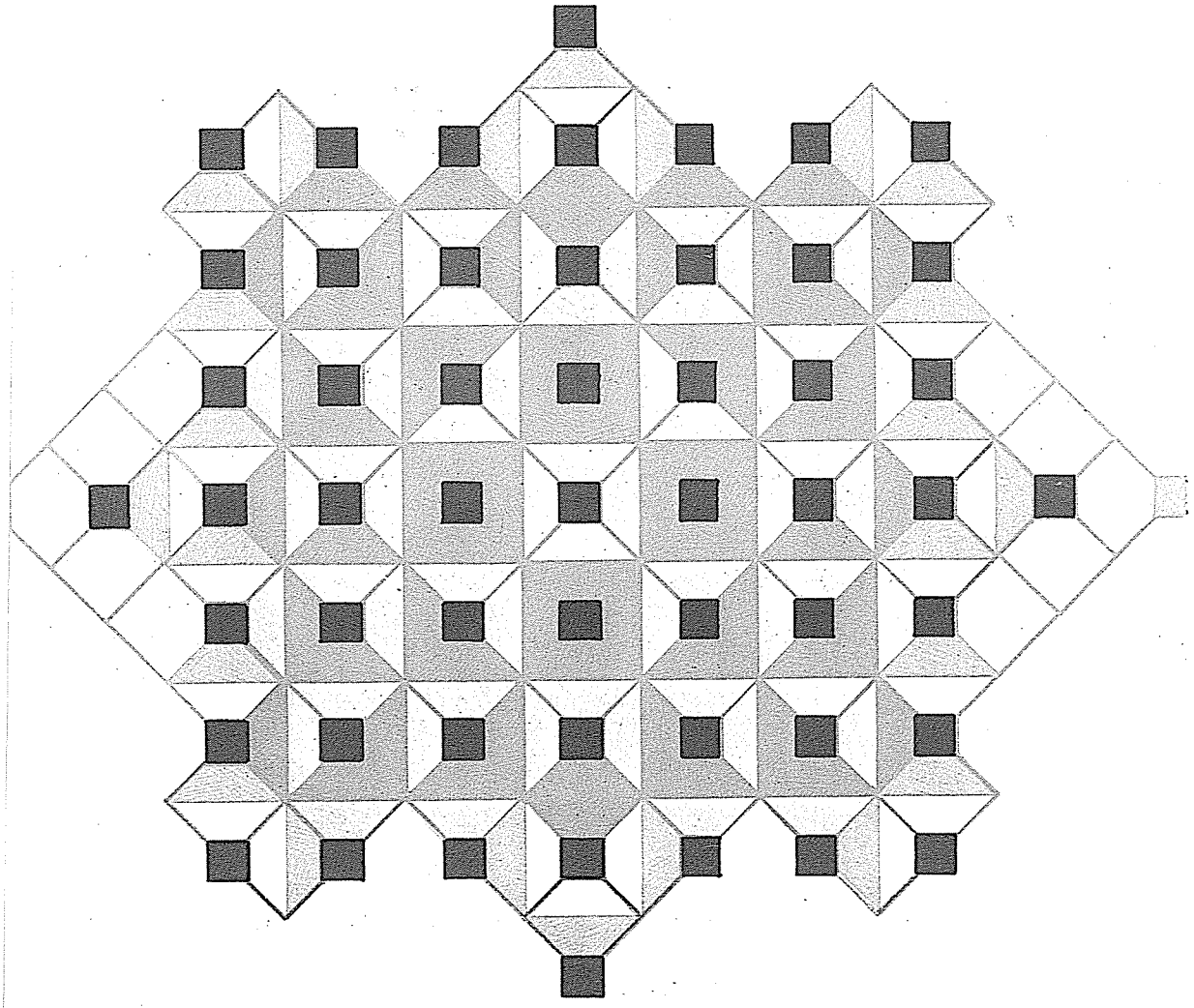
ontario mathematics gazette

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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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(Assistant Editor: Book Reviews, effective December, 1975)

EDITORIAL

This piece is mainly about books. I had hoped to receive some critical comments on the first issue, to which a response could be made. Unfortunately, the postal strike has cut off our feedback. Perhaps you won't receive this issue until long after it seems to be newsworthy.

We have added a section on book reviews, which will be supervised by Bill Higginson. One of our articles taken from Delta-K of the Alberta Mathematics Council makes some harsh statements about textbooks. I couldn't agree more. Its time the publishers and the authors smartened up and started writing to the real needs of our students. Textbooks should not be rushed into print without preliminary trial of materials and much re-writing.

This issue of the Gazette features some articles about the concerns of elementary school teachers. I hope they serve your needs. That reminds me to mention the latest yearbook of the National Council of Teachers of Mathematics MATHEMATICS LEARNING IN EARLY CHILDHOOD, which will be reviewed in the next issue. One of the chapters, "Problem Solving", is written by two Canadians, Doyal Nelson and Joan Kirkpatrick of the University of Alberta.*

Finally, let me express on behalf of all Ontarians interested in mathematics education, very hearty congratulations to John Egsgard on his election to the post of president-elect of N.C.T.M. This is a well-deserved honour. John has served Ontario and Canadian mathematics education with exceptional dedication and will surely strengthen the American organization.

As you know, O.A.M.E. has a working relationship with our American cousins. Isn't it time that N.C.T.M. changed its name to "North American Council of Teachers of Mathematics"?

Arn

**Joseph N. Payne (editor) Mathematics Learning in Early Childhood thirty-seventh yearbook of National Council of Mathematics, 1906 Association Drive, Reston, Virginia, 22091. $12.00*

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LETTERS TO THE EDITOR

The editorial staff of the Gazette welcomes your letters. Herewith is our first.

The Editor
Ontario Mathematics Gazette

While reading a bit of Plutarch I came across his statement that the ancients had some mystical ideas concerning the number 17 because it separated 16 and its epogdoon 18, the only two numbers which could represent the area and the perimeter of a plane rectangular figure.

It can be proved.

Let $A = xy$ and $P = 2x + 2y$

$$\therefore xy = 2x + 2y$$
$$xy - 2x - 2y = 0$$
$$xy - 2y - 2y + 4 = 4$$
$$(x-2)(y-2) = 4$$

if $a = x-2$ and $b = y-2$

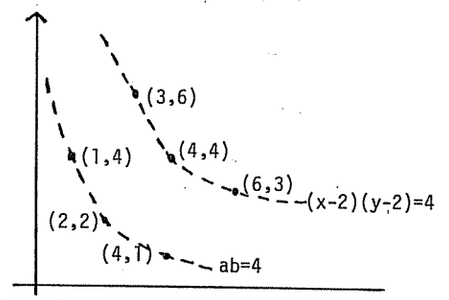
then $ab = 4$

For integral positive solutions

a	b	x	y
4	1	6	3
2	2	4	4
1	4	3	6

the solutions are a 4x4 square and a 3x6 rectangle, all of which Plutarch was aware, and according to his standards they were correct in stating only two solutions.

However, by today's standards they missed a few.



All of this still leaves me with the question, what does "epogdoon" mean?

Darrel Kennedy
John F. Ross C.V.I.
Guelph, Ontario

"SORRY, YOU CAN'T USE THE CALCULATOR IN THE CLASSROOM."

Wes Keener, Mathematics Consultant, London Board of Education

Pocket-size electronic calculators are sprouting everywhere, even in the pockets of school children, and some teachers are undecided what to do about them.

I have heard from concerned elementary and secondary mathematics teachers who feel many of their students have not yet mastered the basic operations with whole numbers, and some critics of calculators argue that their use will create a generation of computational illiterates.

There is little question as to the usefulness of a calculator in upper school or university, but the issue becomes more clouded at the elementary level.

John Del Grande in his booklet 'Pocket Mini-Calculator Revolution' states, "In educational circles, the calculator can provoke as much controversy as the Spadina Expressway, the price of Canadian oil or M.P.'s salaries." He reminds us that we may be on the verge of repeating the ridiculous flap that was staged over the use of ballpoint pens in our classrooms around 1950. Many of us recall the pressures exerted to try and keep these pens out of the classes because the kids wouldn't learn how to write properly.

CALCULATORS ARE HERE TO STAY

Drill in basic arithmetic and computational skills will still have to be learned and understood. A calculator cannot tell a student when to multiply or divide and in what order the operations should be performed. Pressing the wrong key or obtaining an incorrect answer because of a run-down battery are common problems, and if the student doesn't know what he's doing or what to expect, he's got problems.

The calculator cannot replace the understanding of numbers.

Practical Teaching Ideas at the Junior/Intermediate Levels

Following are simply a few of the ways that I used a set of calculators with different groups of children ranging from Grade 4 to Grade 8.

1. Pattern Work (done with a Grade 4 class)

Children are provided with a calculator and a dittoed worksheet similar to the one below while I used an overhead sheet which was the same as their dittoed copy.

Numeral	Multiplied By	Product
426	10	-------
426	100	-------
426	1000	-------
82	100	-------
926	1000	-------
.	.	.
.	.	.
.	.	.

After being instructed on how to use the calculator, they were asked to use the instrument to find the products for questions 1 and 2. I entered their products on the overhead and they were encouraged to look for a pattern and to fill in the next product(s) by themselves using the calculator to check. Many discovered the pattern quickly and enjoyed checking their assumptions. Some children took longer to discover the pattern, but at the end of the lesson, my own intuitive feeling was that this enhanced more pupil involvement and learning than the traditional method.

Another day a similar exercise was initiated dividing whole numbers ending in zeros by 10's, 100's, and 1000's (care was taken not to introduce quotients with decimals at this stage).

2. Fractional - Decimal Equivalence

With a Grade 6 class, I continued the dividing process until I arrived at:

Numeral	Divided By	Quotient
.	.	.
.	.	.
.	.	.
100	10	10
10	10	1
1	10	0.1

I was able to use the children's knowledge of fractions that 1÷10 was the same as 1/10, and at this point they used the calculator to find 1/10=0.1 (printout on calculator conforms with SI notation as shown in the Ministry's Metric Styleguide). The pattern was developed estimating first then using the calculator to check into hundredths and thousandths.

With a "slower" group of intermediate students, it was encouraging to see students taking 5/8 and dividing 5 by 8 on the calculator to obtain the decimal equivalent instead of doing it the other way around.

3. Addition, Subtraction, Multiplication, Division Using Decimals (used at the Grade 6 level)

a) Addition

$$\begin{array}{r} 30 \\ +40 \\ \hline 70 \end{array} \rightarrow \quad \begin{array}{r} 3 \\ +4 \\ \hline 7 \end{array} \longrightarrow \begin{array}{r} 0.3 \\ +0.4 \\ \hline ? \end{array}$$

They were encouraged to use a pattern, estimate their answer, then use the calculator to verify their results. The above example was the easiest pattern used but it did

lead into the more difficult additions. Obviously, subtraction was handled in much the same way.

b) Multiplication and Division by 10's, 100's, 1000's

Number	Standard Numeral
5.2 ÷ 10	-------
0.7 ÷ 10	-------
0.035 x 100	-------
4.03 x 10	-------
2.403 x 1000	-------
0.025 x 100	-------
.	.
.	.
.	.

Question	Product	Number of Figures to the Rt. of the Decimal in the First Factor	Second Factor	Product
5 x 25	125	0	0	0
0.5 x 25	12.5	1	0	1
0.5 x 2.5	1.25	1	1	2
0.5 x 0.25	----	1	-	-
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

The children were able to see the pattern develop for determining the number of decimal places in the product, come up with this pattern, and check it out using the calculator with further examples.

Dividend	Divisor	Fraction	Quotient
156	13	$\frac{156}{13}$	12
15.6	13	$\frac{15.6}{13}$	1.2
15.6	1.3	$\frac{15.6}{1.3}$	12
.	.	.	.
.	.	.	.
.	.	.	.

Which led to questions such as:

"What do you notice about the quotients for $\frac{156}{13}$ and $\frac{15.6}{1.3}$?"

"Why is $\frac{15.6}{1.3}$ equivalent to $\frac{156}{13}$?"

Which in turn, I believe, led to a better understanding

of questions similar to:

$$3.708 \div 2.06 = \frac{3.708}{2.06}$$
$$= \frac{3.708}{2.06} \times \frac{100}{100}$$
$$= \frac{370.8}{206}$$
$$= 1.8$$

After each of the above type exercises, I asked students to answer many questions using the traditional pencil and paper technique using the calculator to check. I really believe that the calculator helped many children see the patterns involved in decimal operations.

Further Uses

1. At the Grade 6 and 7 levels, I found the calculator as a useful aid in teaching rounding off (the calculators had a floating decimal and decimal shift key).

e.g.

First Factor	Second Factor	Product	Product Rounded Off to 1 Decimal Place
3.9	4.3	16.77	16.8

.
.
.

Once again the children were encouraged to look for a pattern. i.e. "When the product is 16.77 and you push D.S. 1 on the machine, why does it become 16.8? (whereas with 16.43, why does it become 16.4?)"

2. In a Grade 8 class, I found it useful when teaching Newton's Method for finding square roots correct to one decimal place (or more). It eliminated a lot of the tedious calculations.

i.e. $9 < \sqrt{83} < 10$

Since it is closer to 9, the possibilities become 9.1, 9.2, 9.3, 9.4, or 9.5.

We were able to learn more mathematics by using the calculator at this stage to check these out.

i.e. $9.1 \times 9.1 = 82.81$
$9.2 \times 9.2 = 84.64$
$9.3 \times 9.3 =$
$9.4 \times 9.4 =$
.
.
.
and find that $\sqrt{83} \doteqdot 9.1$

The list goes on. Calculators can be effectively used for:
- checking work (even fractions by using decimal equivalents

and a knowledge of rounding off)
- grocery list
- per cents
- powers

Will our students become "mathematical dummies" if allowed to use them? As W. Judd pointed out in his article "A New Case for the Calculator", if a student uses a machine to find 603 x 404, then he needs to know something about estimating and rounding off.

i.e. 603 x 404
$\longrightarrow$ 600 x 400
which means he still has to know 6 x 4

Place value understanding is essential:
i.e. 200 ÷ 0.03 "Is the answer very big or very small?"
200 x 0.03 "Is the answer very big or very small?"

Many children now have mini-calculators and many more will have them as the ten dollar calculator becomes a reality. Children see their mothers using them when they go grocery shopping and their fathers when they do the family budget. Can we simply tell children "Sorry, you can't use the calculator in the classroom"?

FRAMES: A NEW APPROACH TO THE TEACHING OF NUMBER

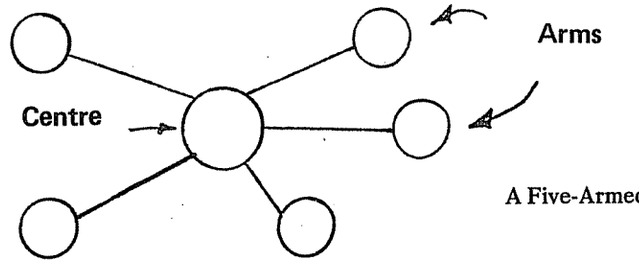
John Bates and Bill Higginson

John Bates is Principal of Davenport Junior School and a former Mathematics Consultant for the City of Toronto.

Bill Higginson is Assistant Professor of Mathematics Education at Queen's University.

One of the most difficult steps for pupils of elementary school arithmetic to make successfully is the transition from the manipulation of concrete objects and related numerical questions to the standard formal symbolic representation of these actions. An unfortunately large number of children never successfully connect the concrete situation of five apples from which two are removed to its related abstract representation 5 - 2. All too many learners at this early stage become committed to the dichotomous and deleterious view Robert Davis has called 'mathematics about something and mathematics about nothing'.

In an effort to develop a method which directly relates manipulative play to the standard operations of basal mathematics programs, teachers in several downtown Toronto schools are presently using an algorithmic form called a frame. The frame is a simple spatial construction which allows children to actively utilize space, while considering arithmetical concepts and logical problems. Frames consist of a central circle and arms attached to this centre.

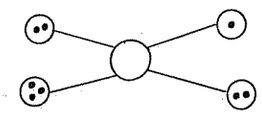


A Five-Armed Frame

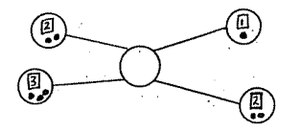
With very young children who are still using counters and developing recognition skills, some teachers use a system of 'pie plates' or paper cutouts which allow children to count, reassemble and describe the arithmetic activities in which they are engaged.

Teachers of five year olds, for example, ask children to place counters in the pie plates which are the arms of the frame. Although many behaviours are exhibited when this challenge is put forth, one of two responses seems to appear most often. Children either load the frame with many counters or they tend to match one counter to each frame arm. Presently, teachers who observe the first behaviour are moving children away from frames and allowing them to continue to manipulate objects outside the frame format. If children exhibit the matching activity, they are encouraged to continue with it. Although terminology differs from teacher to teacher, children are basically asked how many counters they think there would be if they were moved from the arms to the centre. Invariably, children guess an amount and do not, at first, understand that the number of counters does not change as the counters are moved into the centre and back out to the arms. By observation and experimentation teachers lead children to understand that quantity is conserved in this operation.

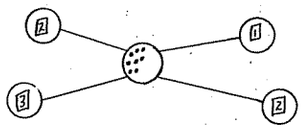
Children are encouraged to operate on frames with counters for a time and then numerals on small cards are used with the activity. Children then proceed by placing counters in the arms of the frame;



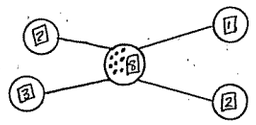
placing numerals with each set of counters



moving counters to the centre,



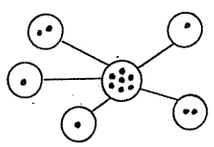
and placing a final numeral card with these,



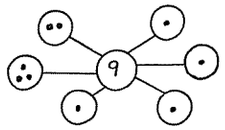
When children have reached this stage, they are encouraged to "draw" their frames on paper and a process of logical challenges begins.

Although there are many children who vary from any set sequential development when they use frames and choose their own numbers for arithmetic, a fairly consistent pattern of development does seem to emerge.

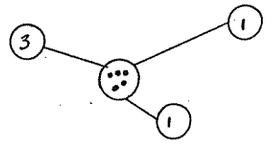
Children who place frames on paper tend at first to do one of four things. They either create dot frames,



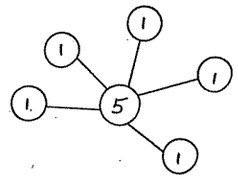
frames with dots in the arms and a numeral at the centre,



the converse of this,



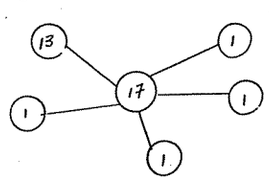
or frames that are completely numerical.



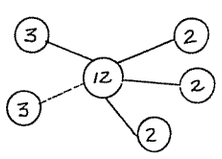
Whatever the case, two qualities are apparent in the frames: many arms are preferred to few arms and repetition of the number one is extremely common and children seem to see this as a counting exercise.

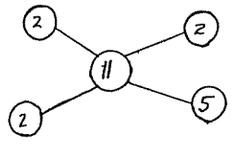
It is interesting to note that very few children at this stage use fewer than four arms and some use as many as thirty. There seems as well, to be some status related to the creation of frames with large numbers in the arms.

After children have had several weeks to become comfortable with frames on paper, teachers begin to challenge them to make changes in their frames. Since one is used so frequently, many teachers simply say "Could you make me some frames that do not have all ones in their arms". With this challenge children tend to do one of two things. The first is they use a concept described as continuance, (where they simply begin somewhere in the positive integers and continue their counting strategy).

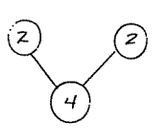


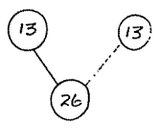
The second common response is to use integers between two and four with frequent repetition.

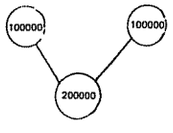




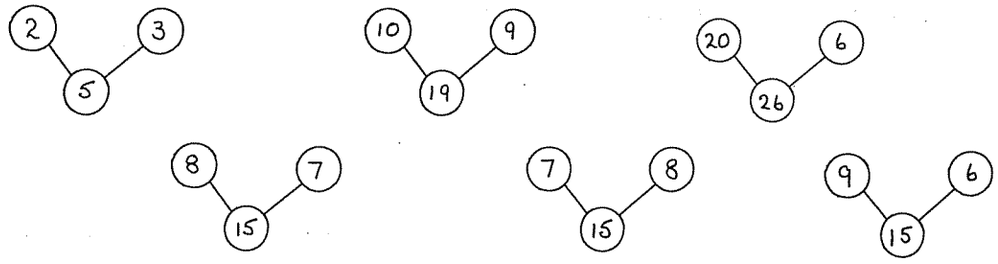
Development continues through several distinct stages. These are characterized by an increase in number size and a decrease in the number of arms in the frame. Eventually children begin to create two-arm frames with the same number in both of these arms.







When challenged to use unequal numbers in the arms of their frame they tend to reduce their number size but quickly begin to create frames of the following type.



Although directly convergent number tasks are not supplied to children at this stage, teachers can control the classes of frame which can be developed by describing:

1) number size in arms,

2) number size in centres,

3) number of arms to be used.

Examples of each of these three would be:

1) do not use numbers smaller than three

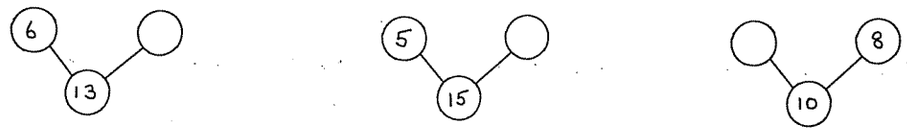
2) make frames for 10 today

3) use only 4 arms or do not use more than 5 arms on your frames

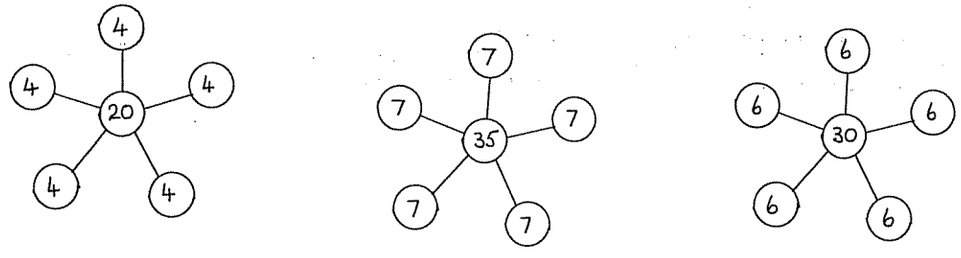
To obtain operations other than addition, teachers use various frame formats and simple challenges.

For instance, to generate concepts in subtraction, they simply ask children to complete frames which have been partially developed.

Finish these:

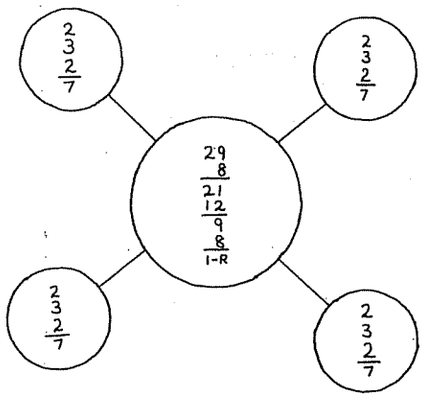


To begin multiplication, teachers challenge children to use "five" arms and place equal numbers in all of these.

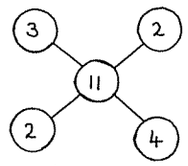


Upon examination of these frames children discover they can only generate certain numbers. (i.e. multiples of 5.)

A typical division algorithm looks like this.



Teachers have commonly allowed children about 20 weeks with frames before juxtaposing standard algorithms. When this juxtaposition takes place, children are encouraged to use both algorithms for a time, before using the standard algorithm exclusively for each operation.



3 + 2 + 2 + 4 = 11

3
2
2
4
11

Although no long term study of the efficacy of the frame method has yet been completed, there are some preliminary results which are quite striking. Recently an examination of some fifteen hundred frames generated by pupils in six primary classrooms at Davenport Junior School in Toronto revealed fewer than ten arithmetic errors.

It is not yet entirely clear why children should find this method so tractable. The frame format certainly makes the inter-relations among the four basic operations quite evident and the opportunity to revert to a concrete representation of a question is always easily available. Related findings in the area of cognitive development suggest that the non-linearity of the format may be another critical component. Further research should reveal more detailed answers to these questions.

TODAY'S LEADERS TOMORROW'S MATHEMATICS

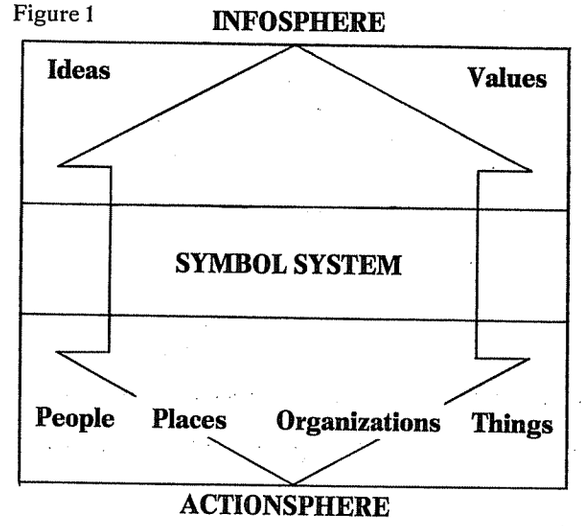
Alexander L. Norrie, Coordinator of Mathematics, Peel Board of Education

Michael A. McDanield writing in the Tofler edited "Learning for Tomorrow" introduces a discussion on curriculum by quoting Thomas Jefferson who stated "I like the dreams of the future better than the history of the past." This philosophy is particularly relevant to those educators involved in the continuing evolution of mathematics curriculum.

It is an understatement to say that mathematics education is in a continued state of flux. Today's and tomorrow's students will be affected by the answers to questions such as: What curriculum change is necessary when metric measurement becomes a reality? Why geometry and what geometry? How are electronic calculators going to be used? What should every man know? Mathematics Educators would be wise to effectively consider a model which stresses the relationship between subject and society. Mr. McDanield's ideas of the infosphere and the actionsphere seem very appropriate.

At a recent conference for leaders in mathematics, sponsored by the Ontario Association for Mathematics Education, the following model was suggested as a base for curriculum change.

Figure 1



An explanation of the model referred to constant change which meets the needs of the actionsphere and the assimilation of appropriate influences from the infosphere. In particular the focus was "what kind of curriculum will meet the needs of students who study mathematics in the future?" This was generated by the viewpoint that current curricular are inadequate and do not meet the needs of today's or tomorrow's students.

The seminar participants were given the task of identifying five important mathematical ideas that would be appropriate in a mathematics curriculum for the year 2001. Eighty-nine experienced teachers of mathematics each presented their own choice of topics and the results of this exercise are presented in Figure 2.

Figure 2 **CURRICULUM TOPICS FOR 2001 A.D.**

The ten topics chosen by most participants N = 89	
1. Problem Solving	71
2. Basic Operations (+,−,x,÷)	41
3. Understanding Number	39
4. Order, Patterns and Relationships	27
5. Measurement	22
6. Mathematics and Electronic Machines	21
7. Geometry Concepts	16
8. Why, When and How to Compute	13
9. Estimation/Logic	11
10. Computer Programs	10

Obviously these figures reflect the current "back to the basic" philosophy that is so prevalent throughout our educational system and a certain bias caused by the re-emphasis on measuring skills creeps in at several places. However, it is most encouraging to note the overwhelming positive swing toward problem solving.

After receiving the group consensus on the inclusion of problem solving, it was necessary to

attempt an enlightened definition of this very diverse topic. The definition, "a kind of divergent thinking which arises when a question causes the individual to consider alternatives and necessitates the use of additional information", did not satisfy most the the respondents and the group went on the define problem solving with phrases such as: logic and skills, conceptual awareness, translation of a symbol system, critical thinking, decision making, concept of number, comprehension, analysis and synthesis...

It is not important that we reach a definition acceptable to all. What is important is the repercussions that will follow for curriculum development if we acknowledge the need to make the necessary changes suggested by these leaders.

Where do we stand? Will the infosphere and the actionsphere continue to be separated by ineffectual curriculum? Or shall the ever beautiful language of the symbol system that is mathematics be utilized to help people, and organizations explore the ideas and values relevant for a future?

INDIVIDUALIZING THE MATHEMATICS PROGRAMME FOR PRIMARY CHILDREN

David McPhail and Sharon McPhail

Individualizing is such a common term in education today that at times we wonder if the true concept of the word is not lost in the meaningless mishmash of 'Educational Jargon'. Too often, educators toss the term 'Individualization' around like the wind playing with leaves. However, it does not stop here.

Trustees proclaim its use within their schools, the administrators boast of the practice under their jurisdiction, and all too often the classroom teacher is told that he/she must take part in this trend, never having been told---

(a) What is individualization?
(b) Why should we individualize?
(c) How do we individualize?

In every school that one might enter, educators can be seen attempting to individualize in some way. For example, a tremendous use of "Math Labs" has come into play in recent years. In this setting the student has the freedom to choose an area of work that is to be covered and after consultation with the teacher goes ahead and works through the various cards one after the other, at his own pace, to work in every way as an individual. Is this individualizing? The student is obviously doing his own thing and working as an individual.

We wonder what happens to this student when he meets frustration along the way? Is he merely referred back to a certain card and then expected to work slavishly through the cards again? Or is he given a truly valuable diagnostic programme by the teacher to assess the difficulty and then move towards remediation of the problem area?

Another approach that has been an attempt to individualize for the student is that of 'Streaming', where students are grouped according to their math ability, as assessed by a standardized test. Each teacher then gets a group of children who have similar abilities in math as per the results of the test. Often this is where the individualizing ends since the students are now treated exactly the same within the walls of their classroom. Although some effort was given to individualizing the programme, like the group approach of the past years too often only 'Lip-Service' was being paid to the true reason for creating the groups, - "The individual ability rather than the collective ability."

Many have used textbooks as the basis for individualizing. This method all too often merely sees the child ploughing through the text page by page, at his own rate perhaps, - but surely mathematics is more conceptually sequential than it is a page by page sequencing.

As one can readily see then, there are many interpretations of how mathematics might be

A SYNTHETIC TEST FOR MAXIMA, MINIMA OR POINTS OF INFLECTION IN POLYNOMIAL EXPRESSIONS

C. Pinnegar, Canterbury High School, Ottawa

One of the most common tasks performed by Level five mathematics students is to test certain functions or relations to discover properties which will be useful in identifying and locating the graph of that function or relation. Unfortunately, the three major texts available to Level five Calculus students, Calculus, Complex Numbers and Polar Co-ordinates, Elliott et al, Differential Calculus, Brainerd et al, and Calculus, Del Grande et al, gloss over interesting exceptional cases. The result is that students (and teachers) often cannot visualize the curve under discussion.

Take for example, the tests proposed by these authors for maxima, minima or points of inflection in polynomial functions.[1] Generally, examples are produced which solve the "run of the mill" question. Totally ignored, however, is the case in which the test for maxima or minima gives a point which is ostensibly also the point of inflection. For example, if $y=x^3-x^6$ we get by their tests a maximum or minimum at $x=0$ and also a point of inflection at the same point $x=0$. Obviously, at least one is incorrect. Del Grande et al simply ignores the problem theoretically, then offers exercises in which the difficulty occurs |6(d)(e) p. 215|. However, they know the problem exists (no. 8 p. 216). Brainerd notes the problem by example (p. 223) but resorts to an unproven theorem relying on the definition of continuous functions (an optional topic in level 5) to close the gap. Elliott, alone of the three, supplies a test (p. 113). But it is difficult to decipher and requires the student to find a third derivative and then substitute, a tedious prospect.

Of course, one can always test points on either side of the critical point to ascertain the path of the given curve. But for level five students, this is not very elegant. To restate the problem then, what does one conclude when the first derivative test for a maximum or minimum reveals a critical point exactly the same as that uncovered by the second derivative test for a point of inflection? The reader can quickly test $y=x^4$ to verify the difficulty.

Let us use three examples as a basis for our discussion. First examine:

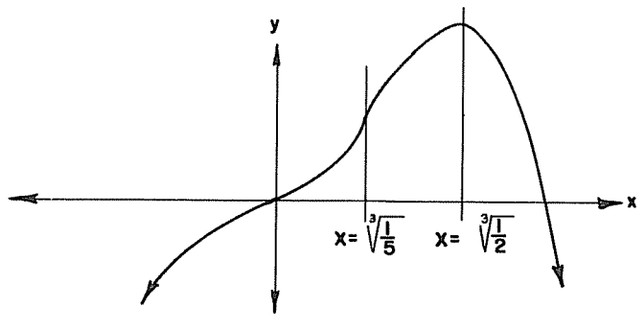
$f(x) = x^3-x^6$; $y'=3x^2-6x^5$; $y''=6x-30x^4$

$=3x^2(1-2x^3)$ $=6x(1-5x^3)$

---

[1] The following conjecture is not necessarily true for trigonometric functions. For example, if $f'(x) = (x-\frac{\pi}{2})^6 \sin x$ the conjecture does not follow. This example was provided by Professor Kenneth Fryer.

A sketch of this curve is shown below:



$\sqrt[3]{\frac{1}{5}}$

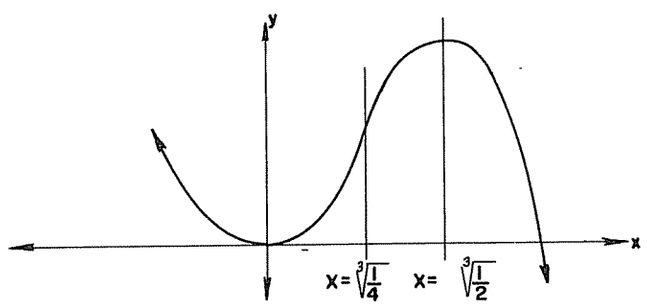
By the second derivative test for points of inflection, they should occur at x=0 and x = $\sqrt[3]{1/5}$ (see for example Del Grande p. 207). But there are also stationary points (implying a maximum or minimum) at the same x=0 as well as x$\sqrt[3]{1/2}$. From the sketch, you can see that point of inflection at x=0 is the correct conclusion.

Now similarly, examine:

$h(x)=x^4-\frac{8x^7}{7}$; $h'=4x^3-8x^6$; $h''=12x^2-48x^5$

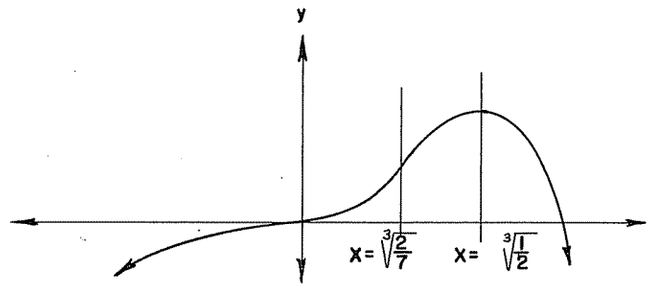
$=4x^3(1-2x^3)$ $=12x^2(1-4x^3)$

A sketch of this curve is shown below. Once again there is duality at x=0 but in this case a minimum, not a saddle point, is the correct interpretation.

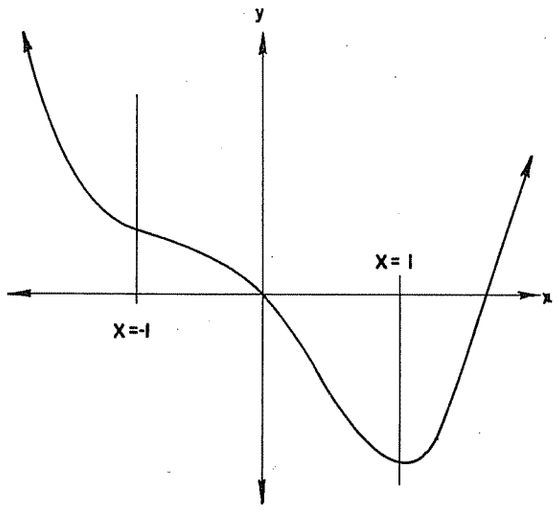


Now try $k(x)=\frac{4x^5}{5}-x^8$; $k'=4x^4(1-2x^4)$; $k''=8x^3(2-7x^3)$

A sketch is shown below. Compare this diagram with Figure 1.



Examine the successive first derivatives: $f'(x)=3x^2(1-2x^3)$, $h'(x)=4x^3(1-2x^3)$, $k'(x)=4x^4(1-2x^3)$. If we ignore the factor $(1-2x^3)$ which was produced artificially, we see immediately that the corresponding equation not only have multiple roots, but also that these multiple roots correspond to the critical points at which duality seems to exist. From the diagram, we may conjecture that if the multiple roots of $y'=0$ are of even order, there exists a point of inflection and not a maximum or minimum at the corresponding critical point, but if the multiple root is of odd order, there is a maximum or minim but no point of inflection at that point. Thus if some $y'=(x+1)^2(x-1)^3$, I would predict at least a point of inflection at $x=-1$ and a maximum or minimum at $x=1$. For your convenience a sketch follows:



P.S. I have tried to prove this algebraically, but so far have failed. Perhaps this might titillate you.

Editor's Note: The author has kindly answered several of my questions about this conjecture. We would welcome your comments.

MATERIALS AVAILABLE FROM OAME CHAPTERS AND OTHER SOURCES

A Handbook of Trigonometry

Grand Valley Mathematics Association, c/o R. G. Scoins, Faculty of Mathematics, University of Waterloo.

Graphing

Hamilton Board of Education, c/o Bob Robinson, 100 Main St. W., Hamilton.

Eureka

Newsletter of Carleton - Ottawa Mathematics Association, c/o Leo Sauve, Editor, Algonquin College, 282 Echo Drive, Ottawa K1S 5G2.

Straight Line Activities

By T. Yit and M. Miletic, Lambton-Kent Secondary School, Dresden, Ontario.

YOUR CONFERENCE IS COMING

See the details on Page 37

A PROPOSAL FOR A COURSE OF LINEAR ALGEBRA FOR HIGH SCHOOL STUDENTS

By A.S. Cooper and C. Fong
Department of Mathematics, University of Toronto

Motivated in part by J. Dieudonne's opinion that linear algebra could and should be taught at the high school level, in part by a strong feeling for the importance of the subject, we have prepared a rough draft of a text suitable for such a course.

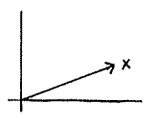
Concepts in linear algebra are usually strongly motivated from geometrical considerations. Hence a first course of linear algebra should emphasize the interplay between its algebraic contents and its geometric pictures. Almost all textbooks use the axiomatic approach which is algebraic in nature. However, many geometric facts are so simple and intuitively so clear that they are easily understood by beginners, while their algebraic consequences are non-trivial and subtle. Therefore we should be flexible. In each step we should adopt whichever of the two languages (algebraic and geometric) is more naturally associated with the concept being introduced, and translate it into the other which sounds foreign to us.

Because of the length of our draft for the course, we only lay down a rough sketch in the following self-explanatory fashion.

algebraic contents (implicit) ⟷ (visualize / theorize) geometric pictures (explicit)

vectors

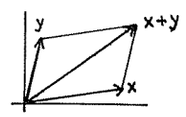
ordered pair $x = (x_1, x_2) \in R^2$ ⟷ directed line segment



operations

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

$a(x_1, x_2) = (ax_1, ax_2)$ ⟷ parallelogram law



linear mappings

first version:

A: $(x_1, x_2) \longrightarrow (y_1, y_2)$ is defined by

$y_1 = a_{11}x_1 + a_{12}x_2$

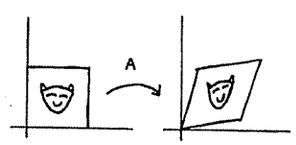
$y_2 = a_{21}x_1 + a_{22}x_2$

second version: ⟶ A sends a parallelogram to a parallelogram

A is linear if

$A(x+y) = Ax + Ay$

$A(ax) = aAx$

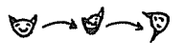


products

matrix multiplication ⟵ composition

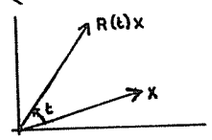
$R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2$

AB



rotations

$$R(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$



(R(t) is linear because of the second version of linear mappings.)

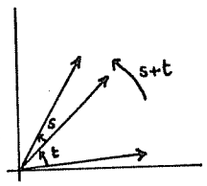
(I)

addition formula

$\sin(s+t) = \sin t \cdot \cos s + \cos t \cdot \sin s$

$\cos(s+t) = \cos s \cdot \cos t - \sin s \cdot \sin t$

$\longleftarrow R(s+t) = R(s)R(t)$

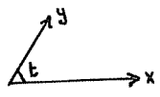


(II)

inner products

$x.y = x_1y_1 + x_2y_2 \longrightarrow x.y = |x||y| \cos t$

$|x|^2 = x.x$



orthogonality

$x.y = 0 \longleftarrow x \perp y$

(III)

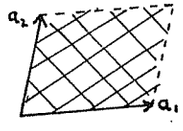
determinants

absolute value of ⟵ area of the parallelogram spanned by a_1 and a_2:

$\det(a_1,a_2) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$a_1 = (a_{11},a_{21})$

$a_2 = (a_{12},a_{22})$



sign of $\det(a_1,a_2) \longrightarrow$ orientation of the pair (a_1,a_2)

determinant of a linear mapping

$\det A = \det(Ae_1,Ae_2)$

where $e_1 = (1,0)$

$e_2 = (0,1)$

(IV)

absolute value of $\det A \longrightarrow$ power of magnification

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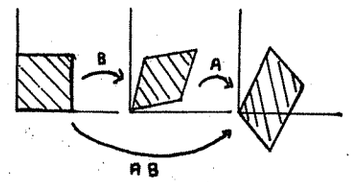
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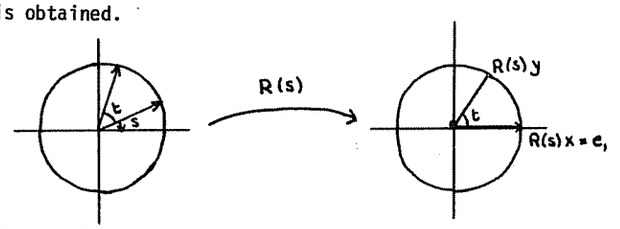
	sign of det A → (>0): orientation-preserving (<0): orientation-reversing (=0): collapse.
	(V)
determinant of a product	det(AB) = det A . det B ←



Explanations:

(I) The formulas for sin(s+t) and cos(s+t) are obtained by comparing both sides of R(s+t) = R(s).R(t). The latter identity is intuitively clear. We think that this is the best way to obtain the addition formulas for sine and cosine.

(II) It is straightforward to check that $x.y = 1/4(|x+y|^2 - |x-y|^2)$ (polarization identity). Since rotation preserves the length of a vector, it preserves inner products. By normalization, we may assume both vectors in the expression x.y are unit vectors. Rotate x by s to $e_1 = (1,0)$, then $x.y = R(s).R(s)y = \cos s$. (See the following figure.) Therefore $x.y = |x||y| \cos t$ is obtained.



The usual way of defining the inner product is by $x.y = |x||y| \cos t$. But the bilinearity of this expression is not obvious and hence $x.y = x_1y_1 + x_2y_2$ is a bit awkward to obtain. The path we have followed, on the other hand, is more pleasant, as it involves the simple and fascinating idea that lengths and angles are preserved under a rotation.

We do not think that we should tell the students that the inner product is motivated by the physical concept of 'work' since the latter is itself only a formal definition and often conflicts with our intuitive understanding of the word. For example a man may walk a mile carrying a bucket of water and still do no work in the physical sense. We should not evade our responsibility for explaining a mathematical concept by passing it over to physics - especially when the physical concept involved is itself often a source of confusion for the students.

(III) We can make use of an inner product in the proof that the area of the parallelogram spanned by $a_1 = (a_{11},a_{21})$ and $a_2 = (a_{12},a_{22})$ is given by $[a_{11}a_{22}-a_{12}a_{21}]$
For we have
Area $(a_1,a_2) = |a_1|(|a_2|\sin t)$

Hence

$$\text{Area } (a_1,a_2)^2 = |a_1|^2|a_2|^2(1-\cos^2 t)$$
$$= |a_1|^2|a_2|^2 - |a_1\cdot a_2|^2 = (a_{11}^2+a_{21}^2)(a_{12}^2+a_{22}^2)$$
$$- (a_{11}a_{22}+a_{21}a_{12})^2$$
$$= (a_{11}a_{22}-a_{12}a_{21})^2$$

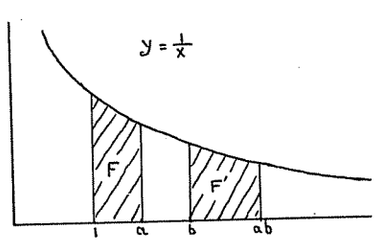
Having given the expression $a_{11}a_{22}-a_{12}a_{21}$ a geometric meaning we then use it to define $\det(a_1,a_2)$, (rather than defining the determinant algebraically without supplying any motivation.)

(IV) At this point in the text we derive that if F is a region in R^2 and F' is its image in R^2 by A, then

area of F' = $|\det A|$ area of F.

This is why we call det A the power of magnification of A.

Two interesting applications: (1) The area bounded by the ellipse $a^{-2}x^2+b^{-2}y^2 = 1$ is πab. (2) Applying the linear mapping defined by the matrix $\begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}$, (see the following figure) we obtain $\ln ab = \ln a + \ln b$.



(V) The formula det AB = det A . det B can then be obtained from the above geometrical interpretation of determinants of linear mappings. This is also the result of a straightforward computation which is somewhat less than inspiring.

We have taken the 2-dimensional space for our basic model. There is no difficulty in carrying the idea over to the one-dimensional or 3-dimensional space. We stay in the 2-dimensional case because it is suitable for demonstration; in the one-dimensional case things turn out to be trivialities and in the 3-dimensional case pictures are too complicated to draw (but we should encourage the students to do so).

The introduction of complex numbers fits well into this program. This can be done in the following way. We start by observing that the product of two rotations is a rotation, while the sum of two is generally not. A matrix represents

Let n be the number of sides of each face $n \geq 3$.

$\therefore$ The total number of edges $E = 1/2\ nF$
(1/2 since each edge is counted from each side)

$$\text{or } F = \frac{2E}{n}$$

Let k be the number of edges at each vertex $k \geq 3$.

$\therefore$ The total number of edges $E = 1/2\ kV$
(1/2 since each edge is counted from each end)

$$\text{or } V = \frac{2E}{k}$$

$$\therefore \frac{2E}{n} + \frac{2E}{k} - E = 2$$

$$\therefore E\left(\frac{2}{n} + \frac{2}{k} - 1\right) = 2$$

$$\therefore E\left(\frac{1}{n} + \frac{1}{k} - \frac{1}{2}\right) = 1$$

$$\therefore E = \frac{1}{\frac{1}{n} + \frac{1}{k} - \frac{1}{2}} > 0$$

$$\therefore \frac{1}{n} + \frac{1}{k} - \frac{1}{2} > 0$$

$$\therefore \frac{1}{n} + \frac{1}{k} > \frac{1}{2}$$

$n \geq 3,\ k \geq 3$

n	k	$\frac{1}{n}+\frac{1}{k}$	$>\frac{1}{2}$?	E	V	F	Name
3	3	2/3	✓	6	4	4	Tetrahedron
3	4	7/12	✓	12	6	8	Octahedron
3	5	8/15	✓	30	12	20	Icosabedron
3	6	1/2	x				
4	3	7/12	✓	12	8	6	Cube
4	4	1/2	x				
5	3	8/15	✓	30	20	12	Dodecahedron
5	4	9/20	x				

PROBLEMS

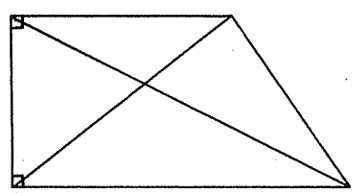
14-2-1
Up to Grade 9

How many games are needed to produce a winner in a knock-out tournament with (a) 8, (b) 27, (c) 47 and (d) n teams?

14-2-2
Up to Grade 12

Is it possible for a knight in chess to start at one corner of a chess board and move to the opposite corner landing exactly once on each square of the board?

14-2-3



What are the lengths of the sides and diagonals of the above figure if all are of integral length?

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SIX CHAPTERS OF OAME

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THE STRATEGY OF CONCENTRATED FIRE POWER

W. Eames, Lakehead University

Dr. Eames is the former editor of the Gazette

1: **INTRODUCTION.** With the increasing mathematization of the social and natural sciences, most math teachers find they must drastically modify and update their course contents, in order to maintain at least a semblance of relevance. Gone are the days when students would be content with only physical and geometrical applications of the calculus, for example! Now most high school calculus courses must contain applications to economics, biology, sociology and psychology, even if they are of an extremely idealized nature.

Thus, the following very simple application of calculus to warfare might appeal to a good student, although it may be difficult to persuade him/her it is really of practical value. At the very least, the main result is both easy to obtain and completely unexpected - two good qualities rarely occurring together!

2: **A SHORT HISTORY OF WARFARE.** We deal with battles between two opposing forces. Battles in primitive times probably consisted of a lot of individual hand-to-hand fights; the essential point for us is that the weapons - fists, clubs, spears - were not long range weapons: it was impossible to concentrate the power of many soldiers on one opponent. Thus if, for example, we had two such armies, one with 100 men and one with 200, we would expect - if the soldiers are equally proficient - that the result would be annihilation of the first army and 100 survivors from the second. The battle could simply be fought by letting half the second army act as spectators, while everyone else is killed.

Nowadays things are different and undoubtedly much improved. Long range weapons enable armies to concentrate their fire power and, as we shall see, this can lead to unexpected advantages. There are some pretty obvious examples: perhaps the battle of Agincourt was the most spectacular - the long range weapons of the English (bows and arrows) annihilated the larger French army, whose weapons were short range.

Nearer our own time, the use of field artillery in the Russo-Japanese war, and now, the anticipated use of inter-continental ballistic missiles, indicate how far we have come from clubs and swords. Interested history students might profitably analyse some modern battles, using the concepts we will now introduce.

3: A MATHEMATICAL MODEL FOR MODERN WARFARE.

We consider two opposing forces, the red and the blue armies, of sizes r(t), b(t), respectively, at any time t. We will measure time from the beginning of the battle, so r(0), b(0) are the initial sizes of the armies. Of course, we could measure the sizes, not in terms of soldiers, but in terms of battalion, ships, etc., whatever is convenient. We assume that each soldier (battalion, ship) in the red army is equally effective in warfare, and that he can concentrate his fire power in any way, so that, throughout the battle he kills R blue soldiers every minute. Similarly, each blue soldier kills B red soldiers every minute; R and B are fixed numbers.

Consider what happens in a short time interval from t to t+h minutes during the battle. The number of red soldiers

killed is, by the definition of r,

$$-(r(t+h)-r(t)).$$

On the other hand, if the time interval is short, the size of the blue army hasn't changed much in this time, and thus is approximately b(t) - its size at the beginning - so the number of red soldiers killed is

$$b(t)\cdot B\cdot h$$

by the definition of the kill rate B. Thus, for values of h near 0,

$$r(t+h)-r(t)\approx -b(t)\cdot B\cdot h$$

so

$$\frac{r(t+h)-r(t)}{h}\approx -b(t)\cdot B.$$

Since this approximation improves as h approaches 0, we have

$$\frac{dr}{dt} = -b\cdot B$$

(Calculus!) Similarly,

$$\frac{db}{dt} = -r\cdot R$$

and so, after a bit of algebra

$$b\cdot B\cdot\frac{db}{dt} = r\cdot R\cdot\frac{dr}{dt}\ .$$

If we integrate both sides with respect to time, we get the unexpected result:

'b^2B-r^2R is independent of time'.

Let us call b^2B the strength of the blue army, and r^2R the strength of the red army. We have shown that *the difference in strength is constant throughout the battle.*

Examples: 1. Consider a battle between 10 machine gunners (the red army) and 100 infantrymen (the blue army), and assume the kill rate of the machine gunners is 16 times that of the infantrymen. Let us measure time so the rates are 16 and 1. Initially, the strengths are:

machine gunners = $16\cdot 10^2 = 1600$

infantrymen = $1\cdot(100)^2 = 10000$

and so, at any time,

$$b(t)^2 - 16r(t)^2 = 10{,}000-1{,}600 = 8.400.$$

The battle will end when one side is vanquished - when b(t) or r(t) is 0. Clearly, b(t) will never be 0:

$$b(t)^2 = 8{,}400 + 16r(t)^2$$

so, at the end, r(t) = 0 and $b(t) = \sqrt{8{,}400} = 91$. Only nine infantrymen have been killed, and all 10 of the machine gunners die in spite of the fact that they are, individually, 16 times more effective in killing! A rather surprising result.

2. The above example was rather an uneven contest - the machine gunners were quite overwhelmed. Let's consider a more equal match: 10 machine gunners against 60 infantrymen. The infantry is the stronger,

by 2000, so a battle would result in it winning, with 45 infantrymen left.

But suppose the machine gunners use a bit of strategy, and manage to break the battle into a series of battles, in each of which they all attack 20 infantrymen. The results are:

(i) 10 mgs vs 20 inf: 8 mgs survive
(ii) 8 mgs vs 20 inf: 6 mgs survive
(iii) 6 mgs vs 20 inf: 3 mgs survive

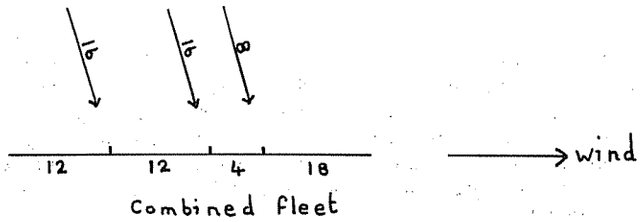
The machine gunners have won!

Thus we see that a bit of strategy can win against what, at first glance, appears to be overwhelming force. Of course, the enemy general has to be rather stupid to allow his army to be picked off piecemeal, but this is what often occurs in practice. A small force is sent to divert a large part of the enemy; their job is simply to pin down a lot of soldiers while their comrades defeat the small remainder. Then they keep regrouping until they win. A naval battle, during the days of sailing ships, would be a good example of these - in that no diversionary force is needed if the wind is right - the attacking admiral hits his opponents part way down the line and lets the wind carry off a portion of the enemy fleet. By the time it has been turned around to re-enter the fight, the next phase has started. Nelson used this ploy at Trafalgar; this has been analysed by Lankester in his classic paper (3). We will give a slightly different interpretation.

4. THE BATTLE OF TRAFALGAR RERUN. Up to 1782, there was no 'naval strategy' - it was 'line up, and attack ship-to-ship'. But in 1782 the British admiral Rodney, probably by accident, broke through the enemy line and by concentrating on his centre and rear, won a victory. This prepared the military mind for Nelson's strategy at Trafalgar. It was known that Nelson would favour Rodney's strategy: the French admiral, Villeneuve, stated before the battle that 'the British fleet will not be formed in a line-of-battle parallel to the combined (ie. the French and Spanish) fleet according to the usage of former days. Nelson will seek to break our line, envelop our rear, and overpower with groups of his ships as many as he can isolate and cut off'. Why the French and Spanish admirals let Nelson dictate the strategy is a matter for military historians; the answer may lie in the fact that the French believed their reverses at sea were due to the introduction of tactics, which one of their admirals had castigated as a 'veil of timidity'. Also, due to the British blockade, the French sailors at that time were so untrained they couldn't manouveur their ships - but they were more accurate at gunnery, so long-range warfare

should have been congenial for them.

At any rate, Nelson and his aides, Collingwood and Keats, planned the battle as follows: They assumed they would have 40 ships, and the enemy would have 46. The British fleet was to attack in 3 columns, as below.



Assuming the wind carried the 18 ships out of the battle, and that it broke into 3 separate battles:

16 British vs 12 Combined
16 British vs 12 Combined
8 British vs 4 Combined,

resulting in 27 British ships surviving, and then the final battle of 27 British against 18 Combined, the net result is that 20 British ships are left. If the usual 'line of battle' attack had been used, the combined fleet would have won, with 20 ships, and most of us would speak French.

In fact, these tactics were changed because of Nelson's fleet had to accompany a convoy to Malta, and in the battle he only formed two lines; the battle is described in great detail in (1).

5: **MORE APPLICATIONS.** It would be interesting to apply these ideas to modern battles, to see if the military strategists could have been using - perhaps in an intuitive way - the quantitative results we can get. A 'dictionary of battles' - there are quite a few in print, and they should be available in the reference section of most school libraries - is all that would be needed.

There may be applications to other situations too - polymer chemistry and molecular biology might contain interesting applications. Also, the mathematics of the situation has not, to my knowledge, really been investigated. The length of a battle can be found - the answer involves the hyperbolic functions. The strength of an army consisting of serveral different types of weapons is certainly not that given in (3), and appears to depend on the opposition, but I have not found even an approximate explicit form for it. What is the size of an overwhelming force? - one that will win no matter what strategy is used. How can we find a best strategy? Other mathematical problems may suggest themselves; I would be glad to hear of them.

REFERENCES

(1) Major General J. F. C. Fuller; The Decisive Battles of the Western World, Vol. 2, a Paladin paperback.

(2) Major General B. P. Hughes; Firepower. Weapon Effectiveness on the Battlefield 1630-1850, Arms and Armour Press.

(3) F. W. Lanchester; Mathematics in Warfare; The World of Mathematics, Vol.4, Simon and Schuster.

INTERNATIONAL MATHEMATICS EDUCATION

THIRD INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION

Karlsruhe, Federal Republic of Germany, will be the site of the Third International Congress on Mathematical Education (**ICME**), 16-21 August, 1976. The Congress is being organized by the West German Subcommittee of **ICME** in collaboration with the Local Organizing Committee. The program will include plenary sessions, survey reports, panel discussions, special projects, and commercial exhibitions. further information and application forms may be obtained from Professor H. Kunle, 3s 'international Congress on Mathematical Education, University, d 75 Karlsruhe, federal Republic of Germany.

N.C.T.M. ANNUAL MEETING

The next annual meeting of the National Council of Teachers of Mathematics will be held in Atlanta, Georgia, April 21 - 24, 1976. An outstanding program is being planned, featuring the largest number sessions ever planned by N.C.T.M. There will be 425 sections, with over 500 participants, representing 47 states and several foreign countries. Plan to attend!

N.C.T.M. NAME -OF-SITE MEETING - DETROIT

You are reminded of a nearby Name-of-Site meeting of N.C.T.M. in Detroit, March 4 - 6, 1976. At this date details of the program have not arrived, due to the postal strike. Southwestern Ontario members of O.A.M.E. are especially urged to consider attending.

CANADIAN EDUCATION SHOWPLACE MOVES TO NEW TIMES AND PLACE

Toronto -- Canadian Education Showplace (CES), Canada's annual national exhibition of resources and communications for education and industry, will move to a new time and place in spring 1976, Lloyd Haines, show manager, has announced.

New dates are March 2, 3, and 4, 1976, and the site will be the Automotive Building, Exhibition Place, Toronto. For the past several years, CES has been a fall event in the Coliseum, Exhibition Place.

"Visitors and exhibitors alike will find the March date is better timed to budget cycles and programs of school boards, colleges and universities," said Mr. Haines. "And the change avoids conflict with a growing number of fall meetings and conferences".

Choice of the Automotive Building means that the show returns to the site of a number of highly successful previous CES productions.

CES is co-sponsored by the Canadian Institute of Management, Educational Digest, National Audio-Visual Association of Canada and the Ontario Association of School Business Officials. It continues to be one of the most comprehensive exhibitions of instructional resources in North America. Exhibits range from early childhood to adult education and include more than 300 classifications of equipment and materials.

Admission to CES is free to professional educators. For registration forms and program information write to Canadian Education Showplace, 481 University Ave., Toronto, Ont. M5W 1A7.

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

THIRD ANNUAL CONFERENCE

MAY 14, 15, & 16, 1976 - ST. CATHERINES, ONTARIO

A THEME APPROACH TO CONFERENCE PLANNING

OBJECTIVES: This format presents an opportunity for theme participants to share, as a common experience, their concerns, ideas, and reactions in more depth than is possible in sectional meetings.

Each theme starts with a keynote person of outstanding reputation and then develops into a number of Workshops, Expository Lectures, etc., at the various levels of education, K to University.

The Themes are:

1. Where Have We Been and Where Are We Going?
2. Standards, Achievement and Evaluation
3. Geometry: Approaches and Influence
4. Meaningful Arithmetic/Algebra
5. Remedial Mathematics
6. Mathematical Models Representing Reality
7. Calculating Devices, A Help or a Hindrance?
8. What about Statistics?
9. Computers
10. Metric Measure and Mathematics
11. Beautiful Mathematics
12. Motivating Mathematics

AN EARLYBIRD PROGRAM

Starting Friday Morning - May 14, 1976

Friday a.m. - (i) Mathletics Competition Finals: (a) Secondary
(b) Elementary
(ii) School Visitations In Region 6, Lincoln, Niagara South and Hamilton

Friday p.m. - (i) 1:00 p.m. - Curriculum Development Presentations by Superintendents of Curriculum
(ii) 3:00 p.m. - Informal Discussions with our Guest Speakers
(iii) 4:30 p.m. - "Warm-up Session" - an Informal Collection of Mathematical Demonstrations and Humour
(iv) Displays, Exhibits and Trading Post
(v) Mathematical Games Room

TRADING POST

Keep us in mind now, run off 20 pages of your best handouts for trading purposes, which entitles you to 20 pages of handouts from someone else.

Look for the pre-registration package and complete program information next month.

Register Early:
1. To obtain sufficient budget funds.
2. Because accommodations should be in demand during the week of Blossom Sunday, particularly in the St. Catharines Centennial Year. (Accommodations at the Brock Campous are limited to 350 beds and are on a first come, first served basis.)
3. Get your name on the list of attendees if your pre-registration form and fees are received by February 27, 1976.

BOOK REVIEWS

GAMES AND PUZZLES FOR ELEMENTARY AND MIDDLE SCHOOL MATHEMATICS: READINGS FROM THE ARITHMETIC TEACHER
SMITH, SEATON E. JR. and BACKMAN, CARL A. (EDS.)
Reston, Virginia: National Council of Teachers of Mathematics, 1975
Paperback, 280 pp., $4.50

This publication is a collection of 105 articles which appeared in The Arithmetic Teacher. The book consists of nine chapters: Using Games and Puzzles in Mathematics Instruction, Whole Numbers, Numeration, Integers, Rational Numbers, Number Theory and Patterns, Geometry and Measurement, Reasoning and Logic and Multipurpose Games and Puzzles. The majority of the articles outline activities suitable for junior division pupils but the range is wide and teachers from primary to intermediate divisions could find this book extremely useful.

Many of the games and puzzles described require the use of simple aids such as dice and cards. While some of the 'games' are rather thinly disguised versions of standard mathematical results ('A Pythagorean Puzzle', 'An Adventure in Division') and of dubious intrinsic interest to children, ('Diffy', Arithmetic Card Games'), others are ingenious and well-designed activities which are also mathematically sound.

The editors of this volume have clearly worked hard to assemble a range of articles which bear the stamp of classroom authenticity. Although the collection would have benefitted from a more detailed coverage of the Why and How aspects of Games in the classroom, it can be unreservedly recommended as a basic volume for the personal libraries of any teacher of elementary mathematics. Unfortunately, as is the case with too many of the paperbound NCTM publications, the binding is of poor quality and after any moderate use one often finds pages falling out of the book.

W.H.

MATHEMATICS: INSIGHTS AND APPLICATIONS, BOOK 1
JOHN J. DELGRANDE, PATRICK T. JONES, LORNA MORROW
Gage Educational Publishing Limited, P.O. Box 5000, 164 Commander Blvd., Agincourt, Ont.
Published May 1975, 438 pages, $6.68 net price to schools

This text is the first in a new series designed for students who need a solid background of mathematics for university entrance, but without the emphasis on theory that characterized some of the previous texts. The format of the book and the method of development are intended for the active, participating Grade IX student.

The chapter titles give an accurate idea of the contents and order of material in the text and are as follows:

1. Sets and Logic, 2. Relations, Mappings and Graphs, 3. Transformations and Coordinates, 4. Number in Mathematics, 5. Equations and Inequations, 6. Polynomials, 7. Factoring Polynomials, 8. Geometry and Transformations, 9. Euclid and Transformations, 10. Measurement, 11. Ratio and Proportion, 12. Properties in Mathematics.

The book has a bright, attractive format with coloured illustrations and lively cartoons. The method of presentation for each section is to give a very brief introduction which the student should be able to follow. There are ample questions in each exercise and the initial idea is further developed as the students work through the exercise. Thus the book is suitable for the teacher who believes in maximum student involvement or in individual progress. Most of the questions are at an appropriate level, and, as one teacher using this book put it, it teaches the student to read.

Many excellent teaching strategies are used. A new one which should be helpful in the work in polynomials is the use of coloured strips and squares. Also the use of the "multiplication computer" should make the concept of multiplying polynomials much easier to grasp.

In solving equations, students learn to use the inverse operations and flow chart as well as how to build equivalent equations. In general, the topics are very well presented, although in some cases, e.g. mapping, the ideas are carried too far.

The 27 pages of puzzles, logic games, constructions and other interesting side trips into mathematics should add interest, variety and challenge for the Grade IX student. The twenty pages of extra review or practice exercises will be handy for the teacher for remedial or review purposes. The book is completely metric with all units of measurement in SI.

While none of the concepts presented are beyond the scope of a good Grade IX student, the reviewer believes that too many ideas are presented and some would be better left to be introduced in a later Grade. Also, in the reviewer's opinion, more emphasis could have been placed on the applications part of the title. Secondary students need more familiarization with the business and economic world and the four pages on per cent could have been made into a profitable chapter. The work on measurement, particularly volume, could have been expanded, and the excellent section on measurement in science could have been put to work in a practical way.

One wonders if the Grade IX students are ready for some of the rather sophisticated ideas in mappings in Chapter 2 and if some of the ideas on Geometry, particularly similar figures, could be left to have something new for Grade X or X1.

A preface outlining the philosophy and aims of the text, with some suggestions as to methods of using it would have been helpful, particularly to the new teacher. Also, since it is unlikely most classes will cover the entire book, a list of sections which might be deleted would be appreciated by most users.

The authors have succeeded in making an attractive text, which is well suited to use by the student. The reviewer recommends that all teachers of mathematics take a look at this series, both for teaching ideas and for possible adoption.

D. R. Campbell

MATHEMATICAL RECREATIONS AND ESSAYS, 12th EDITION
W.W. ROUSE BALL AND H.S.M. COXETER
University of Toronto Press, Toronto, Ontario M5S 1A6
Paperback $5.00, Hardback $12.50, 427 pp.

This twelfth edition of W.W.R. Ball's delightful classic, which was first published in 1892, retains the essential features of the original edition while updating some problems and adding new topics. Included are new sections on polynomials and colouring maps on multiplyconnected surfaces, as well as a new chapter which provides an introduction to combinatorics theory.

The book provides an excellent source of recreational mathematics activities for teachers in the intermediate and senior grades. Topics range from familiar and elementary, geometrical and arithmetical recreations to more difficult exercises on such types as cominatorics, polyhedra design and construction, chess-board recreations, magic squares, mapcolouring problems and others. The extensive coverage of topics results in some explanations being brief and requiring a background knowledge in the topic. The chapter on calculating prodigies would interest the majority of students.

The text is highly recommended for the mathematics library; it can serve as a resource text for teachers interested in recreational mathematics activities or for students researching a recreational mathematics topic.

G.B. Lawson

INGENUITY IN MATHEMATICS
HONSBERGER, ROSS
New York:Random House (Singer School Division), 1970
Paperback, 204 pp., Bibliography, $4.00
Now distributed by The Mathematical Association of America, 1225 Connecticut Ave., N.W. Washington, D.C.

This book is number twenty-three in the New Mathematical Library, an excellent series of monographs sponsored by The School Mathematics Study Group. This series is "written by professional mathematicians in order to make some important mathematical ideas interesting and understandable to a large audience of high school students and laymen". Professor Honsberger, formerly Head of the Mathematics Department of Northview Heights C.I. in North York is presently a faculty member at the University of Waterloo.

In this book of "short, self-contained essays on elementary mathematics". modelled on The Enjoyment of Mathematics by Rademacher and Toeplitz, Dr. Honsberger has attempted to "describe work of outstanding mathematicians and show the incisiveness and ingenuity so characteristic of first rate mathematical work". Much of the material in the nineteen chapters of the book would be accessible to an able senior secondary school student. Ten of the chapters include exercises which range considerably in difficulty. Answers to all exercises are included in the book. Chapter titles include: Probability and Pi, Odd and Even Numbers, Mascheroni and Steiner, The Isoperimetric Problem and The Theorem of Barbier.

Representative excerpts:

I can remember reading years ago that the probability of two positive integers, chosen at random, being relatively prime is $6/n^2$. .(pg.3)

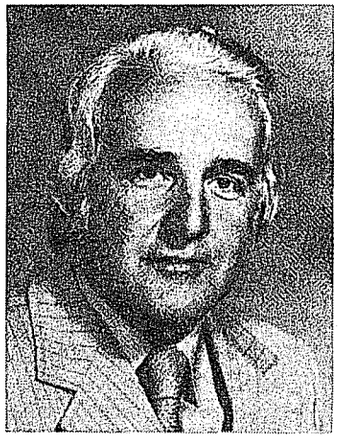
Three counters, A,B,C, are coloured red, white and blue, but not necessarily respectively. Of the following statements, one only is true. A is red. B is not red. C is not blue. What colour is each counter? (pg. 23)

First choose a positive integer N, say 6. Next determine the divisors of N; for N = 6, they are (1,2,3,6). Finally determine the number of divisors - in the case at hand (1,2,2,4) - and you will have a set of numbers with the desired property:

$$1^3 + 2^3 + 2^3 + 4^3 = 1 + 8 + 8 + 64 = 81 = 9^2 = (1 + 2 + 2 + 4)^2$$ (pg. 72)

This first class piece of mathematical exposition should be in every high school library and all secondary school mathematics teachers should consider adding it to their personal collections. It is an excellent source for 'classroom nuggets' and one of the best places for an able young mathematics student to get a glimpse of real mathematics

W.H.



John Egsgard

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APRIL
21-24
1976

THE TEACHING AND LEARNING OF SECONDARY SCHOOL MATHEMATICS

H.W. Vanbrummelen, Principal, Edmonton Christian High School
Re-printed with permission, from DELTA-K, published by the Alberta Teachers' Association Mathematics Council

INTRODUCTION

I am concerned about the state of mathematics teaching and learning at the high school level in Canada today. The majority of students graduating from our high schools have little if any idea of the role which mathematics has played in the development of our culture; they have not been shown the ways in which mathematics is used in our modern society; students in college-bound programs, in particular, view mathematics as an isolated, self-sufficient body of knowledge.

My concern is not of recent origin, although it has grown in degree over the last ten years. When I had just started teaching nine or ten years ago, I remember raising some questions about one of the first experimental programs being introduced in our district in Ontario.* I recognized then - as I still do now that the attempt to give students a thorough understanding of a concept (rather than using a "cookbook" approach) is a valid one. However, my questions about our goals in teaching mathematics ("Why do we teach what we teach?" "How can we make mathematics relevant for the 90 percent of the students who will use very little mathematics after they graduate from high school?") were brushed aside as "too philosophical." This was the era of reform - and I, along with many others, was swept along by the tide of enthusiasm for the many improvements which the new programs *did* contain.

However, my misgivings increased over the years. It became clear that the development of new programs and courses was not guided by any comprehensive concept of the process of education. Rather than starting with the question *why* we should teach certain topics, the foremost question was *how* we should teach what rather uncritically was accepted as "modern mathematics." Instead of giving positive direction in teaching students to grapple with the problems dealing with the warp and woof of today's culture, today's mathematics curriculum is abstract and removed from the student's everyday experience. Rather than preparing him for *life*, the present mathematics courses by and large prepare him only for even more specialized courses in universities or at technical institutes. My views on the present state of mathematics in high school parallel those of Professor Morris Kline in many ways, although I do feel that in his latest book he has neglected the fact that many programs contained worthwhile improvements over the "traditional" texts they replaced.

What a person believes about the meaning and purpose of life determines his philosophy of education, his views on the nature of the child and of learning, and his goals for teaching mathematics (whether these be explicit or implicit). In this brief I will show that my basic faith commitment influences *what* mathematics I teach as well as *how* I teach it. At the same time, I recognize that persons with a different philosophy of life may reach other conclusions - such freedom exists in a democratic society. I believe that a school is accountable to the parents in formulating its philosophy of education and its curricular objectives. Parents should then be able to choose the type of school for their children whose philosophy of education agrees with their own sense of values.

For the sake of brevity, I have not substantiated every statement. For a more detailed exposition of my views, I refer you to my chapter on curriculum in *To Prod the Slumbering Giant* [2] and to the mimeographed proceedings of a seminar on *Mathematics in the Christian School.*[3]

GOALS OF CHRISTIAN EDUCATION

The curriculum of a school is the plan for learning that translates what one believes about man and

* This was the experimental version of *Mathematics* 9 (Coleman et al), used in York county at that time.

his place in our society into a specific program of courses in the school. In our school, the curriculum aims at preparing the student for a Christian life of service in all areas of today's culture. Specifically, our school has the following four objectives:

1. The student's understanding of the unity and rich variety of the created world will be developed. He will systematically learn about matter, plants, animals, and man in society in order to gain knowledge of the structure, function, and interrelationships in creation.

2. The student will be directed to realize that he has a historic place in the world, to live obediently in the god-man covenantal relationship, and to fulfill the God-given task for man. The student will be guided to be a reforming influence in his society to help develop a Christian culture and life style according to the norms of God's Word.

3. The student will be directed to assume responsible discipleship of Jesus Christ in developing skills, knowledge, and insights to live the full Christian Life as a church member, citizen, family member, neighbor, friend, worker, consumer.

4. The student will be led through a general, foundational program of studies. Sufficient competence will be developed so that the student can continue his education in vocational training, a community college, a Christian college, a technical institute, a school of nursing, or a university.

I mention these objectives because I am convinced that a definite philosophy of life, when worked out in a curriculum, will affect a specific course of studies in a substantial way - and this applies to mathematics just as much as to any other subject. Conversely, if one pays lip service to a philosophy of life but does not work it out consistently in a curriculum, one may inadvertantly end up teaching a course based on a philosophy at odds with one's own.

MATHEMATICS CURRICULUM IN TODAY'S SOCIETY

The dominant role of science and technology in our culture has led to material wealth but spiritual proverty. The paradox of modern society is that man has constructed a complex social machine to administer the technical machine he has built, but his whole "creation" stands over and above him and manipulates him. As a result, youth became alienated from our society during the '60s. A reaction against our sterile culture set in - breakdown of authority, skepticism, protest, drugs, astrology, and reactions against scientism and individualism. The youth of 1974 is not as rebellious as that of five years ago, but it shows the continuing alienation in the appalling apathy among a large part of our high school generation.

The "modern" mathematics curricula at best have not counteracted, and at worst have abetted these phenomena. On the whole, the gospel (usually implicit) of modern mathematics texts is this: mathematics should be done for the sake of mathematics; mathematicians do not concern themselves with real life or with the moral problems that society may face; you learn mathematical concepts mainly so that you can use these in developing more advanced mathematical concepts, but whether these are relevant in today's society or are important for historical reasons is immaterial. It is remarkable that the vast majority of even those students disliking mathematics "play the game" to please the teacher or to get their marks; however, when they come to physics class, most of them can't solve an equation because they have only "played the game" with the x's and y's in totally abstract situations. More serious is that most texts (and, therefore, most teachers) seldom show that mathematics *is* relevant, that mathematics *does* have a tremendous impact on our culture, that mathematicians *must* make moral choices in how mathematics is used (it is not without cause that computer centers were prime targets of violence several years ago).

In our school, we try to take the student and make his profession of Christianity a significant one. This means also the mathematics curriculum must lead the student to a deeper understanding of our modern society. He must be made aware of the historical roots of our civilization as well as of the present value systems, the aims and ideals, the ultimate loyalties and allegiances of Western culture. This may not be left to the English and social sciences departments; mathematics also shares in this responsibility.

THE RELATIONSHIP OF MATHEMATICS TO OTHER SUBJECTS IN THE CURRICULUM

One major weakness of the present high school curricula is that students are given almost no insight into the interrelationship of the various disciplines. A worthwhile curriculum is not a hodge-podge of ideas thrown together, but it must provide the student with a sense of unity and purpose, a sense of his responsibility to God, to his community, to society as a whole. It is not the task of the high school to create specialists. Some students will undoubtedly become specialists in their later life, but they must be taught to relate specific knowledge to the overall situation. The curriculum - also in mathematics - must teach them to ask: "How does my specialty affect other areas of knowledge and the whole of life? How can I use my specialty in contributing to the enrichment of human culture? How does my discipline contribute to a meaningful outlook on the future of society?"

The present mathematics curriculum is concerned mainly with techniques and the teaching of facts for the sake of cataloguing knowledge. The danger of such specialization can be seen from the way most texts handle a topic such as linear programming: here is a technique to maximize profit; learn it and apply it to the following seven or eight problems. Unlike most exercises, these problems at least deal with situations the students can visualize occurring in everyday life. On the other hand, the implication of such a section is that the mathematician is concerned *only* with the mathematical technique and not at all with the other aspects of the situation. That a company should maximize its profit by hook or crook is a tacit assumption. The mathematician does not concern himself with the physical and biological aspects of the situation. It is not mentioned that maximizing profit might mean our resources are depleted unnecessarily or that it might upset the ecological balance. Nor does the mathematician concern himself with the psychological or social aspects (might he create unnecessary tensions between workers by putting this into effect?) or the legal aspect (might he not break the spirit, it right to expect workers to have to work overtime or to be laid off at will? By maximizing profit, is the worker being reduced to a semi-robot who cannot do justice to his humanity?).

Our society needs people who know the techniques of mathematics - and these techniques must be taught. But such knowledge may never be taught as an end in itself - our society needs men and women who are aware of the consequences which the use of mathematical and scientific techniques may have, and who are able to make sound decisions on the basis of such knowledge.

The importance of mathematics lies in its applicability to other fields - not only to the physical scienes and technology, but also to biology, psychology, economics, and political science. A mathematical structure can often serve as a model for many seemingly unrelated problems. Therefore, we must include applications of mathematics to other fields in our curriculum. If this is done at the student's level, it will not only deepen his understanding of the mathematical concepts and techniques, but it will also help to make his studies in mathematics more relevant, leading to better motivation. While a subject such as economics cannot be reduced to mathematics alone, students must become aware of the usefulness of mathematical models in such subjects. Almost all major fields of human endeavor and innumerable situations in everyday life are likely to lead to significant applications of mathematics. We must find problems which are complicated enough to represent a situation honestly, but simple enough so that students have some chance to solve it. I admit that this is not an easy task - as the large number of artificial and insignificant problems in most texts indicate. Pollak discusses this in his *Applications of Mathematics* and states that applications are best chosen from classical analysis, linear algebra, probability and statistics, and computer science if we keep in mind which fields will be of major importance in the future.

PSYCHOLOGICAL CONSIDERATIONS

Human development is not an automatic, natural process, but requires pedagogical influence and interaction and the exertion of formative power. Education always implies a deliberate attempt on the part of the educator to lead the student in a particular direction according to certain norms. At the same time, education requires a fundamental respect for those we seek to educate. A child does not *develop* into a person: he *is* a person from the start, though an immature one.[5]

Mathematics must be taught so that the student is shown how the subject helps him to take on a meaningful calling in life, and it must enable the student to be a full, responsible human being who is actively involved in the educational process: the student must participate, cooperate, and be

given opportunities to initiate. Teaching mathematics well requires liveliness, inspiration, stimulation, care and genuine concern for a person's development. A student's opinions and reactions must be respected even if not always approved. To educate means to give direction to the development of a person's life.

However, such development is not a "linear" one (cf. Relevance in Teaching Mathematics")[6]. The student must be able to explore, to follow up hunches, to start with a problem that's meaningful and try to solve it using his intuition, to go off on a tangent, to choose topics himself from time to time that interest him.

Unfortunately, few mathematics texts are written as if mathematics is exciting, as if it is a fascinating journey with beautiful, useful, and "relevant" results. If Euclid had been introduced to geometry in the same way as my Grade X text[7] approaches it, he would likely never have been excited about mathematics. Pages and pages are spent on definitions and postulates and on seemingly endless similar exercises of proofs - and there are almost no results in the chapters that we "cover" in Grade X which the students didn't already know before they started. This year, after discussing the historical background of geometry before the Greeks, including some of the incorrect results used by the Babylonians, I started deductive geometry with a theorem I'm not "supposed" to take until Grade XI: a dissection-type proof of the Pythagorean theorem. We proved the theorem to the satisfaction of everyone involved, and then I pointed out some of the "gaps" in the proof. After that we worked backward to fill the gaps, taking five weeks to learn the same material that "officially" takes five months. So we now have more than three months to inestigate other aspects of geometry.

Textbooks are not written for students; they are written for teachers (with the exception of such books as the SMP series[8]). And they are dull. The typical section has a couple of examples followed by a selection of similar exercises - and sometimes a written description which is usually too difficult to read for all but the best students. There is no attempt to put into practice what well-known educators have held to be psychologically sound, whether this be the use of Ausubel's "advance organizers", the intuitive approach of W. W. Sawyer, or the fact that Dienes has shown that the present sequence for teaching structure in mathematics is backward[9]. Perhaps not all of what these educators have to say is valid, but our textbook writers seem to think that none of it is. And while there are a great many creative teachers, the majority of teachers do not have the time or the ability to design their own curriculum and, therefore, usually depend on their texts.

We must structure the curriculum in such a way that the students are given the emotional freedom to respond to the teacher's guidance in their own unique way: to state their own view, to experiment, to investigate, to search and probe for answers. This means that general mathematics courses cannot and may not be just "more of the same." If students still cannot divide decimals properly when they come to Grade X, don't give them worksheets; give them challenging problems in an everyday context and let them use a calculator to work it out; maybe some of them will be motivated to see the usefulness of dividing decimals, and they may even want to learn the technique themselves. In the college-bound course, don't bind the students in a straitjacket of terminology, symbolism, and axiomatics. Give the students' intuition and imagination free reign; the good students will learn the correct symbolism and correct formal structure in due time, the poor students only get turned off by a too-early inroduction to formalism.

Too often we treat students as if they were all mathematicians - with the exception that we don't even allow them the time needed by mathematicians to grasp ideas intuitively and work with them before attempting to become precise and rigorous. Calculus was used for many years before it was put on a rigorous logical basis. Logic and proof are useful tools in mathematics but do not form its essence. In our teaching, we must make clear that our everyday integral experience is the foundation of mathematics: everything in the world is subject to the cosmic law order. An intuitive approach to new topics with many different intuitive considerations is sound both from a philosophical and psychological point of view. We need problems which read: "Here is a situation - think about it - what can you say?"[10] Both teaching and learning in this way will be difficult at first, but it will also prove to be much more rewarding and meaningful than the stereotyped approach usually used now.

THE HIGH SCHOOL MATHEMATICS CURRICULUM

To be effective, a high school curriculum must be unified in purpose and direction. Our staff is working toward a model as described below. This model tries to stress the interdependence of all disciplines, for it is impossible to go very deeply in any one discipline without striking at the roots of another. All courses around the inner "core" are related to this core, and the objectives of each course tie in directly with the objectives of our school.

There are several "mandatory" subjects, including a course in mathematics. We hope to develop such a course during the next few years. Rather than *stressing* technical skills (although some basic ones would be taught!) the main goal would be to have the student come to an understanding of some of the basic structures of mathematics, as well as the place of mathematics within the structure of creation, how it developed through history, and how it is used and misused in today's society. Thus the mandatory course would emphasize the development and place of mathematics in Western coulture as well as its relationships to the physical, biological, economic and aesthetic aspects of reality.

This course would be mandatory for all students - whether it be made part of our present courses or taught independently. With the present curricular structure in Alberta, it would probably become part of the present courses we offer, both at the general and matriculation levels. Eventually, however, we hope to teach this as one compulsory "module" in each of Grades X, XI, and XII (such a module could be individualized to a large extent) with a number of optional modules available to the students each year - ranging from algebra to transformations and vectors to calculus to business mathematics. The optional modules would develop topics in much greater depth for students having a special interest in mathematics in general or in one or more of the topics covered by the modules.

With this type of structure, we hope that *all* students will learn to recognize the rightful place of mathematics as a science which describes and investigates in detail the numerical and geometrical aspects of the universe around us. Students will gain comprehension to differing degrees, but it is our aim that all will learn to see mathematics as a functional tool to develop and unfold our society and our world. Specifically, we have set the following goals:

1. The student must gain an understanding of the concepts of number and space and their interrelationship so that he can abstract mathematical properties from concrete situations, so that he can theorize about and develop such properties, and so that he can apply the results of such theorizing to new situations. For example, a mathematical model in physics is a mathematical abstraction that can be worked with in order to clarify and gain more insight into the physical situation.

2. The student must realize that mathematics serves as a functional tool in solving our everyday problems. The student must see that mathematics does not exist independently of other disciplines but contributes to the unity of all aspects of creation and helps him to analyse the quantitative and spatial situations in other disciplines.

3. The student must realize that mathematics is a developing science, and that throughout history it has influenced and, in turn, has been influenced by cultural forces.

If mathematics is taught to meet these objectives, the student will gain respect for the law structure of our universe and its dependability, and he will discover the order, patterns and relationships that exist in creation. By having a better understanding of these ideas, the student will also appreciate more fully the aesthetic aspects of mathematics. The realization of our objectives may be helped by the development of the student's techniques and skills. The degree of facilitation should be individualized in relation to the student's abilities and direction.

CONCLUSION

It will be difficult to change the direction of mathematics education with such massive curricular reform behind us. Yet, considering (1) the shortcomings of present texts as pointed out in this brief (texts which the majority of teachers depend on) and (2) the dissatisfaction with the present "new mathematics" courses by parents, students and teachers (a change in the elementary curriculum in Edmonton public schools enabled the *Edmonton Journal* to play up the fact that "the new math is dead"), it is clear that fundamental reform in the teaching of mathematics is a necessity.

Mathematics may not be seen in abstraction from the rest of life. An intuitive rather than an axiomatic approach is called for until the student has a thorough grasp of concepts: mathematics starts with *situations*, not with theorems. Of course, the problem is not just one of writing courses with a different philosophy and using sound psychological methods based on an understanding of the nature of learning, but it also involves having such programs implemented at the classroom level.

How can this be achieved? First, new curricular materials must be developed. I hope that *Mathematics Canada* will be able to initiate and perhaps even sponsor workshops where groups of teachers and educators who are in agreement about their philosophy of education and also about the direction of mathematics education will write materials for the classroom, particularly for a

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BRIEF TO MINISTER OF EDUCATION

Copies of a brief on the qualifications and preparation of Teachers of Mathematics (K-8) are available from M. MacGregor, 247 Bright Street, Sarnia, N7T 4E9. This brief was prepared from comments from OAME members and Faculties of Education who were sent a preliminary draft. The brief has now been approved by the OAME Council and sent to the Minister.

REVIEW OF SENIOR DIVISION CURRICULUM

The Council of OAME invites responses to the report of the Committee published in the September, 1975 Gazette. They should be directed to Neil Williamson, 2329 Bridletown Circle, Agincourt, Ontario.