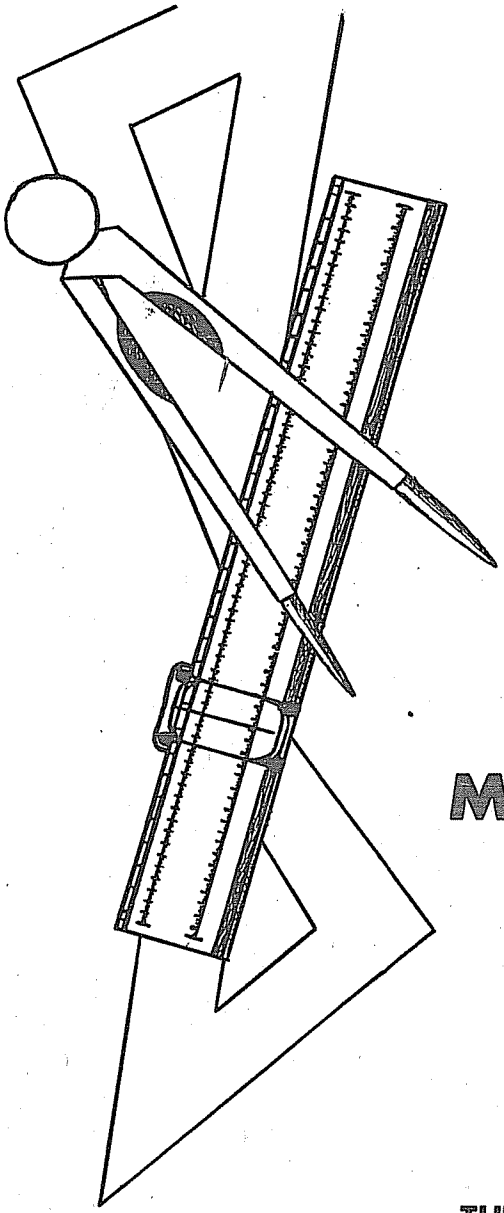


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EDITORIAL

Entering the seventies, the Gazette has a new editorial board, and a new editor. However, the purpose of the Gazette is unchanged, namely, to promote and advance mathematics, and mathematics education in our primary schools, secondary schools, and community colleges.

Most of the recipients of the Gazette teach mathematics at one of the above levels. We select material accordingly.

Main Articles. We seek discussive and expository articles reflecting the content or mode of current mathematics or mathematics teaching.

Mathematical Notes. We seek short papers which give new insights, new proofs of old theorems, applications (in any sense of the word) of standard material, which are suitable for classroom presentation or for student projects.

Problems. We seek easily stated problems, particular problems in that part of modern mathematics which is now finding a place in the 'new' curricula.

Book Reviews. We seek reviews of textbooks and collateral material. Critical as well as informative reviews of textbooks are welcomed, especially those based on classroom use. Thus reviews of established texts are acceptable, not only of new books.

Notices of Meetings. Since the Gazette is a medium for the exchange of information, we wish to inform our readers of relevant meetings and conventions. Thus we welcome announcements of such.

Notes on Meetings in Mathematical Education. We seek reports of such meetings.

The Editor invites criticisms, complaints and suggestions. Hopefully we could institute a letters section.

T. M. K. Davison, Editor.

UNTITLED TALK*

Professor W. W. Sawyer, College of Education, University of Toronto

I was not able to provide you in advance with a title for this talk, as I was not too sure just what the trend of this conference was going to be and how much ground would be covered by George Duff+. I wanted to get some feeling for the direction of your discussions before finally deciding what to say myself. One thing Professor Duff said very strongly, rang a bell with me. He spoke of the need for collecting relevant information and making it available to people in the schools. During my seven years in the states, at the time of the great upheaval there, I was very much struck by the total lack of any philosophy ordering the whole thing - any rational philosophy - that is. People would say, "The new mathematics is extremely important" and if you asked, "For whom?" or "For what purpose?", there was a sudden silence. I remember at one time the mathematicians quoted an engineering journal on the importance of modern mathematics. I looked up this journal to see what the engineers were using it for, but it simply quoted mathematicians who had said it was important. The statement, "This is important", seemed to mean anything from "I have three Ph.D. students and they must all know this theorem" to "This is something the entire population will have to understand soon." Thus the information

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* A Speech delivered to the tenth anniversary meeting of the Ontario Mathematics Commission, September 20, 1969.

+ See number 1 of this volume "Here come the seventies".

provided to teachers did not have the kind of precision that would be needed for reasoned judgements.

Now of course I am not maintaining that the whole of mathematical education ought to be geared to what people are going to work at; however, this is one of the factors we have to bear in mind. We ought to know if a certain type of calculation has become totally obsolete, or, on the other hand, if - say - all students are going to have to use an electronic computer later in life, then we should react to this to the extent that we think it important. We certainly should have the information.

Things are changing all the time and a nation needs something that corresponds to the sensory equipment of an animal, so that we are aware of these changes. Some time ago in Britain I was talking to the Institute for Mathematics and Its Applications and I said to them, "Suppose in some branch of practical life a certain type of mathematics is becoming increasingly important; through what channel do you think that information would reach school teachers, and how long would it take?" I did not get any coherent answer. In fact our national nervous system is non-existent. Generally speaking, an engineer feels he gets far more prestige by solving some particular, immediate problem rather than by surveying the uses of mathematics in a particular area, and trying to predict what will be used in the future. This prediction is essential, since an educational system has enormous inertia. Usually by the time something gets into a textbook it is already out-of-date. We need some arrangement by which we can estimate trends and direct our textbooks towards what seems most likely in the foreseeable future. We need a scheme by which teachers in schools and universities would receive an

annual report from practical people, from practising people as well as from research workers. "Such-and-such a development has occurred in pure science; it seems significant but is not yet being put to practical use. People are beginning to make use of a certain mathematical technique; at present this activity is like a tiny island in the sea, but it seems to be growing and may become a continent."

Whenever a change is being advocated in mathematics education, the evidence for it should be presented. Perhaps a recent discovery is beautiful, entertaining and stimulating. That is a perfectly sound reason for teaching it. Or it may be good for the strong mathematical student but totally irrelevant and uninteresting to others.

I have spent some time discussing practical uses and recent mathematical advances, but I would not like these to be our exclusive concerns. Many ancient things are still interesting and important. There is no connection, as far as I can see, between the date at which a particular piece of mathematics was discovered and its present interest or utility.

I would like to stress too that practical interests are not identical with commercial interests; in fact some of the most important practical applications of mathematics and science lie in stopping commercial interests from doing various things, such as poisoning us by air pollution or in other ways rendering the globe uninhabitable. In a book "Science, Fact and Fiction" Isaac Asimov calculates that the increase of carbon dioxide in the atmosphere could cause the polar cap to melt and the sea to rise some 180 feet, which would be particularly unwelcome in coastal areas. An even more drastic suggestion has been in the Globe and Mail recently, namely that insecticides going into the sea could

could destroy organisms there that produce 80 per cent of the oxygen in the atmosphere and thus extinguish all higher life on earth. Scientific knowledge is coming to be involved in every aspect of daily life, - as for instance in the question whether colour television sets are producing radiation that could upset our genetic processes. Humanity today has a problem it never had before. The past survival of the human race has been due more to its helplessness than either its goodness or its wisdom. That is no longer so; this safeguard is gone; today we have to know what we are doing. Conceivably the most important application of mathematics today is to give average students sufficient confidence and familiarity with mathematical symbolism for them to acquire a good general knowledge of science, and be able to follow some argument involving radiation physics or genetics or bio-chemistry. We also have to produce some extremely competent and public-spirited scientists to take the lead in such discussions.

Various strands in our discussions seem to be centred around the Hall-Dennis Report. It is important to remember that Hall-Dennis is not a global doctrine that you have to accept or refuse in its totality. When I was at school myself, which is roughly half-a-century ago, we had some elements of Hall-Dennis. I would like to emphasize that it was not an arty sort of school; it was a boy's boarding school, run on more or less military lines; six of the best on your behind if you broke the rules and so forth. But there was freedom of work at your own pace, and the programme was adapted to the needs of the student. In mathematics we had a rather ruthless system, by which the whole school did mathematics at the same time; what mathematics set you were in had

nothing to do with your age or achievement in other subjects. There were about ten mathematics sets. The English school year is divided into three terms. If, after a term in a set, you had sufficient mastery of the mathematics taught to that set, you moved to the set above for the next term. If you did not understand enough, you stayed there and did the same work again. Thus the individual student was not held back by slower companions, and the teacher of any set was sure that students entering that set knew what they were supposed to know. This latter point is very desirable. I remember a very good mathematics teacher in Ontario who told me he was tempted to get out of teaching altogether, because he had students in Grade 13 who did not have Grade 10 knowledge of algebra and this created an impossible situation. Incidentally is it truth or myth that in Ontario teachers were forced to promote 80 per cent of their students? This is part of the general folklore, and I have never been able to decide whether to believe it or not.

As to flexibility in the school I attended - in England there is no graduation from high school. You can go right through school without having any certificate to show at all. Some students did, if they had a comfortable family job waiting for them. If you wanted to prove you knew something you offered yourself for some public examination, of which there were a great variety. Students used to go to the Principal and say they wanted to be actuaries, or take Bachelor of Commerce degree eventually, or get a Cambridge scholarship, or whatever it was, and he would make up an a la carte menu for them. It would often happen that, for part of the time, they would simply be parked in a room where some class was being taught, and would read independently. The public examinations, by the way, were never labelled "Grade 13" or anything

like that. They were intended to show that you were fit to be an actuary or a doctor or to enter university or whatever; for the great majority of these, your age was totally irrelevant. You presented yourself for examination when you felt you were ready for it. In England, except in some of the darkest days of the public elementary schools, there has never been a central authority laying down what is to be covered in any particular grade.

The system at Highgate, the school I attended, of teaching all mathematics classes simultaneously, was, as I have said, rather ruthless. It is not easy for a school to find the required number of good mathematics teachers. In 1961 I visited a number of English schools which had the reputation of organising their mathematics well. In the appendix to the Ontario Report "Geometry, K-13" I described the system at Marlborough. I do not know how many people read reports, so I will repeat the description. For background it is necessary to understand the English system of admission to university, which is based on excellence in any one field of endeavour. Thus a school welcomes a brilliant classical scholar who is not clear what 2 + 2 makes, or a brilliant mathematician whose Latin is terrible, and the school has to be organised to accommodate such individuals. Marlborough gave tests to entrants and divided them into three large groups, which I will call "tribes". Tribe A consisted of those who were brilliant in some subject; Tribe C consisted of those who had least intellectual power, and Tribe B came in between these. Each tribe filled several classrooms. Someone who entered in Tribe B would reach the A classes after one year; starting in Tribe C, he would be in a B class his second year, and an A class his third year. But, in

any particular subject, within each tribe there would be variation. All of tribe A did mathematics at the same time. There were 8 classes, A.1 to A.8. The best mathematicians were in A.1. Those whose brilliance lay in other subjects might be in A.8. These classes were taught by eight well qualified teachers, either mathematical specialists or perhaps physicists with strong mathematics. Perhaps in the next period, after the A classes, six of these same teachers went on to teach mathematics B.1 to B.6. And in yet another period, they would teach mathematics C.1. to C.6. Thus the best teachers were made available to all levels of students. I stayed at this school for two weeks and taught the classes usually taken by Douglas Quadling, the head of the mathematics department. The boys in A.1 were just on the eve of an important public examination involving calculus. Quadling said to me, "They will sail through it. You can forget about the examination. Teach them anything you feel like." This I did. Quadling also taught C.6; he was thus in contact with the quickest and the slowest students, which is an excellent practice for the head of a department. With C.6 I did a very gentle introduction to drawing the graph of a straight line, which I believe they enjoyed. Now in both A.1 and C.6 there were boys 14 years old in their first year at this school. If these two groups had been mixed together I would have had, if not an impossible task, at any rate one that called for superb organisation and improvisation - which is a lot to require of every teacher.

Quite apart from any humanitarian considerations, on the most conservative educational philosophy of achieving the maximum academic standards, there is a strong case for permitting students to advance at different rates.

The strongest case however comes from a consideration of what the lockstep does to the children. It is equally disastrous for those who are quicker than the average and for those who are slower. I heard of a boy who came to school already knowing how to write. The other children in the class had never held a pencil in their hands and, reasonably enough, were working at pothooks. The boy who could write, instead of being allowed to read extra books or write or occupy himself in some other way, was forced to do pothooks also. This completely killed his interest in school work generally. On the other hand the slower child used to begin his career by being informed that he had "failed Grade 1". This was damnable. All it meant was that he took, say, a year and a quarter to learn what most children could learn in a year. To label this failure is arbitrary in the extreme. Equally unhappy is the case of the child who is not pronounced a failure, but is allowed to proceed, always a year or two behind the rest of the class, in a state of increasing bewilderment and hopelessness. The lockstep has the advantage of administrative convenience; it has no educational advantage whatever.

As I understand it, the Hall-Dennis dictum that "children should not fail" is directed against the idea that a child should be labelled a failure simply because he does not fit neatly into some over-rigid administrative scheme. This is neither a revolutionary nor a new idea. In Britain in 1919 the Mathematical Association produced a report in which it was mentioned as desirable that children should have the experience of succeeding. Mathematics, it said, was particularly suitable for this. "The tasks can be graduated nicely to the powers of the worker; there is never any necessity to set hopeless tasks .... if the work is too

hard for the pupil the fault is with the teacher, not with the subject."

Some people seem to think the spirit of Hall-Dennis forbids any kind of testing or examining. This I regard as quite mistaken. When I was in New Zealand I ran a mathematics club for 11 year olds in which we gave something like Scout badges. There might be a badge for say "understanding what x means". Note that this badge is given for understanding a concept, not for performing a computation. I would like to see an arrangement by which each child had a record sheet, on which was entered the date when he or she first mastered any particular idea. You will notice the word "mastered". In such a test, the pass mark is 100 per cent, not 50. You cannot build a house out of half-bricks. But of course the testing would be straightforward and informal. For example, one space on the record would indicate whether the child was able to recognize a situation calling for addition, and similarly there would be spaces for an understanding of subtraction, division and multiplication. There would be a space to indicate whether the child understood the system of numeration and could explain and demonstrate the meaning of 243 without actually counting from 1 to 243. Usually difficulties in arithmetic are due to the teacher trying to build on a concept that is not there (or trying to teach without concepts at all). In a very good elementary school I came across a girl who was unsuccessfully trying to find 137 x 4. There were bundles of sticks available, and in the course of trying to get the girl to show me what the question meant, I discovered that she had not realized that ten bundles of ten put together made a hundred. Naturally, it was impossible for her to understand the computation 137 x 4. With the help of the concrete materials she soon saw

what it is all about. (There are two principles in connection with the use of concrete apparatus; children should continue to use concrete materials so long as it helps them to perceive what a computation means; they should never be forced to go on using apparatus once they understand the principle and find the apparatus a bore.)

All over the world you find children trying, and failing, to handle fractions by means of elaborate rules which they do not understand. Before any formal work with fractions there should be testing to make sure that children understand what a fraction is, that they can fill a jar two-thirds full or give you three-fifths of a piece of string. Similarly, in more advanced mathematics, one could have tests of basic understanding as preliminaries to tests of manipulative skill. In the Ontario Report K-13, we envisaged that a brilliant child would leave school with a record of many concepts and processes mastered, a slower child with a record of relatively few mastered. But both would have experienced the satisfaction of mastering something, and both would be happier than they would have been in a lockstep system.

There are two parts of European educational experience on which we could usefully draw in Canada. Both are things that have actually happened - not theoretical speculations.

The first is in the education of the academically abler children. When I first came to the American continent, the thing that struck me most was the intellectual vacuum in Grades 4 to 8. Now this is very important, because it is in these grades that lasting interests are most easily formed; they are the key to the success of education as a whole. In U. S. A. I was once asked to advise a private school which

had both elementary and secondary stages. I said, "Put your very best teachers into the elementary school. Get the children interested then and it will not matter if you have very few teachers in the secondary school; the children can read their mathematics on their own, with an occasional word of advice."

In Europe the general tendency is to encourage children to go ahead as fast as they can; on this continent they seem to be afraid children might strain themselves, so they hold children back. Remember I am advocating a flexible system, and what I am saying at this moment concerns able students only; I would hate to see it forced on all students. For the future university student particularly the rythm of education here is all wrong. In the early stages he has much time to think and nothing to think about; at university, which ought to give much time for thought and discussion, he has a lot to cope with and no time to think about it. To make this more definite we can compare the age at which traditionally a student began a serious treatment of algebra and geometry here - round about 14 years of age - with England, where 11 years of age is customary for the academically inclined student in the state system and 8 or 9 years in the numerous private schools. The student here is thus forced to cram into 8 years what his counterpart overseas does in 10 or 12 years. In the U. S. A. I taught for several years at one of their good colleges; the pressure of work reminded me more of a penitentiary than a university. It was a desperate and self-defeating attempt to make up for what should have been done in a more civilised manner years earlier in the student's life.

The great difficulty, of course, is to find elementary school teachers who know enough to teach more advanced topics to their quicker students.

This is how things usually are; it is the adults, not the children, who are the limiting factor. People have all sorts of complaints about the younger generation, but each generation is the logical consequence of the generation before it. A few centuries ago it would have been regarded as absurd to expect every child to learn to read. But as soon as reading became part of the accepted adult culture, it became something that children could acquire, and were expected to have.

The second European experience on which I want to draw is concerned with exactly the opposite type of child, the physically restless, practically minded child for whom most academic subjects are either incomprehensible or utterly boring - at any rate with the traditional, abstract, verbal presentation. In the last decade or two there has been a remarkable movement of elementary school teachers in Britain to teach children mathematics and other subjects through children's own interests and activities. You will find a description of this in the booklet "Mathematics in Primary Schools" published by the Ministry of Education, and in the film "I do and I understand". The idea is not new. I knew a wonderful public elementary school which operated from 1920 on in the slums between Manchester and Bolton, and achieved some notable academic results as well as providing a real education for all the children. But until recently such schools were exceptional. The most striking thing about the present trend in England is that it seems to be a genuine mass movement of elementary teachers.

In Canada there is enormous scope for someone interested in developing this idea. North American is supposed to be the home of democracy, but this is an illusion - at any rate, so far as applying imagination to the education of the less academic student is concerned. This remark

applies both to the U. S. A. and here. I sometimes meet an appalling snobbery in attitudes to the four year student. Certainly no one has worked out a cirriculum corresponding to his point of view, which arrives at words and ideas through the medium of things and activities.

I have been trying for some time to give practical effect to these ideas with the help of students at the College of Education. This year we are involved in three small clubs for elementary school students. These clubs meet once a week, after school hours. One is at Huron Street School, and provides a serious introduction to algebra for gifted fifth and sixth graders. Another is a Science and Mathematics Club, which Mr. John Buchanan provides at Central Tech for the benefit of children in nearly elementary schools. The third is at King Edward School, and is for half-a-dozen or so restless boys selected by the Principal. The original idea was that this club should meet out of doors whenever possible and engage in all kinds of athletic activities, with great recording of averages, graphs of individual progress and so forth. It may well evolve in some other way. All three clubs are welcomed by the children. The members of the Central Tech club were heard bemoaning the Thanksgiving Holiday, because it would cause them to miss their club on that day. The spirit in the other two clubs is similar.

This welcome from the children concerned makes such little clubs pleasant to run. The club, outside school hours and free from all regulations, provides an excellent situation in which to learn the techniques of a new approach to education.

MATHEMATICS IN THE SECOND HALF OF THE TWENTIETH CENTURY*

A. J. Coleman, Head, Department of Mathematics, Queen's University

We are here today to celebrate the opening of the Mathematics Building at Queen's. In so doing, we honour Ralph Jeffery for his signal contribution to Queen's University and to Mathematics in Canada.

At the same time, we should acknowledge the wisdom of the Government of Ontario which places education so high on its priorities. Jeffery Hall was one of the last buildings for which financing was approved before the current capital formula was introduced. It has been predicted that the incidence of austerity means that Jeffery Hall is the last example of "monumental" architecture at Queen's. If this proves to be the case, the mathematicians may rejoice to have just got under the wire! In more sombre mood, it is a widespread - almost universal - observation that when an active Department, which has been scattered in a variety of sub-standard sites, enters a handsome new building, it immediately begins to decline. My colleagues and myself will face this challenge with the famous Queen's fighting spirit.

A few decades ago it was the custom for a professor, upon his installation, to deliver an Inaugural Dissertation in which he was expected to justify his particular discipline situating it in relation to other diciplines and demonstrating its extraordinary importance. When I came to Queen's from the University of Toronto in 1960 as the Nathan Dupuis Professor and Head of the Mathematics Department, no such task

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* An address given at the opening of Jeffery Hall, Queen's University, on October 17, 1969.

was required of me. However, the opening of these splendid facilities cannot fail to pose the questions "What is Mathematics?", "Is Mathematics sufficiently important to mankind, and specifically to homo sepiens Ontarioensis, to be worth the $2,300,000 spent on Jeffery Hall?"

These blunt questions lead on to a host of other problems of wide-ranging significance none of which have simple or agreed-upon solutions. For example, to what extent is the increasing application of Mathematics in all aspects of society the cause of the depersonalization of life which leads so many people - both young and old - to intense frustration which strikes out in wanton acts of destruction? Or, on the educational front, have mathematicians been so gripped by the delights of their esoteric knowledge that in undergraduate and graduate teaching they have concentrated exclusively - or, at least, chiefly - on reproducing their own species and failed to develop a curriculum which would best serve the common good?

In my view, a professor who does not profess is a dull, and uninteresting non-entity, so that it would be a form of trahison des clercs if I failed to face these questions squarely and to declare myself on them. At the recent observance of the tenth anniversary of the Ontario Mathematics Commission, six of the "founding fathers" were honoured and in particular, I was commended for bringing to the early days of the OMC "an infectious and unrelenting enthusiasm". If I now approach my present task with that unrelenting zest which my friends and colleagues must find altogether unbearable, you should not infer that the opinions I express are, in fact, held as dogmatically as they sound:

In 1664, Isaac Barrow ascended to the Chair of Mathematics as Lucasian Professor at Cambridge. His lectures, which Newton attended, contained essential ideas preparatory to the discovery of the differential Calculus. He resigned the Chair at an early age to make way for Newton and in order to devote himself to the study of theology. His Inaugural Oration contains the following statement:

"The disciplines of Mathematics serve to increase and corroborate the mind to a constant diligence in study. They wholly deliver us from credulous simplicity, most strongly fortify us against the vanity of scepticism and perfectly subject us to the government of right reason. I affirm that while the mind is abstracted and elevated from sensible matter, distinctly views pure forms, conceives the beauty of ideas; the manners themselves are sensibly corrected and improved, the affections composed and rectified, the fancy calmed and settled, and the understanding raised and excited to more divine contemplation".

In a moment, we shall return to an exegesis of Barrow, but, first, contrast the passage we have just heard with the following excerpts from a lecture delivered by Herbert Marcuse at the University of California in July, 1966. As you are aware, many of the New Left activists look to Marcuse when they seek intellectual respectability for their social criticism. He writes:

"The proposition I want to put forth is the following ... the scientist is responsible for the social consequences of science .. the scientist is no longer the dissociated, withdrawn researcher but has become the pillar of the established institutions and policies. ... At the same time, the gap between pure and applied science is narrowed; the most

abstract and formal achievements in logic and Mathematics are translatable into very concrete and material values ... Today there is no conflict between science and society (the established society); they propel each other in the established direction of progress; a direction that appears increasingly dangerous to humanity."

Barrow's defense of Mathematics is concerned with its effects on the individual student. Doubtless if he were here today, he would be accused of a paternalistic attitudes and mental habits are good for everyone and in particular for students. He holds that scepticism (presumably about Christianity) is vanity. He is convinced that studying Mathematics provides a discipline for ones emotions that leads to an improvement of our behaviour and even prepares us to receive the beatific vision - or, as a Zen Buddhist might put it, to experience satori. However, the phrase which I like best is "credulous simplicity". Certainly none of us would wish to be accused of credulous simplicity. You will recall that it is into the mouth of Simplicius that Galileo, in his famous Dialogues on the Two Chief Systems of the World put the most absurd arguments, - which were identical with those of the reigning pope. According to one version of history it was this naive indiscretion which got Galileo into trouble: The Pope did not relish being portrayed as the mouthpiece of credulous simplicity:

Barrow's thoughts perhaps ran as follows. Anyone who has spent hours vainly trying to solve a well-posed problem in Mathematics, if he has any sense at all, will not imagine that the vague, ill-defined and important problems of real life can be resolved by some simple-minded panacea. Currently, in North America, students seem to be fleeing from the "hard" disciplines of physical, mathematical and engineering

sciences into those of psychology and sociology. I think we should sympathize with this trend in so far as it reflects an awakening of social sensitivity by contemporary young people. However, viewed from the dour point of view of the older generation of scientists it would appear that this trend can result only in confusion and frustration. The vast majority of young people today have had no taste of the kind of discipline of mind and "fancy" (emotions) of which Barrow spoke. The problems of our complex technological society will not be resolved by naive optimism and credulous simplicity. In my youth I was greatly attracted by Marxism and still find Marx's emphasis on the importance of economic factors in determining the social process most illuminating. However, in my view, the naiveties of myself and my friends in the mid-thirties now appear mature and tested convictions compared with the emotive and ill-considered puerilities that issue from the New Left or a North American Maoist.

Much of the ill-directed rage which characterizes contemporary protest movements stems from the frustration of young people who imagine that the problems of social organization are susceptible of a quick solution by men of good-will. When a facile solution does not drop from the skies they are convinced that there is a lack of good will, - that they deal with men of evil will.

In his marvelously allusive - almost poetic - essay on "Mathematics and the Good", A. N. Whitehead attempts to explicate the role of pattern in experiences which we call "good". The highest achievements of value presuppose complex patterns manifest in the inter-connectedness of events. Hence for Whitehead Mathematics is "the intellectural analysis of types of pattern".

Perhaps it was Benjamin Peirce with his definition, in 1870, of Mathematics as "the science that draws necessary conclusions" who first realized that Mathematics is totally abstract. It is never concerned with any particular real entity. In the chapter on Mathematics, in Science and the Modern World Whitehead says:

"When we think of mathematics, we have in our mind a science devoted to the exploration of number, quantity, geometry, and in modern times also including investigation into yet more abstract concepts of order, and into analogous types of purely logical relations. The point of mathematics is that in it we have always got rid of the particular instance, and even of any particular sorts of entities. So that for example, no mathematical truths apply merely to fish, or merely to colours. So long as you are dealing with pure mathematics, you are in the realm of complete and absolute abstraction. All you assert is, that reason insists on the admission that, if any entities whatever have any relations which satisfy such-and-such purely abstract conditions, then they must have other relations which satisfy other purely abstract conditions.

"Mathematics is thought moving in the sphere of complete abstraction from any particular instance of what it is talking about."

"Nothing is more impressive than the fact that as mathematics withdrew increasingly into the upper regions of ever greater extremes of abstract thought, it returned back to earth with a corresponding growth of importance for the analysis of concrete fact."

The abstract character of Mathematics is already evident in elementary arithmetic. What precisely do we mean by the numbers one, two, three? It is easy to understand the numerals one, two, three -- they are marks,

signs, symbols. But the numbers one, two, three are of quite a different order. They are abstract. The number two is "present" or "participates" in every pair of natural or supernatural entities; but the number two can never be identified with any particular pair.

The power of these abstract entities to control, organize, dominate our life manifests itself even in arithmetic - one is perhaps constrained to say - especially in arithmetic. These beings one, two, three appear everywhere - in family life in commerce and industry, in science, in war in our attempts to conquer interplanetary space.

I seize upon this example of number to illustrate the abstract nature of Mathematics merely because of all mathematical objects, numbers are the most familiar. However, it is a grave error to think that Mathematics is concerned solely, or even chiefly, with numbers and geometrical figures. One of the most characteristic aspects of the modern development of Mathematics is that since the middle of the nineteenth century Mathematics has broken out of its classic arithmetic - geometric chrysallis.

A fact of capital importance for the philosopher interested in the relation of mind to matter is the extraordinary rapidity with which esoteric entities present first in the imagination of pure mathematicians find application in the physical, engineering, social and biological sciences. In one issue of the Journal of Mathematical Physics, I noted applications of algebraic topology and of von Neumann's theory of continuous integrals of hilbert spaces to study and solve some problems in quantum mechanics. A few years ago everyone would have said that these two theories would forever remain the toys of far-out pure mathematicians and would never be applied to mundane subjects like physics.

One final point concerning the invasion of all aspects of our life by Mathematics: The effective discovery of electronic computers fifteen or twenty years ago gave man an almost inconceivable increase in his arithmetic powers. However, the true significance of the computer is not that it gives us a method of making arithmetic calculations quickly - though this was their first and most obvious application. Rather, the importance of the computer derives from its capacity to make comparisons, to decide and to choose one next step rather than another and to solve logical problems. This capacity opens myriads of new possibilities for the application and realization of the purely abstract ideas of mathematicians.

In truth, the computer has forced a transformation of the form of our life more radical than that effected by the industrial revolution. But whereas the industrial revolution took centuries, the computer revolution has required only a decade to give rise to effects of comparable moment.

In order for man of the twentieth, and future, centuries to feel at home in his mathematized world he needs to understand the nature of Mathematics and the manner in which it is employed to study and control social, economic and technical affairs in the modern world. This means that increasingly it will be the duty of mathematicians to convey to their contemporaries, and to the young people who succeed us basic ideas about Mathematics without which they will feel surrounded by an incomprehensible and terrifying world.

This is the context in which we can best see the significance of the reform of school Mathematics. This reform which is happening simultaneously in all industrialized countries does not result from some absurd

whim of mathematicians. Rather it is a response to a deep human need. In order to understand the world in which he lives, contemporary man must understand the nature of Mathematics at a much deeper level than that normally reached by the traditional programme. Formerly it was enough to teach elementary rules of arithmetic and the simplest properties of geometric figures. Now, men wish to understand the true meaning of Mathematics and how it can exercise so great practical influence.

Possibly it was Sputnik which roused popular attention to the importance of Mathematics. At Queen's, in any case between 1957 and 1969 there has been an extraordinary increase in demand for Mathematics by students in all parts of the University. The figures quoted in the Programme Notes (viz: 4, 6, 9, 40 full-time faculty in the years 1920, 1952, 1960 and 1969 respectively) clearly illustrate this growth which I attribute chiefly to the increased awareness throughout society that understanding of Mathematics is desirable. The increase in staff at Queen's is not proportional to student enrollment since teaching-load has been lowered in order to free the staff to devote more time to research. This was necessary in order to support the Graduate programme and also because the greater volume of Mathematics which now exists demands greater time to master.

The quotation from Barrow, dealing with the effect on an individual studying Mathematics, conjures up a rather idyllic atmosphere. That of Herbert Marcuse, accuses scientists, and specifically mathematicians of buttressing the establishment and contributing to a form of "progress" that "appears increasingly dangerous to humanity".

A few years ago I was startled by an attack on mathematicians as a group. Richard Bellmann - a well-known American applied mathematician -

was addressing a group of mathematicians and engineers in an elegant Santa Monica Hotel at the beginning of a course in Control Theory and Dynamic Programming. I paraphrase his opening remarks: "Gentlemen, there is no use crying over spilt milk. We must simply recognize the objective situation. Ninety-five percent of Mathematics Departments in North America have opted out of civilization." He went on to blame this horrible situation on the group of French authors who publish under the pseudonym Nicholas Bourbaki, whom he accused of excessive abstraction and of taking a supercilious view of applied Mathematics. Bourbaki has made a marvelous contribution to bringing order and coherence into contemporary Mathematics, and as an ardent devotee I challenged Bellmann's attack on Bourbaki. However, his main point is essentially correct. The attitudes in the North American Mathematical community which Bellmann criticized existed long before Bourbaki was born.

One of the crucial points at which these attitudes become clear is the content of the curriculum for Mathematics Majors.

Honours Curriculum. In Genesis, God is reported to have decided "Let us make Man in our own Image". There is evidence that His attempt has not been an unmitigated success! However, where God failed, university professors have succeeded. In the physical, biological and mathematical sciences the Ph.D. process is largely given over to a reproduction of the species. The highest ambition of the Ph.D. student is to be a university professor.

In discussion of Curriculum, it is almost impossible to persuade professors to delete anything. Everything which they learned is ipso facto essential for the student. The effective definition of the essential core of any discipline is always precisely the union of all that the

Professors and Associate Professors learned at the corresponding stage in the educational process!

The first week that I arrived at Queen's this point was brought home rather vividly to me. A fourth year student who was majoring in Mathematics and minoring in Physics had a time-table clash between Algebra and Thermodynamics. Since, in our view the Algebra course accounted for about one-half of his fourth year major, I entered into negotiations with the Physics Department to discover if the student could sutstitute another Physics course for Thermodynamics. This, I was assured, was quite impossible. It was unthinkable that a Physics minor should graduate without a course in Thermodynamics! My gentle chiding that the great thermodynamicists Boltzmann and Planck probably never had a course in Thermodynamics but got through life fairly well, had no effect. Our student has had to manage to go through life devoid of fourth year algebra. I dare tell this story only because at present the Physics and Mathmatics Departments at Queen's enjoy the most cordial relations! Analogous situations arise in all Departments. There are mathematicians who will battle all the way to the stake defending the proposition that it is unfair to allow a Math. Major to go naked into the world unclothed by Sylow's theorem, or a Fourier Integral, or the Stone-Weierstrass Theorem etc.

The mathematicians (and my remarks apply almost equally well to most of the established disciplines) have lavished their love and attention on their prospective children so that the backbone of the curriculum is aimed at producing university mathematicians. After this high aim has been attended to, such interest and energy as remains - is directed to

the so-called "service courses" for engineers and General Course students who, normally, therefore only get the leftovers, or are subjected to a programme which is not conceived as a whole. This situation can be effectively redressed only when the attitudes of mathematicians towards the intending non-mathematician changes. We need to combine with an interest in our subject-matter a larger concern for the social importance of Mathematics and a deep respect for the needs of individual students.

I do not favour abandoning the Honours Programme since it is very much in the national interest to produce a steady flow of men and women who have the greatest possible grasp of Mathematics. Among our universities in Ontario we should be creating experts in every field of Mathematics.

Happily a good spirit of co-operation exists among the Chairmen of Ontario Mathematics Department, many of whom are here today, and whom it is our privilege to welcome for their regular meeting tomorrow. However, the Ontario universities have certainly not yet found organs of co-operation of the power and delicacy which we sorely need if we are to best serve the national interest.

Though at Queen's we have tended to conform to the general North American ethos as far as the Ph.D. in Mathematics goes, our undergraduate programme has been relatively satisfactory in my view. The Honours programme is quite flexible so that the degree of difficulty and abstractness can be varied within quite wide limits to suit the abilities, interests and professional intentions of the students. Until two years ago our offerings for General Course students were quite weak. However, the splitting of the traditional first year Calculus into four types of courses incorporating various amounts of algebra, statistics or computing

enables us to serve the needs of different groups of students more adequately. With respect to service courses to engineers it has been the practice at Queen's for over twenty years to have a senior member of the Department who was an engineer in background - first Morris Wood and now Jack Hogarth - to act as an interpreter of the needs of engineering students to the mathematicians and to advise the engineering faculty to decide what offerings in Mathematics would be suitable for engineers. It was Professor Hogarth who conceived our Course J, - a programme in Mathematics and Engineering in which the student devotes one-half of his time to Mathematics.

Currently, Mathematics is in high repute with the public and so mathematicians are paid well above the average of our society. There are obvious dangers in a situation where high salaries are awarded for the prestige attached to mastering of material about which it is difficult or impossible to provide objective criteria of value. Doubtless a similar problem occurs in some other sciences. However, the relatively esoteric quality of most current research in pure Mathematics makes us especially vulnerable.

To my mind it is salutary that under formula-financing in Ontario the payments are related to teaching-load, since this is an objective criterion and recognized as related to social usefulness. In Ontario this economic fact will surely mean that good teaching will gain in prestige.

The mathematical community must also find ways of giving greater prestige to two other activities, (a) the attainment of broad and comprehensive mathematical scholarship and (b) the development of attitudes and knowledge which will enable a mathematician to collaborate with colleagues

in other disciplines. If we learn to do this and if we find ways of eliminating the trivial, mathematicians will be able to fulfill Whitehead's vision of the significance of Mathematics and, hopefully, escape Marcuse's stricture that we are a danger for humanity.

THE HIGH SCHOOL MATHEMATICS SPECIALIST

Marc A. Laframboise, Associate Professor of Math. Ed., University of Detroit

Is the proportion and quality of teachers with a Type A specialist certificate in mathematics satisfactory in Ontario? Would the province like more of them?

The role of the teacher is vital in the successful carrying out of any curriculum organization and orientation. The high school teachers must be trained to teach the subject matter in the best tradition of modern mathematics.

Teachers of mathematics who were not graduates in honours mathematics, but who are eager to keep abreast of developments in the field, must seek to do so through summer courses or extension courses. What is there at the universities for them? Advanced courses are quite often designed for research-oriented students. These courses are unsuitable for teachers returning to university after an absence of five or ten years.

New courses should be designed to meet the needs of secondary school teachers. In summer and regular sessions, more courses might be offered which bear immediately upon the content of the high school curriculum rather than, say, upon the staple calculus sequence.

In some districts, the number of teachers desiring the same course might make it both feasible and economical to bring a college instructor into the school district. It might thus be possible to give suitable modern college courses to the teachers under favorable conditions, in their own schools, during the regular academic school year. There

should be no reason why the caliber of these college courses could not justify university credit.

What of future graduates (non-specialists) in the field of mathematics who will pursue teaching? They should not find it necessary to fill gaps in their training. Undergraduate programs of teacher-producing institutions should provide a sound background of contemporary mathematical material for the non-specialist and produce teachers adequately equipped to deal with new curricular patterns.

Decisions for or against teaching as a career are frequently made during the late secondary or early college years. If mathematics courses are given mechanistically to future teachers, there is a good chance high school classes may be so taught. Universities do give careful consideration to what mathematics is taught and how it is taught to prospective high school teachers.

Now what of the Type A specialist in mathematics? Certainly, the province would like as many as possible. However, four full years of mathematics from grade XIII seems perhaps extravagant for high school teaching. Would it help somewhat if special (shorter by one year) honour courses for specialists in mathematics were inaugurated?

The academic requirement for the specialist Type A certificate in mathematics is roughly 8 honour courses (50 hours each) in mathematics, with at least a grade B average and some 5 or 6 full science courses with at least a grade C average. This minimum is, in substance, equal to the first three years of the four year course from grade XIII. Teachers render invaluable service to the community and a teacher's influence is felt long after youth has left school. In view of the importance of the teacher's position, it seems reasonable to expect

a very good preparation.

Could not, with mutual benefit to teachers and students, the Type A requirements become a 4-year successful study from grade XII? Graduates of such a program would, of course, have to spend two more years to satisfy M.A. requirements, should they desire to proceed to a higher degree.

This writer would have in mind a program like the one which follows. The course titles are, in most instances, self-descriptive. A course involves 50 hours of class work, i.e. about 3 hours weekly for a semester of about 16 or 17 weeks.

Grade XIII (Year I)

(1) Math. A and B (Algebra, Geometry, Trig.)

(2) English

(3) French or German

(4) Physics

(5) Chemistry

(6) History or Biology

Year II

1st Sem.			2nd Sem.		
Hrs. Wkly			Hrs. Wkly.		
3	Calculus (Apostol)	50 hrs.	3	Calculus (Apostol cont'd)	50 hrs
3	Alg. Geom.(Mat. Det. Vect. Transf.)	50 hrs.	3	Alg. Geom. cont'd	50 hrs
4	Physics & 3 Lab hrs. Mech. Prop. & Structure of Matter	64 hrs.	4	Chemistry & 3 lab hrs. Struct. & Prop. of Atoms & Molecules	64 hrs
3	Numerical Analysis & Computer Science	50 hrs.	3	Mechanics (2-Dim. Vect. Alg. & Calculus)	50 hrs
3	English	50 hrs.	3	Modern Histroy	50 hrs
16	hrs. & lab.		16	hrs. & lab.	

Year III

1st Sem.

Hrs. Wkly.		
3	Intermediate Calculus & Diff. Eq.	50 hrs.
3	Modern Alg. Number Th., Groups, Rings, Ideals, Fields, Galois Th.	50 hrs.
3	Economic Theory	50 hrs.
3	Physics Electromagnetism & Optics	50 hrs.
3	Observations in high school and methods of tchg. math.	50 hrs.
15	hrs.	

Year III

2nd Sem.

Hrs. Wkly.		
3	Linear Alg.	50 hrs.
3	Mod. Alg. (cont'd)	50 hrs.
3	Money, Banking, Nat'l Inc.	50 hrs.
3	Atomic Physics	50 hrs.
3	Observation & Methods of Tchg. Math. (cont'd)	50 hrs.
15	hrs.	

Year IV

1st Sem.

Hrs. Wkly.		
3	Real Analysis	50 hrs.
3	Rrobability & Statistics	50 hrs.
3	Foundations of Math.	50 hrs.
3	General Psychology	50 hrs.
3	Practice Teaching in High School and Methods in math.	50 hrs.
15	hrs.	

Year IV

2nd Sem.

Hrs. Wkly.		
3	Complex Analysis	50 hrs.
3	Mathematical Statistics	50 hrs.
3	Topology	50 hrs.
3	Philosophy (Ethics)	50 hrs.
3	Practice Tchg., etc., cont'd and Mathematical Methods	50 hrs.
15	hrs.	

The first and greatest factor in creating and maintaining interest in mathematics is a sympathetic, well-informed, competent, confident, enthusiastic and inspiring teacher. This kind of teacher is the real sine qua non.

Candidates who are successful in the above course, would certainly be of high academic character, would no doubt develop into trusted and admired teachers, and would be ready for service one year sooner.

POSITIVE AND NEGATIVE

A CROSSROAD IN NOTATIONS

R. A. Stall, University of Waterloo

A short time ago the reading of the minus sign was a simple matter - it was always read the same way. For example

$$-4 - \sin(-\theta)$$

was read as

minus four minus the sine of minus theta.

The words "positive" and "negative" were reserved exclusively for such phrases as "a positive number" and "a negative number".

This oral terminology is still with us, being used exclusively by virtually all who left school ten or more years ago, except possibly for teachers. We now have with us another terminology as well, in which we read some (or all) minus signs as "negative" (some use "opposite" instead): This terminology may have been intended as a temporary expedient for teaching purposes, but indications are that in most cases it is used to the end of high school.

In this article I will try to convince the reader

(a) that the older terminology is a far better compromise between practical convenience and the avoidance of possibly confusing ambiguity of meaning

(b) that the simultaneous presence of both terminologies allows for almost total confusion, in the sense that the various readings have become almost completely interchangeable

(c) that this matter is well worth doing something about, being more serious than the relatively slight ambiguities in the original terminology, concern for which resulted in the new language

(d) that the best remedy is to return to the original terminology, properly presented

(e) that the side effects of implementing (d) could be highly beneficial.

I do not accept the commonly offered rhetorical arguments that a return (essentially) to an older practice is "reactionary" because the practice is older and is <u>therefore</u> bad, that to (essentially) banish a terminology (the new one) is a "negative" policy and is <u>therefore</u> bad, and that there is "nothing new" in my arguments and that <u>therefore</u> we shouldn't bother seriously to consider them. The more respectable argument that we would have to be extremely fussy in implementing (d) because then we would have to rigidly proscribe certain practices is rejected because, apart from a short transition period, occurrences to be proscribed just wouldn't occur, any more than we now have students saying "reverse five" or "gain three".

Let us begin with (a) by noting some characteristics of the older terminology. The sense of "minus" in "minus five" is that of "opposite") (otherwise there would be complications in setting $x=5$ in $-x$, where the sense of "opposite" is necessary). Minus five is a negative number, not because "minus" carries the sense of "negative" - it certainly doesn't - but because five is a positive number and the opposite of a positive number is a negative number. "Opposite" is a relative term, whereas

"negative" is absolute: This is an important distinction which is scrupulously (but without fuss) maintained. Perhaps the crux of this terminology is that signs of (absolute) quality are not used: when operating with particular numbers, such numerals as "5" and "$\frac{1}{2}$" are always interpreted as positive numbers (that is to say, as names of positive numbers), and when using variables one has to indicate positiveness or negativeness in some other way. One does not carry along in one's terminology the difference between the natural number 5 and the positive integer 5 any more than one carries along the difference between the positive integer 5 and the real number 5.

If "opposite" is what is meant, then why not say it? The starting point of the recent fashion may have been concern over using "minus" in the two senses of "opposite" and of subtraction. Let us humbly confess this ambiguity, and at the same time point out that new ambiguities have arisen in trying to eliminate this old one, and that it now seems to be recognized that ambiguities perhaps aren't always serious anyway. The blow of this particular ambiguity can be softened considerably - the reader will be asked to settle for $1\frac{1}{2}$ meanings, rather than 2 - by noting that

$$0 - n = -n.$$

This turns up in such practical manipulations as

$$\frac{4}{2} - 2 - 3 = -3.$$

May we make it quite clear that we are not recommending that this point be hammered relentlessly home as an official dogma, reinforced by innumerable exercises designed to create the reflex action of instantly and always reading "$-n$" as "$0 - n$". We recommend merely that the student (who probably couldn't care less about the ambiguity anyway) be rather casually

informed that "minus enn" for "- n" does at least make sense for the above reason. The born worrier can, if he persists, also be reminded that "-" appears in two rather different contexts in "a - b" and in "- b": it would, in fact, take a most ingenious individual to commit an actual error because of remaining ambiguity. Yet another reason underlying the terminology of "plus three" and "minus three" can be suggested: the exact role of this reason would be very hard to establish, but it seems highly plausible that it was not negligible. The interpretation of positive and negative integers as credits and debits, or gains and losses is most important in the history of our number system. Under this interpretation, +3 is interpreted as the addition of three, and -3 as the subtraction of three, and this leads naturally (I am not saying "inevitably" or "by overwhelmingly concincing logic") to the readings of "plus three" and "minus three".

To sum up, in the terminology which was generally considered standard until recently, and which is still standard among mathematicians, "positive" and "negative" were each used in precisely one sense, whereas "plus" and "minus" were each used in, strictly speaking, two senses. These two senses were, however, very closely related both in meaning and via convenient manipulation.

To describe currently fashionable terminology is harder, because practice varies considerably. We will describe what seems to be near the norm.

One peculiarity has made it very hard to discuss this matter rationally. The following imaginary dialogue may serve to bring this out: it is an essentially faithful representation of some of my own experiences.

N. (new terminology person) The former way of treating this topic was not rigorous. It was ambiguous - you confused symbols of quality with symbols for operations. We now distinguish between the counting number five and the integer positive five, and we refer, for example, to negative five rather than minus five because "minus" is used for subtraction.

O. (old terminology person) But the practice which has in fact resulted also involves ambiguities - in many cases more of them. (Shows this.)

N. You worry too much about ambiguities: they don't really bother most students anyway.

(No attempt will be made to deal with this point of view - if it is one.)

The new usage seems to have begun with readings such as "positive three" for positive integers and "negative three" for negative integers, written "$^{+}3$" and "$^{-}3$", respectively, the raising of the signs being considered enough of a difference to avoid confusion with "+" and "-". A total revolution being out of the question, it was felt necessary to make room for "-3", this now being regarded as "$^{-}3$" with the minus sign moved down, and read as "negative three". It was also realized that "^{-}x", as originally explained, just didn't make sense (some early New Math texts apparently weren't aware of this) so that "-x" had to be accomodated (also for the same reason as "-3"). Now the picture gets cloudy, for "negative eks" doesn't make sense in the original sense of "negative": some respond by saying "opposite eks", while others, beginning to sense complexities, resolutely stick to "negative eks", admitting when cornered that "negative" now has, strictly speaking, two meanings. At this point yet another complication occurs, for (in a way that has not

yet been explained) the practice has become widespread of using "negative" instead of "minus" for actual subtraction. Some must now admit to three uses of "negative".

Rather than try to weigh the relative popularity of various new terminologies (a rather meaningless task), we present two of the current practices, between which the others all seem to lie.

$$-3 - \sin(-x) \text{ is a negative number}$$

is read as

1) negative three negative sine of negative eks is a negative number

or as

2) negative three minus the sine of opposite eks is a negative number

1) which is distressingly widespread, excels even the older terminology in the possession of the feature whose criticism started the whole business. It is entirely possible to be unconcerned about a certain amount of ambiguity, and yet be very much concerned about this much. By changing the first three "negative"'s to "minus"'s, the really important distinction is maintained (or restored), and the three occurrences of "minus" can, if suitably presented, by considered to refer only to the two (or $1\frac{1}{2}$, if our arguments were successful) senses explained above. The result is then the old terminology.

2) is the most meticulous version, but I submit that this imposes a non-trivial burden in everyday practice. For example "-x" is read as "opposite eks", but on putting $x = 5$ we get +-5", read as "negative five". An improved version might be to change the first "negative" to "opposite": thus, "negative" would be used only in such contexts as "negative number".

This would be a great improvement over what is now going on and in theory it seems optimal. The reason for giving the old terminology the edge is that it is already established (at least outside of the schools), whereas the improved 2) is not well-established, even in the schools. This is not to be taken lightly.

Here the case rests for point (a).

Point (b) should now be clear. "-5" is read sometimes as "minus five", sometimes as "opposite five" and sometimes as "negative five". "5 - 3" has two common readings (a third, "five opposite three" is very rare). The terminology met in the outside world cannot be ignored, nor can that in which the subject is first learned. One must consider them together, and the result abundantly demonstrates the validity of point (b).

Point (c) must be admitted if one believes at all that consistent use of a good terminology is helpful. Confusion of "negative" with "opposite+ (which is one of the outstanding new features) is particularly risky: for example, there are no "negative" complex numbers, yet all complex numbers have opposites. We now have not only different meanings for the same words, but different words with the same meanings. If a teacher wants to help a student who is having trouble, say with absolute values, then a clear understanding about terminology is essential, and if the student comes from another school, some preliminary work may have to be done just to find out in what mathematical language he was taught. It is fine to be indifferent about multiple terminologies - until one tries to remove misunderstanding.

If one admits to point (a), then a return to the original terminology is clearly desirable - providing such a return is practicable in present circumstances. Our bridges have not been burned behind us in that the old

terminology is still very much alive: in fact, if one considers the total mathematics community, it still dominates. Another factor which makes such a change relatively easy is that, except for a few cases where readings are given when new symbols are introduced (usually just one sentence per symbol per book), textbooks are neutral in such matter - the oral practice is established entirely by the teacher. The most serious problem will be the extent to which the treatment of the integers in the textbooks can be made to fit our recommendations: This depends very much on the text under consideration. In nearly all cases the problem vanishes after raised signs are eliminated, and as long as raised signs are used only as a temporary device, and not incorporated into the final working notation, there is little cause for concern. In any case, the dependence of teachers on details of textbook treatments is being discouraged. Certainly dependence on any one book is vanishing, and a small pamphlet should suffice for those that would need help. Compared with changes that have taken place already, the proposed change in terminology is trivial.

In the case of a terminology such as the improved version of 2) outlined above, which some may prefer, the practical problems of implementation are more impressive. One would not be using something which is already established - one would be trying to abolish it.

The matter of side-effects mentioned in point (e) is perhaps an after-thought, but may be of some interest. Two issues are involved - that of our concern for the quality of treatments given to various topics, and that of our willingness to change what is new, as well as what is old. Many changes have taken place in a relatively few years.

To what extent are we interested in doing a good job in implementing these changes, rather than just getting new programs started? If we are interested in workmanship, then we must be prepared to respond to the imperfections which are inevitable in any new program, especially if it is carried out in a hurry. Many individuals have responded very well, but opportunities to emphasize the relative importance of workmanship are of considerable value. A large-scale attempt to fix up our ailing terminology would be most beneficial in this regard. It would also demonstrate that what is new (and has been given a fair trial) is no more sacrosanct than that which is old.

SOME ELEMENTARY PROGRAMMING ASSIGNMENTS

The following is an excerpt of a report presented by the sub-committee for project 3 of the Advisory Committee on Colleges of Applied Arts and Technology, Ontario Mathematics Commission. The members of this sub-committee are

C. D. McKay, E. N. McKeown and A. I. Vertesi.

Introduction

This set of programming assignments is designed to provide a sequence of graded problems for students just learning programming. The problems have been formulated with Fortran in mind as the programming language but with some adaptation could be used with other high level languages of the scientific type.

With a few students it is only necessary to provide them with a set of notes and access to a computer but for most it is recommended that the lectures, or better briefings, should cover only the material needed to do the assignment they are being given. It is also highly desirable that a programming consultant be readily accessible to the students at least at the beginning. It is very easy to underestimate the magnitude of the communication gap between beginning students and the computer.

One problem that causes some difficulty, but the assignments do not cover, is the difference between A**J and A**B. Depending on error diagnostics available at run time and whether or not a Fortran readable interval timer exists a good assignment could be designed to illustrate the differences between these two forms as regards round off error, timing and what happens when A is negative.

It is hoped, however, that instructors will find this set of problems useful.

Royal Military College
Kingston, Ontario,
January 1969.

C. D. McKay,
For Sub-Committee, Project 3.

Problem 1

1. Write a programme which

   a. assigns the values 1/7, 1/10 and -1/3 to variables called A, B and C respectively using the statements

```
A = 1./7.
B = 1./10.
C = -1./3.
```

b. and prints out the values and A, B and C on one line using the format statement

```
100 FORMAT(1X,5F14.8).
```

c. and using three names of your own choosing calculates and prints the values of 7. * A, 10. * B and 3. * C, using the format statement of b. part.

d. and then calculates and prints (using the format statement of b. part).

```
DD = 5. * A + B * 8. + C
EE = C + 5. * A + 8. * B
FF = A/B * C
GG = A/(B * C)
HH = A/B/C.
```

2. Modify the programme above by repeating the print statement of 1.d four times but changing the format statement number to 200, 300, 400 and 500 and then add the format statements

```
200 FORMAT(1X,2F14.8)
300 FORMAT(1X,2F14.6)
400 FORMAT(1X,F20.8)
500 FORMAT(1X,3E15.8)
```

3. Alter the programme of part 1 to read a data card to assign the values to A, B and C instead of the statements A = 1./7., B = 1./10., C = -1./3.. The data card should be punched

.14285714	in columns 2 to 10 inclusive
.1	in columns 12 to 13 inclusive
-.33333333	in columns 21 to 31 inclusive.

The format statement to be used with the read should be

101 FORMAT(3F10.8).

Problem 2

1. a. Write and run a programme that will calculate and print on one line the values of I, K, J, L, JL, A, N and M, as given in the expressions below, for I = 1 initially to I = 25 by increments of 1. (Note that A is a real or floating point variable)

K = I/5

J = 3 * I/5

L = 1/3 * 3

JL = I - L

A = I

N = A/3. * 3.

M = 3. * A/3.

b. Examine the results and explain the differences in J, L. N and M. Note the quantities L and JL and explain how these two statements may be used to convert I feet into so many yards plus so many feet.

c. Explain how I - I/2 * 2 could be used to test whether I is even or odd.

d. Give the coding necessary to convert IF inches into so many feet plus so many inches, and IS seconds into so many hours, minutes and seconds.

Problem 3

1. The range of a projectile fired at an angle θ above the horizon with a speed V feet per sec. and neglecting air resistance is given by

$$R = \frac{V^2}{32.2} \sin 2\theta.$$

Write a programme to tabulate R versus θ for $\theta = 0$ to 90° in 5° steps when V = 1000 feet per sec. (Remember that θ should be converted to radians for use in the Fortran sin function.

θ rad $= \frac{\pi}{180}$ θ deg.)

If the range at 90° does not come out to be exactly zero as it should can you suggest any reason for the error?

Problem 4

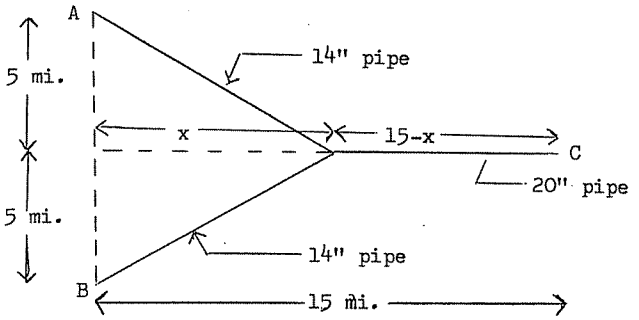
1. A. A man deposits \$1,236 in a savings account which increases the amount by $\frac{1}{2}\%$ of what ever the amount is each month. (That is, each month the amount is mult. by 1.005 starting 1 month after the deposit is made (including the last month's interest).

B. Change the programme above to continue the calculation for 15 years printing the amount and the year for each year.

C. Change the programme to round the amount to the nearest cent each month. (Hint: if A is the amount in dollars and decimal fractions of dollars then N = A will be the amount in integer dollars, set B = N and then M = 100. * (A - B) + .5 is the number of integer cents after rounding. Combining B and M properly will give A rounded to the nearest cent.

Problem 5

1. Two towns A and B, of about equal size, are to be fed water from a lake at C which happens to lie on the right bisector of the line AB. It is decided to build three pipelines, one of 20" pipe from C to a point D on the right bisector of AB and two of 14" pipe from D to the towns A and B. If 20" pipe costs $16 per foot to purchase and lay and 14" pipe $12 where should D be located to minimize the total cost of the three pipelines?
See diagram below



To solve this problem write a computer programme to calculate and print the total cost of the three pipe lines as a function of x, where x is the distance of D from the line AB, for x varying from 0 to 15 in increments of .5 miles. Examine the results to determine where (to within .5 miles) D should be placed for lowest cost.

2. Change the programme so that it prints the change in total cost at each step as well as the present cost. That is the printed line (except for the first) will contain x, total cost and the total cost minus the previous total cost. Arrange the programme so that it exits as soon as the change in cost becomes positive.

3. Change the programme of part 2 so that before exiting the increment is changed to -.05 from .5 printing each line as before and exiting when the change becomes positive again. Examine the results to determine the placing of D to within .05 miles for lowest cost.

Problem 6

1. A class of 15 students take the same 7 subjects. Their marks are punched on cards in the following format:

Card Columns:	Data:
1 - 3	Student number
5 - 7	Mark in course 1
9 - 11	Mark in course 2
etc	etc
29 - 31	Mark in course 7

Write a programme to read these cards one at a time, using a 7 element one dimensional array for the marks, and to print a line with the same information plus the student average in columns 33 - 37 inclusive (2. decimal places).

2. Modify the programme of 1 part to calculate a weighted average using an array of weights WEIT(I) which add to W and which are to be read in from a card preceding the mark cards.

That is

$$AVG = \sum_{I=1}^{7} (WEIT(I) * CMARK(I))/W.$$

3. Modify the programme given above by adding an array CAVG(7) in which the course marks are to be accumulated and, when all students' cards have been read, the course averages are to be printed by dividing the sum of the marks in each course by 15.

4. Modify the programme above to read the students' names from columns 40 to 60 on the data cards using H format or 'quotes' and print the name as well as the mark and average date.

5. Do 4 part using A format and print the line giving the student name first, followed by the marks and then the average.

6. Finally alter the programme so that all cards are read before any lines are printed. This will require the use of a two dimensional array for the marks (CMARKS(7,15)perhaps), an array of 15 elements for students averages and a 7 element array for course averages. The print out should be in order of averages. The line for the student with the highest average should be printed first, then the line for the student with the second highest average, and so on.

Hint: Scan column of averages for the maximum average. Print the corresponding line and then replace the average by zero. Scan again for the next highest average and print and then zero the average. When the highest average left is zero all lines have been printed. This is a simple method of sorting but it also too inefficient to use with very large numbers of entries.

Data

I	1	2	3	4	5	6	7
WEIT(I)	1	1.5	.5	2	2	.5	1

W = 8.5

85	74	53	58	62	73	65	J. DOE
72	61	45	38	75	53	63	R. ROWT
85	73	62	59	84	91	60	S. HOW
91	82	61	52	74	85	72	M. HAHA
52	43	78	65	86	92	81	D. DUCK
71	66	75	53	58	75	66	M. MOWS
39	65	48	72	54	66	68	P. RABT
74	76	73	72	77	70	85	R. FOX
84	67	65	66	82	81	74	S. ZOE
75	61	82	45	58	81	75	K. MIT
75	62	89	76	54	58	65	J. MOP
68	72	83	71	55	68	54	N. BOZ
48	31	35	45	60	51	55	J. BSPFT
90	84	65	70	81	66	95	I. YAX
71	85	64	53	52	53	48	B. GLUG

Mathematical Notes

A METHOD OF FINDING TANGENTS TO A PARABOLA

J. C. Barton, Head of Mathematics Department
Port Perry High School

In my Grade 13 of 1968-69 we discussed the problem of finding the equation of a tangent line to a parabola through an exterior point. This was in October after having done some calculus but no problems of this type. Three students did arrive at a solution, two by the standard method using calculus and one by a very unusual and baffling method. The student was Brian Donnelly and he will be entering Queen's University this fall. I have enclosed this method of solution along with my own solution of the general case and would appreciate any replies that will shed some light on just why this works.

PROBLEM: Find the equations of the tangents to the curve $y = x^2$ from the point (2, -1).

He reasoned that to reach the curve vertically from the point (2, -1) required an increase of 5 for y. Substituting in $y = x^2$ the change in x must be $\pm\sqrt{5}$ and the x co-ordinates of the points of tangency will be $2 \pm \sqrt{5}$. From here the solution is easy.

At this point I gave him the equation $y = x^2 - 4x + 5$ which has a vertex at (2,1) and asked for the equations of the tangents through (1,-2). He thought for about one minute and, without knowing it, applied a translation to the parabola and exterior point so that the vertex of the image parabola was at the origin. From the image point (-1,-3) to reach the image parabola vertically requires an increase in y of 4 and a corresponding change in x of $\pm$ 2. This gives the x

co-ordinates of the points of tangency as 3 and -1.

By this time I was stumped and brought in reinforcements who were also quite baffled as to why the method worked. We worked out the general case as follows.

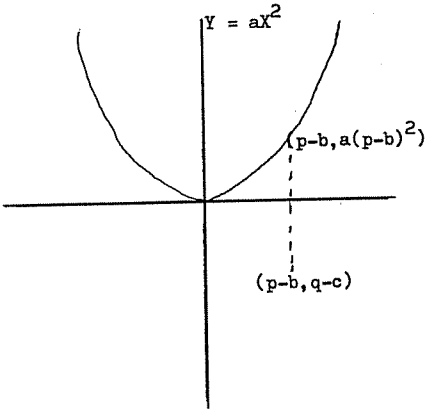
The general equation of a parabola is $y = a(x-b)^2 + c$ and has vertex (b,c). Let the exterior point be (p,q). The problem is to find the equation of the tangent to the given parabola and passing through (p,q).

Apply the translation $(x,y) \longrightarrow (x-b, y-c)$

and $(p,q) \longrightarrow (p-b, q-c)$.

The equation of the image parabola is

$Y = aX^2$.



When $X = p-b$ $Y = a(p-b)^2$

$\therefore Y = \left| a(p-b)^2 - (q-c) \right|$.

If Y changes by this amount then X changes by $\pm \sqrt{\left| \frac{a(p-b)^2 - (q-c)}{a} \right|}$

The radicand will be positive if (p,q) is an exterior point.

X changes by $\pm \sqrt{\frac{a^2(p-b)^2 - (q-c)a}{a}}$

and the points of tangency will have as x co-ordinates

$$p \pm \sqrt{\frac{a^2(p-b)^2 - a(q-c)}{a}}$$

If we had used the calculus, the slope of the tangent at (x,y) on the parabola defined by $y = a(x-b)^2 + c$ is $2a(x-b) = \frac{y-q}{x-p}$ which reduces to:

$$ax^2 - 2apx + 2abp + q - ab^2 - c = 0 \text{ and}$$

$$x = \frac{2ap \pm \sqrt{4a^2p^2 - 8a^2bp - 4aq + 4a^2b^2 + 4ac}}{2a}$$

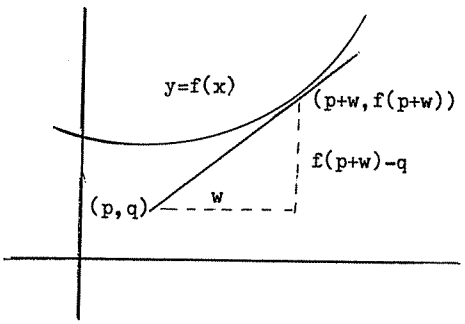
$$= p \pm \sqrt{\frac{a^2(p^2 - 2bp + b^2) - a(q-c)}{a}}$$

$$= p \pm \sqrt{\frac{a^2(p-b)^2 - a(q-c)}{a}}$$

giving us the same first co-ordinates as was obtained using transformations and far less algebra.

Editors Comment.

Let us see how this method fares for a "general" function $y = f(x)$. Let (p, q) be a point not on the curve. Referring to the diagram below, we see that we have to find w.



We know that the slope of the tangent at $(p+w, f(p+w))$ is $f'(p+w)$, where $f'(x)$ is the derivative of $f(x)$. However, from the diagram this is

equal to $\frac{f(p+w)-q}{w}$, so we obtain an implicit equation for w:

$$f(p + w) - wf'(p + w) = q.$$

For example if $f(x) = x^{n+1}$ with $n \geqslant 0$, we obtain

$$(p + w)^n(p - nw) = q,$$

and so one has to solve an equation of degree $(n + 1)$ for w. Moreover, if $n = 1$ this takes the particularly simple form $p^2 - w^2 = q$.

THE COLOURED PENCILS PROBLEM

A. Smith

"How many distinct sets are there of 16 pencils if the pencils, otherwise identical, are of 4 different colours?"

A good approach to this problem, and indeed to many problems, is to commence with a simpler problem of the same kind. For example, "How many distinct sets are there of 5 pencils, these being of two different sorts?" The answer could be arranged as a set of ordered pairs, $\{(5,0), (4, 1), (3,2), (2,3), (1,4), (0,5)\}$, the number of sets being 6, which is 1 greater than the number of pencils because of the inclusion of zero. In general, if n pencils are chosen of two different sorts, the number of sets is (n+1).

Extend this now to pencils of three kinds: "In how many ways may 7 pencils be chosen from 3 boxes containing pencils of different sorts?" If we select 0 from the first box, then we select 7 from the remaining two boxes, and the number of sets in this case is 8; if we select 1 from the first box, then there are 7 pairs obtainable from the other two. Proceedings thus, if we select 7 from the first box we have only 1 way of completing the triple set, that is by selecting (0,0) from the other two boxes. The total number of sets is therefore $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$, i.e. 36 different sets. This is also the number of ordered triples obtainable from the number 7, zeros being included when required, such as (0,0,7), (0,1,6), etc. In general, n pencils of 3 different sorts may be chosen in $1 + 2 + 3 + \ldots + n + (n+1)$ ways.

There are (n+1) terms in this arithmetic sequence, and hence the summation may be compressed into the formula $\frac{1}{n}(n+1)(n+2)$, which also happens to be the combination formula C(n+2,2).

When n=7, the number of triple sets is $C(9,2) = \frac{9.8}{1.2} = 36$, as was found by counting.

Proceeding now to a selection of n pencils of 4 different sorts, the first place in the ordered set of 4 may be filled successively with 0, 1, 2, 3, ... up to n pencils. Then the remaining three places may be filled respectively in C(n+2, 2), C(n+1, 2), C(n,2) ... ways, down to C(2,2) ways, i.e. 1 way of obtaining the last set (n, 0, 0, 0). Thus the total number of quadruple sets is

$$C(2, 2) + C(3, 2) + C(4, 2) + ... + C(n+2, 2), \text{ or}$$
$$1 + 3 + 6 + 10 + ... + \frac{(n+1)(n+2)}{2} .$$

We now have to find a formula for this summation which will be recognised as a sequence in Pascal's triangle. Pascal's theorem states that C(n, r) + C(n, r+1) = C(n+1, r+1). Applying this to the above summation we may arrange in column form and add:

$$\begin{aligned} C(2,2) &= C(3,3) \\ C(3,2) + C(3,3) &= C(4,3) \\ C(4,2) + C(4,3) &= C(5,3) \\ C(5,2) + C(5,3) &= C(6,3) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \underline{C(n+2,2) + C(n+2,3)} &\underline{+ C(n+3,3)} \end{aligned}$$

Adding, and subtracting the members common to both sides, we obtain

$$\sum_{0}^{n} C(n+2,2) = C(n+3,3).$$

Thus when n = 16 pencils, the number of different 4-sets = $C(n+3,3) = C(19,3) = 969$.

This answer (969) could also be obtained by listing and counting, but such counting needs to be done very systematically in order to avoid omissions or duplications. The foregoing solution does suggest a systematic approach to counting when small numbers are involved. It also suggests that the formula may be generalised. Thus, n pencils of 1 sort may be chosen in 1 way only, and this = $C(n,0)$. When there are n pencils of 2 sorts, there are $C(n+1, 1)$ sets.

Similarly for 3 sorts there are $C(n+2, 2)$, and for 4 there are $C(n+3,3)$, suggesting that for n objects of k different kinds there are $C(n+k-1, k-1)$ different selections.

Stating this result in broader form, any psotive integer n may be resolved into ordered k-tuplets, using zeros where necessary, in $C(n+k-1, k-1)$ different ways.

LINES WITHIN A TRIANGLE

G. G. Steel, Hillfield-Strathallan College, Hamilton

1. A geometric relation that was once well-known among high school students in mathematics is that the bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle.

One is led to wonder about other similar relations involving such lines as altitudes, medians, or, in fact, any line from the vertex of a triangle to the opposite side. These relations seem to have received little attention at high school level. Perhaps this is because geometry and trigonometry are not brought together very much at high school. This seems to be a pity, for many rather interesting geometric relations can be examined with the use of simple trigonometry.

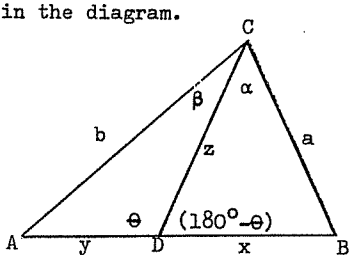
Without in any way underestimating the value of the study of trigonometry in terms of functional relations, it is in fact true that the high school student encounters very few experiences in which he can put his knowledge of trigonometric functions to use. In some high school trigonometric courses the aspects of trigonometry involving relations between sides and angles of a triangle are somewhat neglected: one sometimes has the feeling that such aspects are regarded as poor relations. For students intending to enter technological courses such studies are of great value; for students who have an interest in geometric relations, trigonometry can be a useful handmaid.

This article is intended primarily to suggest possible trains of thought, rather than discuss the relations involved in detail. Many teachers are seeking ideas for independent investigation by their students. It is possible that here is a compact set of ideas that are not generally discussed in texts, and which capable senior students, given a little initial instruction and some general guide-lines, might well develop themselves. Such open-ended projects have values beyond exercises in which the student is required to reach well-defined goals.

Returning to lines within a triangle, one tends, naturally, to examine special cases, such as altitudes, first, and then search for general solutions. In this monograph the general case will be examined first, and a few special features noted afterwards.

2. The General Case

ABC is a triangle. CD is any segment from C to AB, with lengths and angles as indicated in the diagram.



In Δ CAD, $\frac{y}{\sin \beta} = \frac{b}{\sin \theta} = \frac{z}{\sin A}$ (i)

In Δ CBD, $\frac{x}{\sin \alpha} = \frac{a}{\sin(180^\circ - \theta)} = \frac{z}{\sin B}$ (ii)

from (i) $y \sin \theta = b \sin \beta$

from (ii) $x \sin(180^\circ - \theta) = a \sin \alpha$

$\therefore$ $x \sin \theta = a \sin \alpha$

whence $\frac{x}{y} = \frac{a \sin \alpha}{b \sin \beta}$ (1)

Since $\frac{a}{b} = \frac{\sin A}{\sin B}$

it follows that $\frac{x}{y} = \frac{\sin A \sin \alpha}{\sin B \sin \beta}$ (2)

using the cosine law,

$$2bx \cos \beta = b^2 + z^2 - y^2$$

and

$$2az \cos \alpha = a^2 + z^2 - x^2$$

whence

$$\frac{b \cos \beta}{a \cos \alpha} = \frac{b^2 - y^2 + z^2}{a^2 - x^2 + z^2} \qquad (3)$$

or, using (1), from which $\frac{b}{a} = \frac{y \sin \alpha}{x \sin \beta}$

$$\frac{y \sin \alpha \cos \beta}{x \sin \beta \cos \alpha} = \frac{b^2 - y^2 + z^2}{a^2 - x^2 + z^2} \qquad (4)$$

$$\left(\text{or, } \frac{y \tan \alpha}{x \tan \beta} = \frac{b^2 - y^2 + z^2}{a^2 - x^2 + z^2}\right)$$

It may also be noted from (i) and (ii) that the length of the segment CD (= z) is given by

$$z = \frac{b \sin A}{\sin \Theta} = \frac{a \sin B}{\sin \Theta} \qquad (5)$$

and since, also from (i) and (ii)

$$\sin \Theta = \frac{b \sin \beta}{y} = \frac{a \sin \alpha}{x}$$

$$z = \frac{y \sin A}{\sin \beta} = \frac{x \sin B}{\sin \alpha} \qquad (6)$$

One might well enquire whether z can be expressed in terms of parts of the triangle and just the segments x, y or just the angles α, β.

Clearly the former can be achieved directly from the cosine law.

$$z^2 = b^2 + y^2 - 2by \cos A = a^2 + x^2 - 2ax \cos B.$$

The latter follows directly from (5). Since

$$\theta = 180^\circ - (A + \beta), \quad \sin\theta = \sin(A + \beta)$$

and also $\theta = \alpha + B$, $\sin\theta = \sin(\alpha + B)$.

$$\text{Hence, } z = \frac{b \sin A}{\sin(A + \alpha)} = \frac{a \sin B}{\sin(\alpha + B)} \tag{7}$$

3. The Special Case of the Median

If z is a median, then x = y, and from (1) we obtain

$$\frac{a}{b} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin A}{\sin B} \tag{1}'$$

whence

$$\sin A \sin\alpha = \sin B \sin\beta \tag{2}'$$

Since $z^2 = b^2 + y^2 - 2by \cos A$

$$z^2 = a^2 + x^2 - 2ax \cos B,$$

and since x = y,

$$2z^2 = a^2 + b^2 + 2x^2 - 2x(b \cos A + a \cos B)$$

$$2z^2 = a^2 + b^2 + 2x^2 - 2x \quad (AB = b \cos A + a \cos B = 2x)$$

whence follows the familiar result

$$2z^2 = a^2 + b^2 - 2x^2.$$

Using this with (3), we obtain

$$\frac{b \cos\beta}{a \cos\alpha} = \frac{2b^2 - 2x^2 + 2z^2}{2a^2 - 2x^2 + 2z^2}$$

$$= \frac{2b^2 - 2x^2 + a^2 + b^2 - 2x^2}{2a^2 - 2x^2 + a^2 + b^2 - 2x^2}$$

$$= \frac{3b^2 + a^2 - 4x^2}{3a^2 + b^2 - 4x^2}$$

$$= \frac{2b^2 + b^2 + a^2 - c^2}{2a^2 + a^2 + b^2 - c^2} \qquad (c = 2x)$$

$$\frac{b \cos \beta}{a \cos \alpha} = \frac{2b^2 + 2ab \cos C}{2a^2 + 2ab \cos C} \qquad (c^2 = a^2 + b^2 - 2ab \cos C)$$

whence,

$$\frac{\cos \beta}{\cos \alpha} = \frac{b + a \cos C}{a + b \cos C} . \qquad (3)'$$

4. The Special Case of the Angle Bisector

In the case of the angle bisector, $\alpha = \beta$ and from (1) we at once obtain the familiar result

$$\frac{x}{y} = \frac{a}{b} \qquad (1)''$$

whence,

$$\frac{x}{y} = \frac{\sin A}{\sin B} . \qquad (2)''$$

From (3), if $\alpha = \beta$

$$\frac{b}{a} = \frac{b^2 - y^2 + z^2}{a^2 - x^2 + z^2}$$

$$ba^2 - bx^2 + bz^2 = ab^2 - ay^2 + az^2$$

$$z^2(b - a) = ab(b - a) + bx^2 - ay^2$$

$$z^2 = ab + \frac{bx^2 - ay^2}{b - a} \qquad (\text{if } b \neq a)$$

(if b = a, of course, x = y).

Since

$$\frac{x}{y} = \frac{a}{b}$$

$$\frac{x^2}{y^2} = \frac{a^2}{b^2}$$

$$\frac{bx^2}{ay^2} = \frac{a}{b}$$

$$\frac{bx^2 - ay^2}{ay^2} = \frac{a - b}{b}$$

$$\frac{bx^2 - ay^2}{a - b} = \frac{ay^2}{b} .$$

Hence,

$$z^2 = ab - \frac{ay^2}{b} .$$

So, $z^2 = ab - xy$. (Since $\frac{a}{b} = \frac{x}{y}$) (3)"

5. The Special Case of the Altitude

In the case of the altitude β and A are complementary and so are α and B.

From (2) we have

$$\frac{x}{y} = \frac{\sin A \cos B}{\sin B \cos A} = \frac{\tan A}{\tan B}$$

and

$$\frac{x}{y} = \frac{\cos \beta \sin \alpha}{\cos \alpha \sin \beta} = \frac{\tan \alpha}{\tan \beta} = \frac{\tan A}{\tan B} \qquad (2)''$$

(These results can, of course, be obtained at once by examining the two right angled triangles).

From (3), since $b^2 - y^2 = a^2 - x^2 = z^2$

we obtain $b \cos \beta = a \cos \alpha$ (These are, of course each values for z).

If C is a right angle, it is well known that $z^2 = xy$. This turns out to be a special case of the following:

$z = y \cos \beta = x \cot \alpha$

$z^2 = xy \cot \alpha \cot \beta$ (Which becomes xy when α and β are complementary)

or $z = y \tan A = x \tan B$

$z^2 = xy \tan A \tan B$ (which again becomes xy when A and B are complementary).

PROBLEMS AND SOLUTIONS

Edited by W. Fraser

Problems for inclusion in the problem section are welcomed by the editors. Please submit both your problems (preferably accompanied by a solution), and your solutions to published problems to Prof. W. Fraser, Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario.

Problem 70.01. The irrational number $1/\sqrt{2}$ is approximated by a rational number with denominator $q > 0$, and numerator p such that p/q is in its lowest terms and p/q differs from $1/\sqrt{2}$ by less than any other fraction with denominator q. Let $E_q = \left| 1/\sqrt{2} - p/q \right|$. Show that $E_q > \dfrac{1}{2\sqrt{2q^2} + q}$

Problem 70.02. Given x, y, z such that

$$\sin x + \sin y + \sin z = 0$$

and

$$\cos x + \cos y + \cos z = 0,$$

prove that

$$\sin 2x + \sin 2y + \sin 2z = 0$$

and

$$\cos 2x + \cos 2y + \cos 2z = 0.$$

Problem 70.03. Show how to construct a circle which is to pass through a given point and touch two given intersecting straight lines.

A GLIMPSE AT THE FIRST INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION, LYON, FRANCE, AUGUST, 1969

John C. Egsgard, C.S.B., President, CAMT, St. Michael's College School
Toronto, Ontario

One is always on the lookout for an excuse to visit another country - at least I am. The First International Congress on Mathematical Education gave me a reason for my first visit to Europe. It seemed to me that Canadian mathematics teachers should be represented at this congress. In what better way could this be done than by the president of the Canadian Association of Mathematics Teachers? Since CAMT has very little funds it was up to me to approach other groups for assistance. The Canadian Teachers Federation and the Canadian government were unable to help me but I did get a grant from the Canadian Mathematical Congress to cover 80% of my charter fare from Toronto to London, England and return. But a president of a young and financially poor association has to be prepared to make monetary sacrifices so I decided to go ahead. I was not disappointed.

The First International Congress on Mathematical Education was organized by the International Commission on Mathematical Instruction (ICMI). Every four years the parent body of this commission, the International Congress of Mathematicians, holds a meeting primarily for the discussion of the mathematics being developed throughout the world.

The next meeting is in Nice in the summer of 1970 and the last was in August, 1966 in Moscow. Only a very minimal part of the program of these meetings has been devoted to mathematics teaching. Because the leaders of ICMI felt that ideas on mathematics instruction should be shared among all countries, they decided to hold an international congress on the teaching of mathematics. Canada was represented, in the group who made the decision, by Lloyd Dulmage of the University of Manitoba. This is part of the background behind the First International Congress on Mathematical Education that took place from August 24 to August 30, 1969 at the Palais des Congres in the city of Lyon in the south of France.

The meeting drew about 800 mathematics educators from around the world including representatives from the different countries of Europe, several countries from Africa and the Middle East, Japan, Cuba, Australia, Russia, India, United States and Canada. Canada was well represented with 16 from Quebec, 6 from Ontario and 1 from Manitoba. The Ontario members, besides myself were Doug Crawford, Don Mumford, Ram Rai, Brooks Rapley and George Scroggie. There were about 120 educators from the United States.

The Congress itself was divided into three parts - general sessions, of which there were 21; free communications, of which there were 45: and panel discussions.

The addresses at the general sessions were delivered by distinguished mathematics educators from various countries: 3 from France, 3 from the United States, two each from Russia, Belgium, Great Britain and Germany and one from each of Canada, Denmark, Rumania, Poland, Italy and Switzerland. Zoltan P. Dienes of Sherbrooke, Quebec gave one of the

more interesting talks on "La mathematique a l'ecole primaire" in which he described, using a film, some of the experimental work he is doing teaching logic through games to very young children. On the whole I thought that the lectures were rather disappointing but I feel that I shall gain more from them when they are published this fall. Anyone interested in obtaining information about the lectures should write the secretary of the congress, M. Glaymann, 43, boulevard du 11 Novembre 1918, 69 VILLEURBANNE, France. The best lecture was delivered by Henry Pollak of the USA on the topic "How can we teach applications in mathematics?". Other excellent papers were delivered by Madame Frederique Papy of Belgium on the "Mini-computer (apprentissage du calcul numerique et strategie)", and by Mrs. Emma Castelnuova of Italy on "Differentes representations utilisant la notion de barycentre". George Scroggie and Brooks Rapley, both of Ontario, have most of these talks on tape. One of the general sessions was devoted to a panel on international cooperation. Panelists were from the United States (Julius Hlavaty, the president of NCTM), Great Britain, Russia, Belgium and Poland. The audience took part in the discussion so I made a plea for a greater participation in future congresses by those who are on the action line, namely the classroom teachers.

The free communications gave the ordinary members of the congress an opportunity to express some of their ideas on mathematics education. Each communication lasted 15 minutes and had to be submitted to the congress secretariat about three months before the meeting. The first publication that members of the congress received on arrival was a copy of these free communications. Of the 45 papers delivered three were from Canada.

Roland Brossard of the Universite de Montreal spoken "L'Emploi des Structures paralleles comme Outil Pedagogique"; Andre Ronveaux also of the Universite de Montreal talked about "Une Experience Candienne, Equations differentielles au secondaire"; while I delivered a paper on "Some Ideas in Geometry that Can Be Taught from Grades k-6". This later paper will be published in an extended version in Educational Studies in Mathematics, a journal edited in Utrecht, Netherlands, by the president of the congress, Hans Freudenthal.

The panel discussions were last minute additions to the meeting and were not as satisfactory as they would have been if some preparation had been made for them beforehand. One of the panels purported to discuss the beginnings of geometry in the primary school. Instead the panelists spent their time trying to go one up on each other in describing the abstract theorems their children studied in geometry in elementary school.

Accompanying the congress was an exhibit of text books from several countries, most notably Great Britain, France, Germany. Unfortunately, no general request for text books had been made so that most countries were not represented. As well, the Association of Mathematics Teachers from England gave demonstrations of group activities and learning using elementary school children from England.

In the final session of the congress on Saturday several resolutions were adopted encouraging the development of cooperation internationally in the teaching of mathematics. These resolutions can be found at the end of this article.

All in all I think that the congress was worthwhile. It gave the conference members an opportunity to discuss educational problems with

people from other countries. We all began to see that there are many common problems in introducing the new curriculum. As I see it Canada is near the top in this development. The next congress can be an improvement on this if an attempt is made to use classroom teachers at the elementary and secondary level in panels and even at general sessions. Canada should continue to have representation at these congresses for we have much to learn and also much to contribute. I am looking forward to attending the Second International Congress in Mathematical Education in 1973.

This Congress draws attention to the following resolutions embodying matters of great importance to mathematical education:

1. In all countries, the modernisation of the teaching of mathematics should be pursued as vigorously as possible, both in the content of syllabuses and in the manner of presentation. Content and methods are inseparable, and should be kept continually under scrutiny.

2. Mathematical ideas are inherent in many other disciplines (physics, biology, economics, sociology, ...). Much mathematics arises from the construction of mathematical models of real situations, and the teaching of mathematics must recognise this. Collaboration between teachers of mathematics and those of other disciplines should therefore be encouraged.

3. International cooperation should be further developed. Information on the teaching of mathematics can be exchanged at conferences, in publications, and by visiting lecturers. Each country should be more fully informed of activities in the other countries. In particular, the

"advanced" countries should continue to collaborate with the developing countries, in the search for solutions appropriate to them.

4. The rapid development of the content and methods of mathematical education makes it necessary for the teacher of mathematics to given opportunities to pursue further professional study during his employment.

5. The theory of mathematical education is becoming a science in its own right, with its own problems both of mathematical and pedagogical content. The new science should be given a place in the mathematical departments of Universities or Research Institutes, with appropriate academic qualifications available.

The First International Congress on Mathematical Education makes the following recommendations to I.C.M.I.:

1. to study the problems of international information on mathematical education in the various countries, in particular that of the establishment of international information centers and that of the creation of an information bulletin.

2. as regards the form of the next congress; to pay more attention to pre-school education, elementary education, mathematical education for the whole of the young people, adult education.

MEETINGS

The Seventy-Ninth Annual Convention of OATM, will be held from Sunday, March 22 to Tuesday, March 24, at the Chateau Laurier Hotel in Ottawa. The featured speaker will be: Mr. David Taylor, School Mathematics Project, University of London and Professor Kenneth May, College of Education, Toronto. A complete program for wives has been planned. Further details may be obtained from Jack Le Sage, Publicity Representative, 9 Mount Batten Road, Barrie, Ontario.

The Canadian Association of Mathematics Teachers is holding an ETV mathematics conference on Friday and Saturday, April 17, 18, 1970 at the Skyline Hotel in Toronto. At this conference representatives from the provinces will explain what is being done in the mathematics classroom at the present time and what will be done in the future. Each province will show selections from tapes produced by its ETV department. Teachers from across Canada who are using TV or TV recorders in a novel way will be given the opportunity to describe their work. Commercial TV equipment will be on display.

Anyone interested in attending this conference or any teacher who would like to describe his or her novel endeavors in ETV is asked to contact John C. Egsgard, 1515 Bathurst Street, Toronto 349, Ontario.