condition that these two texts impose in order for it to be called a function. This case is illustrated in figure 2. The domain is a subset of A but need not be all of A . In fact, if $B \cap C = \emptyset$ then the domain of $g \circ f$ is $\emptyset$.

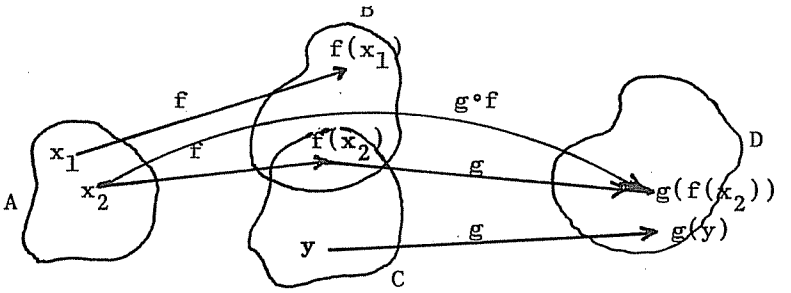


Figure 2

Whether or not a certain composite function is defined may depend on the particular book we are reading. But, as in the other cases we have considered, the essential ideas of composition will be clear in the use that is made of the concept in various branches of mathematics.

In summary we repeat that in all this diversity of approach to the study of functions it is impossible to say that one way is right and another way wrong. The choice is a matter of personal preference and may be influenced by the use the author intends to make of functions later in the book or in more advanced work.

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The following are alternative or improved solutions for a few of the problems, supplied by Professor P.B. Chapman, University of Toronto.

* * * * * *

Q3. Find a polynomial equation of degree eight, whose leading coefficient is unity, whose other coefficients are integers, and which has a root $(\sqrt{2} + \sqrt{3} + \sqrt{5})$.

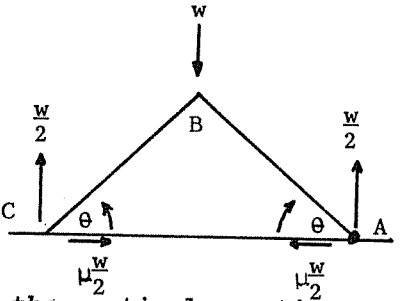
The polynomial is obtained by making all possible choices of sign in the expression

$$(x \pm \sqrt{2} \pm \sqrt{3} \pm \sqrt{5})$$

and forming the product $P(x)$ of the resulting eight factors. It is necessary to prove that the coefficients have the required properties. This can be done as follows.

$$\begin{aligned}
P(x) &= (x^2-(\sqrt{5}+\sqrt{3}+\sqrt{2})^2)(x^2-(\sqrt{5}-\sqrt{3}+\sqrt{2})^2)(x^2-(\sqrt{5}+\sqrt{3}-\sqrt{2})^2) \\
&\quad (x^2-(\sqrt{3}+\sqrt{2}-\sqrt{5})^2) \\
&= [(x^2-10)-2(\sqrt{15}+\sqrt{6}+\sqrt{10})][(x^2-10)-2(\sqrt{10}-\sqrt{15}-\sqrt{6})] \\
&\quad [(x^2-10)-2(\sqrt{15}-\sqrt{6}-\sqrt{10})][(x^2-10)-2(\sqrt{6}-\sqrt{10}-\sqrt{15})] \\
&= [(x^2-10)^2-4\sqrt{10}(x^2-10)+4(10-(\sqrt{15}+\sqrt{6})^2)] \\
&\quad [(x^2-10)^2+4\sqrt{10}(x^2-10)+4(10-(\sqrt{6}-\sqrt{15})^2)] \\
&= [(x^2-10)^2-4\sqrt{10}(x^2-10)-4(11+2\sqrt{90})] \\
&\quad [(x^2-10)^2+4\sqrt{10}(x^2-10)-4(11-2\sqrt{90})] \\
&= [(x^2-10^2)^2-4\sqrt{10}(x^2-10)-4(11+2\sqrt{90})] \\
&\quad [(x^2-10^2)^2+4\sqrt{10}(x^2-10)-4(11-2\sqrt{90})] \\
&= [\{(x^2-10^2)^2-44\}-\{4\sqrt{10}(x^2-10)+8\sqrt{90}\}] \\
&\quad [\{(x^2-10^2)^2-44\}+\{4\sqrt{10}(x^2-10)+8\sqrt{90}\}] \\
&= \{(x^2-10^2)^2-44\}^2 - \{4\sqrt{10}[(x^2-10)+6]\}^2
\end{aligned}$$

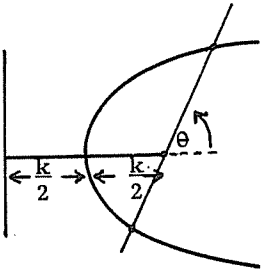
Q4. Two rigid rods AB, BC of negligible mass, each of length ℓ, are joined at B with a frictionless hinge. The end A of one of them is hinged at a point on a fixed horizontal table, the axis of this second frictionless hinge being in the plane of the table, parallel to the axis of the hinge at B. The end C rests on the table, and the coefficient of friction between the rod and the table is μ. If a vertical force W is applied downward at B, find the angle θ between a rod and the table at which slipping just occurs.



By symmetry, the vertical reaction at each point of contact with the table must be $\frac{W}{2}$, all θ. When the free rod is about to slip the static friction law gives the horizontal force at C to be $\mu \frac{W}{2}$. By symmetry, this is also the horizontal component of the reaction at A. Since the resultant reaction at either A or C must coincide in direction with the direction of the rod at that point

$$\tan \theta = \frac{\frac{W}{2}}{\frac{\mu W}{2}} = \frac{1}{\mu}$$

Q5. Suppose the distance between the focus and the directrix of a parabola is k. Show that the length of a chord which passes through the focus making an angle θ with the axis of the parabola is $2k \operatorname{cosec}^2 \theta$.



The polar equation of the conic is

$$r \sin^2 \theta/2 = k/2 \ .$$

Let the points of intersection of any focal chord with the parabola be

$$r_1, r_2 \qquad \theta, \theta + \pi \ .$$

The length of the focal chord is

$$r_1 + r_2 = \frac{K}{2 \sin^2 (\theta/2)} + \frac{K}{2 \sin^2 (\frac{\theta}{2} + \frac{\pi}{2})}$$

$$= \frac{K}{2 \sin^2 (\theta/2)} + \frac{K}{2 \cos^2 (\theta/2)}$$

$$= \frac{K(\sin^2 (\theta/2) + \cos^2 (\theta/2))}{2 \sin^2 (\theta/2) \cos^2 (\theta/2)}$$

$$= 2K \operatorname{cosec}^2 \theta \ .$$

Q10. (a) Verify that if the cubic equation

$$x^3 + px^2 + qx + r = 0$$

has a pair of equal roots at $x = \beta$, then $3\beta^2 + 2p\beta + q = 0$.

(b) Find two equations which express the condition that the line $y = kx + b$ is a tangent at a point (x,y) to the curve

$$x^3 + y^3 - 2xy = 0 \ .$$

(c) What is the equation of the asymptote to the curve

$$x^3 + y^3 - 2xy = 0 \ ?$$

(a) If the equation

$$f(x) = x^3 + px^2 + qx + r = 0$$

has a pair of equal roots at $x = \beta$ then the graph $y = f(x)$ has a horizontal tangent at $x = \beta$. Therefore

$$\frac{d}{dx}f(x)\Big|'_{x=\beta} = 3\beta^2 + 2p\beta + q = 0 \ .$$

(b) If the given line $y = kx + b$ is a tangent to the curve

$$x^3 + y^3 - 2xy = 0$$

then the equation

$$x^3 + (kx+b)^3 - 2x(kx+b) = 0$$

has a pair of equal roots at some point X say. Thus the two equations to be simultaneously satisfied are

$$(1+k^3)X^3+(3k^2b-2k)X^2+(3kb^2-2b)X+b^3 = 0 \qquad (1)$$

and, from part (a),

$$3X^2 + \frac{2(3k^2b-2k)X}{(1+k^3)} + \frac{3kb^2-2b}{(1+k^3)} = 0 \ .$$

(c) If the curve $x^3 + y^3 - 2xy = 0$ has an asymptote, the two equations in (b) must be satisfied for $x \to \infty$, and therefore the coefficients of X^n , $(n = 1,2,3)$ must be zero in (b)(1)

$$\therefore \quad k^3 + 1 = 0 \quad \text{or} \quad k = -1$$
$$b = -2/3 \ .$$

Q14. Show that all points on the graph of the function

$$y = 6x^5 - 20x^3 + x$$

at which the second derivative of y with respect to x vanishes are collinear.

The function $f(x) = 6x^5 - 20x^3 + x$ is an odd function of x (i.e. $f(-x) = -f(x)$), and therefore symmetric in 0. Its second derivative

$$f''(x) = 120\,x^3 - 120$$

is zero at points $x = 0, \pm 1$. A straight line must pass through $(0, f(0)), (1, f(1)), (-1, f(-1))$, because the function is odd.

THE O.A.T.M. EASTER MEETING

The annual Easter meeting of the Ontario Association of Teachers of Mathematics will be held on March 27, 28, and 29, 1967. The place of meeting will again be the INN-ON-THE-PARK, Toronto, where we were so hospitably received last year. This year's programme promises to be every bit as good as last year's, so make your plans <u>now</u> to attend.

Our two key speakers will be Miss Edith Biggs of England, and Dr. Sidney Lindstedt of Calgary, Alberta. Miss Biggs is Mathematics Consultant for the Schools Council of Great Britain, and has achieved great fame in the British Isles as an organizer of in-service training sessions for teachers. She brings a wealth of experience at both secondary and elementary levels, and should have a great many ideas for all convention visitors. Her primary interest is in changing the approach to mathematics, not the content.

Dr. Lindstedt is Head of the Department of Curriculum and Instruction at the University of Calgary. He is also a member of the provincial subcommittees for elementary and secondary mathematics curriculum development. We are sure that he also will have much to contribute to our meetings.

Kicking off the gala affair will be the traditional Early Bird Reception, hosted by those two partners in the preservation and/or development of mathematics, Dr. K. Fryer and Dr. J. Coleman. Other features will include a demonstration lesson by Dr. W. Sawyer, lectures by Dr. K. May and Dr. A. Elliott and a series of lectures, seminars and discussions by other prominent persons. Also on the programme is a tour of the IBM plant, an

IBM-sponsored luncheon, our usual Tuesday evening banquet, and the annual display of winning entries from the Math Fair. Clearly this is the most ambitious programme yet.

Also on the agenda, of course, are activities for wives, including a shopping tour. Other details are not yet finalized, but there will be good times for the ladies! So make your plans now to attend the greatest convention the O.A.T.M. has ever had - and bring your wife along!

PROGRAMME (subject to change)

MONDAY, MARCH 27

P.M.	Transportation available from INN-ON-THE-PARK to general O.E.A. meetings.
5:00	Exhibits and Math Fair to be set up.
7:00	Registration
8:00	Early Bird Reception - Hosts: Dr. Coleman, Dr. Fryer.

TUESDAY, MARCH 28

8:00	Registration
9:00	Opening Session
9:15	Miss Biggs
10:15	Exhibits and Math Fair
10:45	Lectures, Seminars, and Discussions
12:00	Luncheon, sponsored by I.B.M.
	Chairman, Dr. Fryer
1:45	a) Dr. K.O. May - topic to be announced
	b) Dr. H.A. Elliott - Geometry: K to 13
	c) Dr. W.W. Sawyer - Demonstration lesson
2:45	Exhibits and Math Fair
3:15	Dr. Lindstedt
7:00	Banquet - Chairman, Mr. G. Kaye

WEDNESDAY, MARCH 29

9:00	Business Meeting
9:15	Dr. Lindstedt
10:15	Exhibits and Math Fair
10:45	Lectures, Seminars, Discussion
12:00	Luncheon - Chairman, Dr. Coleman
1:45	Miss Biggs
2:45	Presentation of prizes and awards
3:00	Tour of I.B.M. plant

Lectures, Seminars, and Discussion.

1. Elementary Geometry - Mr. J. McLean
2. Computers - Mr. D. Alexander
3. A Grade 13 Topic - Rev. J. Egsgard
4. Graphs and Relations in Elementary Schools - Mr. E. Pipher
5. Ont. Math. Commission Experimental Grade 10 Course for 4-year Students - Mr. R. Dunkley.
6. Math Labs in Secondary Schools - Mr. N. Sharp
7. Unit 1 from the New 5-year Grade 10 Course - Mr. D. Mumford
8. A Topic from Grade 11 or 12 - 5 -year Program, - Mr. J. McKnight
9. Discussion of Preceding Lecture (with speaker as a resource person) - Mr. D. Campbell

THE MATHEMATICS DEPARTMENT

AT

THE COLLEGE OF EDUCATION, UNIVERSITY OF TORONTO

by

D.L. Mumford

A new agreement concerning the relationship of the College with the University of Toronto and the Department of Education has been completed. Among other things this has resulted in

(i) changing the name of the college to The College of Education, University of Toronto

(ii) a much closer relation between the College and the University.

The desire for a closer relationship with the University Department of Mathematics led, in July 1965, to the appointment of Professor W.W. Sawyer, jointly to the College Mathematics Staff and to the University Department of Mathematics.

A second such appointment was made in July 1966 when Dr. K.O. May joined the two staffs. Dr. May received his degrees from University of California at Berkeley, having studied also at the University of London and the University of Paris. His most recent appointment was Chairman of the Department of Mathematics of Carleton College, Minnesota. He has just completed a two year period as a visiting scholar at the University of California. He has

been associated with most phases of the development of new mathematical programs in the United States and has taken part in Institutes for high school teachers and has been a member of N.C.T.M. and S.M.S.G. committees on new programs and teacher training. He has written many books and articles on a variety of topics in the field of mathematics.

The close association with the Department of Mathematics makes it possible for the type B students in accordance with their mathematical background to take, as an adjunct to their professional training, either a Survey Course in Mathematics, given by Dr. May, or an Introduction to Higher Mathematics, given by Professor Sawyer. Each of these courses is of immediate value to the students and each is a three-credit course toward certificate endorsement.

The Department of Mathematics of the University is in the process of establishing a Master of Arts for Teachers (M.A.T.) in mathematics specifically designed for persons in the teaching profession or in their professional training year at the College of Education. Professor Sawyer is presently giving one of the courses proposed for this degree on Saturday mornings at the College.

Because of the new integration of mathematics programs from grades 1 to 13, the Department has assumed responsibility for the mathematics training of students taking the Elementary School option at the College. There are 70 students in this group.

In the school year 1965-1966 the Mathematics Department of the College, under the sponsorship of Professor W.W. Sawyer, established the Metropolitan Mathematics Society which meets six times a year. Each high school in the metropolitan area is invited to send to the meeting four of their senior students

who are particularly able and interested in mathematics. Professor Sawyer conducted the sessions which were attended by 200 to 250 students. Similar meetings have been arranged for this year.

Part of our responsibility is contributing to the in-service training of teachers and trying new ideas in methods and curriculum.

Mr. D. Alexander taught an experimental course in computer training in 1965-66 at the University Schools. This course was an integral part of an S.M.S.G. experimental program.

Last year Professor Sawyer conducted an eight-session course in calculus attended by about 200 teachers.

This year by arrangement with the Toronto Board of Education and the Principal of Bloor Collegiate, Toronto, the College of Education Mathematics Staff is collaborating with the Director of Mathematics for the Toronto Board of Education and the Mathematics Staff of Bloor Collegiate in the trial of class room methods designed to obtain maximum student participation in the discovery and development of mathematical concepts. The methods include individual, small group, and large group participation.

Arrangements can be made to observe this work in progress by addressing requests to the Mathematics Department, The College of Education, or to the Director of Mathematics, Toronto Board of Education.

On an individual basis, the members of the Mathematics Staff have spoken to various groups and assisted in in-service programs arranged by groups outside the College. This is a continuing practice.

COMPUTER COURSES IN SECONDARY SCHOOLS

by

D.W. Alexander
University of Toronto Schools

There are several areas of study in the secondary schools which may be concerned in some way with computers.

1) In the technical stream students can study circuitry, testing, and repair.
2) In the commercial stream students can study the application of computer methods to business problems (data processing).
3) In the academic stream students can study computer science as a separate subject or can make use of computer methods to solve problems arising in present academic classes.

This paper is concerned with a course which makes use of computer methods to further the student's knowledge of mathematics. Two possible techniques can be used. The emphasis can be placed on teaching a computer language with little training on the basic techniques of problem solving which are adaptable to computer implementation, or the emphasis can be placed on the methods of problem solving thus dealing only incidentally with the computer language. The first type of course has a place in the education of students who are interested in acquiring a skill which has vocational value. The second type of course is one that provides a skill but also has further academic value in that it gives experience in a different type of problem solving technique.

An experimental text for such a methods oriented course was used in a grade 12 class at University of Toronto Schools during the 1965-66 school year. The text was the preliminary edition of Algorithms, Computation, and Mathematics by the School Mathematics Study Group. The text is composed of three separate but related sections: a main text, a FORTRAN supplement, and an ALGOL supplement. This is an attempt to influence commercial texts to provide a similar format so that new languages can be included in a basic text without a complete revision being necessary. The class involved used ALGOL since they were using a Burrough's installation (B5500) made available through the generosity of Burrough's Business Machines Ltd. and K.C.S. (a Toronto firm of consultants in operations research and computer applications). Much of the key punching for the students was done at the Ontario Institute of Studies in Education.

The course is based on the study of algorithms.

An algorithm is defined as any unambiguous plan for carrying out some process in a finite number of steps. This is a familiar concept illustrated by:

1) a recipe for a cake;
2) instructions for assembling a "prefab" cottage;
3) the procedure for subtracting a polynomial;
4) the formula for the square of a binomial;
5) the formal method for determining the square root of a number;
6) Newton's method for determining the square root of a number;
7) the procedure for obtaining the product of two matrices.

A useful algorithm has certain special characteristics. Besides 1) being unambiguous; and 2) having a finite number of steps; it should 3) be applicable to a set of problems rather than an individual problem; and if it is to be suitable for implementing on a computer it should 4) involve a set of repetitive steps.

A clear method for describing an algorithm is a FLOW CHART as illustrated below.

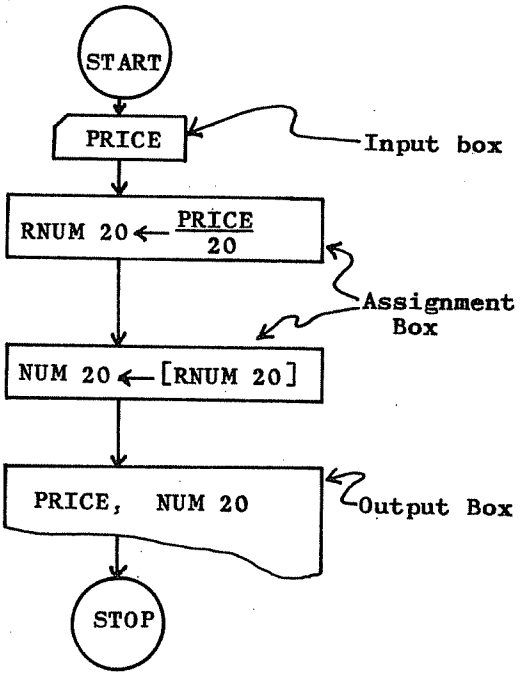
Example 1. A farmer sells a cow for $75.75. Assuming that he receives this amount in 20 dollar bills, as far as possible, calculate the number of 20 dollar bills received.

Solution. Number of 20 dollar bills

$$= \text{Integral part of } \frac{75.75}{20}$$

$$= \left[\frac{75.75}{20}\right]$$

Definition: Greatest integer function is the function such that $x \to [x]$, $x \in R$, where $[x]$ is the greatest integer which does not exceed x.



The price of the cow is read in

The variable RNUM20 is assigned the value of $\frac{PRICE}{20}$

The variable NUM 20 is assigned the value of [RNUM 20]

The value of PRICE and NUM 20 is printed.

The example illustrates the use of boxes of different shapes to represent different types of instructions in a flow chart and also the symbol ← as a symbol used

when assigning a particular value to a variable.

From the flow chart above, with some instruction on the computer language being used, it is possible for the students to make up a program for running on a computer.

A major feature of the examples in the text is the way in which one problem is expanded upon in exercises from chapter to chapter. An illustration of this is indicated by the following example.

Example 2. A carnival wheel has 32 sections numbered consecutively in a clockwise direction 0, 1,2,...,31 . The sections are coloured blue, green, red, and yellow consecutively. The wheel is spun in a counter clockwise direction and points are awarded by the colour of the section opposite the pointer when the wheel stops according to the following table.

Section colour	Blue	Green	Red	Yellow
Points	-30	-10	10	30

Make up an algorithm to simulate this game.

Hint: Use S as original sector number
M as number of sectors spun through
K as a member of $\{0,1,2,3,\}$
where $0 \Leftrightarrow$ Blue, $1 \Leftrightarrow$ Green,
$2 \Leftrightarrow$ Red, $3 \Leftrightarrow$ Yellow.
P as the number of points.

Discussion. This problem requires several days of study and discussion to arrive at a satisfactory algorithm. A variety of interesting topics are introduced in the discussion: modular arithmetic; curve fitting; expectation.

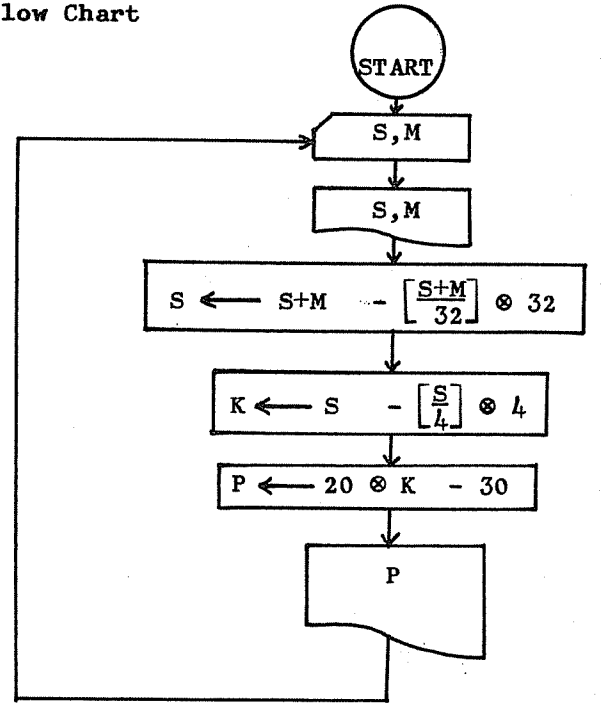
A study of particular values of S and M indicates that S_1 the sector number at the completion of the spin is the remainder after obtaining the integral quotient of $\frac{S + M}{32}$.

This may be represented symbolically by $S_1 = S + M - \left[\frac{S+M}{32}\right] \times 32$. It is clear that the value of k is obtained by a similar process where 4 is the divisor. Thus

$$K = S_1 - \left[\frac{S_1}{4}\right] \times 4 .$$

To assign point values corresponding to a specific value of K it is necessary to express P as a function of K (if a condition box is not to be used). A study of the graph of $\{(K,P)\}$ suggests that the function is linear. It follows that $P = 20 \times K - 30$.

Flow Chart



This simulation of the game provides a loop so that successive "spins" may be considered. Obvious extensions of this problem are dealt with as the

students learn more programming techniques:

1) Point values are introduced which are not determined by a simple function. This requires the introduction of condition boxes.
2) Point values are changed for different sets of spins. This requires the introduction of subscripted variables.
3) A cumulative total of points scored on successive spins is kept. This requires the introduction of a loop or iteration box.
4) A count is kept of the number of spins required to reach a certain value for the total points. This requires an iteration box.

The extensions of the problem open various topics of discussion and provide for experience in the concept of the probability of an event. The students enjoyed this simulation type of problem and went far beyond the required assignments in building up better simulations of the game.

For further details of the course see Algorithms, Computation, and Mathematics, S.M.S.G., (McGill University Press, Montreal, Quebec).

In Ontario action is already being taken at the Departmental level on the question of how the curriculum should be modified to recognize the impact of computers on society.

1) A revision of the business and commerce curriculum has modernized this course introducing a study of data processing.
2) A committee is producing a computer language for high schools.
3) A committee of the Curriculum Division has been set up "To Study the Impact of Computer Science on the Curriculum".

This committee has (1) prepared guidlines for the type of computer suitable for use in Secondary Schools; (2) entered into a study of experimental programs being carried on in the U.S. and Canada; (3) sponsored an experimental use of the S.M.S.G. material in 14 schools.

It should be noted that this experimental course is being conducted in 14 schools which were selected by the committee. If further experimentation is required all schools will be given the opportunity to participate. Requests to join the experiment should be witheld until such a general announcement is made.

From our own experience with this course we feel that for many students such a course can give an added dimension to their study of mathematics. It would be a mistake to say that all students will enjoy such a course. Even some students who enjoy mathematics find the attention to detail demanded in this computer work to be discouraging.

Any teacher who contemplates introducing a course in computer methods should consider:

1) What computer facility will be used?

To avoid long turn around times it is essential that the installation be powerful enough for the problems you wish to do. A small installation will limit the work to such simple problems that the students will feel that the additional work involved in programming is not worth the trouble. It is doubtful if any computer which does not use punched card input and line printer output will be versatile enough for any more than an introduction to the techniques of programming. If it is expected that the students should have a chance to actually work out problems of some depth using computer methods then a fairly powerful installation is essential. It is not essential that students see every one of their programs processed. We found that having programs run overnight was a very feasible method.

2) What language will be used?

This will be determined by the installation used. Some variation of ALGOL has the advantage over present forms of FORTRAN that the position of statements on the cards is not important. This allows corrections to be made fairly easily, an important consideration when dealing with students who are not adept at key punching.

3) What special procedures must be followed to run a program on the particular installation used?

Different installations have different procedures. It is essential that the teacher work with the installation chosen so that he can determine for himself the special procedures that must be taught the students in addition to the text material.

4) What method will be used for key punching cards? There are 4 possibilities.

1. have the students key punch their own cards
2. have the students use mark sensing equipment to produce their own cards
3. have the key punching done commercially
4. have students receiving special training as typists or key punch operators do the key punching.

A student who does not know how to type can waste hours trying to key punch his programs. Mark sensing equipment does not appear to be perfected to the stage where it can process the cards for a class of students (or a school) without breaking down just when it is needed most. Commercial key punching adds an extra cost. The best approach would seem to be to work out a cooperative effort between students programming and students learning typing or key punching so that most of the key punching would be done by those students who will actually need this skill. The working out of the problems involved in such a team effort would be an excellent experience for the students.

The curriculum of the secondary schools is already feeling the effect of the influence of computers. It is important that we as teachers of mathematics be aware of how this may affect future courses and how we can best use the computer to further our two main goals of teaching students and teaching mathematics.

THE ONTARIO MATHEMATICS COMMISSION

EXPERIMENTAL GRADE 10 PROGRAM

by

R. Dunkley
St. Mary's District C.V.I.

This past September saw the introduction of a new experiment in twelve schools scattered throughout the province to test a mathematics course designed for all three branches of the four year course at the grade 10 level.

The course was initiated by the Ontario Mathematics Commission as a result of the 1965 summer study group which produced the booklet "Mathematics, The Four Year Programme". The O.M.C., using funds obtained from O.I.S.E. commissioned a team of five writers to produce text material, based on the findings of this summer study, which would be applicable to all students in four year courses in grade 10. Those forming the writing team were Mrs. Mary St. Louis, Garson-Falconbridge High School, David Campbell and George Lawson, Althouse College of Education, Gordon Ullerick, Wheable Secondary School, London, and Ronald Dunkley, St. Mary's District C.V.I.

The aims of the experiment are

1. to produce text material involving core plus options to determine its suitability in depth, method of presentation, and whether the topics chosen are good ones for this type of student;
2. to help prepare a core of teachers ready to use the new Department of Education course of study.

For example, the business mathematics section was written so that at least part of it could be studied by

technical students, on the argument that many of these students need an understanding of basic business mathematics principles. Two chapters of the material were prepared in semi-programmed form to test the suitability of this type of material for four year students.

The course, in many ways, is similar to the Department of Education course of study, now under preparation, but reflects the findings of the O.M.C. study group, and also contains many contributions of the authors in subject matter. It has been written with the idea in mind that the four year student should study a few topics, with suitable time allotment so that he can become familiar with these topics.

It is, of course, too early to assess the course. However, meetings of participating teachers have been held to discuss the first two chapters, which deal with the slide rule, calculating machines, and arithmetic, and the use of these in mensuration problems. Yes - believe it or not - the slide rule taught to B and C students at the grade 10 level! And more wondrous yet - most of them even enjoyed it a little! Maybe the B and C student is capable of doing some mathematics. At least, this is the belief of those involved with the experiment.

Major points to come out of the first two chapters are:

1. Some participating teachers feel that the slide rule should be introduced as a tool, with no explanation as to why it works. Others feel just as strongly that the better way is to introduce it by doing work with exponents, so that the student will have a little understanding of its operation.
2. There is a strong feeling that the use of calculating machines adds much to the interest of numerical calculation, and since there is a real problem in getting machines for mathematics classrooms, those

persons interested should start to press for these right now.

3. Technical teachers are strongly in favour of changes in the mathematics - particularly the introduction of slide rules in Grade 10.

Further reports on the experiment will appear in future issues of the Gazette. As well, discussions of this experimental course will be on the agenda of the Easter convention of the O.A.T.M.. Interested teachers are invited to bring questions concerning the course to those meetings.

"WHERE ARE WE GOING? WHY ARE WE GOING THERE? AND HAVE WE GOT TO THE RIGHT PLACE EVEN IF WE GET THERE?

by

G.G. Steel
Hillfield-Strathallan Colleges, Hamilton[1]

The majority of men are by nature conservative in outlook, not readily directed into new patterns of thought and behaviour unless acted upon violently. Hence we have revolutions - and revolutions nearly always end by being destructive rather than constructive: though they do not usually begin in this manner. In fact they usually arise out of sound and sincere efforts to improve existing conditions. Resistance to these efforts - or more commonly, indifference to them, leads to the more violent methods of attack. The forces of the attackers are augmented by protagonists of new ideals whose end is the creation of a new world rather than the remodelling of the old. Thus the milder disturbance of change gives place to the outburst of revolution. We have recently witnessed a revolution in the teaching of Mathematics, - in fact the battles are still raging about us. If we look carefully we can see all the usual participants from those who stoutly maintain that what was good enough for them is good enough for the students they teach, to those whose only concern seems to be for the esoteric or for sophistication for its own sake.

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[1] Address given at Queen's University, November 11, 1966.

It is not my intention this afternoon to condemn the new programs in Mathematics - or any programs for that matter. It is all too easy to find fault with any educational program. In fact a good educational program must be shortlived because it will develop a new environment with new problems and situations requiring a changed educational program. But I do wish to ask a few questions about the direction that our present courses in Mathematics are taking.

We must begin by asking ourselves what we are trying to accomplish when we set up a syllabus of Mathematics. Our prime concern, as educators, must be with human beings, rather than Mathematics, at least for the most part of a student's school life. We have to help the child develop as well as possible those facilities with which he is endowed so that he can best adapt himself to his environment. Clearly human beings differ widely in their interests, abilities and tastes, and a sound educational policy must recognize these differences. It is generally accepted on this continent that early specialization is not in most cases a good thing, and a good Mathematics syllabus must be sufficiently flexible (even for potential University students) that it can satisfy a variety of needs and abilities. Certainly, for many reasons, such a program must be built around a firm core of basic mathematical concepts. It follows that a syllabus which is dominated by pure mathematical thought follows too narrow a path in a general educational system.

In Canada particularly, we must bear in mind that the conditions for education differ from those of most other countries. Despite the laudatory attempts of the Robarts Plan in Ontario, a greater percentage of High School students here attempt to pass through the 5 year Arts and Science Course than is the case in Europe or in

the United States. In Europe there is generally a far more rigorous selective system in operation. Relatively few students complete their High School lives in purely academic courses. South of the border only two years of High School Mathematics are required of students who do not wish to continue in that field. In practice, in Canada, most 5 year students take 4 years of Mathematics at least.

I do not believe that our present programs include sufficient work in applied Mathematics. Not only is this a weakness because of the deficiency, but also it is a flaw in that the student has fewer opportunities to develop his problem solving capacities. This is surely a serious weakness. One of the few areas in the 13 Algebra Course in which students are confronted with problem solving situations is that of permutations, combinations and probability. My own experience this year leads me to suspect that students brought up on the new program can handle the language of subsets and r - arrangements very well, but when required to use this in the solution of problems they have at least as much difficulty, if not more, than students of former years.

I greatly fear that our present crop of students by and large, that is, excepting the good ones who will do well, whatever the system, can write better-looking Mathematics on paper but are less aware of the underlying significance of what they are writing, and less able to apply their knowledge.

I am increasingly tempted to ask if we <u>are</u> still a problem-solving people. There is an increasing tendency today to let 'other people' solve the problems, and I am not sure that educationally we are doing a great deal to stem this tendency. It seems easier today to develop the technical processes needed to put people on the moon than

to solve the human problems of crime, waste, pollution and transportation.

Some years ago, noise of significant changes in mathematical education reached us from points abroad, particularly from the United States. There was a developing reaction against a common form of teaching mathematics that was mechanical and conceptually weak. It was argued, and rightly so, that the teaching of mathematics should be meaningful, that students must be led to a clearer understanding of the processes of mathematics and at the same time be able to see something of the coherent structure of mathematics. In fact, of course, good teachers had always taken this point of view; so there was nothing really new about this. At the same time it was recognized that some things which had once been of importance pragmatically, had ceased to be of great value, whereas new modes of thought compatible with a world becoming increasingly influenced by machines were not being developed.

To help bring about the changes required, a number of the relatively recent developments in higher mathematics were introduced into High School (and even now in Primary) education. In so far as these aided in the clarification of basic mathematical concepts they were of the greatest value. As teachers we must surely agree that anything, whether new or old, that helps the student towards a proper understanding of what he is trying to do should be used, and that whereas one point of view will help one student, a different one brings the light to another. Now some of these new ideas - sets, symbolic logic, some of the more sophisticated concepts of structure proved interesting in themselves (there is considerable evidence that students find the so called 'new maths' more interesting than the 'old') and in too many cases the temptation to pursue these for their own ends was not

resisted. To make room for new topics some seemingly fairly essential ones have suffered. Have you noticed any great improvement amongst your average students in their ability to add and subtract algebraic fractions with compound denominators? How is their factoring these days? - and it is still rather necessary that they be able to factor. In the old 'new' course the major work in factoring was taught in Grade Ten. I see the topic is not mentioned in Cur. I. 12D. though students are expected to begin the study of Groups and Fields.

This brings me to my deepest cause for concern about where we are heading: the increasingly abstract nature of and approach to what we are teaching. We live in a world in which the abstract plays an increasingly significant part. The vogue is for abstract paintings in which line, form and structure are the significant aspects; music is abstract with its interest again in line and tonal pattern. Even food is abstracted; and although not too many people prefer a steak pill to the real thing, instant coffee is popular. These are the trends today, and if we are not careful we shall cease to be human beings. After all, abstraction is entering the field of human relations. Spouses are selecting their mates by computer. The next logical development in this process is that instead of trying to find a wife who is a perfect mate, the prospective husband will be supplied with the abstraction of a wife. I, for one, would find such a situation less than satisfying.

It is not too difficult to teach students the use of abstract language - in fact it is often easier to teach than the solution of systems of quadratic equations in two variables. It is highly debatable whether they comprehend what lies behind this language. We must bear in mind that we are <u>not</u> providing a mathematical course

just for the bright student, or for those whose ultimate concern is mathematics and science at university level. The process of learning is from the particular to the general and then to the abstract; and most abstractions are realized only after long examination of the former two and much trial and error. To teach the other way round, i.e. from the abstract or general to the particular seems to be a contradiction of the natural sequence of events. The average high school student finds even generalization difficult. I question whether abstraction should form any significant part of his thinking in the middle Grades.

I fully recognize that most authors are really concerned about this problem, but the courses are so vast, so much material has to be covered, that they have to take the easy way out, and can only make sporadic efforts to develop the natural processes of learning. Teachers have the same problem. I do not think I am being too sweeping about this. How many times do we encounter the situation where a student has a problem, but if you lead him, step by step through the processes necessary to solve the problem he recognizes that he could have done the same thing himself, but didn't. The point is that we try to teach too much mathematics at the expense of teaching students how to learn mathematics. Knowledge! Understanding! Comprehension! These are three different things. We are certainly trying to develop the second of these, perhaps at the expense of the first, but we have not squarely faced the matter of real comprehension of mathematical thought. Would not even Mathematics Departments at Universities be better pleased to receive students who really knew how to learn mathematics and who operate in a spirit of enquiry, investigation and experiment, rather than those who know how to do mathematics, and certainly those who know how to talk about it and can neither do

nor learn to do it! I believe from a human point of view, and this is still the most important consideration, that we would be giving a better service to our students.

It is not our first responsibility as educators to teach students all about life, it is to prepare them to meet life; it is not our first responsibility to teach students as much mathematics as we can, but to prepare them to handle as much mathematics as they will encounter.

I surely cannot be alone in feeling that the present courses of study are too long for the average student. Topics flit past as we try to keep up with the schedule necessary to complete the course on time - glancing over our shoulders from day to day to see how closely our progress is being watched by one above. The good students rarely get a chance to savour what they are learning, enjoy it and apply it. The average student is left hanging on by his fingernails, insecure and uncertain in his knowledge.

We have for some time been subject to the theory of spiral learning - a learning technique that consists of a number of glances at a topic separated by a period of a year. Too often this leads to a thickening crust of superficial familiarity only; it provides little or no opportunity for a student to become immersed in a topic, and only by immersion can he understand, know, comprehend - and <u>retain</u>. I believe that some serious investigation should be done to compare the relative merits, both with regard to time saving and achievement level of spiral learning and learning in depth followed by review, and application.

I have spoken earlier of basic principles which should form the core of the High School Mathematics course - and so far have left this as an undefined term; but I realize that upon the definition hinges much of the course content. It is not my purpose here to outline a model course - that

is matter for another time - but before all else a competent mathematician must be able to perform all operations with rational algebraic expansions, and some irrational ones - and do so with ease and quickly. A student must be able to factor well (factoring apart from its practical value is itself a study of pattern) and solve equations - at least up to two variables and up to 2nd degree, and have a good understanding of ratio and proportion in all its aspects. He should have good knowledge of (not be aware of) the more common properties of standard geometric figures in two and three dimensions and have a working knowledge of Trigonometry. He must be aware of the nature of irrational numbers, and be familiar with some types of these.

He must be able to write correct mathematics - not merely because he is told that such and such is right and the other wrong, but knows why.

Even more important than content is the nature of the student's approach to mathematics. The vast majority of students still feel that mathematics involves cut and dried procedures; that there is a right way of solving a problem which he either knows or does not know. There is far too little realization that mathematics is still largely a matter of trial and error. We are so busy teaching our students mathematics that there is practically no time to teach them how to learn about mathematics - and the latter seems to me to be the more important.

I do not wish to put up a defence for many of the old type 'word' problems - though they had value. Many of them are no longer relevant, and partly for this reason, partly because of the pressure of other matters, practically no time is devoted to problem solving. Now it seems to me the spirit of enquiry followed by investigation are fundamental aspects of education. Is not this what education is all about? It is too little present in our current

courses. (As a side issue I think more of it might produce better mathematicians!) To my knowledge very little work has been done in developing an approach to the solution of problems in our present environment on this continent. The field is a vast and interesting one not restricted to physics and chemistry.

In addition to the above, one must include, of course, a growing awareness of structure and function which are common to so many fields of contemporary thought. Only when students are thoroughly familiar with a number of different groups - not restricted to their occurrence in mathematics do I believe that the abstraction should take place, and at once the search should be begun for other examples not already studied. The process: concrete to general to abstract does not end with the abstraction but leads at once to further investigation of the concrete if it is to be reinforced and become really significant.

I have already spoken too long, and I know that you want to have your say. I cannot guarantee that what is said here will have any effect on our courses of study, but it is sometimes satisfying to say what one feels; I think it good to have an opportunity to exchange views amongst ourselves; and I believe there are some here present who do have some influence on the setting up of our courses.

I end therefore with a plea. More time! Let us teach less mathematics, but teach what we teach much better. Rather than small doses of a given topic from time to time, let us have the opportunity of getting our students thoroughly involved with a topic and its application. They may then really comprehend, understand, and know something and be able to use their knowledge.

To accomplish this let the course content be reduced by 25 - 30%, so that the slower student can have a chance

to attain confidence in what he is doing, and the better students can develop greater depth of understanding and pursue their own studies in related and other topics. I visualize a textbook not just with graded exercises at the end of a section containing questions of increasing difficulty, but rather, after questions related to the basic topic, more of the same kind for the weakest students, some of greater difficulty, some involving related problems and the application of principles in diverse areas, and suggestions of projects for further research. And adequate time to do these things properly!

J. Vincent Robinson, Modern Algebra and Trigonometry, McGraw-Hill, New York, 1966.

The author of this book states that its primary aim is to provide a suitable foundation for modern courses in analytic geometry and calculus. To this end he has made an excellent choice of material including sets, relations, trigonometric, polynomial, exponential and logarithmic functions, matrices, sequences, permutations. The text is, on the whole, lucid, readable, and logically developed. It would be impossible to cover so much material absolutely rigorously without making the book prohibitively long, yet sufficient attention has been paid to rigor to justify the word "modern" in the title.

There are, of course, some faults, but none of them seriously impairs the book. Your reviewer would prefer:

1. That on p. 4 equality of sets be defined by $A = B$ if and only if A and B are the same set, in the same way that equality of numbers is defined on p. 14.
2. That the postulated properties 1 to 5 of numbers listed on p. 15 be treated as deductions from the substitution property 6 in the same list.
3. That proofs requiring "if and only if" arguments be more carefully explained (e.g. p.31).
4. That arc length be defined before it is used in trigonometric definitions, using the completeness of real numbers axiom 9 on p. 32.

5. That the need for induction be specifically mentioned each time it is required in a definition or a proof (e.g. pp. 303, 342).

On p. 203 the graph of $2\sin x-1$ has been labelled "2 sin x" .

Techniques have not been neglected. For example, there is a chapter on solving triangles, and on p.239 ff. is an account of synthetic division. There is just the right number of well graded exercises.

S.M. Trott

W.L. Duren, Jr., The Maneuvers in Set Thinking, Reprinted from The Mathematics Teacher, Volume LI, No. 5, May, 1958.

The author begins by considering the place of sets in the curriculum. In his opinion, basing mathematics on sets is not a fad as some conservatives insist. "The intimate connection between sets and logic insures that the language of sets will continue to be more and more necessary to the understanding of the language of mathematics."

By defining function in terms of set, he indicates that set is a more basic concept.

The "maneuvers" referred to in the title are the following.

(a) The subset maneuver: 2^X is defined to be the set of all subsets of a set X containing n elements. The system $(2^X, \cup, \cap, ^\sim)$ forms a Boolean Algebra.

(b) The maneuver of reduction to cases: an important method of proof. An example is

given using a tree diagram.

(c) The matching maneuver: the set of all matchings of a nonempty set onto itself leads to the idea of permutation and then to groups.

(d) The mathematical induction maneuver: leads to sequential counting and the formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) .$$

(e) The classification maneuver: involves determining the number of classifications of a set X of n elements into two cells, the first containing r elements and the second containing $n-r$ elements.

(f) The function maneuver: function is based on the concept of set.

(g) The measure maneuver: is shown to be related to probability.

(h) The distribution maneuver: leads into statistics.

An excellent pamphlet with material related very closely to our new Grade 13 curriculum. A valuable reference source for teachers.

M.J. Liebovitz

T.E. Hull, Introduction to Computing, Prentice-Hall, 1966.

This short text (208 pages) provides an excellent introduction to the fundamental concepts of solving problems by means of a computer. The concept of an algorithm and a flow-chart are introduced and employed throughout the text. The features of a stored program

computer are discussed with enough emphasis on a machine language to allow the reader to appreciate the process of implementing a language such as FORTRAN. The features of FORTRAN IV are discussed clearly but more examples of complete programs would make it easier for a reader to produce programs using this text without additional assistance. An excellent summary of FORTRAN is provided in an appendix and could be used as a programming handbook. Problems which occur in the flow-chart exercises are used as exercises for programming practice in the FORTRAN section. Complete answers are not provided but suggestions are made to assist in solving the early questions. The last chapter of the text considers the logical basis for the definition of the term algorithm. An excellent Bibliography is provided.

This text is an excellent reference for any mathematics teacher who wishes to become familiar with the concepts involved in computer programming. It could also be read with understanding by better than average Grade XIII students, but should not be considered as sufficient to provide them with enough direction to start using a computer without extra assistance being given in the form of additional texts or individual instruction.

D.W. Alexander

PROBLEMS AND SOLUTIONS

In this issue we begin a new department in which we plan to print problems proposed for solution by our readers. These problems must be comprehensible to high school students and/or teachers, and we solicit solutions from both groups as well as problems for solution. The proposer need not know a solution (or even if the result is true, in which case it should be so worded), but if he does have a solution, he should submit it. One or more of the solutions received will be published in subsequent editions.

(To avoid any possible confusion, we emphasize that there is no connection between this new department and the Department of Education, Grade 13 Problems Paper.)

Send problems and solutions to:

The Editor, Problems and Solutions,
Ontario Mathematics Gazette,
c/o Department of Mathematics,
Queen's University,
Kingston, Ontario.

PROBLEM NO. 1 (D.A. Clarke, University of Toronto)

Let $P(x)$ be any open sentence where x ranges over some set. Let Q be a sentence. Let $\forall x$ be the universal quantifier "for all x ..." and $\exists x$ the existential quantifier "there exist an x such that...". Show

$$\forall x(P(x) \Rightarrow Q) \Leftrightarrow ((\exists x P(x)) \Rightarrow Q) ,$$

and

$$\exists x(P(x) \Rightarrow Q) \Leftrightarrow ((\forall x P(x)) \Rightarrow Q) \ .$$

PROBLEM NO. 2 (G.F.D. Duff, University of Toronto)

Let $[x]$, $x \in R$ denote the greatest integer not exceeding x ; for example $[\frac{11}{3}] = 2$, $[\pi] = 3$, $[7] = 7$, $[-\frac{3}{2}] = -2$.

a) Sketch the graphs of $y = 2 \sin x$, and $y = [2 \sin x]$, for $0 \leq x \leq 2\pi$. How many isolated (i.e. single) points are there on the latter graph?

b) Given that $y = n[\frac{1}{n} f(x)]$ yields a "steps and stairs" approximation to $y = f(x)$, where the rise in each step is $\frac{1}{n}$, and the steps all lie <u>below</u> the graph of f , find the equation of a similar "steps and stairs" approximation to $y = f(x)$ that lies <u>above</u> the given curve.

This is another new department in which we will publish short notes reporting mathematical discoveries that students have made on their own.

Send notes to:

The Editor, Notes from Students,
Ontario Mathematics Gazette,
c/o Department of Mathematics,
Queen's University,
Kingston, Ontario.

NOTE NO. 1 - Colin L. Swan, Windsor

The following table yields the values of $P(n,r)$ in a manner somewhat analogous to Pascal's triangle for $C(n,r)$. It is based on the fact that

$$P(n,0) = 1 \quad , \quad n \geq 0 \ ,$$

$$P(n,r) = n\,P(n-1,r-1) \ , \quad n > 0, \ r > 0 \ ,$$

so that each row is obtained from the preceding one by a simple multiplication.

$n \backslash r$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	2						
3	1	3	6	6					
4	1	4	12	24	24				
5	1	5	20	60	120	120			
6	1	6	30	120	360	720	720		
7	1	7	42	210	840	2520	5040	5040	
8	1	8	56	336	1680	6720	20160	40320	40320

WHAT CAN YOU DRAW WITH A RULER?

by

Norman M. Rice
Queen's University, Kingston

Let us begin by trisecting the angle $\angle CBA$ of fig. 1 below. To do this, draw a semicircle with center B and any convenient radius r, cutting BC in the point D. Next mark two points on the ruler a distance r apart, and then slide the ruler, keeping the two points respectively on AB extended and the semicircle, until the ruler passes through D. If the points E and F are as indicated on the diagram, then it is easy to see by considering isosceles triangles and exterior angles that $\angle DEA = (1/3)\angle DBA$.

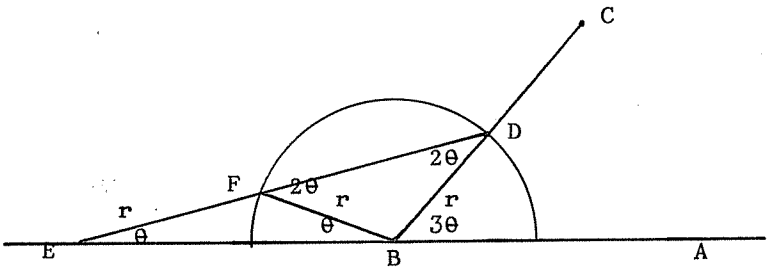


Fig. 1

Does this construction conflict with the well-known theorem that in general it is impossible to trisect an

angle using only straightedge and compass? No, because in the construction we used the ruler not just as a straightedge, but as an instrument with which we could slide around line segments of fixed length.

The ancient Greek geometers concentrated almost entirely on constructions that could be made by straightedge and compass. (Not entirely though: the trisection construction described above was known to Archimedes.) It is not surprising that by using additional instruments (such as a real ruler rather than just a straightedge) one can draw diagrams that were formerly impossible. What is surprising is the great number of figures that can be drawn using fewer instruments. It is this question that we will consider in the present article.

Let me then, before we continue, point out that the title of the article is misleading. Since a ruler (with its distance markings) can be used both as a straightedge and a compass (and more, as the trisection above shows) a ruler represents more than the classical amount of drawing apparatus. In fact what we will consider is:

(1) What can you draw with a compass? and

(2) What can you draw with a straightedge?

The answer to the first question is that any

geometrical construction that can be made with compass and straightedge can be made using compass alone. (This was proved in the late 18th century by the Italian mathematician Mascheroni.)

An immediate qualifying remark is in order. The answer does not claim that lines can be drawn using a compass. It does claim that any point that can be located by straightedge and compass (such as the point of intersection of a given circle with a line through two given points) can be located using a compass alone.

There is not space in this article to prove completely the assertion, but we can see the techniques used. Let us first of all note just what constructions are basically possible using straightedge and compass:

(a) We can draw a circle with given center and radius.

(b) We can find the points of intersection of two given circles.

(c) We can find the points of intersection of a straight line with a circle.

(d) We can find the point of intersection of two straight lines.

Any geometric construction with straightedge and compass is accomplished by using some finite succession of these basic constructions. Thus it is sufficient to show that these four are possible using only a compass. Clearly the first two are. Let us prove that (c) is also possible.

A couple of preliminary constructions will be useful. First of all, given A and B it is easy to construct (by erecting equilateral triangles ΔABD, then ΔBDE, then ΔBEC) a point C on AB extended with $AB = BC$.

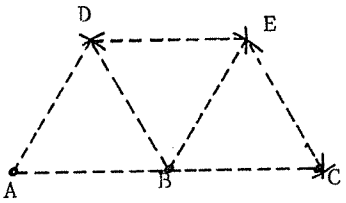


fig. 2

Secondly, given a circle c with center O and radius r , and a point P with $OP > \frac{1}{2}r$, we can construct the inverse of P with respect to the circle c , i.e., a point Q on the line OP such that $OP.OQ = r^2$. To do this (see fig. 3), draw an arc with center P and radius PO to cut c at R and S . Then draw arcs with centers R and S and radius RO to intersect at O and Q . Then Q is the required inverse point, for the isosceles triangles ΔPOR and ΔROQ are similar, and hence $PO/OR = OR/OQ$, i.e., $OP.OQ = OR^2$ as required.

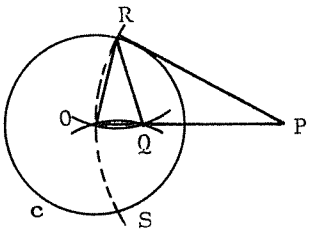


fig. 3a

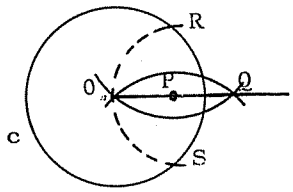


fig. 3b

We should also note that in fig. 3 P is the inverse of Q with respect to c . However, the construction described above does not serve to construct inverses for all points inside c , so the following observation is in order: given a point P inside c (fig. 4) we may double the segment OP as many times as necessary in order to get a point P' outside c ; then

the inverse Q' of P' can be constructed, and when the segment OQ' is doubled the same number of times as OP was, the resulting point Q is the inverse of P with respect to c.

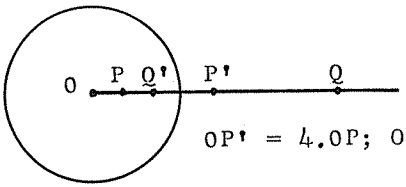


fig. 4

It is this construction of inverse points that is the key to solving (c) as we shall now see. Given a circle c with center O, and two points A,B (fig. 5) draw arcs with centers A and B passing through O and intersecting again at Q. Construct the inverse P of Q with respect to c, and draw an arc with center P and radius PO cutting c at R and S. Then R and S are the points where AB intersects c; for AB is the right bisector of OQ, and $RQ = RO$ (as can be seen from fig. 3), so R is also on the right bisector of OQ, i.e., R lies on AB; and similarly S lies on AB.

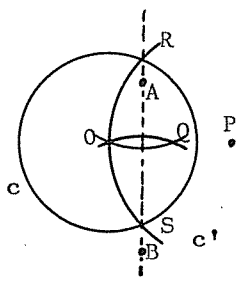


fig. 5

The above solves (c) except for the case where AB passes through the center O . This case requires a special construction, such as the following (fig.6): With center A and any convenient radius draw an arc to cut c in C and D . With centers C and D and radius CO draw two arcs, and with center O and radius CD cut these arcs at P and Q (fig. 6i). With centers P and Q and radius PD draw two arcs intersecting at R (fig. 6ii). Finally, with center P and radius OR draw an arc toccut circle c at S .

Then S is the required point of intersection of AB with c .

(Trigonometry is helpful in proving this somewhat tricky construction.)

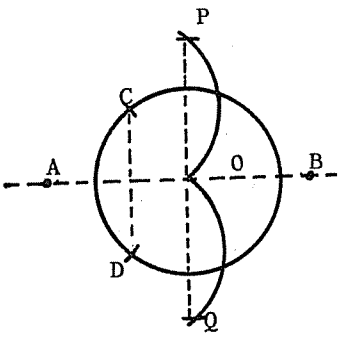


fig. 6i

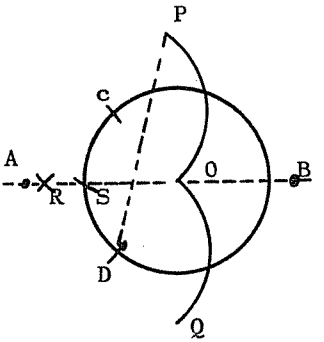


fig. 6ii

It can be proved that in fig. 5 the circle c' with center P and radius PO is the inverse of the line AB , i.e,, is the locus of points that are inverse (with respect to c) to the points of the line AB . Conversely, the inverse of the circle c' is the line AB . One may verify that if the line AB doesn't intersect c then the circle inverse to AB

(constructed precisely in the manner above) will not intersect c either.

With these observations we can now solve (d). Indeed, given four points A,B and C,D (fig. 7) we can take any circle c in the plane (so long as its center O does not lie on AB or CD) and construct the circles c_1 and c_2 inverse to AB and CD (with respect to c). These two circles will intersect at O and some other point P ; the inverse Q of P (again with respect to c) is the point of intersection of AB and CD . Question: What if AB and CD are parallel?

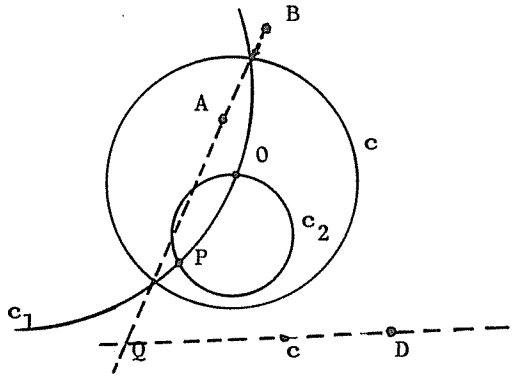


fig. 7

Let us consider briefly, now, the second question: what constructions are possible with a straightedge alone? The answer is: not as many as we might like; for instance, most constructions involving any notions of length or angle are impossible. It can also be proved that even if all the points on a circle are given it is impossible to locate the center of the circle using straightedge alone. However, in 1833 the German mathematician Jacob Steiner published a little book containing the following remarkable theorem: If a circle and its center are given, then every point that can be constructed

with straightedge and compass can be constructed with straightedge alone. (There is also some evidence that the French mathematician Poncelet had previously discovered this result during his stay in a Russian prison - Poncelet was an officer in Napoleon's army, captured during the French retreat from Moscow in 1812.)

To prove Steiner's theorem we should again consider the basic constructions. A circle will now be represented by its center and a point on the circumpherence, and we should construct (i) the points of intersection of a line and a circle, and (ii) the points of intersection of two circles. Let us consider (i).

Again, some preliminary constructions will be useful. First, using the given auxiliary circle c with center O , fig. 8 (constructing the points in alphabetical order) shows how, given any line ℓ not through O , to construct another line parallel to ℓ . (Use fig. 8a if ℓ intersects c , and fig. 8b if it doesn't.)

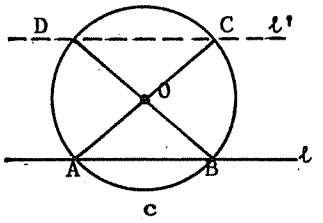


fig. 8a

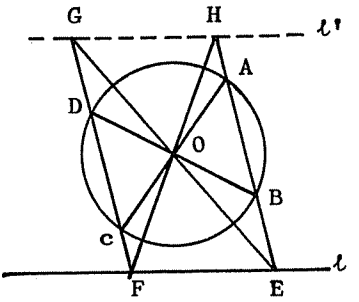


fig. 8b

Next, given two parallel lines ℓ and m , fig.9 (again constructed in alphabetical order) shows how to find three points A,B,F on ℓ with F bisecting AB .

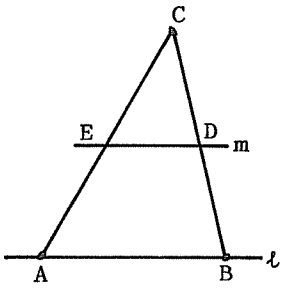


fig. 9i

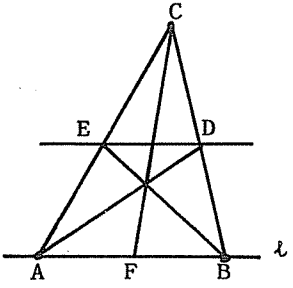


fig. 9ii

Finally, given three points A, B, C on ℓ with B bisecting AC , fig. 10 shows how to construct a line parallel to ℓ through a point D outside ℓ .

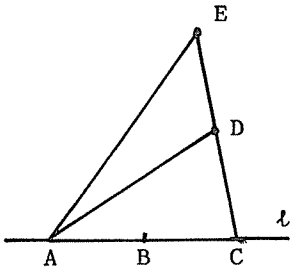


fig. 10i

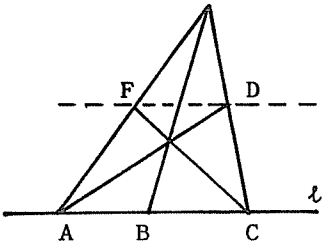
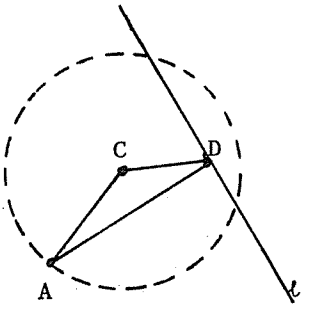


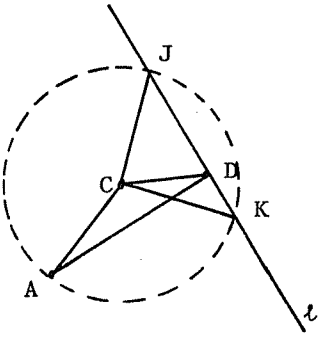
fig. 10ii

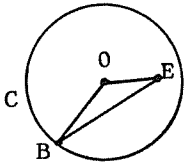
Putting these constructions together, we see that it is possible to construct a line parallel to any given line ℓ and passing through any point not on ℓ . (If ℓ passes through 0 then we are already in the situation of the last construction above.)

Now we shall construct the points of intersection of a line ℓ with a circle having center C and passing through A (fig. 11). If c is the given auxiliary

circle, draw a radius OB parallel to CA . Take any point D on ℓ , and construct lines through O and B parallel to CD and AD respectively, meeting at E (fig. 11i). Through E draw a line ℓ' parallel to ℓ , cutting c at F and G . Then through C draw lines parallel to OF and OG cutting ℓ at J and K , which are the required points (fig. 11ii). (We leave to the reader the task of considering the special cases, where ℓ passes through C or through C and A .)







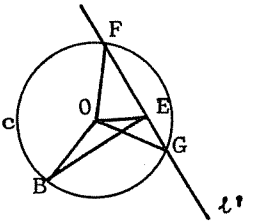


fig. 11i

fig. 11ii

As for the problem of finding the points of intersection of two circles, we can solve it via the following steps:

Firstly, fig. 12 shows how to draw a perpendicular to a given line ℓ through a point P .

Using this, fig. 13 shows how to draw the common chord of two intersecting circles if one of the points of intersection is already known.

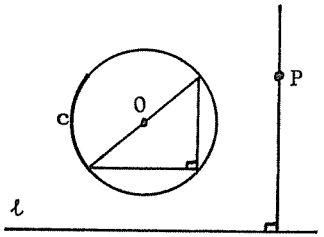


Fig.12

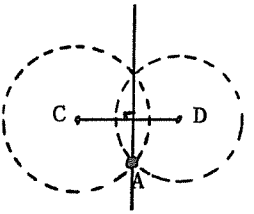


Fig.13

Finally, then, given two circles c_1 and c_2 (fig. 14) passing through A and B , we can construct the midpoint E of AB and consider the circle c_3 with center E passing through A and B . Since c_1 and c_3 intersect at the known point A , we can construct the common chord ℓ of c_1 and c_3 . Similarly we can construct the common chord m of c_2 and c_3 , meeting ℓ at F . But it is known that the three common chords of three intersecting circles are concurrent, so F must also lie on the common chord of c_1 and c_2 . Therefore the line n through F and perpendicular to CD must be the common chord of c_1 and c_2 , and the problem of finding the points of intersection of c_1 and c_2 has thus been reduced to the previously solved problem of finding the points of intersection of a line (n) with the circle c_1 . (Problem: the above construction breaks down if A, B, C, D are collinear; show that this can be avoided by finding more points on c_1 if necessary.)

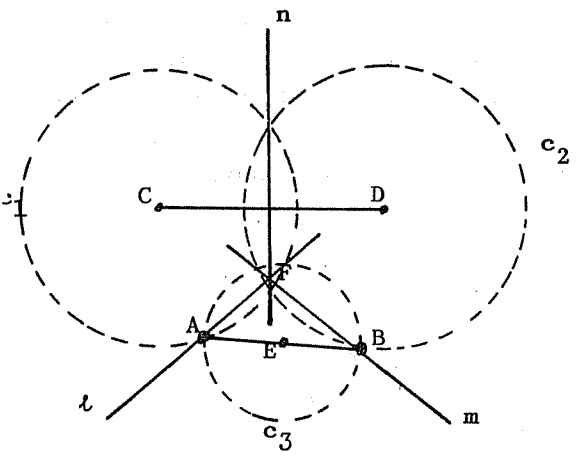


Fig. 14

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2. For inversion in the circle and constructions using compass, see R. Courant and H. Robbins, What is Mathematics?, Oxford University Press, 1941.

3. For construction using straightedge, see Jacob Steiner, Geometrical Constructions with a Ruler, Scripta Mathematica, New York, 1950; or A.S. Smogorzhevskii, The Ruler in Geometrical Constructions, Pergammon Press, London, 1961.

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1967 is an important year for holding a membership in the N.C.T.M. since the August meeting will be in Montreal. Accommodation has been reserved at McGill so that those attending the convention may also spend a few days at EXPO 67. Members of the N.C.T.M. will receive further information about this convention early in the new year.

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