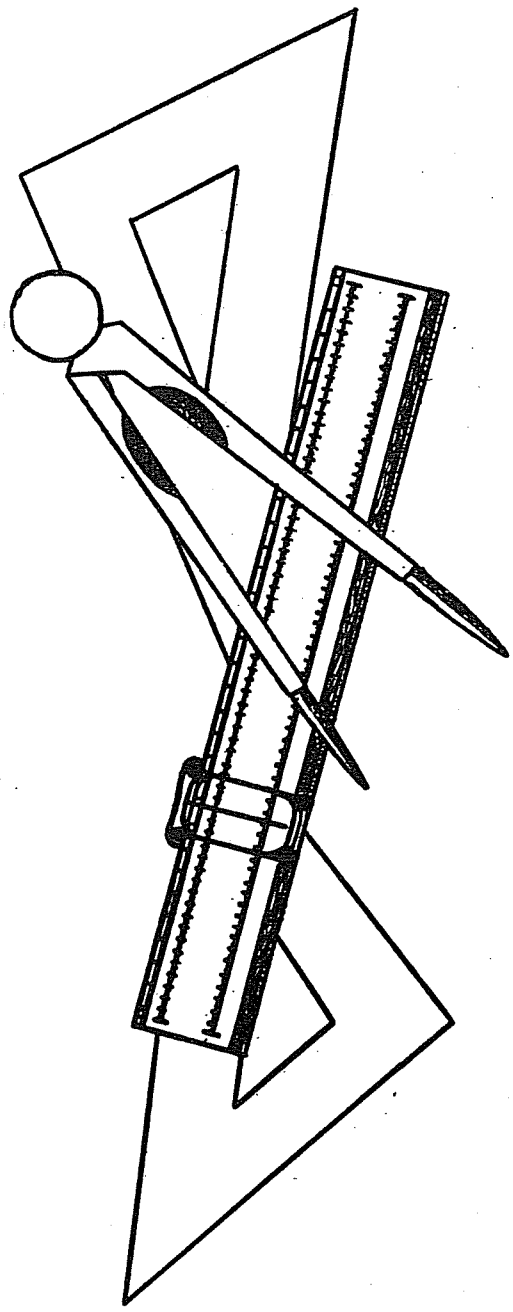
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ONTARIO MATHEMATICS GAZETTE

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ONTARIO MATHEMATICS GAZETTE

TABLE OF CONTENTS

	Page
Editorial	161
An organization for mathematics (a report from the O.A.T.M.-O.M.C. coordinating committee)	163
Proposed Constitution	165
Plan for Action	170
The Re-Organization of the Ontario Educational Association by M. B. MacGregor	172
Le Comité de langue française by J. Quesnel	173
Preliminary Notice: National Council of Teachers of Mathematics Toronto Conference	174
Think Metric A talk by Jack Bell	175
Why not require an 80% performance in mathematics? by D. C. Attridge	179
A discovery approach to $Q \times Q$ by C. D. Gallant	183
Involvement: the Gelfand Club by E. J. Barbeau	188
Solution to problem 71.06	190
Making mathematics interesting for non-college bound students by W. G. Anglin	191
Mathematical induction: a set theoretic approach by C. R. Hirsch, Jr.	200

TABLE OF CONTENTS

Continued	Page
Finger multiplication by G. Vervoort	205
Solution to problem 71.04 by A. W. Walker	207
The Spirograph: an application of modular arithmetic by B. H. Litwiller and D. R. Duncan	208
Book reviews	211
Subscription rates	221
Notice of O.A.T.M. meeting	222

EDITORIAL

Whatever happened to simple, direct English? Surely there was once a time when short words were preferred to long ones, when resource persons were called 'experts' or 'specialists', and when elementary ideas were explained in terse unambiguous sentences that had real verbs in them (and nouns and adjectives and other odds and ends too, all put together in a logical order!).

No, perhaps not, there never was such a period in human history - your editors, now entering into their dotage, imagine it all. But this is no reason why we, as teachers, cannot fight against the insidious enemy - indeed, if our job really is to make ideas intelligible to normal human beings, then we must be in the front lines of this particular battle. Simple, direct English is our best weapon, not closed-circuit television, programmed calculators, computer-aided instruction, or expense-account excursions to conferences on developing interpersonal relationships. We cannot deny (no, to be correct, we dare not deny) that these other weapons are useful, but if concrete objective problems are translated into verbal conundrums which would need a keen legal mind to unravel, then no amount of mindless activity will save the day.

So what can we do about it? First, we must understand what we're doing - the content of a course is important in spite of all educational theorizing to the contrary. Secondly, we must, like the character in 'The First Circle', strive for the 'language of maximum clarity'. Away with such phrases as 'total learning experience' and 'cumulative sequential impact'; 'essentially' and 'substantive' should be regarded with suspicion; and if students could be made aware of the difference between 'imply' and 'infer' a bright light would be lit in a gloomy world. A good dictionary, access to a guide to style like Fowler's 'Modern English Usage', and a teacher who demands clarity and precision are all a student needs. Of course, it might be a help if he also required to read good examples of prose, and these can range from Einstein's popular writings on physics to the Chronicles of Barsetshire, with a few instruction manuals thrown in to leaven the mixture.

Indeed, it might be both amusing and instructive to translate understandable English into administrative or educational gobbledegook. It's not easy - we've tried - so to help us in this worthwhile project, would you send along any truly horrible genuine examples. The best ones will be printed, the

very best may even get a prize! Perhaps we should also run a contest to suggest the best prize for the writers of the winning entry!

The Editors.

A malevolent deity suddenly reduces the Earth's velocity relative to the Sun to zero. How many days will the Earth take to fall into the Sun? What will happen to the Moon?
(from a Cambridge Tripos paper, 1965)

AN ORGANIZATION FOR MATHEMATICS

A Report from the OATM-OMC Coordinating Committee

The cause of mathematics education in Ontario has been served in various ways past and present by a number of groups and organizations.

In its long history to date the Ontario Association of Teachers of Mathematics (OATM) has provided a two-fold service for teachers. It has been a meeting-ground for its members for professional updating, for stimulation, refreshment and information through provincial and regional conferences. And in its affiliation with the Ontario Education Association it has given mathematics teachers some participation in the wider Ontario educational scene. It has been in a position, through its representation on the OEA senate, to influence recommendations to the Minister in the interest of mathematics education.

OATM leadership over the years has grappled with two problems: (1) the building of a larger base of support and involvement through increasing membership (2) development of service to and support from the elementary school teacher with special interest in mathematics. In the 1960's progress was made in both of these areas, but much more is desired.

Now, without detailing its history, let us try to summarize the role of the Ontario Mathematics Commission (OMC). In the late 1950's great concern was felt for the school mathematics curriculum in Ontario. This concern found expression among mathematics teachers at a conference at Lakefield, Ontario, and the resulting organization (OMC) was supported by the Ontario Teachers Federation and the Ontario Department (now Ministry) of Education. This support was both financial and in the form of personnel appointed to the Commission. The Commission was composed of approximately fifty persons appointed from the complete spectrum of mathematics education in Ontario - elementary, secondary, private schools, college, university, teacher training, OATM, and Department of Education. In my opinion, this wide representative nature of OMC was its greatest strength. The Commission plunged into its task of reviewing, designing, testing, writing, proposing changes in mathematics curriculum.

But then, in the late 1960's, after ten years of operation, the OMC executive felt it necessary to examine the Commission's role. Two main factors led to this exercise. One was the removal of Department of Education support, the other was the formation of county boards of education. The former seemed to indicate the opinion that the original assignment was thought to be completed, and the latter, in many ways, put the curriculum game in a new ball park. It speeded up the decentralization of curriculum responsibility and also, in the case of the larger boards, brought about the establishment of mathematics coordinators and other staff whose efforts, quite expectedly, were to be directed to the needs within their own jurisdictions. And individual teachers were now busy coping with the implications of "continuous progress", "credit system", the new H. S. 1, "integrated studies", team teaching, open concept schools and a whole bag of innovations in the context of reorganized administration.

OATM and OMC representatives have had discussions during the past three years with a view to designing a province wide organization to serve these and other needs of mathematics education. Two essential principles, among others, have emerged. Firstly, one organization to fulfil the functions of both OATM and OMC is feasible and desirable. Secondly, that organization should be composed of and controlled by an active, voluntary membership. Those who have a concern for excellence in Ontario's school mathematics program and a desire to assist the teacher of mathematics would be able to exercise this concern and desire through one and the same vehicle. It has been agreed that the geography of our province would necessitate much of the functioning of the organization to be on a practical regional basis. However, in matters of province-wide concern it should be a strong, unified, recognizable entity.

The proposed constitution which follows this article evolved through considerable discussion and it could provide the framework for an effective organization.

The years immediately before us will bring changing teacher qualifications, changing administrative structures, cyclic review of curriculum, increased applications of semestering, more part-time students, and other educational innovations not yet conceived. To provide the right mathematics for the right student, at the right time, in the right manner will be a challenge certainly as formidable as any in the past.

A strong, active, alert organization of those who have the real 'stake' in mathematics education in Ontario is both essential to the challenge and exciting to anticipate.

The decision regarding formation of a new organization will be made on the occasion of the OATM Annual Conference in North Bay - May 11 - 13. It is sincerely hoped that the largest possible representation of mathematics teachers take part in that decision.

PROPOSED CONSTITUTION

(for an organization concerned with Mathematics Education in the Province of Ontario)

ARTICLE I - NAME

The name of the organization shall be "Ontario Association for Mathematics Education".

ARTICLE II - AIM

To promote excellence in mathematics education in the Province of Ontario.

The term "excellence", in this context, refers to the contribution that mathematics education can and should make to the growth and intellectual development of all individuals in our Province, with due regard to the needs of our society.

ARTICLE III - OBJECTIVES

1. To aid the professional growth of teachers of mathematics by proposing, initiating, or co-operating in pre-service and in-service programs designed to develop and promote effective teaching methods.

2. To evaluate mathematics programs.

3. To encourage and to engage in research relating to mathematics education and curriculum design.

4. To initiate and to encourage the planned development and evaluation of experimental teaching materials.

5. To provide co-ordination in mathematics education at all levels as well as effective communication among all mathematics educators in Ontario by promoting the exchange of ideas through publications, conferences, and personal involvement.

6. To provide liaison with other educational organizations in Ontario and with mathematics education organizations in other provinces and countries.

7. To maintain a representative mathematics educator's "voice" in Ontario.

ARTICLE IV - MEMBERSHIP

1. Membership will be open to all individuals or organizations interested in mathematics education.

2. There will be two types of membership:

   (1) Individual membership carries the right to vote, to act on any committee and to receive all regular publications.

   (2) Institutional membership is open to business, commercial and industrial organizations with an interest or involvement in mathematics education. This membership includes the right to representation on specified committees, the right to receive all regular publications, and the right to one vote at any general meeting.

ARTICLE V - ORGANIZATION

1. Chapters

A group of members who wish to organize and act in harmony with the objectives and programs of the provincial organization may be recognized annually as a chapter by a simple majority vote of the members present at the annual general meeting of the organization.

2. The Provincial Executive shall consist of

President
Vice-President (who is the President-elect)
Three Executive Members
Secretary-Treasurer (with no vote if a paid employee)
Past President

3. The Provincial Council shall be composed of forty members of the organization as follows:

(1) All members of the Provincial Executive.

(2) One delegate from each chapter of the organization.

(3) Individual members, or representatives of institutional members, elected at the annual meeting.

4. Committees

In support of the stated objectives of the organization committees shall be appointed by the Provincial Executive from members of the organization. A member of the Provincial Executive will be appointed to each of the committees but may not chair the committee. Each committee shall elect its chairman at its first meeting which shall have been called by the President.

These committees shall have the right to recommend to the Provincial Executive the appointment of sub-committees in their area of responsibility. These sub-committees may include persons who are not members of the organization.

The Provincial Executive shall have the right to appoint such sub-committees.

ARTICLE VI - MEETINGS

1. An annual general meeting of the organization will be held.

2. Special meetings may be called at any time as deemed necessary by the Provincial Executive.

3. The Provincial Council shall meet three times per year as follows:

- one meeting not more than two months prior to the annual general meeting
- one meeting not more than two months following the annual general meeting
- one other meeting.

Special meetings of the Provincial Council may be called by the President or on request of a majority of Provincial Council members.

4. The Provincial Executive shall meet bi-monthly or at the call of the President.

5. Committees and sub-committees shall meet as their function requires.

ARTICLE VII - FEES

Fees for individual membership, institutional membership, and subscription rates for publications will be set annually by the Provincial Council.

ARTICLE VIII - ELECTIONS

1. Elections shall be held at each annual general meeting for the positions of

Vice-President (the President-elect)
Provincial Executive Members
Elected Provincial Council Members
Secretary-Treasurer (except in the case of this officer being a paid employee of the organization. In this case, the appointment will be ratified at the annual general meeting.)

NOTE: In the first year of the organization the President will also be elected.

2. (a) In the event of a resignation of any member of the Provincial Executive during a term of office, the Provincial Council shall elect one of its members to the vacant office.

(b) In the event of a resignation of any member of the Provincial Council during a term of office, the Provincial Executive shall be responsible for filling the vacancy subject to ratification by the Provincial Council.

3. All offices and appointments shall be effective from the time of election or appointment until the end of the following annual general meeting.

4. Elected Provincial Council members, other than Provincial Executive members, are elected for a three year term, one-third of the number to retire each year.

ARTICLE IX - AMENDMENTS TO THE CONST'N

1. Proposed amendments to the constitution must be presented to the members in writing at least two weeks prior to the general meeting in which the amendments are to be discussed.

2. A motion to amend the constitution, in order to carry, must receive a favourable vote of not less than two-thirds of the members present at the meeting and qualified to vote.

BY-LAWS

1. Signing officers of the organization shall be any two of the President, the Vice-President, the Past President, the Secretary-Treasurer.

2. Duties of officers

(1) The President shall preside at all general meetings, meetings of the Provincial Executive and meetings of the Provincial Council.

(2) The Vice-President is the President-elect. He shall assume the duties of the President whenever required. He shall prepare an annual budget in consultation with the Secretary-Treasurer.

(3) The Secretary Treasurer shall keep a record of all meetings of the organization, the Provincial Council, and the Provincial Executive. He shall supervise and record all regular financial transactions of the organization including membership. He shall present a financial report at the annual general meeting. He shall prepare a financial report for each fiscal year.

(4) The Past President shall be chairman of a nominating committee of three members.

(5) Provincial Executive Members shall share the executive responsibilities of the organization in implementing policies established by the Provincial Council and the membership.

3. Duties of Provincial Councillors

(1) To promote the aim and the objectives of the organization.
(2) To represent the interests of the membership.
(3) To serve on committees.

4. Majority Required

A motion to amend the by-laws, in order to carry must receive a favourable vote of a simple majority of the members of the Provincial Council.

5. Quorum

The quorum for a meeting of the Provincial Council shall be one-third of the members of the Provincial Council plus one.

Joint OMC-OATM Coordinating Committee
December 9, 1972

Plan for Action

A constitution describes the objectives and basic organizational plan. But there must be set forth specific operational procedures before the organization can function. The OATM-OMC Coordinating Committee has recognized this and has set up four subcommittees to work out details concerning

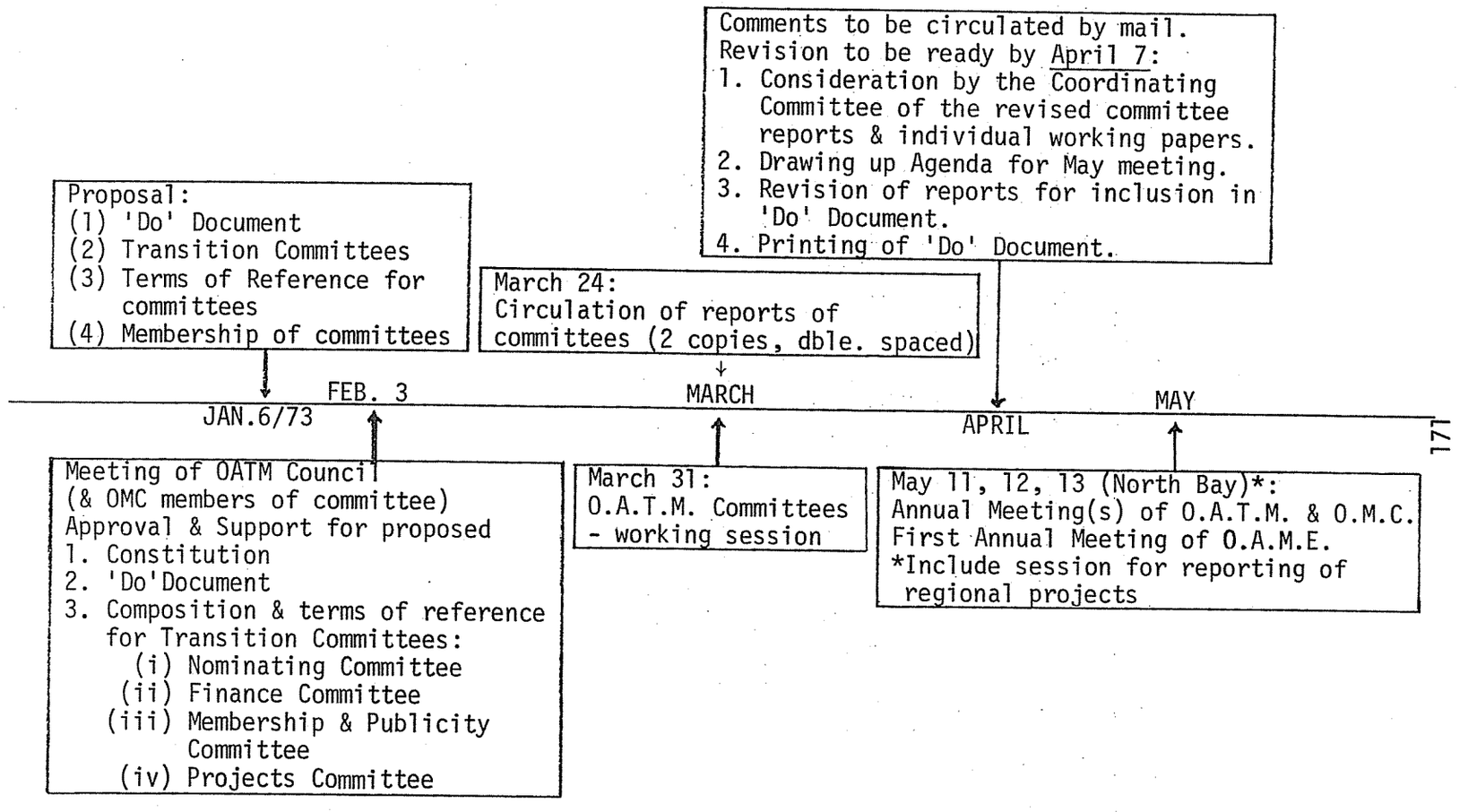
(1) Membership and Publicity
(2) Projects
(3) Nominations and elections
(4) Finance

These committees will be considering these aspects of the organization, gathering information and opinion, and will submit their reports to the Coordinating Committee by March 24th. They will be suggesting By-laws for the organization related to their specific terms of reference. Submissions on any of the four topics would be welcomed by the sub-committees and should be addressed to The Secretary, Ontario Mathematics Commission, 1260 Bay Street, Toronto.

The reports of the sub-committees will be compiled for presentation at the North Bay Conference of OATM, May 13th, along with the proposed constitution. This will give the practical working arrangments and details of operation which will translate the objectives of the constitution into action.

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION (O.A.M.E.)

"PLAN for ACTION" and the Time Line proposed by the Coordinating Committee of the O.A.T.M. and the OMM.C. - as of January 6, 1973



THE RE-ORGANIZATION OF THE ONTARIO EDUCATIONAL ASSOCIATION

M. B. MacGregor, Vice President, O.A.T.M.

Effective Date - August 31st 1973

The Ontario Educational Association's officers believed that the time had come to change the organization of this association. The main changes had to do with the relationship existing between the various sections and the central organization.

The governing body of the central organization was reduced from a senate of forty-eight members to a board of governors of nine members. Not all sections will now have a member on the central executive. These nine members were elected at a general meeting in December. The various sections will now have corporate membership. The Ontario Association of Teachers of Mathematics, provided its membership remains over the four hundred mark, will be entitled to two votes at all duly constituted meetings of the Ontario Educational Association. If the membership should fall below the four hundred mark it will only have one vote.

The Mathematics Section as a corporate member will be required to pay $1.00 per member up to a maximum of $500 to the Ontario Educational Association. Since the Ontario Educational Association will suffer a drastic cut-back in their budget, the services they can offer will need be reduced. However the central office will do work for the sections on a cost basis for labour, materials and postage.

Corporate membership with the Ontario Educational Association will continue to include among other things its support on educational issues, insurance for its members and the placing of the names of its members on a computer mailing list. An annual conference will be held.

The Ontario Educational Association will continue to sell individual memberships for $10.00 per member. It is the hope that elementary school teachers who teach a variety of subjects will purchase these.

This is a very brief report and I suggest that any interested reader write the Ontario Educational Association at Suite S904, 252 Bloor Street West, Toronto 5, Ontario for a revised copy of the By-Laws of the Ontario Educational Association.

LE COMITÉ DE LANGUE FRANÇAIS DE L'ONTARIO MATHEMATICS COMMISSION

Jacques Quesnel, Président

Le comité de langue française de l'Ontario Mathematics Commission (O.M.C.) a comme but de promouvoir l'avancement de l'enseignement des mathématiques dans les écoles de langue française de l'Ontario.

Les objectifs de ce comité pour l'année en cours sont:

1. Faire parvenir un questionnaire aux enseignants francophones leur demandant d'indiquer leurs besoins.

2. Voir à la préparation et à la publications de matériel qui serait de nature à combler les lacunes signalées.

3. Entreprendre les démarches nécessairs en vue d'organiser des "Leadership Seminars" qui seraient présentés en franâis.

<u>Les membres du comité de langue française sont</u>:

Président: Jacques Quesnel, Conseil scolaire d'Ottawa, Ottawa

Mr. Michael Closs, Department of Mathematics, University of Ottawa, Ottawa.

M. Georges Cuerrier, Conseil des Ecoles séparées des Comtés de Stormont, Dundas et Glengarry; Cornwall.

M. François X. Dupont, Formation des Enseignants Faculté d'Education, Université d'Ottawa, Ottawa.

Mr. Don Fox, Ministry of Education, 1825 Woodward Drive, Ottawa.

M. André Ladouceur, Ecole secondaire De La Salle, 501 rue St. Patrick, Ottawa.

Mlle. Rose Routhier, Conseil Scolaire des Ecoles Catholiques de Carleton, Ottawa.

Mlle. Rosaline Séguin, Faculté d'Education, Université d'Ottawa, Ottawa.

PRELIMINARY NOTICE

National Council of Teachers of Mathematics
Toronto Conference

On November 2 - 4, 1973, the Name-of-the-site meeting of the National Council of Teachers of Mathematics will be held in Toronto. The Conference is devoted to provide sessions of interest to teachers of mathematics at all levels. Speakers on the program include:

Lola May, Frederick Mosteller, Robert Rourke, Max Sobel

Sessions will be held at the various levels K-6, 7-9, 10-13, and sections are also included for particular interest to the Colleges of Applied Arts and Technology.

In addition to the many sessions devoted to aspects of mathematics teaching both at the elementary and secondary level, a large number of workshops are provided to accommodate the needs of all participants at the conference.

A full complement of other features related to mathematics teaching is also provided for those attending the conference

- school exhibits display
- current free and inexpensive materials display
- films and other media can be previewed
- publishers display.

Sightseeing and tours are being arranged to help fill in those "off" hours: visit the various unique features of Toronto while you are here. A program of hospitality will be available for wives who attend the conference.

Plan to attend. Mark the date: *November 2 - 4*

Further information on registration will be provided in a later issue of The Gazette.

THINK METRIC

An illustrated talk by Mr. Jack Bell in Arnprior
at the O.A.T.M. Conference, 7th November, 1972.

Mr. Bell treated an audience of about 300 mathematics teachers to a most fascinating preview of the future when Canada changes to the Metric System. To help him he had an intriguing display of weights, measures and other offshoots of the Metricated world and a delightful collection of slides, both humourous and instructive, which would be the envy of any school mathematics department if they could but lay their hands on them.

To begin with, Mr. Bell warned us that there was a growing feeling in the world against North America, the inhabitants of which appeared to be either stubborn or stupid. After all, the metric system of measurement is easily proven as a far simpler and more logical system than any other. The United States has not yet made any public pronouncement of its intentions but Canada it appears, will be making a start very soon to change to the international system which is fast becoming the measuring tool of the world. This is not to say that it will all be accomplished in a weekend. The changeover will take place over a period of perhaps ten years. Equally clearly we should prepare the children early in the decade of change, not at the end.

Toothpaste tubes were held up for the admiration of the assembly with the wry comment that Canada became world champions not only in hockey but in the development of a new metric series of toothpaste containers. There are 29 different imperial sizes at present in the stores. If you care to examine a representative selection you would find one at $1.13 weighing 5 5/8 oz. and another at 69¢ weighing 2 7/8 oz. Which is the better buy? Only a teacher of mathematics might be fool enough to work it out! The new sizes are already coming on the market in the four sizes recommended by the metric marketing authorities; 25, 50, 100 and 150 millilitres (though 75 and 125 ml are permissible). You don't have to be a professor of mathematics to choose the better buy between a tube of splurge at 82¢ for 50 ml and a tube of capfil at $1.70 for 100 ml with very obvious advantages to the customer.

Paper sizes were next for inspection. We were all surprised to discover that an opened out sheet of the Globe

and Mail was half a square metre in area. If the top edge of one of these sheets is taped to the bottom edge of another the area is a square metre and the shape, though not a square, has the interesting proportions such that when folded in half, it assumes the same proportions ad infinitum. In fact it is the folding of this interestingly shaped double spread of the Globe and Mail which gives us the new metric paper sizes ... each reducing in size by half the area yet keeping the same shape.

A-0 is the largest size (1 square metre in area)
A-1 is the opened up Globe and Mail
A-2 is the front page
A-3 folded for the newsboy
A-4 is letter size paper
A-5 a notebook
A-6 a well known brand of toilet tissue ... great delight of the audience when they were advised to metricate their bathrooms.
A-11 is the size of the normal postage stamp.

The great advantage of this system is that all the weird shapes of paper disappear. Foolscap and legal size, etc., will no longer cause storage problems. In schools the stock cupboards will be a dream of tidiness with one pile of A-3 scrapbooks sitting under two piles of A-4 notebooks which in turn are topped with four piles of A-5 jotters. Eventually children will not be issued with 3 ring binders. They will have instead 4 ring binders holding sheets of A-4 paper or two packs of A-5 one above the other.

At present there are 140 nations of the world committed to the metric system. Even the U.S.A. may debate the idea this spring. It is very probable that metric will be in by the early 80s all over North America. However we should move slowly in education. Don't be tempted to throw out all the yardsticks and weights and thermometers in bright-eyed zeal to be 'with it'. As Mr. Bell remarked, some teachers tend to take a kilometre when given a centimetre.

"A gram is a gram, wherever I am" led on to a most illuminating display of variations of imperial measures all over the world. One thing which was new to many was the information that the Quebec foot - called the Paris or French foot, and used in rural areas - was longer than an Ontario foot. The awkward situations which can occur because of these variations were depicted in slide after slide, each

more comical than the last. No words were needed and it was not only an education in mathematics but a beautiful display of the art of putting over a message visually.

Surprisingly, the modernized metric system is only 12 years old! It is based on the old Napoleonic system of 10,000,000 metres from pole to equator but the new standard of length is based on wave lengths. The title of the new regime is "Système International d'Unités" or "International System of Units" so we are going SI rather than metric.

The six base units of measurement will be the following:

m	metre	length
kg	kilogram	mass (not weight, the difference between which will be far more readily appreciated by our children in this age of travel to the moon)
s	second	time
A	ampere	current
K	kelvin	temperature (though not used in everyday parlance. Rather we will use degrees Celsius- °C)
cd	candela	luminous intensity

and recently, the seventh base unit, the mole, has been added.

There will also be some change in the way of writing our numbers. The decimal point will move from a position above the line to a position on the line. Commas will disappear between the groups of three digits so, for example, 93,000,000 will be printed with half spaces between each group of three or typed with a space thus, 93 000 000. This is probably in preparation for a possible switch to the decimal comma, rather than the decimal point.

In preparing for the changeover, estimation must be practised by thinking metric, i.e. "I guess that table will be about 75 cm" and then going over and measuring it. Conversions are to be avoided. At the start we will tend

to say "Someone is about 6 feet tall, which is a bit less than 2 metres, because a metre's a bit longer than a yard, so he must be a bit less than 200 cm"! Much better to practise guessing people's height in cm and narrowing down the errors with successive tries. Clothes will come in sizes based on the centimetre and a beauty show will be sizing up 90, 60, 90 and its variations. In the future, ordering wood for building your house will cause some distress if you insist on ordering 2 by 4. Olympic standards will be metric and a man who reckons he is safe for the breathalyser on a pint had better realize you can't fit a litre into a pint pot. Distances between cities will be measured in kilometres and shorter distances in metres.

It is encouraging to think the children will catch on to the system far quicker than we will. After all, we have to unlearn our system first. Mr. Bell's final word of advice was, "Before teaching metric ... Learn Metric - Think Metric - Go Metric."

The object of mathematical techniques is to economize thought in mechanical processes, so that it may be expended on logical argument. - pupils should eventually be able to explain what they are doing - .

(A report for the Math. Ass., 1957)

WHY NOT REQUIRE AN 80% PERFORMANCE IN MATHEMATICS?

D. C. Attridge, Thornlea Secondary School

That's right. We require the majority of our students to perform in mathematics at an 80% level or better and the kids like it too. When a student passes our courses in mathematics he has done so by obtaining at least an 80% mark in not just some questions but ALL questions on the tests.

Let me explain why we started this idea and what the results have been. Too often we have heard the complaint that our students seem to be coming from previous classes unprepared. What we thought they should know, they didn't. Now what? Well, we had to repeat time and again all of the previous material, not in review but sometimes, it seemed, for the first time, yet we knew that the previous teachers had tried to cover that material before.

The Problems With 50%

I feel that we asked for this problem because, in the past, we only required a 50% performance to pass our courses. We even passed students who didn't know the material but they were nice kids and had worked hard. Sympathetically we felt that if we didn't do this then the students would lose a complete year, and just because of mathematics.

However a 50% performance in any field of study is difficult to assess. A student may "*pass*" the courses but may not know a thing about much of the material. Let me illustrate. Suppose there are ten topics to any course. A student could very easily study only seven or eight of these ten topics and do quite well on them on any test, perhaps obtaining a better than 50% mark on the whole paper. He doesn't know a thing about two or three of the topics and still he passes. Now, because he has passed, the teacher usually does not recognize what topics the student has not learned. The teacher will either concentrate on the poorer students to get them up to a pass, or on the very good students to get them up to scholarship standards. Now where does this leave the average student? Well, the student is pleased because he has passed. The teacher has forgotten about him because he is a statistic that is showing well. But that student still does not know two or three of the

topics, and he may never have to prove himself in them.

I realize that some of the material lost in one year can be reviewed the next year. Likewise a change in teacher, presentation, atmosphere, etc., will also help. But still the problem exists - the students only know some of the material that we expect them to know. Often we will have to review in so much detail that some of the material, which we hope to cover in the new year, may have to be left out. So, what else is new?

A Rewrite System

Well, the mathematics teachers at Thornlea S.S. have the idea that they want their students to learn all the core material (material that must be covered, as compared to optional material). The students realize that they will be tested only on this core material and that they will have to handle it at an 80% performance level. You may ask what happens if this is not achieved. We have a rewrite system worked out and the students have two or three other chances to brush up on the missed material before they rewrite a similar test. There are different versions of this rewrite testing, but my personal method is to retest only that material for which the students did not achieve 80% or better. Let me illustrate:

Suppose there are five ideas to be tested and the student received 10 marks each for a full and correct solution to each. A student could get four questions nearly correct and do only a fair job on the fifth; perhaps he gets 42 out of 50 - a very good mark. Another student may do quite poorly and only do one question well. Perhaps he might get 18 out of 50. Both of these students must rewrite the test - the first student for only the one question missed, and the second will have to do four of the five rewrite questions.

The rewrite test is very similar to the original test in that the type of problems tested are of the same nature as the originals. For example, if we are finding out if the student knows how to solve a quadratic equation, whether we ask him to solve the equation $5x^2-2x+1=0$ on the original paper, or the equation $3x^2+5x-3=0$ on the rewrite paper, the problem is still the same. If he can finally solve quadratic equations, then that is the crux of the

challenge. Again, if he misses it on the first rewrite, he tries another rewrite test until he finally does get it. We do not guarantee that he will remember how to do the problem tomorrow, but who can? However, we at least can state that he could do it at one time. Can any other criteria say that? It should be noted that our school operates on a trimester system which allows us to offer courses that last only 13 weeks or so. Therefore our core material can easily be seen in total so that we know which students are missing certain topics. If we operated on a complete year's programme, I feel that it would be much more difficult to keep tab on the students' progress.

One of the problems that occurs with such a programme is that it requires a great deal of record keeping. We must constantly be aware of the sections of the core material that have been done to our satisfaction (80% or better), and what parts of the core have to be rewritten. Each rewrite test I set includes all the questions that need to be rewritten. When the entire class gets a certain type solved then it no longer appears on a future rewrite paper. With each rewrite paper I just circle the problem types that each particular student must do, thus making each rewrite a personal paper. Testing and retesting takes up a lot of our time, for I test every two weeks and retest on the alternate weeks. Since we have an activity period built right into our school day, often the retests are done at this time, rather than steal valuable class time. I usually allow up to three rewrites on each topic, thus allowing four chances for each type of question. Surely a concerned student should be able to achieve 80% after so many chances.

Another teacher, using this 80% idea, requires that each student must get 80% or better on the complete test or rewrites. That is, if a student does not get 80% on the complete test, then he must rewrite the complete test. The reason here is to allow the student to see the overall concepts that are being tested, rather than the individual concepts in my methods.

Of course, if we are testing only core material at the 80% level of performance, then it is difficult to examine the students on the other material that may not be considered core material. Often we test this interesting material at the bottom of the paper, and do not require any

percentage performance, but perhaps treat it as a bonus.

At Thornlea S.S. there are four categories of rating students; *honours*, *passing*, *unsatisfactory*, and *incomplete*. We do this because we do not always interpret a percentage mark in the same way. For example, what does a 70% mark mean? It could be a very good mark for one teacher and quite an ordinary one for another. However, when one speaks about an honours student, we all understand that interpretation of the student. Our incomplete rating is given to those students who are generally working on an individual progress subject. He may have done quite well in the time for one term, but was unable to complete all of the work for that subject's credit. He is given this incomplete rating which allows him more time to complete the requirements in the next term. He will get his final credit at the end of the second term ... more on this at another time.

Benefits

Now what are the benefits of such an approach? Well, as I have stated before, when a student passes our courses, we know that he once knew *ALL* of the material, that is core, at an 80% performance level. Of course we hope that he retains it, but at least we can say that he knew it well at one time. But, more important, the student himself knows that he knows the material well. This, of course, builds confidence to continue to do a better job in the future, for he feels that he has the potential now to do it. As a teacher I can now expect that the students coming to me will have done well in all of the previous core material.

I feel that the majority of the students like this system. At first they are shocked that we expect so much, and the parents and some of our colleagues are full of questions. But once they see what we are trying to do, and how we give the students two or three extra chances at the same material, they generally appreciate it. I like it because I really know what particular problems are bothering certain students, and I can assist each student with his own particular problems. Since we are able to offer repeats of most of our courses in the following terms, our unsuccessful students may have to repeat the course. Surely after four unsuccessful attempts at a certain type of problem, considered

to be core material, he should repeat. Because of this repeating of courses in a second term, the student will not lose any overall time in school, for he can double up on some future courses.

I want to make it clear that we do this in both our general as well as our advanced courses. So far the students have had few complaints and they find the system quite fair, often praising it. Like all methods it does not work for every student or every teacher, but I feel that we are heading in the right direction as compared to what we did traditionally. At least I feel that our students know the core material to a much greater degree.

Donald C. Attridge received his B.A. from Western and his M.A. from York. He has taught mathematics for seventeen years in various schools, including a Junior College in Switzerland, was Chairman of the Junior Math. Contest, co-authored 'General Mathematics 2' published by Ginn, and is currently president of the Ontario Association of Teachers of Mathematics.

A DISCOVERY APPROACH TO $Q \times Q$

C. D. Gallant

St. Francis Xavier University

Discovery methods in mathematics education have been variously praised, criticized, or even condemned. Much of the criticism, based for example, on inefficiency or ability levels, etc., is likely valid and some teachers hesitate to employ discovery techniques because of a lack of experience with such methodology, particularly in their own preservice education. A majority of teachers, however, are convinced of the tremendous value of the judicious use of discovery methods; surely the experience of discovery is vital to every teacher and should be explored through independent study or inservice institutes.

This article reports some of the results which were conjectured and/or proved by teachers during several guided discovery lessons. The most significant feature of these lessons was the high level of motivation with respect to the formulation of conjectures and their proof or disproof.

The object of each lesson was to discover properties of the set of points $Q \times Q = \{(x,y) : x \text{ and } y \text{ are rational numbers.}\}$

First, some definitions: The elements of $Q \times Q$ are to be called *rational points*. That is, the rational points are just those points in the plane whose coordinates are both rational numbers. The *integral points* are those points whose coordinates are integers. A *rational line* is a line whose equation can be written as $ax + by = c$, where a, b, c are rational. A *rational circle* is a circle whose equation $x^2 + y^2 + ax + by + c = 0$ has $a, b, c,$ rational. Similarly, one can define rational cubics, integral lines, integral circles and so on.

The proof of the following propositions just requires a little elementary algebra:

- *the line joining any two rational points is a rational line;*
- *if two rational lines intersect, their point of intersection is rational;*
- *a rational line contains infinitely many rational points.*

The last property is shown by letting one of the variables take on an infinite number of rational values.

The propositions lead in a natural way to the following suppositions:

- *can a rational line contain only rational points?*
- *are there lines (not necessarily rational ones!) which contain no rational points?*
- *are there lines which contain just a finite number of rational points?*
- *are there lines which aren't rational but contain infinitely many rational points?*

These suppositions - some (but not all) of which are true - can then be varied by substituting 'integral' for 'rational' in various ways"

- *are there rational lines with no integral points?*
- *are there rational lines with infinitely many integral points?*

are two particularly interesting suppositions.

After settling these problems it is natural to consider 'rational circles' and since these are rather more complicated than rational lines some of the proofs will be indicated.

A rational circle has its centre at a rational point: the centre of $x^2 + y^2 + ax + by + c = 0$ is $(-a/2, -b/2)$. But a circle with a rational centre need not be rational, nor need a rational circle have a rational radius.

A rational circle need not have any rational points on its circumference: a simple number-theoretic argument shows that $x^2 + y^2 = 3$ is such a circle. (This leads to the question: *what characterizes such rational circles?*)

Simply by solving the two equations, one can show that the rational circles $x^2 + y^2 = 1$, $(x-1)^2 + y^2 = 1$ do not intersect at rational points, and so the two equilateral triangles with base $[0,1]$ are not rational. *Is there an equilateral triangle whose vertices are rational points?* This was probably the hardest problem we considered. The answer is 'no'. For suppose the construction is possible and denote the rational vertices by (x_0, y_0), (x_1, y_1) and (x_2, y_2). Now observe that $S^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2$ is a rational number. Hence, the area of such an equilateral triangle is $\frac{S^2\sqrt{3}}{4}$, an irrational number. On the other hand,

the area of the triangle is given by

$$A = \frac{1}{2}\begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix},$$

which is clearly a rational number. This contradiction proves the theorem.

The questions posed for rational circles can now be posed for rational cubics, and some of the results are similar. The most interesting result encountered was *'There exist rational cubics having no rational points, finitely many rational points, and infinitely many rational points'*, examples being $y^2 = x^3 + 1$, $y^2 = x^3$, $px^3 + p^2y^3 = 1$, where p is any prime.

The possible areas for further exploration depend on the students' backgrounds. The following topics are suggested for further study. The references can be consulted to provide necessary clues.

1. What are the properties of the set of rational points in the real projective plane? *Look up Bezout's Theorem.*

2. Examine the subgroups of $Q \times Q$, and the quotient groups of $Q \times Q$, both algebraically and geometrically.

3. The set of rational points on a rational cubic in the projective plane form a group.

4. The set of rational points on a rational conic in the projective plane form a field. Examine cubics which degenerate into a conic and a line and imitate the group structure of the cubic.

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INVOLVEMENT: THE GELFAND CLUB

E. J. Barbeau

The University of Toronto

After looking at the geometry of inversion in a circle, why not see what happens with an ellipse or a parabola? Can a method for solving Pythagoras' equation be generalized to a similar equation in n variables? These are two investigations members of the club were led to by the monthly selection. Not for marks, or an award, but for the satisfaction of finding how mathematics fits together.

How does a monthly sheet which briefly describes a mathematical topic and solicits written solutions to a few problems differ from the other resources available to the mathematically-inclined? It is not intended to be recreational, but rather to indicate some of the important ideas of modern mathematics, not merely through reading and copying, but through solving problems, learning to think and communicate mathematically. Some topics may not be found on the standard curriculum, even at college level, but are nonetheless useful additions to one's stock of mathematical lore; other topics directly prefigure what will later be studied, perhaps more abstractly; still others extend schoolwork. Hopefully, there is sufficient depth that the student may be able to realize what are some of the characteristic approaches and problems and may be able to investigate for himself a few of the examples. The student will find the material valuable to the extent that he is willing to test his understanding by committing his ideas to paper and corresponding about them with a mathematician at a University - to enable this is the key function of the Club. Projects, puzzles, competitions, films, reading books and journals, on the other hand, may tend to be more ephemeral, more passive or more freeranging; while they encourage a great deal of mathematical development, they do not generally involve the sort of interaction the Gelfand Club promotes. Two ancillary functions of the club might be noted - to provide for possible contact of students with similar interests, and to build up a stock of material on which teachers and mathematical clubs may draw (there is no copyright, so you may duplicate at will).

We would like to hear from more students. The Club offers excursions into analysis, algebra, geometry and number theory; selections have been prepared by H. S. M. Coxeter, I. Halperin, R. Honsberger, J. Taylor and J. Wilker as well

as the author of this article. Articles by other authors have been promised on graph theory and linear programming. The difficulty varies. But unfamiliarity, rather than difficulty, seems to be the main deterrent to students. The selections on Euclidean geometry and Inequalities had some tough questions, but were among the most popular. Please tell students not to be diffident about submitting incomplete solutions or solutions they feel might be "wrong"; often, they contain unorthodox methods that can be made to bear fruit.

The Club imitates a similar venture launched in the U.S.S.R. in 1964 by Professor I. M. Gelfand, Professor of Mathematics at the University of Moscow and a leading researcher in functional analysis. It was started here in 1969 by Professor Israel Halperin of the University of Toronto. About 35 selections have appeared already and are available at a cost of 10¢ each plus a basic charge of 25¢ to cover duplicating and postage. Any student may obtain current selections by sending stamped, self-addressed envelopes; others may subscribe at a rate of $1.50 per year. The money will be used to partially reimburse the University of Toronto Mathematics Department for use of its duplicating facilities.

Write to: The Gelfand Club of Ontario
c/o E. J. Barbeau
Department of Mathematics
University of Toronto
TORONTO, Ontario
M5S 1A1

or, in North-Western Ontario, write the editors.

The current selections are:

1969: October; November; December.

1970: January; February; March (Finite Differences); April (Number Theory); May (Polynomials); July (Recreational Problems); September (Truth tables); October (2×2 matrices); November (Visualizing geometric figures); December (Mathematical discovery).

1971: January (Incommensurability; Cavalieri's method); February (Combinatorial problems); March (Blocking sets); April (Information theory); May (Frieze patterns); July (Experimental problems); September (pseudoprimes and related numbers); October (Sketching graphs); November (Analytic geometry); December (Inversion in a circle).

1972: January (Inequalities); February (Newton's method for polynomial roots); March (Diophantine equations); April (Integers); May (Factoring); July (Euclidean geometry); September (Gram-Schmidt Orthonormalization Process-vectors); October (Fourier series); November (Some work of Euler); December (Equidistant sets).

1973: January (Sequences, recursion relations, limits); February (Real numbers); March.

Problem 71.06 (O.M.G., vol. 10, p. 119)

Solution by *A. W. Walker, Toronto, Ontario*

At the point $T(at^2,2at)$ on the parabola $y^2 = 4ax$, the slope of the tangent line is t^{-1}, and the equation of the normal is

(1) $y - 2at = -t(x-at^2)$.

Replacing t by $-t^{-1}$, the equation of the perpendicular normal at the point $U(at^{-2},-2at^{-1})$ is

(2) $t^3y + 2at^2 = t^2x - a$.

Lines (1) and (2) meet at the point P with coordinates

(3) $x = a(t^2+t^{-2}+1)$, $y = a(t-t^{-1})$,

and eliminating t, the locus of P is the parabola

(4) $y^2 = a(x-3a)$.

A method of greater interest, however, is to combine (1) and (2) so as to eliminate t^3, giving

(5) $a(2a-x)t^2 - (2a-x)yt + (y^2+a^2) = 0$.

This equation, like (1) and (2) from which it came, must be valid when t is replaced by $-t^{-1}$; hence, regarding (5) as a quadratic in t, the product of its roots is -1, so

(4') $a(2a-x) + (y^2+a^2) = 0$.

(continued on p. 220)

MAKING MATHEMATICS INTERESTING FOR NON-COLLEGE BOUND STUDENTS

W. G. Anglin
Ontario Ministry of Education

There seems to be an attitude toward mathematics on the part of non-college bound students which is best exemplified by Lucy in the comic strip 'Peanuts'. She has no interest in doing mathematics for its own sake and sees no connection between it and her future because "I'm only going to be a housewife" and the only immediate need for it that she can see is to help her on the coming test she has to write.

The purpose of this paper is to try and suggest problems that might appear realistic and worth while to the pupils in the hope that their interest will be aroused and they will see that mathematics really does belong to them in their situation. I feel that this can be achieved within the context of the present courses and does not imply throwing out existing courses and introducing 'practical mathematics'.

The topic of statistics is one which can be made topical for the students by setting aside the prepared problems of the text book and having the students design, conduct and evaluate some survey of their own. There is more to the study of statistics than just taking data from a book, drawing a frequency distribution and then going on to the next problem.

In one school which I have visited the Grade 10 non-college bound mathematics is taught by the head of the Phys. Ed. department. His approach to the presentation and interpretation of data was to poll the entire student body to determine their smoking habits. His classes had to struggle over the design of the survey made, organize the data so that it made some sense to them, present it in some way so that it made some sense to others and draw conclusions and make predictions as a result of their survey. The students became so involved in this survey that the teacher hopes to continue it in future years and perhaps identify trends in smoking habits within that school.

Another teacher in the same school had his students do an analysis of the heights of the student population. The possibilities for this kind of survey are endless: makes of

cars, T.V. favourites, weights, shoe sizes, fraction of the population that is left handed, accessories on cars, number of children in a family, etc. The opportunities to choose the problem, set up the experiment, conduct it, organize the data, draw conclusions, make predictions and criticize the sample chosen would all be worthwhile for students at any level. Their interest is much greater in this type of problem than it is in a rigged problem on Page 84 of someone's text book. Most text book problems are too good to be true. Every sample and experiment leads to a normal distribution. This is not realistic for the size of sample available to the students.

The topic of the graphical and algebraic solution of two first degree equations in two variables is an area in which realistic situations can be found to introduce the topic and perhaps generate some interest in the student.

Several situations which give rise to a pair of first degree equations in two variables are:

(a) straight commission for a salesman or basic salary plus a smaller commission of his sales

(b) team bus rental at a flat rate per mile or a fixed rental charge plus a lower rate per mile

(c) printing charges for programs (or football tickets) at a cost per program or an initial charge plus a lower cost per program

(d) barber paid a percentage on his income or a weekly salary plus a smaller percentage on his income.

Each of these situations can give rise to a graph similar to Figure 1, in which the horizontal axis could represent sales, or number of miles, or number of programs, or number of hair cuts in a week and the vertical axis could represent total income, or total costs depending on the problem considered. It is worthwhile in this kind of problem to consider when one method of payment or income has an advantage over the other (and, also, from whose point of view it is an advantage) and when both methods provide the same cost or income.

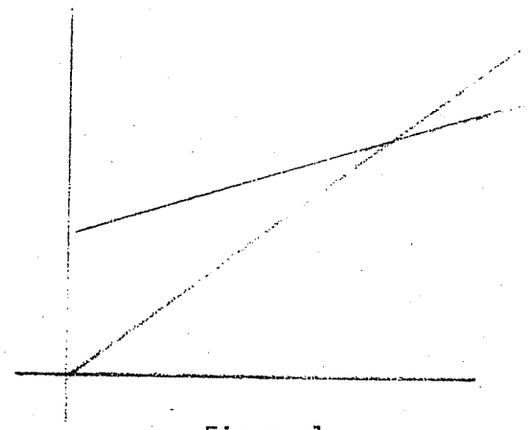


Figure 1

It is easy to justify introducing an algebraic solution to find the point of intersection if a problem is designed for which the coordinates of the point of intersection are not integral. If the coordinates are integral a graph is usually an accurate enough solution to determine the point of intersection but for a non-integral case some method other than graphing can be justified.

Ratio and proportion lends itself to realistic problems and a use of materials, and perhaps, even a trip to the vocational wing of the school to see some applications of ratio and proportion in the automotive or machine shop. Some of the possibilities in this topic are:

(a) actually measuring the circumference (with string or by rolling) and diameter of a variety of circles and spheres and graphing D against C. If D is considered the independent variable the resulting graph should have a slope which can be calculated and will be an approximation to π. If the students have not yet encountered π (which is a distinct possibility, according to many teachers) this could provide a springboard from which a discussion of π would follow.

(b) number of turns on the pedals of a bicycle graphed against the distance travelled. Can the resulting slope be given any significance here?

(c) diameter of pulleys against belt speeds or diameter against R.P.M. of gear. It is in this area that some look at gearing might be worthwhile. Perhaps a trip to an old saw mill might be possible to actually see the uses made of these ideas. Even a trip to the automotive shop to see a car's gear box might help, or a discussion of gearing of a bicycle might be arranged.

(d) consideration of time of travel for a fixed distance at various speeds provides opportunity for discussion.

(e) graph of the length of the side of a square against its area could be introduced here or might be used as a lead-in to squares and square roots.

(f) finding areas of irregular shapes might be considered by looking at a weight against area graph for regular shapes and then weighing irregular shapes cut from the same material.

(g) weight vs. volume graphs for irregular solids.

(h) tracing an irregular shape (palm of your hand, a leaf, the sole of your shoe, etc.) on squared paper, and counting squares can be used as an approximation to the area of the figure. The idea of getting a better and better approximation by taking a smaller and smaller unit might be discussed in this connection. The area of the sole of your shoe could be used as a lead-in to a discussion of pressure.

Squares and square roots might be introduced from the graph mentioned previously or perhaps from a table of values (Figure 2) and a desk calculator. With a desk calculator the relationships among $\sqrt{2}$, $\sqrt{8}$, $\sqrt{18}$, or $\sqrt{3}$, $\sqrt{12}$, etc. might be discovered and further developed.

A way of considering the existence and meaning of rational exponents and perhaps introducing logarithms is to use the fact that $2^{10} \doteq 10^3$. Therefore, $2^5 \doteq 10^{1.5} \doteq 30$,

$$2^2 \doteq 10^{0.6} \doteq 4,$$

$$2^1 \doteq 10^{0.3} \doteq 2.$$

$1 = 1^2$	$6 = ?$	$11 = ?$	$16 = 4^2$
$2 = ?$	$7 = ?$	$12 = ?$	$17 = ?$
$3 = ?$	$8 = ?$	$13 = ?$	$18 = ?$
$4 = 2^2$	$9 = 3^2$	$14 = ?$	$19 = ?$
$5 = ?$	$10 = ?$	$15 = ?$	$20 = ?$

Figure 2

Factoring of trinomials of the form $x^2 + ax + b$ (where b has as one pair of factors c and d and $a = c + d$) can be approached conveniently as a game, once the basic ideas have been introduced. If a class is challenged to make up (without benefit of books, pens or paper) a factorable trinomial which the rest of the class cannot factor the challenge leads to an interesting contest for the class. For the student making up one of these trinomials; keeping track of the original factors, multiplying them and giving the result to the rest of the class demands quite a bit of mental activity and yet they will do it willingly in the spirit of the game.

Purchasing personal property and managing personal property are areas of the course that might lend themselves to bringing in an outsider to discuss insurance costs (both life and car insurance), the stock market, borrowing money (have a bank manager, a finance company manager and a credit union manager discuss their respective operations). Business and industry offer a resource to the schools that has been infrequently tapped in the past. The guidance departments do not hold exclusive rights to outside assistance. It is available to a teacher willing to look for it.

Systems of numeration are introduced into our courses, not so students can gain facility in the arithmetic of other bases, but in order that they gain a better understanding of the base ten system. A base two system can be conveniently introduced using a set of 1, 2, 4, 8, 16 and 32 oz. weights and asking the students to select the necessary combinations to get all the integral weights from 1 oz. to 63 oz. The same kind of thing can be done using the child's construction toy 'lego' in which each piece has a number of small pegs on the top. Pieces with 1, 2, 4, 8, 16 and 32 pegs can be chosen and a table built up as before (Figure 3). Feeler guages used

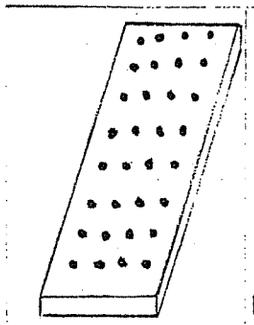
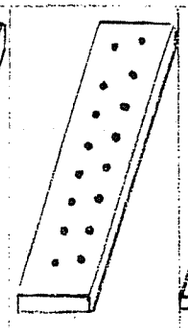
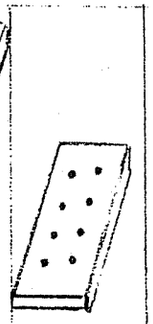
						
1						1
2					1	0
3					1	1
4				1	0	0
5				1	0	1
6						
7						

Figure 3

in the automotive and machine shops use the principle of combining several to get the required thickness so that there are only a small number of gauges required.

Numerical applications of geometry includes Pythagoras with all his possibilities but it can also include a consideration which will neatly tie together number patterns and some simple geometry. If a table is built up (Figure 4) either under teacher guidance or as a student problem many relationships and patterns can be considered. Within this table there are patterns in both the rows and columns.

For some students it would be quite an accomplishment to build up this table one row at a time but there would be some in the group who would be able to see the patterns developing and jump a few rows. There would even be some who

Characteristics of Regular Figures

Number of Sides	Angle at Centre (degrees)	Interior Angle (degrees)	Interior Angle (degrees)	Sum of Interior Angles (degrees)	Sum of Exterior Angles (degrees)	Number of Diagonals
3	120	60	120	180	360	0
4	90	90	90	360	360	2
5	72	108	72	540	360	5
6						
. . . .						
60						
. . . .						
n						

Figure 4

could generalize from their observations and fill in the row for a regular figure with n sides. The opportunity for each student to carry a problem as far as he can should be one of the objectives of our mathematics courses.

Other considerations in the area of geometry might be discussions of symmetry in architecture and in the basic geometric shapes, construction of models, considerations of shapes which can be used in the tesselation of the plane, and three dimensional sketching.

An interesting problem that can act as a motivator for students at almost any level is a tree diagram of the possible outcomes of a 4 out of 7 series. If this is considered at World Series, N.B.A. playoff or Stanley Cup time there is no problem with student interest. A little bit of probability can be introduced through this problem and perhaps pursued with probabilities in simple poker hands or in rolling dice of flipping coins.

Trigonometry is one topic for unmotivated students which is reasonably easy to justify. The use of similar triangles and the applications to inaccessible heights and distances can make this a practical unit, well-suited to outdoor activities in spring and/or fall. Tables of the trigonometric function values can be constructed with a reasonable degree of accuracy by drawing a fairly large unit circle on coordinate axes and reading coordinates off the graph for given angles. Taking a 10° or 15° interval gives two figure approximations of sine and cosine values. A need for a greater degree of accuracy will justify turning to prepared tables.

Quadratic equations frequently seem to be done as an isolated unit independent of the quadratic function. If these two topics are tied together each can be used in explanations for occurrences in the other. The significance of the roots of a quadratic equation can be discussed along with the intercepts of the graph of the corresponding quadratic function and the lack of real roots of some equations can be "seen" when looking at the associated graph.

There seems to be a tendency to limit discussions of sequences in many courses to geometric and arithmetic examples only. This tends to leave many students thinking there are no others when, in reality, there are many interesting sequences which are neither arithmetic nor geometric. Some of the columns

in Figure 4 illustrate a few cases. Others are the cases of triangular (1,3,6,10, . . .), square (1,4,9,16, . . .), pentagonal (1,5,12,22, . . .), and hexagonal (1,6,15,28, . . .) numbers. The many patterns evident in Pascal's triangle can provide challenging material where some discussion of natural occurrences of the Fibonacci sequence might interest some of the students.

The topic of measurement is frequently restricted to problems for which a formula is available and the problem is not one of measurement at all but one of substitution and simplification. How many thousands of students have found the areas of rectangles, circles, parallelograms, triangles and the ends of barns, and the volumes of boxes, cylinders and cones? Have they ever been asked to find the area of a leaf, their hand or their foot, or find the volume of a coke bottle, a rock or a collapsible plastic container? Two treatments of the problem of finding areas and volumes of irregular shapes and solids have been mentioned earlier.

I feel that the main challenge for teachers is to reawaken the interest of students in a subject whose reference to reality is sometimes quite neglected. An opportunity to see the applications of some of the material might be a first step toward catching their interest again.

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MATHEMATICAL INDUCTION: A SET THEORETIC APPROACH

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In view of the emphasis now being placed on mathematical structure, proof, and unifying concepts in the secondary curriculum, it is surprising that most textbooks approach mathematical induction in much the same way as was done over a decade ago. Some of the newer texts give a more lucid explanation than the standard presentation of "pre-reform" years, yet the mathematical and pedagogical approach to the topic is most traditional. Usually, the Principle of Mathematical Induction is stated as an axiom and perhaps illustrated by a situation involving a chain reaction phenomena such as the teacher knocking down the first of a finite set of aligned dominoes, the dominoes being arranged so that when the first one topples against its neighbor, the neighbor falls, knocking down the next domino, and so on, until all the dominoes have fallen. The Principle is next usually applied to an example and then the student is provided with a number of exercises on which to practice. The problems he encounters are usually various sums such as $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. Seldom is any explanation given to the source of these rather mysterious sums.

In contrast, what follows is a description of an attempt to make mathematical induction more plausible and legitimate by relating it to a more intuitively acceptable property of the set of positive integers which most students can easily discover. First, we outline an analysis of the mathematical content. It is based on the fundamental concept of set.

Mathematical Considerations

Notation Let $N = \{1, 2, 3, 4, \ldots\}$ of positive integers.

Definition a is a least member of a set S if and only if $a \in S$ and for each $x \in S$, $a \leq x$.

LEAST NUMBER PRINCIPLE Each nonempty set of positive integers has a least member.

Definition A set T of numbers is hereditary if and only if $x + 1 \in T$, whenever $x \in T$.

Theorem 1 (PRINCIPLE OF MATHEMATICAL INDUCTION)

Let $P(n)$ be a statement that is defined for each positive integer n and let $S = \{n \in N | P(n)$ is true$\}$. If
(1) $1 \in S$
and (2) S is hereditary
then $S = N$.

Proof: Let $P(n)$ be a statement that is defined for each positive integer n and let $S = \{n \in N | P(n)$ is true$\}$. Suppose that $1 \in S$ and S is hereditary. Assume, however, $S \neq N$. Let T be the set of all positive integers which do not belong to S. Since by assumption $T \neq \emptyset$, it follows by the Least Number Principle that T has a least member-say p. Since $p \notin S$ and $1 \in S$ it follows that $p \neq 1$. So $p - 1 \geq 1$. Since p is the least positive integer which does not belong to S and since $p - 1 < p$ it follows that $p - 1 \in S$. But this being so; since S is hereditary $(p-1) + 1 \in S$, that is, $p \in S$. But this is a contradiction. Therefore our assumption that $S \neq N$ is false. Consequently, $S = N$.

Theorem 2 (SECOND PRINCIPLE OF MATHEMATICAL INDUCTION)

Let $P(n)$ be a statement that is defined for each positive integer $n \geq k$, for some $k \in N$ and let $S = \{n \in N | P(n)$ is true$\}$. If
(1) $k \in S$
and (2) S is hereditary
then $S = \{n \in N | n \geq k\}$.

Though Theorem 1 is obviously a consequence of Theorem 2, pedagogically the above ordering is preferable. The specificity of Theorem 1 minimizes the amount of guidance the teacher must give the class as they construct a proof. Moreover, the students' understanding of the proof of Theorem 1 is enhanced by requiring them to later prove Theorem 2 individually on their own.

Pedagogical Considerations

Mathematical induction is a tool for proving the validity of statements concerning the positive integers. Hence, the place to begin is with a closer study of the set of positive integers and its various subsets. A consideration of examples such as {2, 4, 6, 8, ...}, {3, 5, 7, 11}, {18, 19, 20, 21, ...},

{832, 833, 834, 835, ...}, {12, 15, 18, 21, ...}, and {125, 17, 32, 16, 64} soon leads to the student discovery that one characteristic of these sets is that each has a "smallest number" or "first member" results in the formulation of a definition similar to that given above. The Least Number Principle is then a re-phrasing of the students' discovery using the new terminology of least member of a set. Is this also a property of the positive real numbers? No, the set $\{x|x > 3\}$ has no least member.

There are also characteristics distinguishing the above sets of positive integers. For example, most students immediately note that some sets are finite, while others are infinite. Upon a closer inspection of the infinite sets students will observe that in the case of {18, 19, 20, 21, ...} or {832, 833, 834, 835, ...}, each element after the "first" is obtained by adding 1 to the preceding element. We may think of each of these sets generating its successive members. We abstract this notion, and identify sets possessing this property as hereditary sets. Here the terminology is descriptive and most students are attracted by it. Other examples of hereditary sets are $\emptyset$, {integers}, and {positive rational numbers}. Are {1, 2, 3, 4} or {integral multiples of 5} hereditary? No.

Having developed the above preliminary background and reinforced the concepts with appropriate practice exercises we may then begin to motivate the Principle of Mathematical Induction. Convincing students of the need for such a principle is relatively easy, but it is often either overlooked or bypassed in the cause of expediency. The following is a sample of conjectures (some true, some false) that students can be guided to discover and formulate.

1. The number of dominoes in a set which runs from double zero to double n is $\frac{(n+1)(n+2)}{2}$.
2. For each positive integer n, $a - b$ is a factor of $a^n - b^n$.
3. $10^n > n!$ for each positive interger n.
4. If $n \geq 2$ students are entered in a chess tournament and if each plays one game with each of the other students, then the number of games played is $\frac{n(n-1)}{2}$.

5. For $n \geq 3$, the number of diagonals of an n-sided convex polygon is $\frac{n(n-3)}{2}$.

6. If $z = r(\cos\theta + i\sin\theta)$ then for each positive integer n, $z^n = r^n(\cos n\theta + i \sin n\theta)$. (DeMoivre's Theorem).

7. The maximum number of regions formed when n points on a circle are joined in pairs is 2^{n-1}.

If students are given the opporutnity to make discoveries, they will soon come to appreciate the need for some method of verification. How would one verify a conjecture such as 6 concerning the powers of a complex number written in trigonometric form?

Let T be the set of all positive integers for which the conjecture is false. If $T \neq \emptyset$, then by the Least Number Principle, T has a least member-say p. Since the conjecture is true for 1, $p \geq 2$ or $p - 1 \geq 1$. Also since $p - 1 < p$, the least integer for which the conjecture is false, it follows that the conjecture is true for $p - 1$.

That is, if

$$z = r(\cos\theta + i \sin\theta)$$
$$z^{p-1} = r^{p-1}(\cos(p-1)\theta + i \sin(p-1)\theta).$$

So
$$\begin{aligned} z^p = z^{p-1}\cdot z &= r^{p-1}(\cos(p-1)\theta + i \sin(p-1)\theta) \cdot r(\cos\theta + i \sin\theta) \\ &= r^p(\cos[(p-1)\theta + \theta] + i \sin[(p-1)\theta + \theta]) \\ &= r^p(\cos p\theta + i \sin p\theta) \end{aligned}$$

that is, the conjecture holds for p. But this is a contradiction. Hence our assumption is false. $T = \emptyset$ and we conclude that the statement is indeed true for each positive integer n.

Students will find it instructive to use this indirect method of proof to verify a number of their conjectures. It is not long however, before some students will suggest a way of using the notion of a hereditary set to prove their conjectures directly. At this point the Principle of Mathematical Induction can be meaningfully introduced. Moreover, the prior experience with indirect proof tends to place the proof of Theorem 1 and 2 within the reach of most students.

Let us return now to conjecture 6 and use the Principle of Mathematical Induction to prove it directly.

Let $S = \{n \in N | z^n = r^n (cos\ n\theta + i\ sin\ n\theta)$ where $z = r(cos\ \theta + i\ sin\ \theta)\}$.

$1 \in S$ since $z^1 = r(cos\ \theta + i\ sin\ \theta) = r^1(cos(1 \cdot \theta) + i\ sin(1 \cdot \theta))$.

Assume now that $k \in S$. Hence, it follows that

$$z^k = r^k(cos\ k\ \theta + i\ sin\ k\ \theta),$$

and so
$$\begin{aligned} z^{k+1} = z^k \cdot z &= r^k(cos\ k\ \theta + i\ sin\ k\ \theta) \cdot r(cos\ \theta + i\ sin\ \theta) \\ &= r^{k+1}(cos(k\ \theta + \theta) + i\ sin(k\ \theta + \theta)) \\ &= r^{k+1}(cos(k+1)\theta + i\ sin(k+1)\theta). \end{aligned}$$

Thus $k + 1 \in S$ and therefore S is hereditary. Consequently, it follows by Theorem 1 that $S = N$, that is the conjecture is true for each positive integer n.

Given a statement concerning the positive integers the student now has two techniques for establishing its validity. Neither technique, however, will establish conjectures 3 or 7. The reader is encouraged to find appropriate counterexamples. That some of the student conjectures were false serves to emphasize the need for proof and deductive methods.

In conclusion, some work with mathematical induction is quite generally considered to be a part of the content of the algebra courses in the secondary schools. This is reasonable since mathematical induction is one of the main forms of proof used in mathematics, perhaps second only to indirect proof in the number of instances when it is applied. It is safe to say, however, that the majority of the students find mathematical induction to be one of the most difficult topics in the algebra program. It is suggested that mathematical induction can be made more plausible and legitimate if it is related to the Least Number Principle and can be made less mechanical if developed in terms of hereditary sets.

Dr. Hirsch received his doctorate in mathematics education from the University of Iowa, where he worked under Dr. H. Vernon Price.

FINGER MULTIPLICATION

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One of the older forms of multiplication is the so-called finger multiplication. It dates back to before the short interlude during which students were expected to know all multiplication tables - at one time this was considered a rather excessive burden for the innocent child. Instead only the multiplication tables from 1 through 5 were committed to memory. To do other problems such as 9×7 the student used his fingers as follows:

Each number from 0 through 10 was represented by one hand according to the following rules.



A fist represented the number 0.



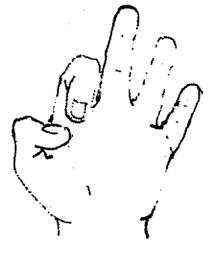
The four fingers down and thumb up was the number 1.



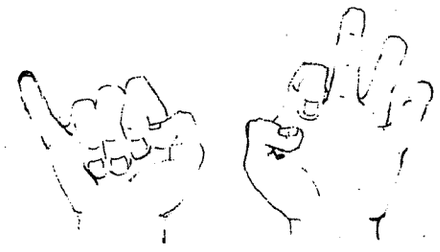
Thumb and index finger up represented the number 2 and so on up to 5.



The number 6 was represented by four fingers up and thumb down.



The number 7 by 3 fingers up, index finger and thumb down, etc.



To do a problem such as 9×7 the student would represent the number 9 with his right hand and the number 7 with his left. The number of fingers down on both hands is $4 + 2 = 6$. This will be the first digit in the answer of the problem ($9 \times 7 = 6_$). The number of fingers up on both hands is 1 and 3 - the product of 1 and 3 is 3 ($1 \times 3 = 3$). This will be the second digit in the answer of the problem, therefore, the answer to 9×7 is 63.

The proof that this type of multiplication works can be given as a challenge to any reasonably competent grade ten algebra student. It is possible to extend this type of

multiplication to such problems as 11×12 by introducing "negative fingers" but the details become so artificial that they're likely to be of interest only to the student who can develop and discover such rules himself.

Problem 71.04 (O.M.G., vol. 10, p. 40, 176)

Solution by *A. W. Walker, Toronto, Ontario*

The short "geometric" proof requested by the proposer is as follows. Denoting by P the meet of lines AB and DG, and assuming only that $ABCD$ is a parallelogram, by affine projection with ray pencils centred at G, centred at H, parallel to BC, and parallel to AC, we have

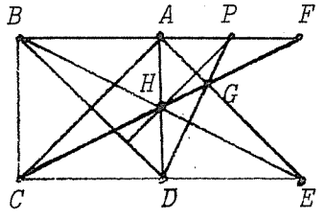
$$\frac{FP}{FA} = \frac{CD}{CE} = \frac{FA}{FB} = \frac{FH}{FC},$$

and PH is parallel to AC. If $ABCD$ is a square, PH and DH are respectivley perpendicular to BD and BP, so H is the orthocentre of triangle BDP and BH is perpendicular to DP.

Comments:

(i) The problem, solved in this way, illustrates very neatly how ratios of lengths are preserved if either the projection bases or the projecting rays are parallel.

(ii) The point H may lie anywhere on line AD (not necessarily between A and D); but if $H = D$, then $G = D$ and line DG is undefined.



THE SPIROGRAPH: AN APPLICATION OF MODULAR ARITHMETIC

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Presumably everyone knows what a spirograph is: in its most elementary form it consists of a plastic wheel with gear teeth on the circumference and a plastic ring with gear teeth on both inside and outside circumferences. To use it, the ring is held stationary on a piece of paper and the wheel is rotated around the inside or outside of the ring --- the teeth mesh, so there is no slipping --- and a pencil inserted in a hole in the wheel traces a design.

We ask: *how many revolutions of the wheel are necessary to trace a closed design?* That is, at what stage does one begin to retrace the design just completed? This will happen when the wheel returns to its initial position. Suppose that, at the beginning, we mark the point of contact of the wheel and ring on each of them. Then the design will be completed when and only when these two marks again coincide, unless we have inserted the pencil in the middle of the wheel. (We assume this exception is not the case since a spirograph wheel does not usually have a hole in the centre!) The wheel is said to have completed a "revolution" between any two consecutive occasions in which a given point on the wheel (such as the point originally marked) touches the ring. Suppose the wheel is revolved x times and that its circumference is n. (We could measure the circumference in inches, feet, etc., but it is most convenient to measure it in 'number of gear teeth'.) Then the wheel moves an amount nx along the ring and so, if we let the circumference of the ring be m (gear teeth), the design will be completed when and only when $nx = m$. That is, x is a solution to the congruence $nx = 0\ (m)$.

A well-known number theory result is: $nx = 0\ (m)$ iff $x = 0\ (m')$, where $m' = m/(m,n)$. We have denoted the greatest common divisor of m,n by (m,n). The proof of this result can be found in most freshmen algebra texts.

Thus, the design is complete after x revolutions, where x is any positive integral multiple of m'; the smallest such x is clearly m' --- if the wheel is rotated more than m' times the pencil retraces the design.

For example, suppose the ring has 96 teeth and the wheel 64 teeth. Then $m' = 96/32 = 3$, so a complete design is traced after 3 revolutions of the wheel. We note that it is immaterial whether the wheel is placed inside or outside the ring, but the relevant circumference of the ring is to have 96 teeth. Furthermore, it does not matter into which hole on the wheel the pencil is inserted.

If the ring has 96 teeth and the wheel 42 teeth, then a complete design requires 16 revolutions of the wheel, since $\frac{96}{(96,42)} = 16$.

Suppose we began to trace a design with a particular tooth on the wheel - call it *tooth* a - completely meshed with the ring. As the design is traced this tooth will mesh with various notches in the ring. We ask: *when a design is completed, with how many different notches has tooth* a *meshed?*

Since a was initially meshed with the ring, it will mesh again in one revolution --- by which time the wheel has travelled n notches along the ring --- again after the second revolution, and so on. Thus is will mesh with those notches which are $0, n, 2n, 3n, \ldots, m'n$ notches distant from the initial position --- and after this it repeats the design. The question is --- *are the notches at these distances all different?* Clearly, two notches are identical iff their distances differ by a multiple of m. Thus the answer to our question will be found by determining how many different modulo m numbers there are in the list $n, 2n, 3n, \ldots, m'n$.

In fact these numbers are all different, modulo m: if two of them pn, qn were congruent modulo m then, because n, m' are relatively prime, m' must divide $p - q$. Since $|p-q| < m'$, this implies that $p = q$.

Thus *tooth* a will mesh with m' different notches in completing a design. Since we could start our design at any stage, we conclude that each tooth meshes with m' different notches in completing a design.

In particular, if m, n are relatively prime, then each tooth meshes with each notch exactly once.

The results of our investigation made use only of the number of teeth and notches of the wheels and rings; it

was not required that the wheel and ring be circular in shape. It is required, however, that the notches and teeth be regularly spaced and may be on any "oval shaped" plane figure. The racks included in the Spirograph set are examples of various types of ovals.

Many recreational type games and activities are based upon significant mathematics and can be beneficially used at many instructional levels. These uses may vary from the stage of the concrete to that of abstract analysis. The Spirograph is but one example of such an activity.

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BOOK REVIEWS

For a list of books available for review, write the editors or H. P. Heinig (Department of Mathematics, McMaster University, Hamilton), or see the December Gazette, p. 144. Don't forget - the reviewer can keep the book!

Polyhedron Models, by Magnus J. Wenninger, MacMillan of Canada, 1970 ($16.50) pp. 208. Forward by Professor Coxeter of the University of Toronto.

I quote first from the author's preface. "This book presents a well-defined set of geometrical solids, the seventy-five (known) uniform polyhedra, together with a representative set of stellated forms. A description of the underlying theory of polyhedra is included to bring out the relationships that exist between the various solids. But mainly this book is simply a set of instructions on how to make models of solids.

The sources in which you can find an account of the mathematical theory of this topic are given at the end of the book. If in the past you found the study of geometry a bit difficult, or if at present you are not particularly attracted by geometry, you may wonder if this topic will hold your interest. The fact is that you really do not need to understand all the theoretical mathematics involved in the original discovery and classification of these solids. On the other hand you cannot avoid all the mathematics, especially the terminology used here and some of the symbolism.

The objective in this book will be to set down an explanation of the solids, at once simple and practical and not too speculative, one sufficient for the purposes of constructing the models. It is really surprising how much enlightenment will come, following the construction of the models rather than preceding it, and once you begin making them you may find that your enthusiasm will grow. You will soon see that each of these solids has a beauty of form that appeals to the eye in much the same way that the abstract mathematics appeal to the mind of a mathematician."

The method of construction alluded to above is the tab method, not the skeletal. The author states that each model, other than the very simplest, took about eight hours to construct.

The book is divided into three sections. Section 1 "The Convex Uniform Polyhedron: The Platonic and Archimedean Solids" has a discussion of, and instructions for, the construction of these best known solids; 5 of the former type, 13 of the latter. Included here are colour schemes for the faces, for the purpose of instruction and decoration. The coloured rhombicosi dodecahedron is especially attractive.

Section II is entitled "Some Stellations and Compounds". A stellation of a solid is a solid obtained by extending facial planes. Included are 24 stellations of the Platonic solids, and 24 of the Archimedean solids.

Section III "Non-convex Uniform Polyhedron" has instructions, and photographs of, 53 uniform polyhedra.

A uniform polyhedron has the same arrangement of regular polygons on every vertex. There are 75 known in all (thought to be a complete enumeration) and models for all are in the book!

This book should be on the bookshelf of every geometer, and available to mathematics clubs and the like. It is a delight!

T. M. K. Davison, McMaster University

Base-X Arithmetic, by Bruce Ewen, McLelland & Stewart, 1970. Price unknown (probably about $2.00). pp. 64.

A very good book to have for reference purposes. The explanations and examples are quite easy to follow yet the author has given this topic quite a thorough treatment. Students could use this as a text-book. There are problems exercises at the end of each chapter. The material is suitable for Grade 12 and upwards.

A few points of interest:

(1) The number 1764 to the base x is the polynomial

$$x^3 + 7x^2 + 6x + 4.$$

(2) A routine method in which to add, subtract, multiply or divide in any base is developed slowly and carefully. Usually one chapter is devoted to each operation.

(3) A strong connection is made to Pascal's triangle.

(4) Decimal fractions become ximal fractions in any base.

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Mathematics for the Elementary School Teacher, (Nichols & Swain), Holt, Rinehart & Winston, Inc. 1971.

This text is a veritable compendium of mathematics for the elementary school teacher. It discusses all of the mathematics which the prospective teacher should have as minimum academic preparation.

There are chapters on logic, sets, and systems of numeration, both ancient (well illustrated) and modern (various bases). Relations and functions are studied as ordered pairs. Number systems through real numbers are treated. There is an introduction to algebra and a study of matrices with the usual elementary fundamental operations. Each chapter closes with a glossary of important terms and review problems.

There is a good deal of geometry treated from the modern viewpoint, areas and volumes with quite a bit of coordinate geometry. There are graphs through the parabola. There are also chapters in number theory and elementary probability.

There are 425 pages of material. Study of the book by a class would require a couple of regular semesters or one year. A judicious selection of topics could provide a good one-semester review for those teachers having had previous contact with the material.

In the chapter on sets, the union of a pair of sets is the set consisting of all those elements which belong to either set; the intersection of a pair of sets is the set consisting of all those elements which are common to both sets. Although the definitions are correct, words mean different things to different people. Symbolic definitions could easily be used here and might be advisable.

$A \cup B = \{x | x \in A \text{ or } x \in B\}$; $A \cap B = \{x | x \in A \text{ and } x \in B\}$

Also in finding GCD and LCM, algorithms are handy and useful when you have frequent use of them, but are soon forgotten otherwise. First principles should be emphasized here (the prime factorization method), since this is what will remain after most of the material is forgotten.

The book material is relevant to the elementary grades. It is a C.U.P.M. type of course for prospective elementary mathematics teachers. A sense of structure is conveyed with emphasis on the rationale of the algorithm. The text is readable by the student and there is a good selection of significant problems for which there are selected answers. Students may even learn to prove theorems.

This book, or one like it, should be in the reference library of each prospective or actual elementary school mathematics teacher.

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University of Detroit

Modern Physics and Quantum Mechanics, by Elmer E. Andersen, Ph.D., W. B. Saunders Company, 1971, pp. 430 value approximately $12.00.

This book is designed for a descriptive course for senior undergraduates preparing for a graduate course in quantum mechanics. The ideas are presented to develop an understanding of quantum physics, rather than by historical order.

Underlying concepts presented in initial chapters on The Special Theory of Relativity, Wave and Particle Properties of Photons, Early Wave Mechanics and The Bohr Theory, set the stage for studies in Schrodinger wave mechanics, quantum and matrix mechanics, eigenfunctions and angular momentum operators applied to the hydrogen atom, orbital angular momenta, relating approximate methods to ideal systems and quantum mechanical scattering theory. The author includes a summary and list of suggested references at the end of each chapter for clarification and further research by the reader.

Understanding of related mathematics is essential since development assumes good knowledge of Linear Algebra, Differential Equation and Matrix Theory.

Considering the level of studies for which it is designed, the text is adequate. Insight should be gained into the physical behaviour of our universe. The text aims to create an appreciation of the conceptual as well as the actual applied physics. The conscientious reader should be spurred to move into further related studies of relativistic quantum mechanics and quantum field theory.

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University of Toronto

College Trigonometry, by F. Richman, C. Walker, and E. Walker, Scott-Foresman, 1970, pp. 198.

It is refreshing to find a book which covers all the essential ground with no fuss, providing an abundance of suitable, straightforward exercises to enhance the theory. Especially does it gratify this reviewer to find a modern book which does not waste a chapter or so on set theory, thereby holding the length to 198 pages with index, instead of perhaps 260 or so.

Other bouquets are the excellent, practical approach to rational numbers as a measuring problem (a little short, though); and the lack of elevated signs of quality. This reviewer has detested the elevated signs ever since he first encountered a class (2-3 years ago) for whom "x - 2 = 7" was pronounced "x negative two equals seven". We congratulate the authors on their having avoided N-N fever (i.e. numeral versus number fever) and on their facing up to the problem of significant digits in a no-nonsense fashion.

We approach the section on complex numbers with some trepidation, but, wonder of wonders!, the authors have rejected the pedagogically arid wasteland of "horrdered pairs" in favour of a straightforward extension of the real number properties.

Now that praises have been sung, let us treat a few points that countered our own pet ideas.

There are no even number answers, which is nice for tests, but bad for extra study. Many would disagree that "every function F may be described by a table in this general

sense" Page 19, or that "with the rational numbers we can measure many more lengths than with integers" Page 3. Some might prefer the more modern "codomain" instead of "range". A word about skew axes before introducing Cartesians would make future exposure to Tensors much less painful (contravariance, covariance). No justification is offered for the use of "smooth curves" through discrete points in curve plotting. (Pages 26, 27, 48). The introduction to trig. functions is not in keeping with the generally practical tone of this book; at least this reviewer fails to see that the concept of function is as easy for a beginner as the law of similar triangles (trig. ratios). But, having chosen this approach, why not use the functions to define the word "angle"? Since the addition formulas have been done by rotation of the plane, it seemed a little strange that a motivation for the quantity i as a 90° rotation operator was not given. Also the multiplicity of roots on the unit circle in de Moivre's Theorem lends itself well to a rotation approach but was not used.

To sum up: We enjoyed this concise little book, and believe that it would be a comfortable companion to teach alongside; its few peccadilloes would be easily pointed out and overcome by an experienced teacher, and we are looking forward with pleasure to the second edition. It would certainly be well worthwhile for a school library, supplementary reading at the Grade XI-XIII level, or for self study.

L. D. Black, Lakehead University

Advanced Calculus, Avner Friedman, Holt, Rinehart, Winston Publishers Inc., pp. 423 + vi, (1971).

This is intended, according to the author, for students who have already completed a one year course in elementary calculus. However, this book has more material than necessary for one year's course. The author's inclination is to omit parts of several sections throughout and some of the topics in the last chapter (10) which contains supplementary material (from the author's viewpoint).

It is divided into two parts. Part I deals with the theory of functions of one variable and the theory of functions of several variables is duly treated in Part II.

In Part I we start off with numbers and sequences dealing with the classical approach to the set R^1, bounded

sets, limits and sequences and the Bolzano-Weierstrass Theorem. Continuous functions and related operations are discussed. Extreme and intermediate values receive a fairly thorough treatment. Here one sees a treatment on semi-continuity which is seldom done in texts of this nature. Also bounded variation is discussed which is all too seldom found in other texts.

Differentiability follows this and the standard treatment is used here with some added attractions, such as left and right differentiability and a good rigorous treatment of the Chain Rule, considering the cases where if $k(x) = f(g(x))$ then we must deal with:

(a) either there is a δ sufficiently small such that $g(x_o) \neq g(x_o+h)$ for all $0 < |h| < \delta$.

(b) or there is a sequence $\{h_n^*\}$, $h_n^* \neq 0$, $h_n^* \to 0$ as $n \to \infty$ such that $g(x_o+h_n^*) = g(x_o)$; assuming $g(x)$ is defined in a neighborhood of a point x_o for both cases (a) and (b).

The Mean Value Theorems, L'Hospital's Rule and extrema are well treated with good examples to follow.

A thorough treatment of integration follows, beginning with the Darboux Integral and then the classical Riemann Integral. Properties of the integral and methods of integration are followed by the Mean Value Theorems and improper integrals.

Closing Part II is a chapter on sequences and series of functions. This was well done in the beginning of this chapter (5), but the author should have tried to be more thorough in the matter of testing for convergence as, in my opinion, this is the heart of any theory of convergence. In the same vein, the treatment of uniform convergence is extremely shoddy (compared to Sokolnikoff's and Fulks' texts for example). A topic such as uniform convergence deserves nothing less than a completely thorough if not exhaustive treatment, otherwise the relevance of this to differentiation and integration of series is lost. Power and Taylor's series are discussed and then improper integrals.

In Part II the author gives a corresponding treatment to the space of several variables as to R and to continuity in the plane and n-dimensions. Vector valued functions and curves are dealt with - a bit too hurriedly in

my opinion. The Heine-Borel Theorem was not dealt with in Part I, but is brought to the students' notice here, whereas the Bolzano-Weierstrass Theorem is dealt with here as in Part I. Arcwise connectedness is treated here, although without any illustrative examples except in exercises.

Partial Differentiation is thoroughly discussed and a good treatment of this is done for the n-space as well. The corresponding Mean Value Theorems and Taylor's Theorems with extrema are introduced here. However to my feeling this latter topic (extrema) needs worked-out examples, which are lacking. Implicit Function Theorems, Inverse Mappings and Jacobians are discussed and dealt with, with a good number of exercises to follow. Vectors are only dealt with lightly at the conclusion of this chapter. Since second year calculus courses traditionally deal with vectors in a somewhat exhaustive manner, I question the book's approach on this topic. The exercises are good and if the student does these conscientiously he should get a good idea of the theoretical approach, but again there is a lack of applications and worked-out examples.

Multiple integration follows with a discussion of content zero, leading to hypersurfaces in R^n. The treatment here is good.

The standard subjects such as double, iterated and multiple integrals follow with uniform convergence in R^n, although the author treats the latter subject lightly by referring the reader to Part I and mentions that the prior results extend almost word for word here. This is understandable.

In comparison with other texts (Fulks, Sokolnikoff), improper integrals are dealt with well and in a more topological viewpoint. Yet the treatment is not as thorough as Taylor's.

The last "compulsory" chapter (9) is devoted to line and surface integrals. Green's Theorem, change of variables, surfaces and area are all treated with some examples following. The Divergence Theorem is followed by a treatment on change of variables in triple integrals. Stokes' Theorem is the last topic in this chapter.

Chapter 10 deals with supplementary topics such as Euler's summation formula, asymptotic series, Fourier series

and the Divergence Theorem in R^n. Outside of the Riemann-Stieltjes integral, this chapter concerns itself with students of applied analysis. This chapter is a good idea and is to be appreciated by the interested math student.

As for an overall conception of the book's merits, I feel that it is in the sad position of being too advanced for the second year student and a little weak for the third year level. I refer you to the Advanced Calculus texts by A. E. Taylor, W. Fulks and I. Sokolnikoff to appreciate my viewpoint of this.

The treatment on vectors was weak and this is inexcusable for second year. The treatment on convergence of sequences and series is good for second year but too weak for third year. A more thorough treatment on point set theory as regards to R^1 is necessary for any standard advanced calculus course.

The book itself is pleasant to read and there is a goodly amount of material but the omissions or weak spots are all too noticeable. In all fairness the book would be an excellent text for reference purposes but certainly not as a sole text for an advanced calculus course.

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Introductory Algebra for College Students, Eugene Nicols; Holt, Rinehart, Winston Publishers Inc., pp. 463 + xii, (1971) $10.95.

Sentence two of the preface appears to be accurate... "it contains all of the topics necessary for studying college algebra"...although it should be borne in mind that American College algebra lies somewhere between Ontario high school and university levels. A student who has studied Mathematics as high as Ontario Grade 12 should find very little new in this book, although going through it might prove a very beneficial review before entering university. On the other hand, he who, having neglected the pristine queen of the sciences in favour of less demanding disciplines, and has become aware of immediate future need of her, should be able to work entirely through this book in a summer.

We regret the whole first chapter wasted on sets, the only purpose of which seems to be to prepare readers to read books which use the set notation. We also should like to see some reference to identities. For example, why not define the distributive law as $a(b+c) \equiv ab + ac$? Obviously here $\forall\, a, b, c \in I : a(b+c) = ab + ac$ works, but what about $\frac{x^2-1}{x-1} \equiv x + 1$? Also, where are the simple diagrams to illustrate some of these axioms? Their utility far outweighs the minor amount of space used up.

The difference between equation and identity is so profound that we feel it should be stressed. However, this may be a personal prejudice, and we would not like this comment to be construed as too sharp a criticism of the book. Rather let it be a reflection of the inevitability of a student's understanding, the completion of which task has never yet been perfectly undertaken by a book.

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Comments on the solution on page 190.

(i) We may also eliminate t^3 by combining (1) and (2) so that a factor t can be removed, giving

(6) $(y^2+a^2)t^2 + (2a-x)yt + a(2a-x) = 0$,

which follows from (5) when we replace t by $-t^{-1}$. Addition of (5) and (6) permits removal of a factor (t^2+1) and eliminates the parameter, giving (4').

(ii) Using familiar properties of the parabola, the coordinates of P may be found without solving (1) and (2). If Q is the meet of the tangent lines at <u>any</u> points T and U on the parabola $y^2 = 4ax$, and M is the midpoint of chord TU, the line QM is parallel to the x axis; and if TQU is a right angle, the point Q lies on the directrix $x = -a$ and M is the common midpoint of the diagonals PQ and TU of rectangle $QTPU$. The coordinates of P as given in (3) follow at once from these facts.

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