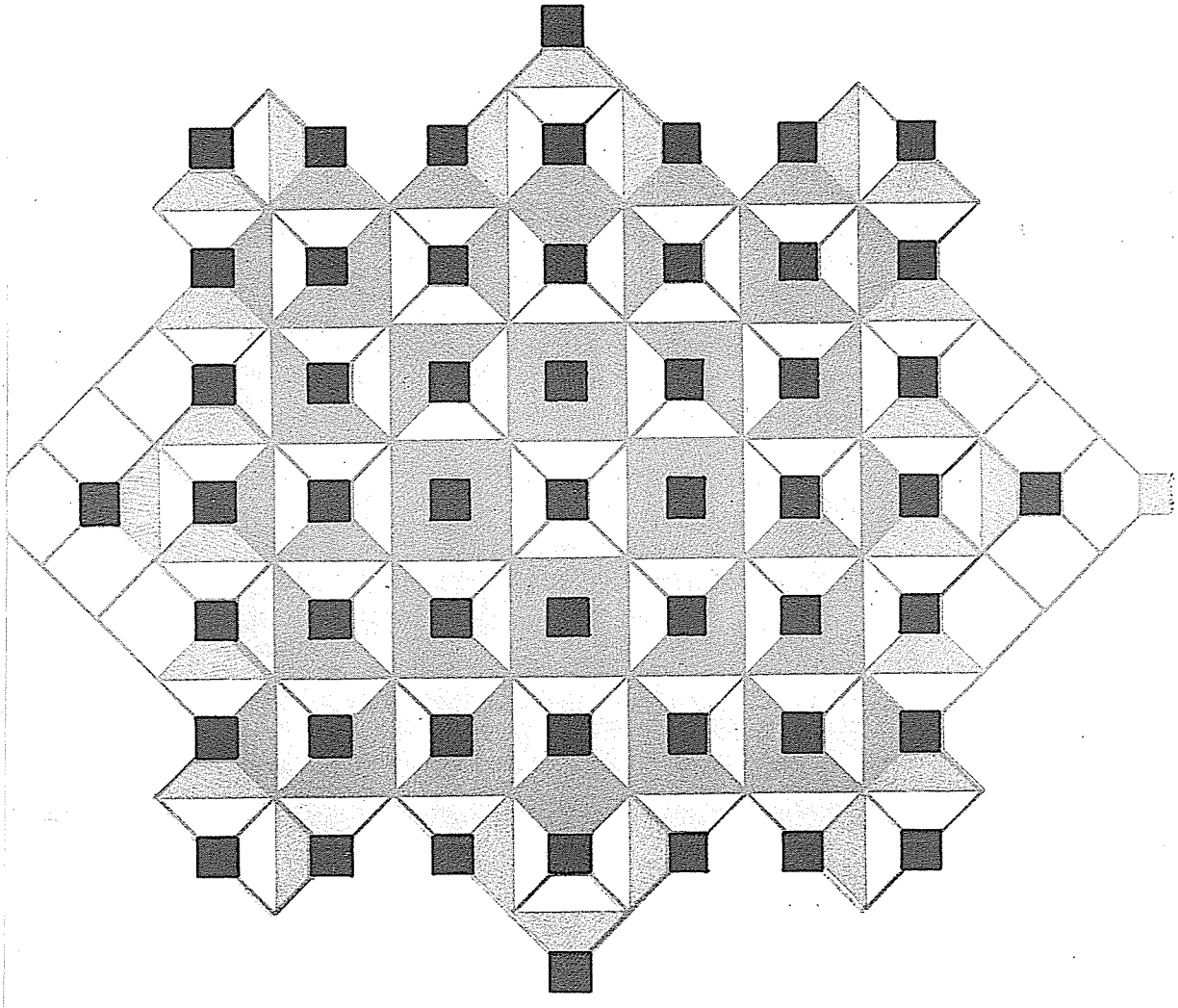
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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Editorial correspondence and articles should be submitted to one of the editors listed below.

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Joan Routledge
is the President of OAME and will be presiding over the sessions at St. Catharines in May.

PUBLICATIONS AVAILABLE

TO O.A.M.E. MEMBERS

	Regular Price	O.A.M.E. Member Price
Handbook of Trigonometry (1975)	$2.00	N/C
A Preparedness Test for Grade Nine Students (1973)	2.00	N/C
Sample Grade 13 Examinations	1.00	N/C
Grade 11 (Level 5) Sample Examinations	1.00	N/C
Reprints of 1974 & 1975 Gauss Contest Papers	.10	N/C

To receive these please contact:

Mr. R. G. Scoins,
Faculty of Mathematics,
University of Waterloo,
WATERLOO, Ontario.

EDITORIAL

One of the members of the Editorial Board has suggested dropping the debate over basics, 3 R's, standards and the supposed mathematical shortcomings of secondary school graduates. He thinks we should be helping to provide constructive solutions. However, the Executive of O.A.M.E. has asked me to bring to your attention a recent analysis of standards in the schools. The Education Committee of the Canadian Chamber of Commerce has published the results of a survey of universities and community colleges on the current state of the 3 R's and remedial action being undertaken.* The report's conclusions and recommendations follow:

1) The educational system has changed drastically, and in surprisingly similar ways, across Canada over the past 5-10 years.

2) Some of these changes have been generally beneficial (e.g. the introduction of subject, rather than grade, promotion, and greater opportunities for student discussion and expression of opinions). Others have not been so good (e.g. the imbalance created by the increasing use of course options at the expense of an emphasis on the "basic skills" of communication and computation, and the lack of objective standards of course evaluation).

3) The key question to be asked is "is there a way to retain and, indeed, improve upon these 'good' changes while, at the same time, eliminating the 'bad'?".

4) A clear majority of the respondents to the survey indicated that: a) students **do** lack proficiency in the "basic skills"; b) there **is** a need to upgrade the teaching of "basic skills"; c) there is also a need to measure where we are, and where we want to be, in "basic skills", particularly in the area of communication and computation; d) they were not sure on what basis this measurement might best be carried out.

5) Returning to the question posed in (3) above, we believe that it is clearly the responsibility of Provincial Departments of Education to: a) develop appropriate measurements for testing students' proficiency in "basic skills", if such measurements are not now available; b) decide upon the "basic skills" standards required either in the community or for purposes of further education; c) develop "action programs" for use within the schools to ensure that students do achieve and, better still, surpass these standards.

6) We commend the use of tests by universities to determine a student's potential in a variety of areas, and suggest that these tests be developed on a national or provincial basis. We realize, of course, that there are many difficulties involved with such tests; nonetheless, we believe that something is needed to help in entrance decision-making.

7) The effective use of these tests, together with a testing program in the basics at the primary and secondary school levels would, we submit, provide a credible replacement for outmoded high school departmental examinations and, at the same time, relieve the universities of the onerous and unwanted task of developing eleventh hour remedial programs in the basics.

8) We believe that there is a real need for ongoing co-operation and co-ordination between educators and the community in these and allied areas so that each may be fully aware of, and be able to effectively deal with, the other's concerns and requirements.

No doubt we can all find much to agree with in the above. There is no space to consider the report in detail. If you are interested, write to Mr. Ronson for a copy. At the risk of appearing to be defensive, it seems to me that this was not a very good survey. The Chamber letters to the institutions tended to pre-judge the situation and invite negative comment. A substantial proportion of the respondents to the statement that "high school graduates are lacking in 'basic' skills" either disagreed or were unwilling to comment. (34% of universities, 39% of colleges). There was inadequate provision for this segment of opinion.

Getting back to basics, it seems to me that the Chamber report did not come to grips with the real root of the problem. Since 1946 the percentage of the 15-19 year age group in the schools has risen dramatically. (From 35% to over 80% in Ontario.) It is this extra 45% we should be thinking

about. Society has decreed that elimination is no longer a satisfactory solution to the problem on non-performing students. Should we use the same metre-stick as before to measure performance? We mathematics teachers used to be fond of saying "You can't add apples to oranges!" Has anyone the temerity to suggest that the extra 45% are worse educated at age 19 today than their counterparts in 1946?

Most of you have read the newspaper accounts of the O.E.C.D. report on Canadian education. (Globe and Mail February 11.) That report states that we neglect our underpivileged, handicapped and native peoples and that we lack a national policy on education. Are these not the "basics" we should be looking at? Do not these problems contribute greatly to weaknesses in the 3 R's? Does the Chamber of Commerce have a policy on these matters? I hope so.

The Chamber's recommendations about testing need to be looked at carefully. It is high time that we in Ontario returned to the external evaluation of secondary school performance. It is not that I don't trust the schools. Far from it! It is because society must show the students once more that what they are doing in school is significant and that how well they do matters. Humans are social animals and cannot be self-directed at all times. We need the encouragement and evaluation of other people. Let the universities, colleges and the Ministry get down to the task and stop stalling around with surveys and studies! My personal hope is that standardized tests will be used sparingly and with great skepticism. The Chamber charge that traditional examinations are outmoded is not a valid one except in the strict financial sense.

The O.E.C.D. report mentioned the lack of communication between provinces. Perhaps the evaluators did not understand the B.N.A. Act. However it is a fair criticism. Thanks to the February 21 Globe and Mail we can find out the latest about Manitoba. There was an article about the deputy minister of education and his attitude toward the 3 R's. I hope you read it. Apparently Hall-Dennis has reached the West. If one is to believe the article Ontario does not have a monopoly on woolly thinking in high places.

My piece, if not a farewell, should be a summary statement. Health (and the O.A.M.E. Executive) permitting, your editor has one more year to serve. However, I go on pension June 30. What does one say to account for thirty-six years spent in or near the classroom? It is not fair to take advantage of a captive audience. Let me just thank my former students and colleagues for an interesting career. The beautiful people I worked with in the classrooms of the schools, the R.C.A.F., the Department of Education, Althouse College, O.M.C. and O.A.M.E. have made it all worthwhile.

For my valedictory, read the quotation from my patron saint on page 6 . Socrates says it better than I can.

Arn

* **Report on Basic Educational Skills,** prepared by the Education Committee of the Canadian Chamber of Commerce, June 1975. (Mr. J. C. Ronson, Chairman, Education Committee, Canadian Chamber of Commerce, 330 University Avenue, Toronto, M5G 1R7).

NOTICE TO CONTRIBUTORS

The editorial staff of the Gazette wishes to thank the many mathematics educators who submitted papers for publication this year. Your support makes our part of the work possible.

Articles are needed for the 1976-77 year. Next year's issues will be published in September, December and March. Deadlines for receipt of material are respectively August 1, November 1 and February 1. You are asked to have your article typed, double spaced, on 8-½ x 11 paper. Diagrams should be prepared with india ink, ready for photo-reproduction. Articles may be sent to the editor or any other member of the staff.

Education

Whom, then, do I call educated?

First, those who manage well the circumstances which they encounter day by day, who possess a judgment which is accurate in meeting occasions as they arise, and rarely misses the expedient course of action. Next, those who are decent and honorable in their intercourse with all men bearing easily and good-naturedly that which is unpleasant or offensive in others, and being themselves as agreeable and reasonable to their associates as is humanly-possible to be; Furthermore, those who hold their pleasures always under control, bearing up under them bravely and in a manner worthy of our common nature; Finally, and most important of all, those who are not spoiled by their successes, who do not desert their true-selves, but hold their ground steadfastly, as wise and sober minded men, rejoicing no more in the good things which have come to them through chance than in those which through their own nature and intelligence are theirs since birth. Those who have a character which is in accord, not with one of these things, but with all of them, these I maintain are educated and whole men, possessed of all the virtues of a man.

An Original Pen : Brush Scroll — *Socrates - 470 - 399 BC.*

HOW WE SPEND OUR TIME

Stolen from the Carleton Elementary Mathematics Group Newsletter

The life expectancy of the average man is seventy years. Of the seventy years, twenty years are spent sleeping, seven years are spent in recreation, and five years are spent in shaving, washing, and dressing. Seven full years are spent seeing films or plays, one year is spent on the telephone, three years are lost waiting, and three years are spent at table!

Each year, thirty hours are lost looking in mirrors; four hours go to wiping the nose. About five months of an average man's lifetime is spent in tying shoelaces. Of 614,000 hours of life, he dedicates 40,000 hours to playing cards!

LETTERS TO THE EDITOR

Lake Superior High School, Schreiber Campus
P. O. Box 339
Schreiber, Ont., POT 2SO

Prof. Arn Harris
Editor, Ontario Mathematics Gazette.

Dear Prof. Harris:

Here is a comment on C. Pinnegar's article in December 1975's issue of the Gazette. The following theorems, I believe, provide the answer asked for in that article. They assume the truth of a few well known results, that are included as lemmas for convenience.

Lemma 1: Let m be a real number.

Let $y=f(x)$ be a function such that f'' is continuous on a neighbourhood of m.

(a) If $f'(m)=0$ and $f''(x)>0$ for all x (except $x=m$) in a neighbourhood of m, then there is a local minimum at $x=m$

(b) If $f'(m)=0$ and $f''(x)<0$ for all x (except $x=m$) in a neighbourhood of m, then there is a local maximum at $x=m$

(c) If $f''(m)=0$ and $f''(x)>0$ for all x in a one-sided "neighbourhood" of m and $f''(x)<0$ for all x in the other one-sided neighbourhood of m then there is a point of inflection at $x=m$

(d) The above three cases are mutually exclusive.

Lemma 2: Let m be a real number.

Let $y=f(x)$ be a function that is continuous on a neighbourhood of m, and such that $f(m)\neq 0$.

Then either (a) $f(x)>0$ for all x in a neighbourhood of m

or (b) $f(x)<0$ for all x in a neighbourhood of m

Theorem 1: Let m be a real number

Let n be a natural number

Let $y=f(x)$ be a function such that $f'(x)=(x-m)^{2n}q(x)$ where q is a function such that:

i) $q(m)\neq 0$

ii) q' is continuous on a neighbourhhod of m.

Then $x=m$ is the location of an inflection point of $y=f(x)$

Proof: $f''(x) = 2n(x-m)^{2n-1}q(x)+(x-m)^{2n}q'(x)$

$= (x-m)^{2n-1}\left[2n\, q(x)+(x-m)q'(x)\right]$

Let $w(x)=2n\, q(x)+(x-m)q'(x)$ (So $f''(x)=(x-m)^{2n-1}w(x)$)

$\therefore$ q' is continuous on a neighbourhood of m, $\therefore$ it follows readily that w is also.

Also, $w(m)=2n\, q(m)$. But $q(m)\neq 0$. $\therefore$ $w(m)\neq 0$.

By Lemma 2, either $w(x)>0$ for all x in a neighbourhood of m

or $w(x)<0$ for all x in a neighbourhood of m.

But $(x-m)^{2n-1}$ changes sign at x=m.
Hence the conditions of Lemma 1(c) are satisfied and we are done.

Theorem 2: Let m be a real number
Let n be a natural number

Let $y=f(x)$ be a function such that $f'(x)=(x-m)^{2n-1}q(x)$
where q is a function such that:
(i) $q(m)\neq 0$
(ii) q' is continuous on a neighbourhood of m.

Then, for $y=f(x)$, x=m is the location of a local
(i) maximum, if $q(m)<0$
(ii) minimum, if $q(m)>0$

Proof: $f''(x) = (2n-1)(x-m)^{2n-2}q(x)+(x-m)^{2n-1}q'(x)$

$= (x-m)^{2n-2}\left[(2n-1)q(x)+(x-m)q'(x)\right]$

Let $w(x) = (2n-1)q(x)+(x-m)q'(x)$. (So $f''(x)=(x-m)^{2n-2}w(x)$)
Proceeding as in Theorem 1, we get that $w(m)=(2n-1)q(m)$
and that either $w(x)>0$ for all x on a neighbourhood of m
or $w(x)<0$ for all x on a neighbourhood of m.
In the former case, noting that $(x-m)^{2n-2}>0$ for all $x\neq m$,
we get that $f''(x)>0$ for all x (except x=m) in a neighbourhood
of m and that $q(m)>0$.
In the latter case, we get that $f''(x)<0$ for all x (except
x=m) in a neighbourhood of m and that $q(m)<0$.
The theorem then follows from Lemma 1(a),(b).

Remarks:

1. These two theorems certainly establish the truth of the conjecture for cases in which f' is a polynomial, for q would certainly meet the continuity requirements. The condition that $q(m)\neq 0$ is essential, however.
Thus, by Theorem 1, $f'(x)=(x-\frac{\pi}{2})^6\sin x$ would certainly have a point of inflection at $x=\frac{\pi}{2}$, for $y=f(x)$
However, given that $f'(x)=(x-\frac{\pi}{2})^6\cos x$, Theorem 1 would be silent.
It would be interesting to attempt to extend the concept of multiple root and then pose Pinnegar's conjecture again. Surely, for example, $\frac{\pi}{2}$ is a multiple root (of multiplicity 7) of $(x-\frac{\pi}{2})^6\cos x$, even though $(x-\frac{\pi}{2})^7$ is not a factor. Certainly such a conjecture holds true for that example.
2. I hope this letter constitutes the sort of comment you were seeking.
3. I would be interested in any information you may run across regarding the idea of extending the definition of multiple roots, as referred to, above.

Respectfully submitted,

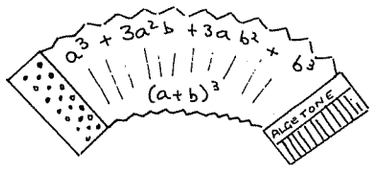
Dale S. Cooper

Editor's Note: I received a second letter from Mr. Cooper the following day with some suggestions about point 3 above. His second submission will be published in September.

P.P.S.: A third submission followed later. The plot thickens!

Editor, The Gazette,
Dear Arn:

CONCERTINA MATHEMATICS MUST GO



Much of the grade 10 curriculum is concerned with factoring and distributing. This alternate squeezing and expanding of algebraic expressions is reminiscent of the concertina player - however the algebraic 'music' is neither interesting nor necessary.

The number of algebraic expressions which have easily discernable factors are very limited, and only exist in the rarefied atmosphere of the school classroom. In reality algebraic expressions are ugly, involved, and to all intents and purposes non-factorable. Any one who has the good fortune to find an algebraic expression in a factored form would be foolish to distribute the factors. The result would probably behave like one of those fiendish wooden puzzles, and never go back together again!

There are many other areas of algebra in which we may develop manipulative skills; in particular it is wise to consider the introduction of simple mathematical modelling into the curriculum. This topic combines algebraic skill with logical thinking and would be much more useful to the student; moreover this would be a definite move in the direction of modern mathematical applications.

Peter Weygang
Levack High School

"LOGIC FOR YOUNG CHILDREN" FOR CHILDREN 4 - 8 YEARS

Wendy Auger, Teacher Consultant in Mathematics, Scarborough

1. **Investigation Stage:**

Children are encouraged to play with materials and by doing so discover their attributes. The children may use them for:

- building material
- picture making
- pattern making.

(Later the teacher starts the pattern and the child completes it. Make sure one complete pattern is given and enough of the next to enable the child to distinguish it.)

2. **General Sorting Experiences:**

Children sort according to size, shape, or colour at this stage. Use a variety of sorting materials, eg. cars, plastic models, buttons, etc., so the children do not associate an underlying mathematical idea with a particular piece of apparatus.

3. **Use of "Logiblocs" (Dienes)** The blocks have four attributes:

- colour (red, yellow, blue)
- shape (triangle, circle, square, rectangle)
- size (large, small)
- thickness (thick, thin)

These blocks are the main tool in the logic games to follow.

4. **Specific Sorting Experiences:**

Sorting According to Colour:

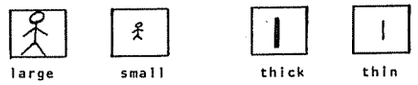
Provide three hoops. The children sort the blocks into a RED set, a BLUE set, and a YELLOW set. The sets can be labelled with a word card [red] or a symbol card with a splash of the correct colour [●]

Sorting According to Shape:

Provide four hoops. The children sort the blocks into sets of triangles, circles, squares, and rectangles. The sets are labelled either with a word card circle or a symbol card o

Sorting According to Size & Thickness:

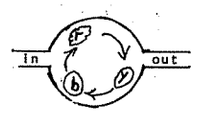
The same procedures are repeated as in the two experiences above. The appropriate cards are as follows:



5. **"Factory Games" (States and Operators)**

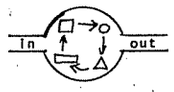
(a) Colour change.
(b) Shape change.
(c) Combined operations (colour and shape).
(d) Combined operations (colour, shape, thickness).
(e) Combined operations (colour, shape, thickness and size).

(a) Colour Change:



This game involves changing a block in one way only.
eg: a red circle is put on "in"
and a yellow circle must be put on "out".

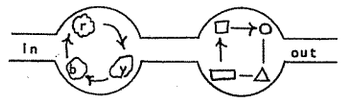
(b) Shape Change:



In this game, the block put on "in" can be changed in shape only.

eg: a red square is put on "in"
and a red circle must be put on "out".

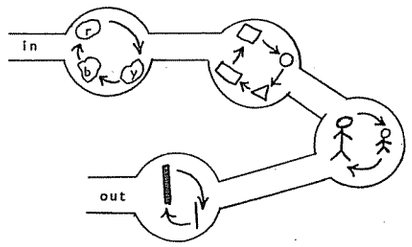
(c) Combined Operations (shape & colour)



The same procedures are repeated as in games (d) and (b).

(d) and (e) These operations combine the other attributes of size and thickness with shape and colour.

eg:



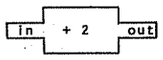
Later these activities can be done without using blocks - the child works this out as a mental process when he is able to handle abstract ideas confidently.

These games can be thought of as a prelude to states and operators.

eg: $2 + 3 = 5$

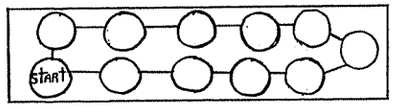
We start with 2, our starting state; we add 3, our simple operation; we finish with 5, our finishing state. We could think of this as a machine operation.

eg: "add 2 machine"



6. (a) **One Difference Game:**

Any block can be placed on the next circle as long as it differs in one way only.

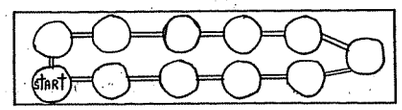


The single solid line between the circles signifies "one difference".

The added challenge comes when the last block must be placed so that it differs in one way from its preceding block and the original block at the "start" position.

(b) **Two Difference Game:**

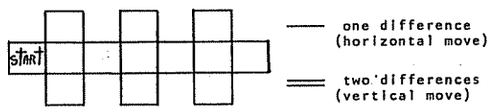
The rule for this game is very general - any block may be placed on the next circle as long as it differs in two ways only.



The double lines between the circles signifies "two differences".

(c) **Combined Difference Game:**

The two previous games are combined in this game.

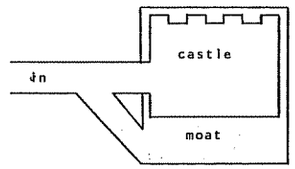


Children take turns placing blocks in the squares until the board is full.

7. **Negation:**

This is the simplest logical connective - whether an object is a member of a set or not.

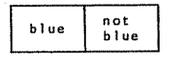
(a) Soldiers and Castle Game



A decision must be made as to which soldiers will go into the castle and which will go into the moat. As a challenge, see how many different decisions you can make in this game.

(b) Pandora's Box (partitioned box).

eg:



We put all the blue blocks in one section of the box and all the "not blue" blocks in the other - these are obviously red and yellow blocks but it is simpler to call these "not blue".

A simple "Not ..." statement is called **Negation** in logic.

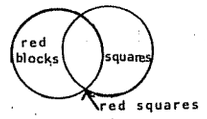
Work cards could then be used.

eg: Find all the blocks that are **not** circles.

8. **Conjunction:**

This involves determining the simplest logical connective between two attributes. Using a Venn Diagram. sort the blocks into 2 sets - put all the red blocks in one side of the Venn Diagram. Put all the squares in the other side. Determine what red blocks could be moved to centre section where the circles overlap.

eg: Venn Diagram -



The red squares may be moved into the centre section because they satisfy the membership attributes of both sets.

The statement "both ... and ..." is used in conjunction to show the connection of two attributes.

Later, a three-hoop Venn Diagram can be used.

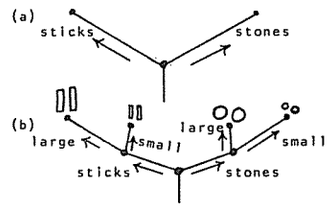
9. **Decision Trees:**

The basic idea behind these games is that of making systematic decisions which will result in an orderly pattern at the top of the "tree".

Whenever there is a "branching" in the decision tree, a decision must be made concerning the direction in which the objects will be sorted.

The children should be encouraged to investigate what happens when the decisions are interchanged.

"Sticks and Stones Games"



These games can be extended by gradually making the decision trees more complicated. Other materials ("Logiblocs") can be used as well. However, children seem to enjoy games which involve materials from their environment. They could be encouraged to collect materials and devise their own games.

REFERENCES

1. "Learning Logic and Logical Games" - Dienes (E.S.A.)
2. "Sets, Numbers, and Powers" - Dienes & Golding (E.S.A.)
3. "Modern Mathematics for Young Children" - Dienes (E.S.A.)
4. "Let's Play Math" - Holt & Dienes (Walker & Co.)
5. "Logic for Young Children" - Hewitt & Postill
6. The Bulmerske Mathematics Acitivity Cards - Available from Scholar's Choice

CHECK YOUR WORK

C. Manahan, Ontario Teacher Education College - Toronto

Do your children know how to check their work? Perhaps we should take another look at that very old method called 'casting out nines'. It really is quite simple and a little practice makes it possible to check answers speedily and accurately.

Ordinarily we might check a multiplication question by 'doing it again' , or by reversing the factors (commutative property) and then multiplying. In the casting out nines method we find a single digit remainder by adding all of the digits of each number, as follows:-

```
  4 1 2       rem.   7
  X 2 4       rem. X 6
1 6 4 8             42 -- rem. 6
8 2 4
9 8 8 8 -- (33) rem. 6
```

There are certain short-cuts that speed up the process of finding the remainder. In the product above, 9 8 8 8, we might have crossed out the 9 and added only 8 + 8 + 8 = 24, and then added 2+ 4 = 6.

Let's practise a few. Remember to cross out (mentally if you wish) any 9s or numbers that add up to 9 such as 8 + 1; 2 + 7; 5 + 4; etc. What is the remainder for each of these numbers?

1 2 3	___	2 4 7 5	___
4 5 6	___	1 3 0 6	___
7 9 1	___	6 8 1 9	___
4 2 4	___		

(If you had these remainders you were perfect: 6, 6, 8, 1; 0, 1, 6).

Whenever the sum of the digits is 9, we cast that 9 out and call it 0. If the sum of the digits is 10, 20, 30, etc. we cross out the 0 and call the remainder 1, or 2 or 3.

This process of casting out nines to find a rem. is just the same as dividing the number by 9 in order to find a remainder (if any). But the method of adding the digits to find the remainder is much faster than dividing.

Let's try this multiplication question and then check the results by casting out 9s.

4 8 1 3
X 3 4

Results by casting out 9s.

4 8 1 3 --- 4 8 1 3 7
X 3 4 X 7
1 9 2 5 2 4 9 -- 4 9 -- rem. 4
1 4 4 3 9
1 6 3 6 4 2 --- 1 6 3 6 4 2 --- 1 6 4 2 -- rem. 4

I have shown all the steps so that the "/" crossing out would not obscure the numbers. Normally just the remainder is written down, like this:

1 6 3 6 4 2 -- 4.

Since the product yielded a rem. 4, and the product of the remainders of the factors (7 X 7) also gave us a rem. 4 we can assume that the answer checks!

This method of checking is very effective with division also. Usually we have to multiply the divisor times the quotient and then add any remainder to see if we get the dividend. Working with single digit remainders is much easier.

Suppose we had to do this question and then check it in the usual way.

```
      1 6 9 6
24 ) 4 0 7 2 3
     2 4
     1 6 7
     1 4 4
       2 3 2
       2 1 6
         1 6 3
         1 4 4
           R. 1 9
```

Usual check:

2 4 X 1 6 9 6 + 1 9 = 4 0 7 2 3

Now try casting out 9s!

2 4 X 1 6 9 6 + 1 9 = 4 0 7 2 3

6 X 4* + 1 = 7

2 4 + 1

6 + 1 = 7

... (*1+6+6=13 and 1+3=4*) above ...

Casting out nines is also an efficient check for addition and subtraction although some feel that it takes longer to cast out nines than to check in the usual way.

Let's see.

(3)

2 3 1	6
1 9 6	7
4 5 2	2
2 8 5	6
1 1 6 4	3

Did you have trouble adding up the remainders? Remember to cast out 7 + 2. That leaves 6 + 6 = 12; and 1 + 2 = 3. This 3 compares with the rem. 3 of the answer 1+1+6+4=12; and 1 + 2 = 3. Easy!

Let's try subtraction, which is usually checked by adding the difference and the subtrahend to get the minuend.

```
  3 4 0 5
- 1 2 0 3
  2 2 0 2
```

2202 + 1203 = 3405, but compare that with

6 + 6 = 3

and 6 + 6 = 12; 1 + 2 = 3. Same remainder as 3 ¢ Ø ⅝.

Many people like to know why this old trick works. At the first of the article we said that we are really dividing by 9 in order to find a remainder. We don't care how many 9s there are in the number, just what the rem. is.

A number such as 2 3 is really 2 0 + 3, or

2 X 1 0 + 3.

If, through the use of the distributive property, we rename 10 as (9 + 1) we would have this equation:

2 3 = 20 + 3

= 2 X 1 0 + 3

= 2 X (9 + 1) + 3

= 2 X 9 + 2 X 1 + 3

If we divide by 9 there would be no remainder for 2 X 9 (because the 9 would 'go' evenly). We would be left with only 2 X 1 + 3, or 2 + 3. Which is why we can add the digits and find that the remainder would be 5.

Let's try 423!

4 2 3 = 4 0 0 + 2 0 + 3
= 4 X 1 0 0 + 2 X 1 0 + 3
= 4 X (9 9 + 1) + 2 X (9 + 1) + 3
= 4 X 9 9 + 4 X 1 + 2 X 9 + 2 X 1 + 3

When we cast out 9s (divide by 9) there is no rem. for the 4 X 9 9 or for the 2 X 9; because the 9 goes evenly, leaving us with

4 X 1 + 2 X 1 + 3

or 4 + 2 + 3 = 9
= 0.

This explanation is NOT necessary to the performance of the checking method, indeed many children would not understand it!

However, we believe that children should be taught not only how to add, subtract, multiply and divide, but also how to CHECK their answers. And the method should be fast and easy!

P.S. Do you know what to say when someone tells you that there is no way to check a mini-calculator?

1 2 3 4 5 6 7 9 X 7 2 = *8 8 8 8 8 8 8 8 This CHECK uses all the digits (except 0) and the digits in the answer require the most power to 'display'.

Do your children have fun learning the 9 times table? Write to me, at 951 Carlaw Avenue, Toronto M4K 3M2, for some ideas.

GEOMETRY IN KINDERGARTEN TO GRADE 6 - SKILL OR FRILL?

Joan Routledge, York County Board of Education

The 1966 Curriculum Guidelines for the Primary and Junior Divisions said in a footnote to the section on Space and Shape, "This section is an integral part of the course, but it should be treated experimentally with different outcomes expected from different children and different classes." It suggested that "an intuitive approach is best, with children discovering properties of shapes, symmetry, similarity, then dimensions, in a progression of experiences." The document called for research and feasibility studies on the introduction of three-dimensional and two-dimensional geometry ideas to children at the K-6 level. In the past ten years a number of good programs in K-6 geometry have been developed in Ontario.

With the current pressure on schools to get back to the teaching of the basics, there is a danger of again excluding geometry from the K-6 curriculum. Must we concentrate all the children's time and effort on the memorizing of basic number facts and computational procedures? The Ontario Ministry of Education thinks not. The 1975 Provincial Curriculum Policy Statement for the Primary and Junior Divisions, **The Formative Years**,[1] places equal emphasis on the three branches of mathematics - arithmetic, measurement, and geometry. Geometry must not be relegated to an optional or enrichment topic for the favoured few only.

The struggle to include geometry in the elementary school curriculum is not a new or recent development. Mathematics curriculum reformers have been calling for this for a number of years.

In 1909 the English Board of Education proposed that a greater emphasis be placed on intuitive geometry. Only when and if the student was old enough and clever enough should he be required to learn geometry in any strict logical sense. Deductive geometry and formal proofs would be expected from fourteen-year-old Grammar School students, the top 15% of the school population. At about the same time Germany introduced a geometry curriculum that is the forerunner of the transformational or motion geometry advocated in **The Formative Years.**[1]

In 1923, a report of the Mathematics Association of Great Britain placed great stress on a gradual and progressive development of geometry through the grades. There was to be no formal separation between solid and plane geometry, and symmetry should have a larger place in the teaching of geometry than had been usual. Apparently the advice was ignored. In 1934 the English Board of Education declared that the comparative neglect of geometry was one of the main defects in the teaching of mathematics in the elementary schools. It was pointed out that too many people thought of geometry as being synonymous with Euclid.

In its second report in 1938, on the **Teaching of Geometry in Schools,** the Mathematics Association of Great Britain re-emphasized its statements in its 1923 report and suggested three stages in the teaching of geometry. The outline of Stage One stressed a need for starting an experimental and intuitive approach to geometry up to the age of fourteen.

Why hasn't geometry taken its rightful place on the K to 6 mathematics curriculum? The Ontario Institute for Studies in Education's report on **Geometry: Kindergarten to Grade 13**[2] states that "Geometry is the natural language of spatial relationships so common to the real world as to be taken for granted." Perhaps adults tend to forget that a child is not born with an awareness and understanding of his environment. **Education in the Primary and Junior Years**[3] reminds us that "infants learn about shapes through the eyes, the hand, the tongue, and the lips. As they start to move about, they begin to build an awareness of distance, direction, and elevation.---The early years in school are vital in stimulating inquiry and experiment and in making possible the discovery of new spatial relationships and the development of new motor skills in relation to them."

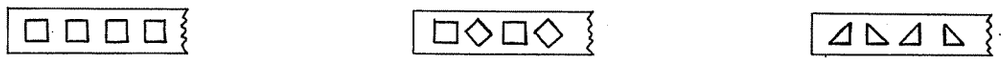
Pre-school experiences are with three-dimensional shapes such as blocks of all shapes, boxes, cans, cones, and balls. These can be touched, felt, tasted, moved, and seen. Two-dimensional shapes are experienced through pictures of familiar objects but it must be remembered that these appeal to the sense of sight only. A young child gains experiences with objects in the environment best through manipulation. Insights are based on motion.

At the primary level a child continues to need experiences with three-dimensional objects, sorting, classifying, organizing, ordering, and abstracting mathematical notions. Given a box of odds and ends, a child will spend many happy hours sorting into sets according to colour, size, shape, use, likes, dislikes and so on. Left to his own devices, a young child seldom sorts and classifies according to some cardinal number attribute. Apparently he comprehends in a qualitative sense before he understands quantitatively. Later on he becomes interested in ideas of how many, how many more, and how many fewer.

Toys, blocks, and counters are the child's first experiences with symbols to represent real objects. For example, a block on wheels may symbolize a real car. A rectangular block with a triangular one on top may represent a house. Children need many experiences using counters to represent number ideas before they can use numerals and mathematical signs with understanding.

Much of the work in geometry has aesthetic as well as intellectual value. Symmetrical patterns, designs, and other notions in motion geometry appeal to the artistic side of children. As the report **Geometry: Kindergarten to Grade 13**[2] points out, "A child is probably aware of the symmetry of a butterfly before the concept of distance has become fully clarified." Children enjoy constructing a symmetrical pattern of beads to make a necklace, or coloured blocks to make a wall. Simple strip patterns and ones involving slides, flips, and turns can be comprehended before more abstract number patterns.

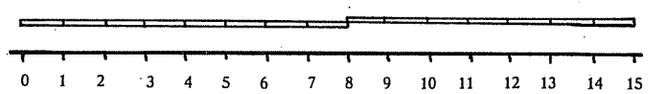
Strip Patterns using



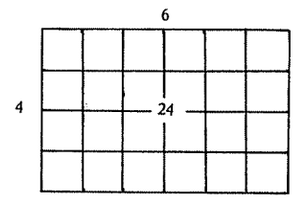
slides **turns** **flips**

Intuitive ideas of symmetry can be developed through the construction of paint 'blob' or ink 'devil' pictures, paper cutting about one or two folds, and the drawing of reflection images with the aid of a plexiglass mirror. An understanding of spatial relationships is enhanced through picture puzzles, simple tangrams, and tiling patterns. Young children can demonstrate that two figures are congruent (same size and shape) be tracing one figure on acetate and sliding, turning, or flipping the tracing onto the second figure.

A single experience often can serve to develop ideas in geometry, arithmetic, and measurement. Looked at from a geometric point of view, addition and subtraction appear linear, while multiplication and division are two-dimensional and rectangular in shape.

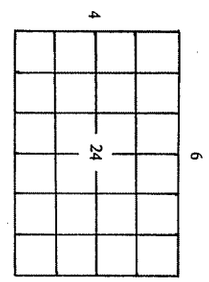


8 + 7 = 15 15 - 7 = 8



4 x 6 = 24

24 ÷ 6 = 4



6 x 4 = 24

24 ÷ 4 = 6

After a one-quarter turn, the rectangle illustrates the commuted fact. The perimeter of this rectangle is 20 units, and its area is 24 square units. Children enjoy searching for other rectangles with the same area or the same perimeter.

Children find it easier to memorize basic multiplication and division facts if they have some insight into the symmetrical relationships evident on the multiplication chart. When they place a mirror along the diagonal, they discover that this diagonal is a line of reflection. Matching squares about this diagonal contain the same numbers. They are excited by the fact that the numbers on the mirror line are special kinds of rectangular numbers - square numbers. They can find patterns in the way in which the digits in the ones and in the tens in each column change. Some children will discover a pattern when they add the digits in each product in a column or a row.

X	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Many children experience their first feeling of pleasure and success in mathematics through working with geometric concepts. Let us not deny them these experiences by again overemphasizing the basic arithmetical skills to the detriment of the even more basic geometric ones.

1. **The Formative Years** Ontario Ministry of Education Circular P1J1 1975

2. **Geometry: Kindergarten to Grade 13** Ontario Institute for Studies in Education 1967

3. **Education in the Primary and Junior Years** Ontario Ministry of Education 1975 page 74

GEOMETRY IS OUR WORLD

LET'S TEACH IT THAT WAY

David and Sharon McPhail

In the last several years we have seen a greater emphasis on areas in math other than computation. Geometry, for example, has been brought into the primary grades. With the advent of our new provincial document, we see that Geometry is continuing to be emphasized as an important part of the young child's education.

Even with this emphasis, Geometry is one area of Mathematics that is often most neglected. Being a relatively new area for the primary teacher, a large proportion of the teachers feel inadequately prepared to teach the concepts and to teach them well.

Our new document tells us, "Geometry is the natural language of spacial concepts and space relationships in the real world". There are two determinants of intellectual growth; a completely necessary innate potential and a completely necessary stimulating environment. Since these statements are completely supported by prolonged research, we suggest that our attention as teachers should be directed to the importance of the environment.

To the child, Geometry is an exciting study. Children are essentially interested in finding out new things about their environment and through the use of Geometry we as teachers can accomodate and nurture this interest.

The young child has familiarized himself with many of the solid shapes before he has even entered school. He has played with spheres (balls), cubes (his building blocks), cylinders (tin cans), and other solids. Through his play, he has discovered some properties before he has received any formal education.

In the beginning stages of teaching geometry, the young child needs time to explore the physical models through further play situations. It is through the manipulation of things like cans, paper cones, etc. that the young child will investigate and identify significant geometric concepts.

Upon entering school, though, the child's geometry is often taken out of the real world; he is given sets of shapes to handle with no relation to the outside world. Geometry provides a means for children to analyze the geometric properties of their physical environment. Our new document suggests, "Activities with physical objects permit children to investigate and identify significant geometric concepts.". Let us remember that the statement does not mean that we only use plastic shapes.

The link between Geometry and our world affords us an excellent opportunity to stress the skills of observation. We as educators, can continue to increase a child's awareness of geometric shapes in his environment. Our ever increasing creative playgrounds are a prime target for the study of Geometry; the spaces, surfaces, and equipment create an environment for learning. In these playgrounds, the child is using cylinders in the form of telephone poles, large electrical spools and often dome structures.

Geometry surrounds us. Children, when walking through any neighbourhood, will recognize rectangular prisms, triangular prisms and other solid shapes after having been made aware of the shapes. A look at the various traffic signs will produce more geometric shapes. To observe floor tiles will lead us into an investigation of tessellations.

The observations of children will often lead into further inquiries and through these inquiries a wealth of geometric concepts will unfold. The integration of language, through letter writing, will reveal those geometric shapes that are found to be best suited to building structures. The architect, in discussing the sphere, for possible man made structures, may introduce the young child to some more unusual structures (geodesic dome). Further letter writing could possibly lead into the reason for certain shapes being used for packaging of items in the supermarket.

In many instances, young children are being asked to relate Geometry to the environment by searching magazines to locate such things as rectangular prisms. This is extremely difficult for a young child to do since he has been taken from the third dimension. Far better a child use a polaroid camera to record the three dimensions he is able to see. Then, through the integration of math and language, he will be able to produce a booklet concerning those shapes. When looking back at these two dimensional pictures of the three dimensional world he will then be able to relate to their three dimensions more easily.

There will be a place for individual work and there will be a place for the whole class to work together because it will prove more valuable. Activity cards may be designed with the environment in mind.

Many excursions may be taken in relation to this topic. A visit to the industrial arts shop, to the local supermarket, downtown to look at structures will provide the class with many other avenues of investigation. Through involvement with people in the community the children will see the use of geometric shapes.

We must constantly bear in mind that we live in a three dimensional world and that we too often transpose this to the two dimensional world when it comes to teaching Geometry to elementary students. Although some may feel that teaching Geometry using the real world is not as valuable as other methods might be, much research has been developed to support this philosophy. Therefore, until we, as teachers, can show proof that the philosophy is incorrect, we should adhere to it.

Let's use the real world to its greatest advantage and make our students of today more aware citizens tomorrow.

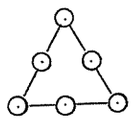
REFERENCES:

Day, Barbara, **Open Learning In Early Childhood,** MacMillan, New York 1975 Ministry of Education, **Education in the Primary and Junior Divisions** Toronto, 1975.

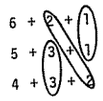
POLYGONAL NUMBERGRAMS

J. Norman C. Sharp, Co-ordinator of Mathematics, Borough of Etobicoke Board of Education

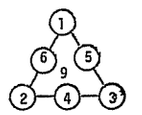
We often look for drill activities that will help to develop numerical computation skills. If these activities have a change of format and offer a challenge, they are usually accepted and used. This activity developed from a simple beginning - use the six digits, 1, 2, 3, 4, 5, 6, to fill the six circles forming one equilateral triangle so that the sum along each side is 9.



To begin with, the student starts by guessing or trial and error. After some consideration you wonder how many ways can I get the sum 9 by using three of the digits? These are:



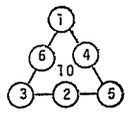
There are no others - remember, a digit must not be repeated in a given sum. This array tells us that the corner numbers are 1, 2, 3, since these are the repeated numbers.



ΣV = 6

1+2+3

The next exercise is to obtain a sum (10) along the side. By the same type of analysis, the numbers can be inserted as shown in the diagram.

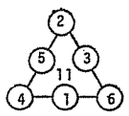


ΣV = 9

1+3+5

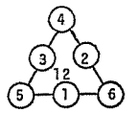
Can this continue indefinitely or is there a limit to the sum along each side? A chart showing the sum along the side with the sum of the numbers at the vertices reveals an interesting sequence and also that the maximum vertex sum is 4+5+6=15.

Sum along the sides	9	10	11	12
Sum of the vertices	6	9	12	15



ΣV = 12

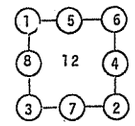
2+4+6



ΣV = 15

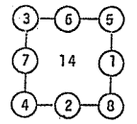
4+5+6

To find extensions to the triangle, I then form a square with the numbers 1,2,3,4,5,6,7,8 to be inserted in the spaces. The diagrams give the sum along the sides as well as the sum at the vertices.



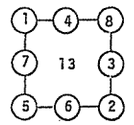
ΣV = 12

1+2+3+6



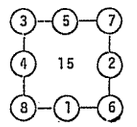
ΣV = 20

3+4+5+8



ΣV = 16

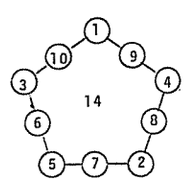
1+2+5+8



ΣV = 24

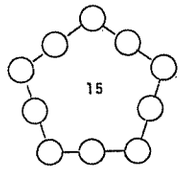
3+6+7+8

The remaining diagrams show the extension to 5,6,7, and 8 sided figures extending the numbers used with each extension. You will notice many number patterns have developed and you may wish to extend this beyond the octagon. There is one bit of unfinished business and that occurs in the pentagon. I have not been able to find the arrangements of 1,2,3,4,5,6,7,8,9,10 to yield a sum of 15 nor 18 along the sides. Maybe you can find it or determine a reason for its non-existence.

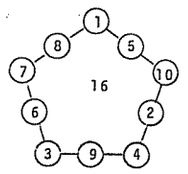


ΣV = 15

1+2+3+4+5

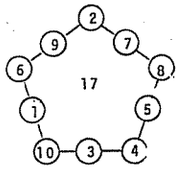


ΣV = 20



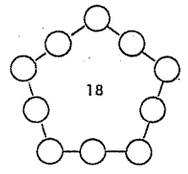
ΣV = 25

1+3+4+7+10

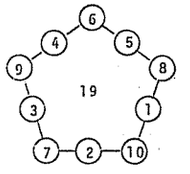


ΣV = 30

2+4+6+8+10

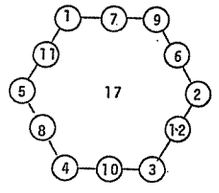


ΣV = 35



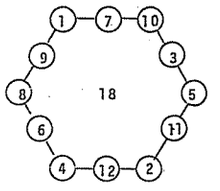
ΣV = 40

6+7+8+9+10



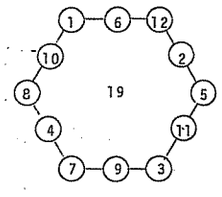
ΣV = 24

1+2+3+4+5+9



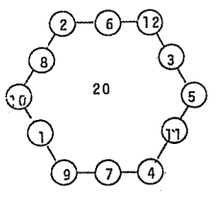
ΣV = 30

1+2+4+5+8+10



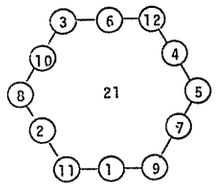
ΣV = 36

1+3+5+7+8+12



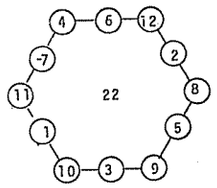
ΣV = 42

2+4+5+9+10+12



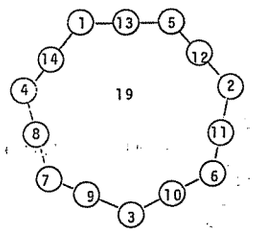
ΣV = 48

3+5+8+9+11+12



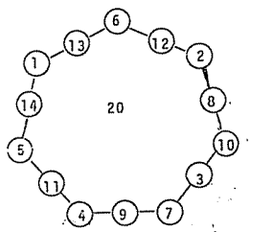
ΣV = 54

4+8+9+10+11+12



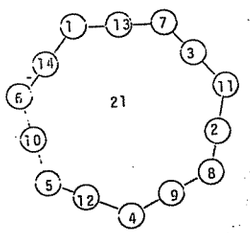
ΣV = 1+2+3+4+5+6+7

= 28



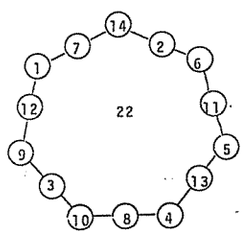
ΣV = 1+2+4+5+6+7+10

= 35



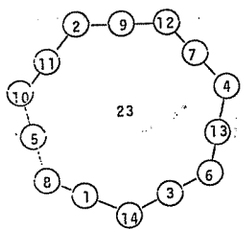
ΣV = 1+4+5+6+7+8+11

= 42



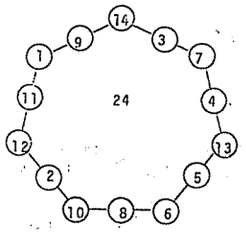
ΣV = 1+4+5+6+9+10+14

= 49



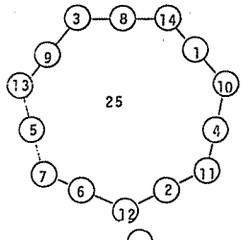
ΣV = 2+4+6+8+10+12+14

= 56



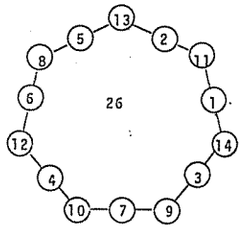
ΣV = 1+6+7+10+12+13+14

= 63



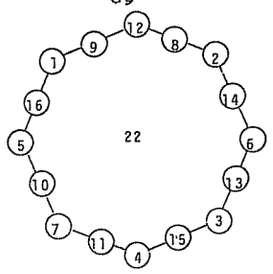
ΣV = 3+7+10+11+12+13+14

= 70



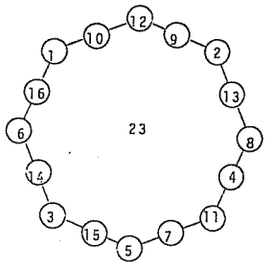
ΣV = 8+9+10+11+12+13+14

= 77



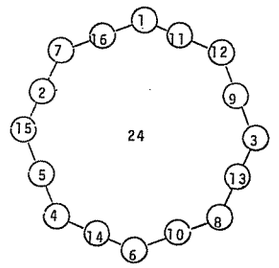
ΣV = 1+2+3+4+5+6+7+12

= 40

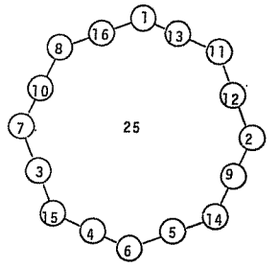


ΣV = 1+2+3+5+6+8+11+12

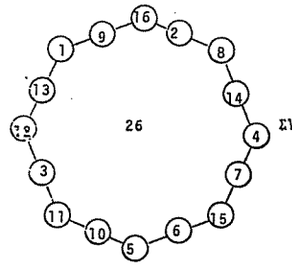
= 48



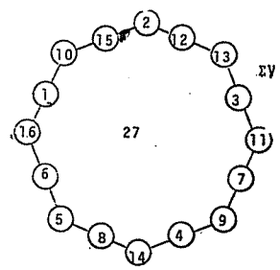
ΣV = 1+3+4+6+7+8+12+15
= 56



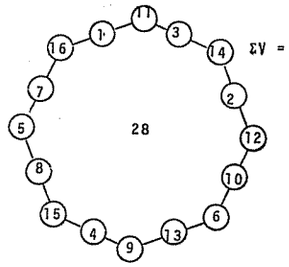
ΣV = 1+2+6+7+8+11+14+15
= 64



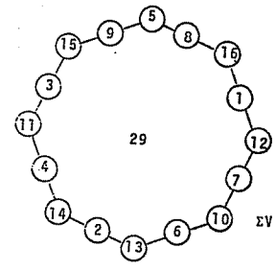
ΣV = 1+4+5+8+11+12+15+16
= 72



ΣV = 2+5+9+10+11+13+14+16
= 80



ΣV = 5+6+9+11+12+14+15+16
= 88



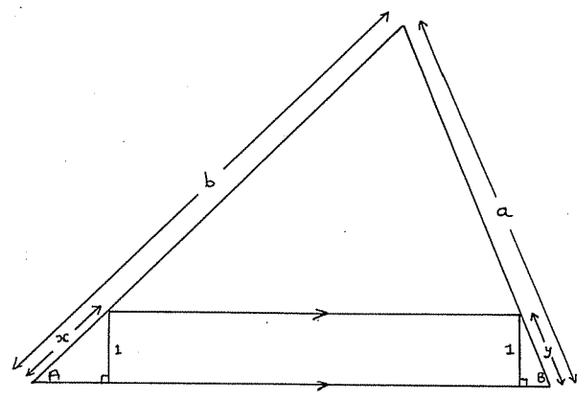
ΣV = 5+10+11+12+13+14+15+16
= 96

AN ALTERNATIVE DEDUCTION OF THE SINE LAW

by Peter Weygang, Levack High School

Most students are familiar with some form of the parallel proportion theorem for a triangle - such as the 'side-splitting' theorem in 'Elements of Mathematics #3; p. 109.

This theorem can be used in a derivation of the Sine Law, as follows:



In the diagram a line is constructed parallel to the base of the triangle, and at a perpendicular distance of 1 unit.

We have $\frac{b}{x} = \frac{a}{y}$ parallel proportion theorem

$\therefore$ $by = ax$

But $x = \frac{1}{\sin A}$ and $y = \frac{1}{\sin B}$

Substitute for x and y

Thus $\frac{b}{\sin B} = \frac{a}{\sin A}$

P. Weygang
Levack High School

TABULAR METHODS IN RECTILINEAR MOTION PROBLEMS

by William J. Bruce, Department of Mathematics, University of Alberta

Dr. Bruce is a regular contributor to Delta-K.

Consider the problem in which a particle moves horizontally on a straight line such that its position or displacement at any time t is given by

$$s(t) = t^3 - 3t^2 - 24t$$

and that the motion of the particle is to be discussed or described.

Upon differentiation, it is found that the velocity, v(t), of the particle is given by

$$s'(t) = 3t^2 - 6t - 24,$$

which factors to yield

$$s'(t) = 3(t+2)(t-4).$$

There are two numbers in the domain of s(t) for which s'(t) = 0, namely -2 and 4. These critical numbers divide the domain of s(t) into subsets and tabulation can be done as follows:

Time	$t < -2$	$-2 < t < 4$	$4 < t$
Sign $s'(t) = v(t)$	+	-	+
s(t)	Increasing	Decreasing	Increasing
Direction of motion	$\longrightarrow$	$\longleftarrow$	$\longrightarrow$

This table indicates that s(-2) = 28 is the relative maximum displacement, whereas, s(4) = -80 is the relative

minimum displacement.

A second differentiation shows that the acceleration, a(t), is given by

$$s''(t) = 6t - 6,$$

which factors to give

$$s''(t) = 6(t - 1).$$

Note that, since $s'(t) = v(t)$, it follows $s''(t) = v'(t)$.

Now there is only one number in the domain of s(t) for which $s''(t) = 0$, namely 1. This critical number divides the domain of s(t) into subsets and tabulation can be done again. The following is a suitable format.

Time	$t < 1$	$1 < t$
Sign $s''(t) = a(t)$	-	+
Velocity, v(t)	Decreasing	Increasing

This table shows the interpretation to be placed on the signs of a(t) relative to the nature of the velocity in the sub-intervals.

The above tables can be used now to make a summary tabulation of the motion of the particle. The format might be

Time	$t < -2$	$-2 < t < 1$	$1 < t < 4$	$4 < t$
Direction of Motion	$\longrightarrow$	$\longleftarrow$	$\longleftarrow$	$\longrightarrow$
Velocity	Decreasing	Decreasing	Increasing	Increasing
Speed	Decreasing	Increasing	Decreasing	Increasing

Recall that speed is the absolute value of the velocity. Now the motion can be described in words directly from this table.

Prior to $t = -2$, the particle is moving to the right but is slowing down. At $t = -2$ it momentarily stops, changes direction, moves to the left, and speeds up. At $t = 1$ it ceases to speed up and then continues to the left while slowing down until at $t = 4$ it momentarily stops again. It changes direction, moves to the right, and continues to speed up.

Note that the first time the particle changes direction it is 28 units to the right of the zero reference point. The second time it changes direction it is 80 units to the left of this reference point.

Certainly the tabular presentation is very concise and quite adequate as a description of the motion. However, should a verbal discussion be desired, this can be precisely stated much more easily with reference to the tabular forms. Furthermore, graphical work can be done very quickly, with a minimum of plotting, from the tabulated information.

TESTING THEORY

W. Longworth, Principal, Willowfield Gardens Public School, Scarborough

Pupil evaluation is one of the main tasks of the teaching process. Since raw test scores taken by themselves have little significance, the ability to analyze data is one of the basic requirements in effective measurement and evaluation. The meaning of raw scores can only be extrapolated with reference to some standard or norm. But what is the standard?

Some have said that measurement by tests is like measuring with a rubber band - just how hard do you stretch the band? The objective in the following is to develop a sound rationale for the setting of realistic standards of achievement and then allowing for the drawing of consistent interpretations of the meanings of raw test scores. Such a system would allow the building in of more uniformity in pupil evaluation from teacher to teacher so that a more consistent pattern of results could be reported to parents from year to year. The following is based on the assumption that class achievement is a more stable indicator of what the level of achievement should be than is teacher judgement about what the level of achievement should be. This is, of course, the methodology used in developing all standardized tests. The test is written and then administered to large groups of people to establish the levels at which people of various ages achieve. This average achievement is then used to define the standard expected for people of that age and those above or below the standard are graded with reference to their position relative to this norm. Position relative to the norm is determined with reference to the standard deviation of the test to take into consideration the varying spread of test scores about the norm for different tests.

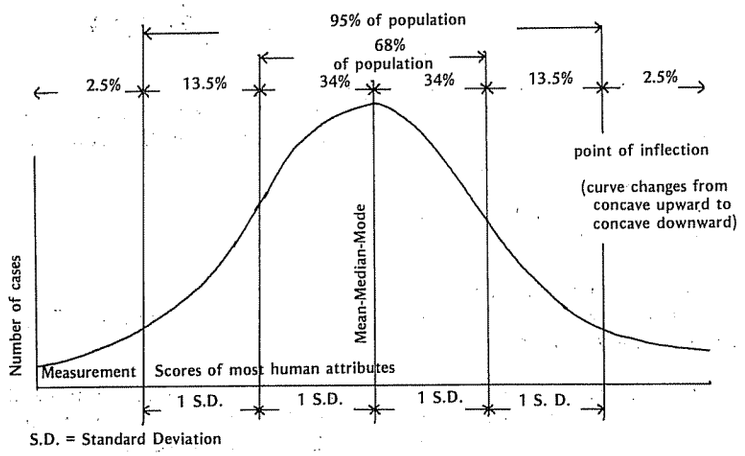
GRADING OF PUPILS

Judgments about pupil progress are relative to many factors - the class, the teacher's standards, etc. Individual standards are good, but we must present a uniform picture of a child's progress from year to year and from teacher to teacher. It is difficult to justify one teacher's "A" rating being equal to another's "B" or "C" rating.

In the interests of uniformity, it is desirable then that teachers generally attack the assignment of grades with similar methodologies.

The Normal Curve - The Bell Curve

The Bell Curve is a mathematically defined frequency distribution that will be found to be true for an infinite number of observations of most human traits. It may form the rationale for our grading practices.



If we assume that our class is a random sample of the total population (not a supportable hypothesis, but close enough for our purposes!) - we can use the theory of the Bell Curve to give rationale to our grading practices. Certainly, there is a built-in error in using this assumption, but every attempt to measure human attributes, no matter how finely planned, results in an error of measurement. It is possible to compute the possible range of the error, but all that is important for us is to realize that it exists. Using this concept, for example, we realize that the real I.Q. is only within $\pm$ 10 points of the score attained on most intelligence tests.

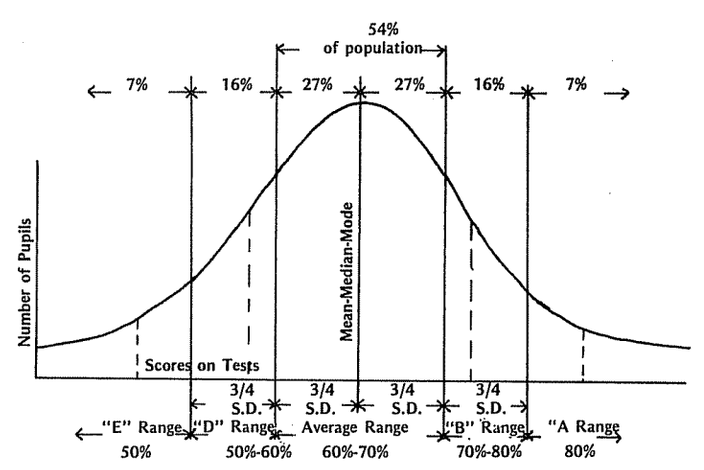
In summary, we must realize that although the science of testing is exact the results are not. We realize that any score, to be realistic, bears a great deal of **interpretation** which must be based on insight of what you are really doing in the testing process.

A suggested methodology for Grading Tests

1. Arrange the scores from high to low and draw a frequency distribution.
2. Compute the average (mean) - This shall be the reference point for grading all pupils. This point is equated with the average mark desired (62-64% in most jurisdictions).
3. Estimate the Standard Deviation using the formula:

$$\frac{\text{Sum of top sixth scores - Sum of bottom sixth scores}}{\text{½ number of students taking test}}$$

4. **Consider:**
   - "A" students to be 1½ Standard Deviations above average
   - "B" students to be ¾ Standard Deviation above average
   - "C" students to range from ¾ Standard Deviation below mean to ¾ Standard Deviation above mean
   - "D" students to be less than ¾ Standard Deviation below mean.
   - "E" students to be less than 1½ Standard Deviations below mean.



5. Compute the parameters for each mark range.
6. Mark these mark ranges on your frequency distribution, but be flexible enough to vary these parameters so as to let them fall between natural breaks in your distribution if possible.
7. **Consider:**
   - "A" students to be in 80 - 100 range
   - "B" students to be in 70 - 80 range
   - "C" students to be in 60 - 70 range
   - "D" students to be in 50 - 60 range
   - "E" students to be less than 50
8. Assign number marks to students falling within each grade range depending on the location of the score within the range.

It is important to note that the above analysis is most suitable for tests to be used for reporting purposes. There are other forms of analysis more suitable for tests having other objectives such as diagnosis. In this case, results can be diagnosed quickly by a show of hands.

Relatedness between scores achieved on different tests

If test scores are going to be compared, the means and the indices of discrimination should be the same. Although not technically accurate, we may compare scores if the test averages are similar. For greatest accuracy, standard scores (z scores) must be computed. These can be found by dividing the distance of the score from the mean by the average variability of the scores (Standard Distribution of the distribution). We will not usually find a need for analysis at this level so that we can overlook this particular concept. We must, however, be wary about comparing results from different tests.

In Conclusion

A profitable analysis of items by a show of hands can be done for all tests so that individual test items can be validated for use in future years. This information can be written on the back of a file card on which the question has been written.

1. **Level of Difficulty Index** - simply the percentage of students that got the question right.

2. **Discrimination Index** - the degree to which the item sorted out the superior and inferior students. (Number in top half of class getting item correct minus number in bottom half of class getting item correct - discard the papers falling on the average.)

STEPS IN CONSTRUCTING A TEST

1. List of objectives for which test is to measure.
   - Many objectives for testing (Diagnosis, pupil ranking for reporting purposes, maintenance of standards, educational tool for the pupil, evaluation of lesson, motivation of pupils, etc.)

   Objectives can be specific as well as general:
   1. How well does the pupil round off numbers?
   2. How well does a pupil estimate answers, etc.?

   - All objectives must be determined before a test can be designed. The objectives must be realistic for the grade level and provide some success to all pupils.

   - Test objectives must be compatible. It would be hard to visualize a good diagnostic test being also a good power test to be used for the grading of pupils.

2. Gather together the content covered in the instruction.

3. Prepare plan for test.
   - Good testing practices reinforce the major learning objectives the teacher had when she set about to teach the material.

   - Utilize a variety of question types in your testing program. (Multiple choice, short answer, fill in blanks, essay, matching, true-false, etc.)

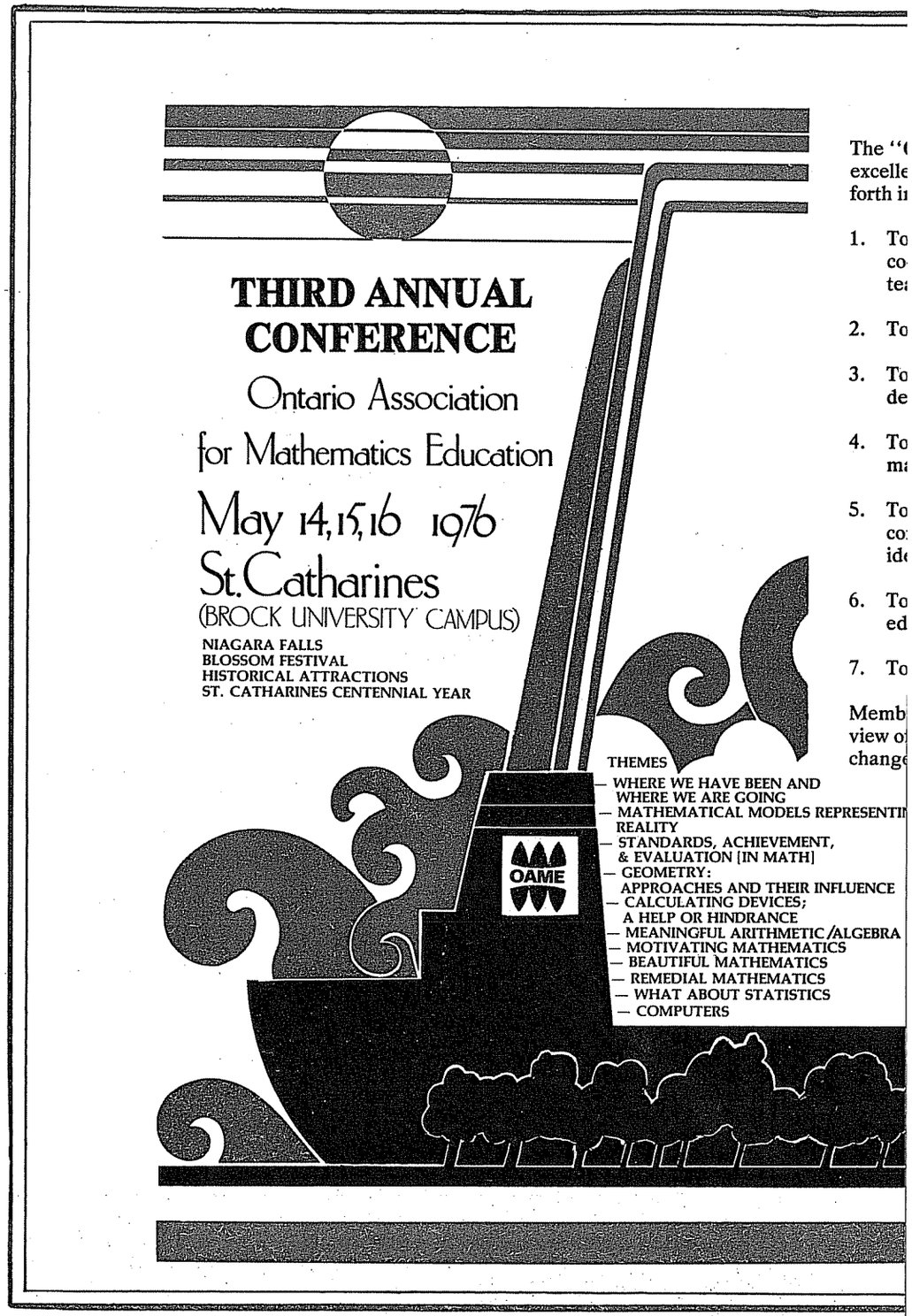
4. Draft the items in preliminary form.
   - Decide kinds of questions - write 25% to 50% more items than will be used - write on 3'' x 5'' file cards so as to easily reshuffle. Discard the weaker items.

5. Decide length of test.
   - 80% to 90% of pupils should complete in allotted time.

   - The greater the number of question, the less the error of measurement.
     e.g. 10 questions, each accounts for 10% of the mark;
     50 questions, each accounts for 2% of the mark.

6. Edit and select the final items.

7. Rate the items for difficulty (1 - 5 scale? teacher judgment)
   - Items should be analyzed for their discriminatory characteristics - there should be easier



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WHY OAME IS OF INTEREST TO TEACHERS OF MATHEMATICS AT ALL LEVELS

ıssociation for Mathematics Education'' (OAME) was founded in 1972 to promote ıathematics education in Ontario. The specific objectives of the association, as set ME constitution are:

: professional growth of teachers of mathematics by proposing, initiating, or ıg in pre-service and in-service programs designed to develop and promote effective ethods.

: mathematics programs.

ıge and to engage in research relating to mathematics education and curriculum

and to encourage the planned development and evaluation of experimental teaching

: co-ordination in mathematics education at all levels as well as effective ıtion among all mathematics educators in Ontario by promoting the exchange of :gh publications, conferences, and personal involvement.

: liaison with other educational organizations in Ontario and with mathematics ›rganizations in other provinces and countries.

n a representative mathematics educator's ''voice'' in Ontario.

open to all individuals or organizations interested in mathematics education. In ent demands for relevance and public accountability in education, and of the rapid ıre imposing on all levels of the teaching of mathematics, we urge you to attend.

For further information please write to

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L2S 3A1

items to provide success for the weaker students, and harder items to challenge the best of students. The test should be composed of enough sufficiently easy items so as to give success to a large majority of the students.

- Sometimes items are included which do not discriminate between those scoring highest and those scoring lowest on the test. In this case, the question should be examined for ambiguity, or responses made by better students might be found to be alternative right answers that the teacher had not anticipated.

8. Rearrange items in order of difficulty.
   - Usually tests should be designed so that easy questions are at the beginning and harder questions are at the end of the test. Some teachers start tests with a question sufficiently easy to insure a correct response by everyone for motivational purposes only. No other item should be included that results in all correct, or all incorrect, responses in your class.

9. Prepare administrative instructions - timed? scoring? examples? budgeting time?, etc.

10. Decide rules for scoring and prepare answer Key.

11. Produce the test.
    - In scoring subjective answers, read entire response and then re-read, assigning marks as previously planned - grade one item on all papers before proceeding to next question.

Class _____ **TEST ANALYSIS SHEET** Name of Test _____

Teacher _____ Date of Test _____

Raw Score	Number of Cases	Gr. Rang	Rev. Gr. Rang.	Mark

NUMBER OF STUDENTS _____

MEASURES OF CENTRAL TENDENCY

1. Median Score (score of middle student, approximate average) _____
2. Mean Score (average score) _____
3. Mode (most frequently earned score) _____

MEASURES OF DISCRIMINATION OR VARIABILITY OF TEST

1. Range (highest score - lowest score) _____
2. Estimated Standard Deviation (S.D.)
   Most Stable measure of variability
   - average distance of all scores from the mean.
   - measure of variability about the mean.
   - widely used in research.

$$\frac{\text{Sum top 1/6 - Sum bottom 1/6}}{\text{1/3 number of students}}$$

COMPUTATION OF GRADE RANGES

A Range Mean + 1½ S.D.= ≥ _____

B Range Mean + ¾ S.D.= ≥ _____ <A

C Range - portion not defined by other ranges.

D Range Mean - ¾ S.D.= < _____ >B

E Range Mean - 1½ S.D.= < _____

NOTE: Grade areas - determined by Standard Deviation and mean only.
Revised grade areas - take into account natural breaks in distribution.

INTERNATIONAL MATHEMATICS EDUCATION

THIRD INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION

Karlsruhe, Federal Republic of Germany, will be the site of the Third International Congress on Mathematical Education (**ICME**), 16-21 August, 1976. The Congress is being organized by the West German Subcommittee of **ICME** in collaboration with the Local Organizing Committee. The program will include plenary sessions, survey reports, panel discussions, special projects, and commercial exhibitions. Further information and application forms may be obtained from Professor H. Kunle, 3s' International Congress on Mathematical Education, University, d 75 Karlsruhe, Federal Republic of Germany.

N.C.T.M. ANNUAL MEETING

The next annual meeting of the National Council of Teachers of Mathematics will be held in Atlanta, Georgia, April 21 - 24, 1976. An outstanding program is being planned, featuring the largest number sessions ever planned by N.C.T.M. There will be 425 sections, with over 500 participants, representing 47 states and several foreign countries. Plan to attend!

N.C.T.M. NAME-OF-SITE MEETING - WINNIPEG

In case you haven't heard, Winnipeg will be hosting a N.C.T.M. Name-of-Site Meeting October 21-23, 1976. For further information contact Roy Spivak, Program Chairman, Grant Park High School, 450 Nathaniel Street, Winnipeg, Manitoba REM 3E3. Phone 204-452-3112.

EIGHTH CANADIAN MATHEMATICAL OLYMPIAD - 1976

Patron: The Governor-General of Canada

Date of Examination

The Eighth Olympiad will be written on Wednesday, May 5, 1976 from 9 a.m. to noon at schools throughout Canada. The examination papers will be marked in May and the results will be circulated to schools as soon as possible thereafter.

Conditions of Examination

All examination materials, whether submitted for marking or otherwise, are the property of the Canadian Mathematical Congress. Decisions of the examiners are final. Cash prizes and scholarships may be awarded at the discretion of the examiners, who may choose to subdivide or otherwise alter the terms of awards.

Eligibility of Candidates

Candidates must be Canadian citizens or landed immigrants, and must be registered in a high school or its equivalent, as follows:

(a) in British Columbia, the Yukon, Alberta, the Northwest Territories, Saskatchewan, Manitoba, Nova Scotia, New Brunswick, the competition is open to students in Grade XII or earlier;

(b) in Ontario the competition is open to students in Grade XIII or earlier;

(c) in Quebec the competition is open to students in CEGEP-1 or earlier;

(d) in Prince Edward Island and Newfoundland the competition is open to students in Grade XII, first year college or university, or earlier.

Nothing in the foregoing is intended to preclude the student's enrolment in advanced mathematics courses, at his own institution or elsewhere.

Nomination of Candidates

All candidates of the Olympiad must be nominated by the chairman of a provincial competition or by a school principal. Copies of nomination forms can be obtained from:

Professor G.H.M. Thomas,
Chairman, Organizing Committee,
Department of Mathematics,
University of Saskatchewan,
SASKATOON, Saskatchewan S7N OWO.

Provincial chairmen may nominate candidates up to the quota limits stated below.

	Quota limit
Newfoundland	5
Nova Scotia	10
New Brunswick	10
Prince Edward Island	5
Quebec	50
Ontario	50
Manitoba	10
Saskatchewan	10
Alberta & the Northwest Territories	16
British Columbia and the Yukon	21

Ontario Chairman: Prof. K. D. Fryer, University of Waterloo.

School principals may nominate not more than three candidates who are unable to participate in a provincial contest. Each nomination by a school principal must be accompanied by an entry fee of $5.

The First Five Olympiads

A booklet containing the questions of the first five Olympiads and solutions to the questions and the names of the prizewinners has been prepared and may be obtained from:

Secretariat,
Canadian Mathematical Congress,
3421 Drummond Street,
MONTREAL, Quebec H3G 1X7

at a cost of $2, including postage.

Prizes

FIRST PRIZE	-	**$1,500**
SECOND PRIZE	-	**$1,000**
THIRD PRIZE	-	**$500**
FOURTH PRIZES	-	**5 prizes of $200**
FIFTH PRIZES	-	**10 prizes of $100**

A few additional prizes may be awarded to those submitting especially elegant solutions to one or more problems.

A certificate of honourable mention is presented to approximately 25 candidates whose rank in the examination follows the 18 prizewinners.

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FOR
MATHEMATICS EDUCATION

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SIX CHAPTERS OF OAME

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OAME LEADERSHIP SEMINAR '75
LOOKING BACK

Another successful Leadership Seminar in Mathematics for Elementary Teachers (K-8) was held at the Park Motor Hotel in Niagara Falls on October 22-25, 1975. Approximately 100 participants from 45 Boards of Education across the province listened to the provocative keynote speakers and after each address broke into groups to discuss the salient points from their talks. Generally, the discussions focused on curriculum development, pedagogy, and classroom management. The four speakers whose topics were discussed were:

Jim MacLean, Education Officer, Ontario Ministry of Education
Topic: The Mathematics of "The Formative Years"

Gerald Rising, Professor, State University of New York at Buffalo
Topic: Curriculum Development

Ada Schermann, Assistant Professor, Institute of Child Study, Toronto
Topic: Child Development

Dan Wentworth, Science Consultant, Hamilton Board of Education
Topic: Workshop Skills

This format provided the participants with an opportunity not only to share their thoughts among themselves but also discuss the presentation with the speaker when he visited each of the smaller groups.

The format of the final day of the seminar was somewhat different in that the participants had a choice of attending one of three mathematical content sessions at the Primary, Junior, or Intermediate levels plus a session on the various resources available to aid the teachers in their mathematics teaching. The speakers for these sessions were:

Shirley McIntyre, Consultant, East York Board of Education
Shirley Robinson, former Consultant, Lincoln County Board of Education
Bob Robinson, Consultant, Hamilton Board of Education
Jim Tennant, Co-ordinator, Frontenac County Board of Education

Throughout the three days, the teachers welcomed the opportunity to visit and discuss mutual concerns with the representatives of the many publishers, manufacturers, and distributors of mathematics materials.

The seminar concluded with a luncheon and address by Joan Routledge, President of OAME who passed on best wishes to the teachers when they return to their school systems to provide the leadership and help in mathematics education that so many are seeking.

Appreciation for the success of Leadership Seminar '75 is extended to the two co-ordinators:

Bruce Jay, former Consultant, Niagara South Board of Education
Morley MacGregor, Secretary-Treasurer, OAME

who were ably assisted by the following staff:

Anne Hummel, Lambton County Roman Catholic Separate School Board, Sarnia
Jack LeSage, formerly Ministry of Education, Toronto
Mike McKenna, Ministry of Education, St. Catharines
Tom Mousseau, Niagara South Board of Education,
Bob Robinson, Hamilton Board of Education,
Pat Willis, Lincoln County Board of Education,
Mona MacGregor, Secretary of the course.

LOOKING AHEAD TO THE OAME LEADERSHIP SEMINAR '76

Based on the results of the questionnaire submitted by the participants at the '75 Seminar this year's seminar plans to be bigger and better than ever. Co-ordinated by Morley MacGregor and Bruce Jay and their able staff a new innovation, "21 Monday Morning Specials" has been added. These sessions will deal directly with the teaching of specific mathematical subject matter. The format has also been changed and the registration fee which covers accommodation, meals and operating costs will remain unchanged at $100. The location has also been switched to the Ascot Inn in Toronto and will be held October 20-23, 1976. Special discussion periods have been arranged and the following speakers have been asked to make presentations.

Leroy Callahan, State University of New York at Buffalo

Art Dayman, Ministry of Education,

Don Fraser, Faculty of Education, University of Toronto

Bob Robinson, Hamilton Board of Education

Letters have been sent to Directors of Education for each board in Ontario describing Leadership Seminar '76.

For more details contact: Mr. Morley MacGregor Sec. Treas. OAME
247 Bright St.,
SARNIA, Ont. N7T 4E9

BOOK REVIEWS

MATHEMATICS LEARNING IN EARLY CHILDHOOD
PAYNE, JOSEPH N. (ED.)
Reston, Va.: National Council of Teachers of Mathematics, 1975
297 pp., $13.00

This colourful, large-format publication is the thirty-seventh yearbook of the NCTM and under present editorial policies one of the last 'single-theme' yearbooks to be brought out by the Council. It is a look at the state of the art of mathematics learning in children between the ages of three and eight.

The book consists of twelve chapters each of which deals with some aspect of early childhood mathematics learning. Six of the sections deal with mathematical content. These focus on; Number and Numeration, Operations on Whole Numbers, Fractional Numbers, Geometry, Measurement, Relations Number Sentences and Other Topics. The other six chapters survey the areas of Learning and Cognition, The Curriculum, Research on Mathematics Learning, Problem Solving, Experiences for Young Children and Directions of Curricular Change.

The intention of the editor was to produce a book which would help teachers of young children make informed decisions about mathematics teaching and curriculum. The book was to be both a source of knowledge and a source of activities in which three themes were stressed; problem solving, relating mathematics to the real world of the child and a developmental point of view toward learning.

It would seem that these aims have been fairly successfully met. The chapters, as one might expect, vary in quality and in the extent to which they are consistent with the three general themes. The chapter on Research on Mathematics Learning by Suydam and Weaver, for example, is well-written and organized around basic questions a classroom teacher might ask, such as, 'What

types of manipulative materials have been found to be effective?'. Such a survey cannot, however, be better than the related research in question and this, to put it charitably, has not always been of uniformly high quality. Publication lag time has meant that the survey is already rather dated. Most of the content chapters give very useful suggestions as to how teachers might structure effective learning situations for their pupils. The Chapter on Problem Solving by Nelson and Kirkpatrick of the University of Alberta is outstanding. They characterize 'good problems' and give a number of examples which illustrate their basic premise that, "The central purpose of any instruction in mathematics at the early childhood level is to help the child see order and meaning in the situations and events that occur in his day-to-day activities." The preliminary research on which this chapter was based showed that young children were capable of solving some surprisingly difficult problems if the problems were structured properly. The final results of this research should prove to be most interesting.

The initial two chapters of the book do not give a strong introduction to, or basis for, the work as a whole. The second chapter, The Curriculum, by Trafton may just be not particularly useful, but the first chapter by Stevenson, Learning and Cognition, seems in places to be quite out of step with the rest of the material in the book. Rather than a developmental viewpoint we seem to get a type of retreaded behaviourism. This shows itself in the emphasis given to terms like 'stimuli' 'remembering', 'components' and in the example of problems of the 'Mary is older than Jane' genre. It is unfortunate that the theoretical underpinnings for so much of the book are left so unclear. The work has a large number of photographs which are attractive and instructive and an equally large number of diagrams and sketches which are neither. Scores of faceless children and large pink 'a's on activity pages do little to improve the aesthetics of the book. On the whole, however, this publication is one which deserves the attention of anyone interested in pre-school and primary mathematics learning.

W.H.

APPLICATIONS IN MATHEMATICS COURSE A AND COURSE B TEACHER'S EDITIONS
JOHNSON, DONOVAN A. ET AL.
Glenview, Ill.: Scott, Foresman, 1972
400 pp. (approximately). Hardbound $6.52

These books are the bound versions of two series of booklets written for the non-academic student of secondary school mathematics. The student booklets in Course A are on the topics of Functions and Graphs, Relations and Geometry, Sampling and Statistics, Estimation and Measurement, Prediction and Probability, and Algebra. The corresponding units for Course B, each of which appears as a sixty-two page paperbound booklet costing $1.44, are on the topics of Computing Devices, Equations and Formulas, Number Explosions, Curves and Graphs, Earning and Spending, and Decision Making. Each booklet in turn is a subdivided into fourteen or fifteen lessons. The lessons usually incorporate several subsections of varying difficulty and an enrichment puzzle or activity section. Support material in the form of duplicating masters and overhead visuals is also available.

While this book series is unlikely to totally solve any school's curriculum construction problem for general mathematics courses, the individual booklets do make very fine 'outside' references. The teacher's editions, in particular, therefore, can be recommended as suitable purchases for the mathematics preparation room libraries.

W.H.

CONCEPTS OF MODERN MATHEMATICS
STEWART, IAN
Harmondsworth, U.K.: Penguin, 1975
Paperback, 315 pp., $3.50

The major problem with this book may well be that it has come ten years too late. The mid-sixties saw many authors labour to explain modern mathematics to parents and laymen. Many of these efforts were heavy on terminology, and light on mathematics and the mythical 'intelligent layman' might well have concluded after slugging through turgid material on bases and Venn diagrams that some sort of practical joke was being played on him.

How refreshing it is then to read this book, the effort of a man who obviously knows and loves contemporary mathematics. In the space of twenty short chapters Dr. Stewart is able to give significant insight into the nature and power of many branches of the discipline of mathematics. The 'new math' stalwarts such as binary notation and DeMorgan's laws are to be found here, but so are homotopy classes, Godel numbers, the eightfold way and catastrophe theory.

Many of the topics covered, such as functions, linear programming, and transformation geometry relate directly to secondary school mathematics curricula. Other material, such as the chapters on Linear Algebra and Counting: Finite and Infinite, could serve well as introductions to university-level courses.

Dr. Stewart is a member of the very active mathematics department at the University of Warwick and has been associated for some time with the lively undergraduate mathematics journal called **Manifold** which emanates from that institution. His writing style, perhaps influenced by his **Manifold** experience, is light and informal as is shown in the following passage where he comments on rigour in mathematics; "Practically everybody possesses some degree of mathematical intuition. A child solving a jig-saw puzzle has it. Anyone who has succeeded in packing the family's holiday luggage in the boot of the car has it. The main object in training mathematicians should be to develop their intuition into a controllable tool. Many pages have been expended on polemics in favour of rigour over intuition, or of intuition over rigour. Both extremes miss the point: the power of mathematics lies precisely in the combination of intuition **and** rigour. Controlled genius, inspired logic."

A SPECIAL NOTICE

To all holders of Membership Cards with expiry date August 31, 1976, OAME records indicate that 300 members purchased memberships prior to the closing of school in June 1975. All 1976-77 Members will receive a Membership Certificate 12" x 9" suitable for framing.

Be an Early Bird Member by using the form printed below and mail to:

Morley MacGregor
Secretary Treasurer OAME
247 Bright St., Sarnia, Ont.
N7T 4E9

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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Mr. ☐ Miss ☐ (Please print) NEW MEMBER ☐
APARTMENT No. _____ RENEWAL ☐

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TOWN OR CITY _____ POSTAL CODE _____

SCHOOL _____
Chapter Affiliation
(if applicable) _____

PLEASE ENCLOSE CHEQUE OR MONEY ORDER
FOR $10.00, PAYABLE TO OAME

The choice of examples to demonstrate mathematical results is excellent and the chapters are profusely and well illustrated. The ambitious scope of the book, given the space restrictions, means that no single section gets covered in depth. The work has, however, a very distinct sense of internal coherence. Dr. Stewart seems to have come close in this work to combining the mathematical range of Courant and Robbins' **What is Mathematics?** with the expository excellence of W. W. Sawyer. All in all, an outstanding effort.

WORKJOBS: ACTIVITY-CENTERED LEARNING FOR EARLY CHILDHOOD EDUCATION
LORTON, MARY BARATTA
Don Mills: Addison-Wesley, 1972
255 pp. $7.00 paperback.

This spiral-bound, illustrated, large format publication in the Addison-Wesley Innovative Series is a first class sourcebook for teachers in the early primary division. The book gives careful instructions to Kindergarten and Grade One teachers about the Materials, organization and implementation of one-hundred workjobs. Half of the workjobs deal with the development of language and the other half with the development of mathematics. At this level of instruction, however, as many primary teachers know, the two subjects are very closely related. **Workjobs** could in fact pass muster quite easily as a math book since the section groupings for language include Perception, Matching and Classification. The mathematics sections are on Sets, Number Sequences, Combining and Separating Groups, and Relationships.

Each workjob is laid out on two facing pages with at least one large photograph illustrating a child involved in the task and comments under the headings of Activity, Getting Started, Ideas for Follow-Up Discussion and Materials.

This publication has to be considered a must for teachers of Kindergarten and Grade One children. It is unfortunate that there are not more publications of this type available to the classroom teacher.

W.H.

TEACHING MATHEMATICS: A SOURCEBOOK OF AIDS, ACTIVITIES AND STRATEGIES
SOBEL, MAX A. AND MALETSKY, EVAN M.
Englewood Cliffs, N.J.: Prentice-Hall, 1975
240 pp. $12.70 ($6.50 paperback)

Max Sobel and his colleague from Montclair State College, Evan Maletsky, have collaborated to produce an attractive and useful sourcebook for teachers of intermediate mathematics. The titles of the six chapters of the book are: The Art of Teaching, Motivational Ideas, Recreational Activities, Laboratory Experiments, Classroom Aids and Models and Audiovisual Facilities, Resources and References.

The strongest feature of the book is its richness in items of high motivational interest. While few of the ideas are novel it is unusual to find so many of them between the covers of a single book. For example, the third section of the chapter on Recreational Mathematics is called Mathematics in Familiar Games and in it are suggestions for extracting mathematical concepts from Tic-tac-toe, Pool, Checkers and Chess, Poker, and Roulette.

In the first chapter the authors state and illustrate what they believe are "the essential ingredients of any mixture that is to produce artistry in teaching". Their 'axioms' are: Ask interesting and exciting question, make provisions for student discovery, make mathematics come alive, start or end the period with something spectacular, make effective use of multisensory aids. The chapter on Classroom Aids and Models contains many fine suggestions about aids for teaching arithmetic, geometry and algebra.

In a sense, the major strength of this book is also its major weakness. The large number of discrete ideas which make it a 'motivation mine' of considerable richness also contribute to making the book as a whole seem rather disjointed and somewhat superficial. Another factor which adds to the

disharmony is the attempt on the part of the authors to have this single book serve not as a sourcebook for practising teachers but also as a 'methods' text for mathematics education students. The authors would have been wise to produce two separate books for the two groups, for while there are a number of areas of overlap, the attempt to put two distinct books into one has meant that two potentially excellent publications have been sacrificed to produce one good one.

For mathematics education students it would be preferable to have more annotated items in the bibliography and also more discussion of the philosophical and psychological assumptions under which mathematics teachers might operate. For practising teachers it would be desirable to have many more of the activities followed through in depth and some indication given as to how some of the activities, as for example in the chapter on Laboratory Experiments, could be approached in a less convergent way.

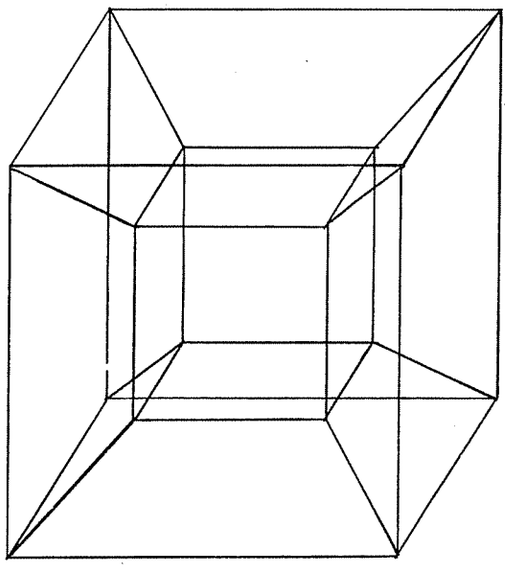
However, applying the 'bird in hand' principle we should accept this publication for what it is rather than for what it might have been. It is a very good book for the mathematics teacher to add to his departmental or personal library. The paperback edition is quite good value for money in these inflationary times.

W.H.

DIVERSIONS

by T. G. Friday

N DIMENSIONAL CUBES



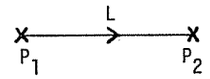
Arthur Clarke's first television appearance was a twenty minute discussion of the tesseract, or four dimensional cube, a diagram of which is drawn above. This fascinating model, although impossible in our three dimensional world, can be studied in a way which involves one of the most important techniques of investigation in mathematics. That is the building of an arithmetic model, the search for pattern and the extrapolation from the known to the unknown, or method of induction. This study, and the extension to n-dimensional "cubes", I have enjoyed and shared with my students at all levels in high school. I think that some of them even enjoyed the work.

The first step in this study is to identify "cubes" of zero, one, two and three dimensions, analyse them, and list their components. The "cube" of zero dimensions is a point. That of one dimension a line; of two dimensions a square and of three dimensions is a regular cube. The analysis consists of counting the number of points, lines, squares and cubes. The result of this is tabulated below.

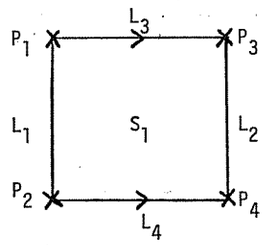
No. of Figure	Points	Lines	Squares	Cubes	Total	Dimensions
point	1				1	0
line	2	1			3	1
square	4	4	1		9	2
cube	8	12	6	1	27	3

Some very interesting patterns are immediately apparent. One is that the number of points is 2^n, and another is that the total number of components is 3^n, where n is the number of dimensions.

The second step of the study is to analyse the progression of one row of the table to the next. The line can be considered as a point moved. The square can be thought of as a line moved perpendicular to itself. The cube can be obtained by moving the square in a direction perpendicular to itself.



When the point P_1 is moved it generates a line L and another point P_2.



When the line L_1 is moved to line L_2 it generates the square S_1. Also point P_1, at one end of the line, when moved to P_3 generates line L_3. Similarly P_2 moving to P_4 generates L_4.

Hence in any move the number of points is doubled, since in each move every point is in its original position and in a final position. Also every moving point generates a new line. In addition to newly generated lines, the lines present in an original position are doubled by moving to a new position. A moving line also generates a square.

Therefore when we move from the square to the cube we start with 4 points and have 4 new ones at the end of the move, for a total of 8. Each of the 4 points generates a line when moved giving 4 new lines. In addition the number

of lines in the square is doubled to 8, 4 original and 4 new, giving the required total of 12 lines.

Now each of the 4 lines of the square when moved generates a square. Again the number of squares is also doubled to 2 for a total number of 6 squares.

So far by analysing our 0, 1 and 2 dimensional cubes and extrapolating to 3 dimensions we have achieved the expected result. The extrapolation from the cube to the tesseract follows exactly the same lines.

The number of points is doubled to 16 in the tesseract.

Each of the 8 points generates a line in moving, and the number of lines in the cube is doubled to 24, giving a total of 32 lines.

Each of the 12 lines generates a square in moving, and the number of squares is doubled from 6 to 12 giving a total of 24 squares.

Each of the 6 squares generates a cube in moving and the number of cubes is doubled by 2 giving a total of 8 cubes in our new figure, the tesseract.

This pattern can be continued to any number of dimensions, and the resultant extended table is shown below.

No. of Figure	Points	Lines	Squares	Cubes	tesseracts	5D	6D	Total	Dim.
point	1							1	0
line	2	1						3	1
square	4	4	1					9	2
cube	8	12	6	1				27	3
resseract	16	32	24	8	1			81	4
5D"cube"	32	80	80	40	10	1		243	5
6D"cube"	64	192	240	160	60	12	1	729	6

etc.

This table can also be generated by expanding $(2+1)^n$ in decreasing powers of 2, where n is again the number of dimensions.

For example $(2+1)^3=2^3+3.2^2+3.2+1$

$=8+12+6+1$ in the case of the cube.

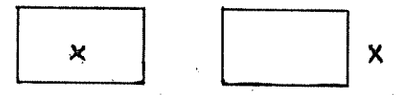
This would seem to be reasonable when we consider that to go from one dimension to the next we take 2x the number of a component present + 1x the new number of the same component. Hence the factor (2+1) appears quite naturally.

The total number of components 3^n present at each stage is also apparent from this expression.

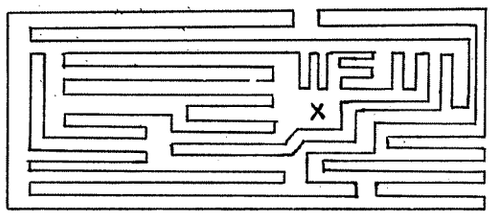
Further Reading: - And He Built a Crooked House
by Robert A Heinlein in an Anthology of Short Stories "Where Do We Go From Here" edited by Isaac Asimov
- Flatland by Abbott
- Sphereland by Abbott

PROBLEMS

14-3-1
Up to Grade 9



Fig(a) Fig(b)



Fig(c)

If x in Fig(a) is inside and in Fig(b) is outside, how can you tell if it is inside or outside in Fig(c) without tracing a path?

14-3-2
Up to Grade 12

a) What is the value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}$?

b) What is the sum of $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots$?

14-3-3

Into how many regions do n-planes divide space if no two planes are parallel, and no four intersect at a point?

The following trick, from Mathematical Recreations and Essays by Ball and Coxeter, illustrates how the trignometric identities might be used in the analysis of a number trick:

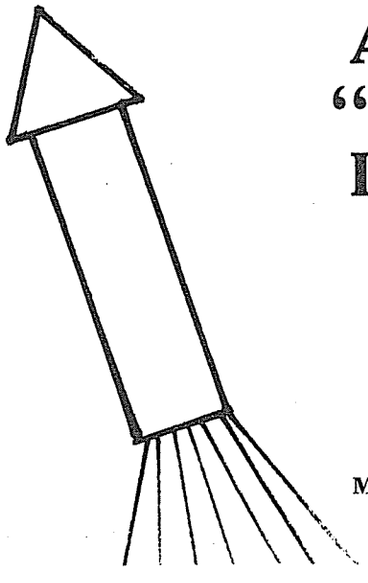
		Example	Algebraic Analysis	Trigonometric Analysis
1.	Write down any number greater than 1.	$\frac{5}{3}$	a	$\sec^2\theta$ ($\sec\theta = \sqrt{a}$)
2.	Form the reciprocal	$\frac{3}{5}$	$\frac{1}{a}$	$\cos^2\theta$
3.	Subtract from 1	$\frac{2}{5}$	$\frac{a-1}{a}$	$\sin^2\theta$
4.	Form the reciprocal	$\frac{5}{2}$	$\frac{a}{a-1}$	$\csc^2\theta$
5.	Subtract from 1	$-\frac{3}{2}$	$\frac{1}{1-a}$	$-\text{ctn}^2\theta$
6.	Form the reciprocal	$-\frac{2}{3}$	1-a	$-\tan^2\theta$
7.	Add the original number	1	1	1

Note: (i) Step 7 could again read 'Subtract from 1' and the result will be the original number.

(ii) If in Step 1 the restriction 'greater than 1' is removed, what effect will this have on the trigonometric analysis?

G. B. Lawson, U.W.O.

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MORE SOMA

Robert Addison, Student, North Lambton Secondary School

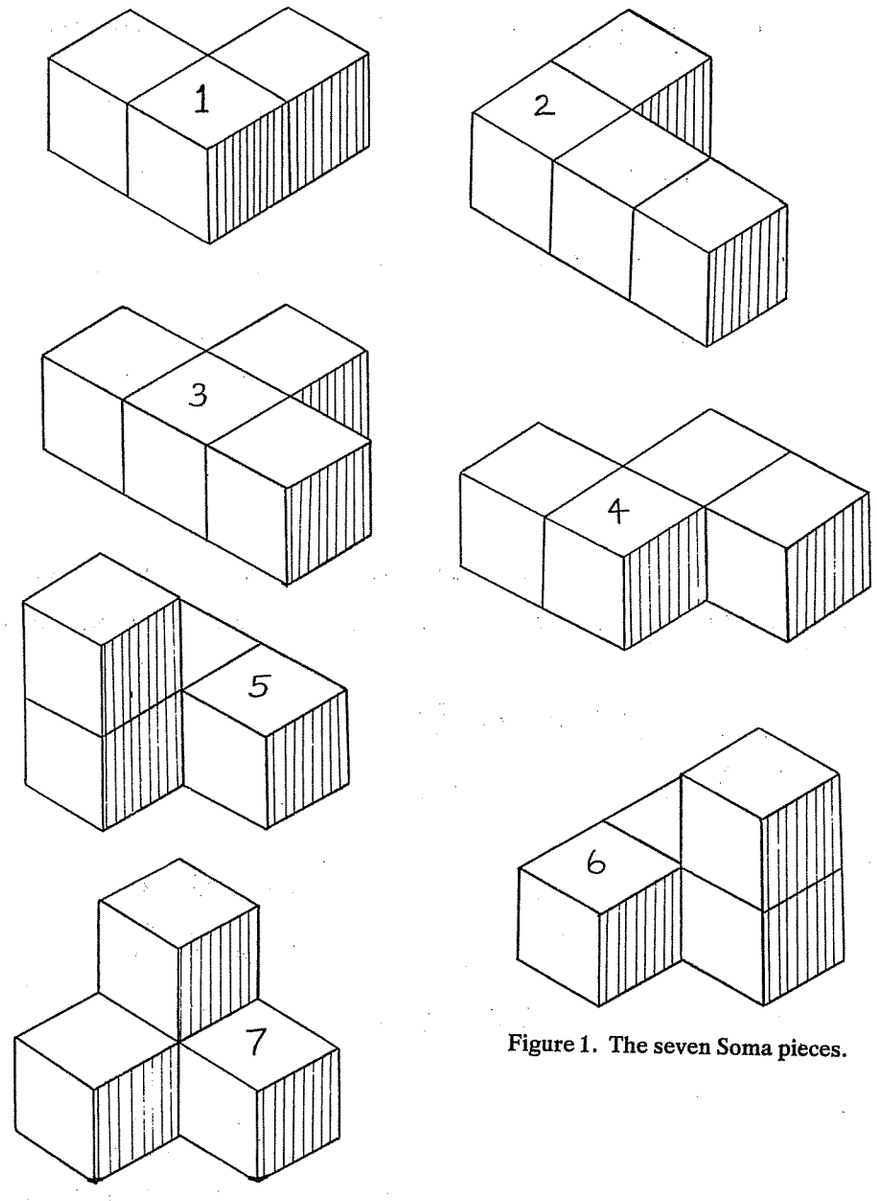
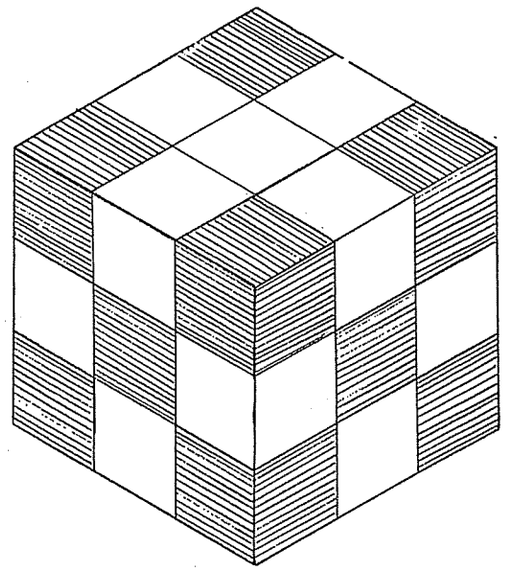


Figure 1. The seven Soma pieces.

This exercise shows that there are many limitations in the positioning of the seven Soma pieces when building the Soma cube.

Suppose the 27 individual cubes which make up the Soma cube are alternately coloured black and white, as in figure 2.



As a result, there are 14 black cubes and 13 white cubes. This is designated (14,13).

Now consider the possibilities of colouring the seven Soma pieces alternately black and white.

Pieces 2,4,5 and 6 each contain two black and two white cubes, giving a total of (8,8). Therefore, pieces 1, 3 and 7 must total (6,5).
Piece 1 may be (1,2) or (2,1).
Pieces 3 and 7 each may be (1,3) or (3,1).

In order to obtain a total of (6,5) with pieces 1,3 and 7, pieces 3 and 7 have to add up to (4,4), and piece 1 must be (2,1). This means that the vertex of piece 1 must be white and therefore cannot be used as one of the vertices of the Soma cube, all of which are black.

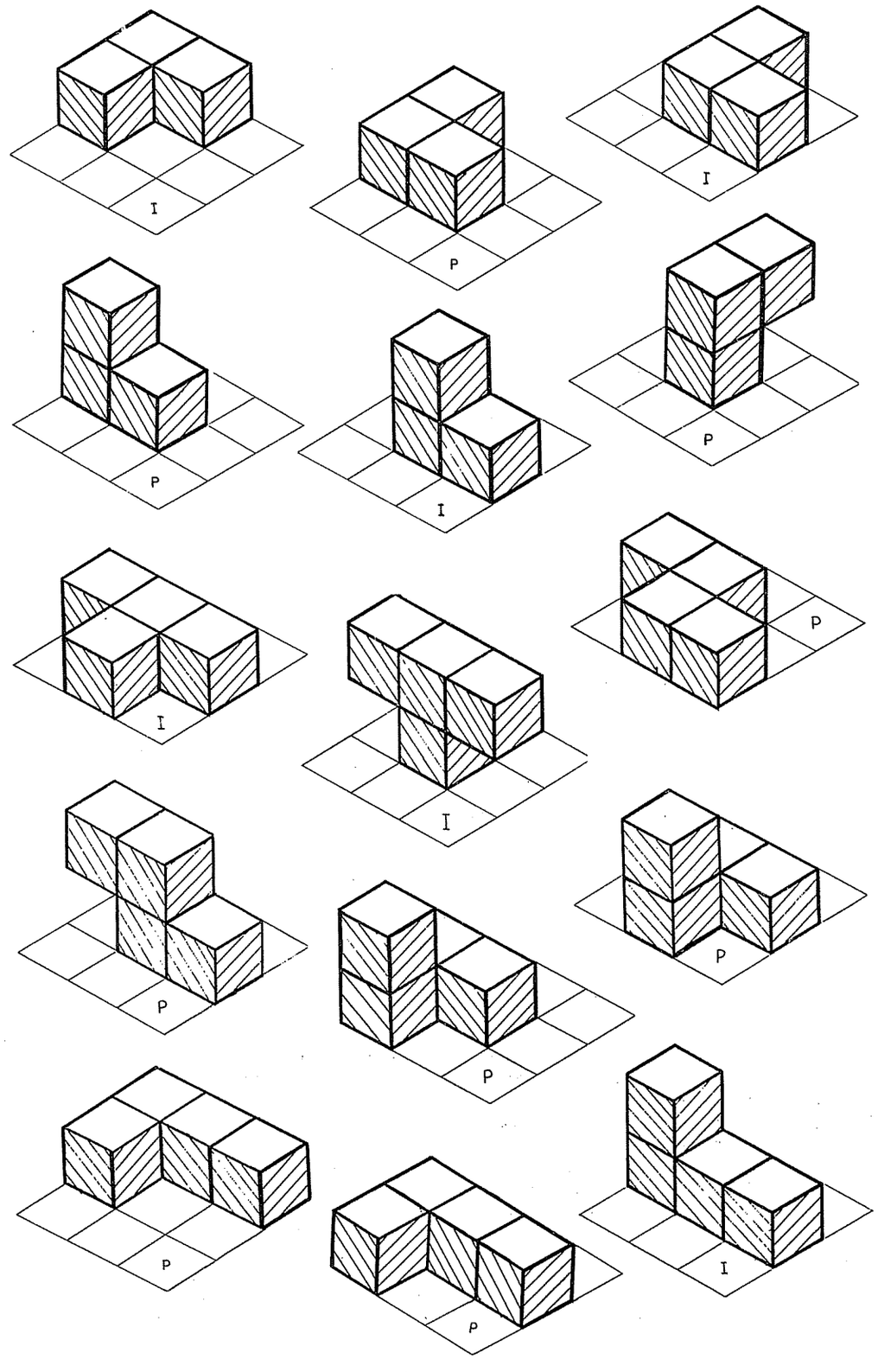
The Soma cube must have eight vertices. Consider the number of vertices which the seven Soma pieces may occupy:

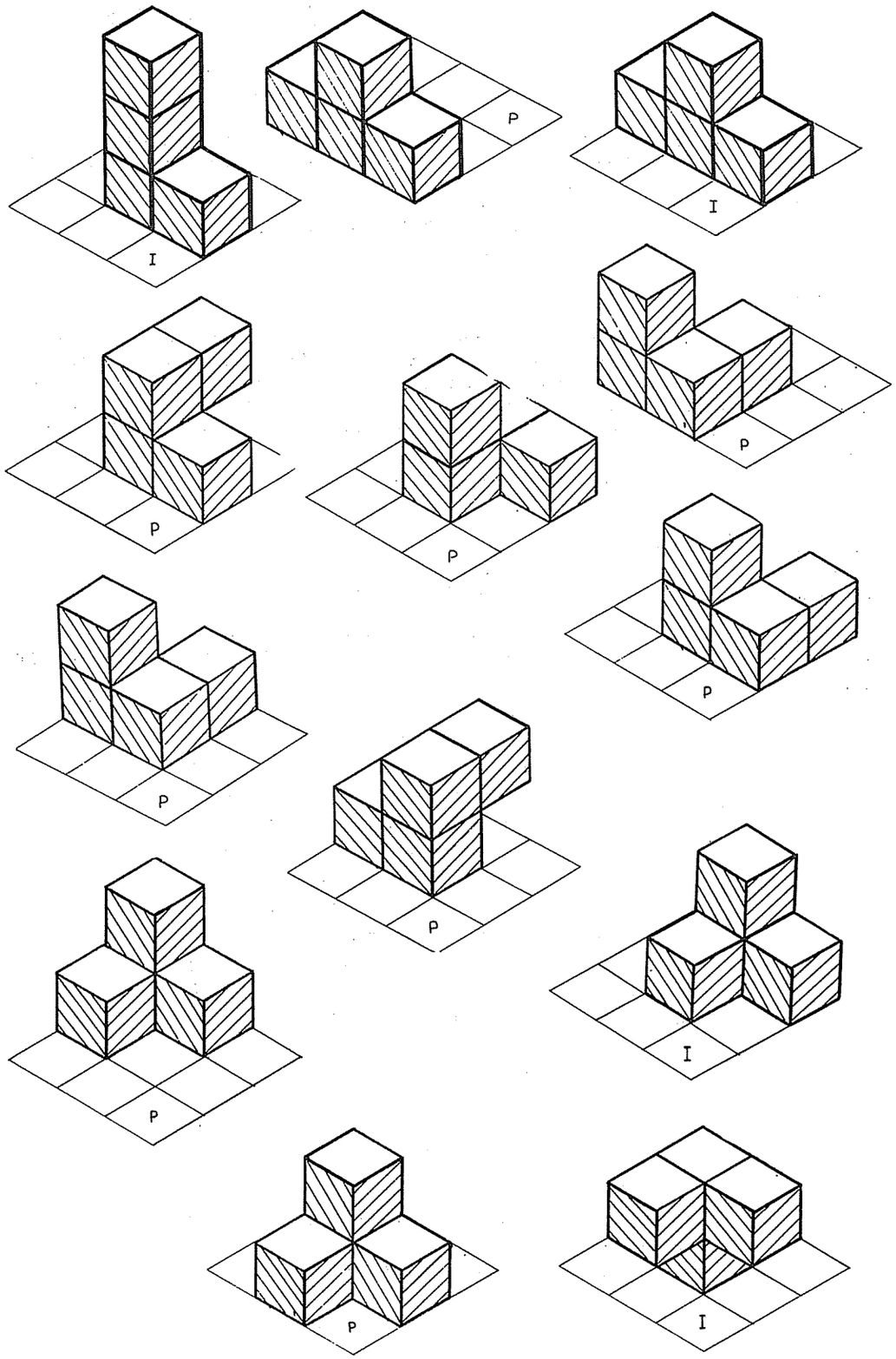
piece 1:	0 or 1 vertex
" 2:	0, 1 or 2 vertices
" 3:	0 or 2 vertices
" 4:	0 or 1 vertex
" 5:	0 or 1 vertex
" 6:	0 or 1 vertex
" 7:	0 or 1 vertex

If piece 3 were to occupy no vertices, then at most only seven vertices could be filled by the six remaining pieces. Therefore piece 3 has to occupy one, hence two, vertices.
Therefore piece 3 must be (3,1).
Pieces 3 and 7 have to add up to (4,4) in order to build the cube.
Therefore piece 7 must be (1,3).
If piece 2 were to occupy no vertices, then the remaining pieces could only fill 7 vertices at most.
Therefore piece 2 must occupy at least one vertex, and may occupy two vertices.

In the case where pieces 2 and 3 both fill two vertices, one of pieces 1,4,5,6 and 7 must occupy no vertices.

Here is a chart of the theoretical positions of the seven Soma pieces in the Soma cube. The positions that are possible are marked with a "P"; those that are impossible are marked with an "I".


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INTEREST - SIMPLE AND NOT SO SIMPLE

David R. Campbell, Co-ordinator, Department of Mathematics,
Faculty of Education, University of Western Ontario

How often we hear students ask "What is the use of Mathematics?" The purpose of this article is to show how considerable Mathematics can result from a problem which appears to be fairly simple. From the newspaper: For Sale - Used Car
Cash Price - $200
Instalment Price $50 down + $60 per month
for 3 months.

Problem: Find the per annum interest rate.

The great teacher of Mathematics, G. Polya, believed we should first guess or estimate the answer. As a first step, the reader is asked to estimate the interest rate if the buyer decides to pay for the car by instalments. Estimate of the interest rate is _____% per annum. The author has asked many groups of student-teachers to estimate the interest rate, with a complete range of answers (from 5% to 80%). Students are then asked to justify their answer and this always provokes a lively discussion.

The following points usually need clarification.

- How much does he actually owe?
- How long does he owe it for?
- What is the interest (cost in dollars) of the debt?
- Is the interest all paid at the end or is it included in the monthly payment?

We shall now proceed to give various solutions for this problem from the simple to the complex. It is suggested that this problem could be profitably discussed at any level from Grade IX through University.

Solution 1

Once it has been established that the actual loan is $150 and the interest is $30, it has been found advisable to find the actual dollar cost of the loan, if the money were borrowed from a credit union at 1% per month and paid off in instalments of $60 per month as in the contract.

Hence, if he borrowed from the Credit Union;

For the first month he owes $150. Interest .01x150=1.50
For the second month he owes $90. Interest .01x 90= .90
For the third month he owes $30. Interest .01x 30= .30
Total cost of the loan 2.70

A cost of $2.70 is at an interest rate of 12% per annum.

∴ a cost of $30 will give an interest rate of

$$\frac{30}{2.70} \times 12\% = 133\tfrac{1}{3}\%$$

This result is a little surprising to most students, so we begin to look for other solutions.

Solution 2

Consider the debt broken up into 3 separate amounts, each owed for 1 month;

For the first month, he owes $150 for 1 month
For the second month, he owes $90 for 1 month
For the third month, he owes $30 for 1 month
In total, he owes $270 for 1 month

$270 for 1 month costs $30
$270 for 12 months costs $360

Interest rate = $\frac{360}{270} \times 100 = 133\frac{1}{3}\%$

These first two solutions could be done at a Grade Nine or Ten level with General Mathematics classes.

Solution 3

Students might now question whether the $30 interest is all paid at the end or whether it should be distributed over the 3 monthly instalments. So now let us work out the problem by considering each $60 payment as $50 principal and $10 interest.

For the first month, he owes $150 for 1 month
For the second month, he owes $100 for 1 month
For the third month, he owes $50 for 1 month
In total, he owes $300 for 1 month

$300 for 1 month costs $30
$300 for 12 months costs $360

$\therefore$ Interest rate = $\frac{360}{300} \times 100 = 120\%$

At this stage, you might like your students to work out 2 or 3 problems involving a greater number of instalments. Many texts "give" a formula for interest rate, but the author believes that an excellent opportunity to introduce some very good mathematical ideas will be missed.

Solution 3(a) - Generalization of Solution 3

Students should realize by now that a formula would be valuable in finding the interest rate on instalment buying.

Let C be the carrying charge or interest.
Let A be the amount of the loan.
Let n be the number of monthly payments.
For the first month, he owes $A for 1 month.

For the second month, he owes $A - \frac{A}{n}$ for 1 month.

For the third month, he owes $A - \frac{2A}{n}$ for 1 month.

For the nth month, he owes $A - \frac{(n-1)A}{n}$ for 1 month.

In total he owes $nA - \frac{A}{n}\left[(1+2+3+ \ldots (n-1)\right]$ for 1 month.

Now we need a formula for the sum of the first n natural numbers. This provides motivation for the study of arithmetic sequences and series, or at least to find the sum of some simple series.

When students have found that

$1+2+3+ \ldots +(n-1) = \frac{(n-1)n}{2}$, they are ready to proceed with the formula.

He owes $nA - \frac{A}{n}\left[\frac{(n-1)(n)}{2}\right] = \frac{A(n+1)}{2}$ for 1 month.

Interest Rate per Annum = $\frac{\text{Interest}}{\text{Principal x Time}}$ =

$$\frac{C}{\frac{A(n+1)}{2} \times \frac{1}{12}} = \frac{24C}{A(n+1)}$$

Applying this formula to our problem C=\$30, A=\$150, n=3.

$\therefore$ Rate = $\frac{24x30}{150x4}$ = 1.2 or 120% per annum.

Solution 4

A rather similar solution that students might discover is to consider the \$150 as three separate \$50 debts.

Thus he owes one \$50 for 3 months.
He owes one \$50 for 2 months.
and he owes one \$50 for 1 month.
In total he owes \$50 for 6 months.
Interest on \$50 for 6 months is \$30.
Interest on \$50 for 12 months is \$60.

Interest rate = $\frac{60}{50} \times 100\% = 120\%$.

Once again, students should be encouraged to generalize this idea and produce a formula.

Solution 4(a) - Generalization of Solution 4

Let C be the carrying charge or Interest
Let A be the amount of the loan
Let n be the number of monthly payments.
Consider the loan of \$A to be made up of n debts, each of the amount $\frac{A}{n}$.

Then he owes one $\frac{A}{n}$ for n months.

He owes one $\frac{A}{n}$ for (n-1) months.

He owes one $\frac{A}{n}$ for (n-2) months.

He owes one $\frac{A}{n}$ for 1 month.

Thus he owes $\frac{A}{n}$ for a total time of 1+2+3+ ... (n-1)+n months
= $\frac{n(n+1)}{2}$ months.

Rate = $\frac{\text{Interest}}{\text{Principal x Time}}$

Interest = C Principal = $\frac{A}{n}$ time = $\frac{n(n+1)}{2}$ months

Rate = $\frac{C}{\frac{A}{n} \times \frac{n(n+1)}{2} \times \frac{1}{12}} = \frac{24C}{A(n+1)}$ as before.

Solution 5

A thoughtful student might suggest that more interest should be paid in the first month because more

is owed at that time. The Ontario Consumer Protection Act (1967) states that in all time payment transactions the lender must provide the borrower, before credit is given, with a clear written statement showing the cost of borrowing, both in dollars and cents and as an annual percentage rate. The Consumer Protection Bureau of the Department of Financial and Commercial Affairs suggests the use of the Direct Ratio formula for calculation of interest rate for High School students. This method assumes a greater share of the interest is paid in the first month. Referring to solution 4, we see that three sums of \$50 are owed in the first month, two in the second and one in the third. Hence

$\frac{3}{3+2+1}$ or $\frac{3}{6}$ of the interest should be paid in the first month, $\frac{2}{3+2+1}$ or $\frac{3}{6}$ in the second, and $\frac{1}{3+2+1}$ or $\frac{1}{6}$ in the third. This gives rise to the table below.

Month	Debt Outstanding During Month	Total Payment	Part of Payment For Interest	Part of Payment For Principal
First	150	60	$30x\frac{3}{6}=15$	60-15=45
Second	150-45=105	60	$30x\frac{2}{6}=10$	60-10=50
Third	105-50=55	60	$30x\frac{1}{6}=5$	60-5=55

Thus, in total he owes 150+105+55=310 for 1 month.
310 for 1 month costs \$30.

$$\text{Rate} = \frac{\text{Interest}}{\text{Principal x Time}} = \frac{30}{310x\frac{1}{12}} = \frac{360}{310} = 1.16$$

Thus this method gives a rate of 116% per annum.

Solution 5(a)

To generalize this solution is a real challenge for secondary students, but it provides the Direct Ratio Method which may be used to check advertisements and gives answers very close to the "correct" or actuarial method.

Let P = Principal or unpaid balance (\$150 in our example).
Let D = Charge in dollars (\$30 in our example).
Let n = Number of payments (3 in our example).
Let m = Number of payments per year (12 in our example).

The regular payment will be $\frac{D+P}{n}$ for n periods m times per year.

This method assumes that in the first month the amount of the payment which is used for interest is

$$\frac{n}{(1+2+3+\ \ldots\ n)}\,D = \frac{2n}{n(n+1)}\,D.$$

In the second month, the amount that is used for interest will be

$$\frac{n-1}{1+2+3+\ \ldots\ +n}\,D = \frac{2(n-1)}{n(n+1)}\,D.$$

This leads to the chart below which is a generalization of solution 5.

Period (each $\frac{1}{m}$ years)	Debt Outstanding During the Period	Regular Payment (at end)	Part of Payment for Interest	Part of Payment on Principal
First	P	$\frac{P+D}{n}$	$\left(\frac{2n}{n(n+1)}\right)D$	$\frac{P+D}{n} - \frac{2n}{n(n+1)}\,D$
Second	$P - \frac{P+D}{n} + \frac{2n}{n(n+1)}\,D$	$\frac{P+D}{n}$	$\left(\frac{2(n-1)}{n(n+1)}\right)D$	$\frac{P+D}{n} - \frac{2(n-1)}{n(n+1)}\,D$
Third	$P - \frac{2(P+D)}{n} + \frac{4n-2}{n(n+1)}\,D$	$\frac{P+D}{n}$	$\left(\frac{2(n-2)}{n(n+1)}\right)D$	$\frac{P+D}{n} - \frac{2(n-2)}{n(n+1)}\,D$
Fourth	$P - \frac{3(P+D)}{n} + \frac{6n-6}{n(n+1)}\,D$	$\frac{P+D}{n}$	$\left(\frac{2(n-3)}{n(n+1)}\right)D$	$\frac{P+D}{n} - \frac{2(n-3)}{n(n+1)}\,D$
Fifth	$P - \frac{4(P+D)}{n} + \frac{8n-12}{n(n+1)}\,D$	$\frac{P+D}{n}$	$\frac{2(n-4)}{n(n+1)}\,D$	$\frac{P+D}{n} - \frac{2(n-4)}{n(n+1)}\,D$
Sixth	$P - \frac{5(P+D)}{n} + \frac{10n-20}{n(n+1)}\,D$	$\frac{P+D}{n}$	$\frac{2(n-5)}{n(n+1)}\,D$	$\frac{P+D}{n} - \frac{2(n-5)}{n(n+1)}\,D$
"				$\frac{P+D}{n} - \frac{2}{n(n+1)}\,D$
"				
"				
nth	$P - \frac{(n-1)(P+D)}{n} + \frac{2(n-1)(n)-(n-2)(n-1)}{n(n+1)}\,D$	$\frac{P+D}{n}$	$\frac{2[n-(n-1)]}{n(n+1)}\,D$	
(n+1)st	0			

Our problem now is to find the total of column 2 which will give the principal which is owed for $\frac{1}{m}$ years, as in our original question. This process gives an opportunity to introduce more worthwhile mathematics. The sum of the P's is nP.

The sum of the next part of column 2 gives us

$$-\left[\frac{P+D}{n} + \frac{2(P+D)}{n} + \frac{3(P+D)}{n} + \ \ldots\ + \frac{(n-1)(P+D)}{n}\right]$$

$$= -\frac{P+D}{n}\left[1+2+3\ \ldots\ +(n-1)\right]$$

$$= -\frac{P+D}{n}\left[\frac{n(n-1)}{2}\right] = -\frac{(P+D)(n-1)}{2}$$

The last part of the column will provide the greatest challenge and introduces a new series.

$$\frac{2n}{n(n+1)}\,D + \frac{4n-2}{n(n+1)}\,D + \frac{6n-6}{n(n+1)}\,D + \frac{8n-12}{n(n+1)}\,D + \ \ldots\ + \frac{2(n-1)(n)-(n-2)(n-1)}{n(n+1)}\,D$$

$$= \frac{D}{n(n+1)}\left\{2n+(4n-2)+(6n-6)+(8n-12) + \ \ldots\ +\right.$$

$$\left[2(n-1)(n)-(n-2)(n-1)\right]\Big\}$$

$$= \frac{D}{n(n+1)}\left\{\left[2n+4n+6n + \ldots + 2(n-1)(n)\right] - \left[2+6+12 + \ldots (n-2)(n-1)\right]\right\}$$

$$= \frac{2Dn}{n(n+1)}\left[1+2+3 + \ldots + (n-1)\right] - \frac{2D}{n(n+1)}\left[1+3+6 + \ldots \frac{(n-2)(n-1)}{2}\right]$$

At this stage, the series 1+3+6+10 ..., will have to be summed, which will lead to the necessity of finding a formula for

$\sum_{n=1}^{n-2} \frac{n^2+n}{2}$. This will lead further to the establishment of formula $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

After showing that $\sum_{n=1}^{n-2} \frac{n(n+1)}{2} = \frac{1}{2} \sum_{n=1}^{n-2} (n^2+n) =$

$$\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]_{n=1}^{n-2} =$$

$$\frac{1}{2}\left[\frac{(n-2)(n-1)(2n-3)}{6} + \frac{(n-2)(n-1)}{2}\right],$$

we continue with the sum of the last part of column 2.

$$= \frac{2D}{n+1}\left[\frac{n(n-1)}{2}\right] - \frac{D}{n(n+1)}\left[\frac{(n-2)(n-1)(2n-3)}{6} + \frac{(n-2)(n-1)}{2}\right]$$

$$= \frac{2D}{n+1}\left[\frac{n(n-1)}{2}\right] - \frac{D(n-1)(n-2)}{n(n+1)}\left[\frac{2n-3+3}{6}\right]$$

$$= \frac{2D}{n+1}\left[\frac{n(n-1)}{2}\right] - \frac{D(n-1)(n-2)}{n(n+1)}\left[\frac{2n}{6}\right]$$

$$= \frac{2D}{n+1}\left[\frac{n(n-1)}{2}\right] - \frac{D}{n+1}\left[\frac{n(n-1)(n-2)}{3n}\right]$$

$$= \frac{D}{n+1}\left[n(n-1) - \frac{(n-1)(n-2)}{3}\right]$$

$$= \frac{D}{n+1}\left[\frac{3n^2-3n-n^2+3n-2}{3}\right]$$

$$= \frac{D}{n+1}\left[\frac{2n^2-2}{3}\right] = \frac{2D}{3}\left(\frac{n^2-1}{n+1}\right) = \frac{2D(n-1)}{3}$$

Then the total for column 2 is

$$nP - \frac{(P+D)(n-1)}{2} + \frac{2D}{3}(n-1)$$

$$= nP - P\left(\frac{n-1}{2}\right) - \frac{D(n-1)}{2} + \frac{2D(n-1)}{3}$$

$$= P\left(\frac{2n-n+1}{2}\right) + \frac{D}{6}\left[-3n+3+4n-4\right]$$

$$= P\left(\frac{n+1}{2}\right) + \frac{D}{6}(n-1)$$

$$= \frac{3P(n+1) + D(n-1)}{6}$$

Thus he owes a total of $\frac{3P(n+1) + D(n-1)}{6}$ for a period of $\frac{1}{m}$ years for which the interest is D.

$$\text{rate} = \frac{i}{Pt} = \frac{D}{\frac{3P(n+1) + D(n-1)}{6} \times \frac{1}{m}}$$

$$= \frac{6mD}{3P(n+1) + D(n-1)}$$

This is the Consumer's Affairs Direct Ratio Formula.

Applied to our problem, m=12, D=30, P=150, n=3.

$$r = \frac{6x12x30}{(3x150x4)+30(2)} = \frac{2160}{1860} = 1.16$$

∴ The rate is 116% by this method as above.

Solution 6

One of the pedagogical benefits of this problem is that it shows the necessity of careful thinking in everyday problems. A very bright student might see that solution 5 is not quite correct, because the interest for the first month should be 150i, where is the rate per month. This would leave 60-150i to be paid on the principal at the end of the first month. Using this more exact method we obtain the table below and solution 6. The interest rate is i per month.

Month	Debt Outstanding	Interest	Payment	Payment on Principal at End of Month
First	150	150i	60	60-150i
Second	150-(60-150i) = 90+150i	$90i+150i^2$	60	$60-90i-150i^2$
Third	(90+150i)-(60-90i-150i)$_2$ = $30+240i+150^2$	$30i+240i^2+150i^3$	60	

Total interest = $270i + 390i^2 + 150i^3$

But Total Interest Charges are 180-150=$30

$$150i^3 + 390i^2 + 270i = 30$$

$$5i^3 + 13i^2 + 9i - 1 = 0.$$

This now brings in the problem of solving a cubic, and it is suggested that the method of successive approximations be used with the help of a calculator. Using this method i=.097 per month, which gives 12 x .097 = 1.16 = 116% per annum.

Solution 7

If students have taken compound interest, and present value, they are ready for the next solution. If not, this would be an ideal time to introduce the principles of compound interest. (See Math. Teacher, April 1972, Pages 310-312).

Let the rate be i per month:

Then the Present Value of the $60 per month for 3 months is $\frac{60}{(1+i)^3} + \frac{60}{(1+i)^2} + \frac{60}{(1+i)}$ which is equal to $150.

This gives rise to a Geometric Series, so a reason is

provided for introducing another important topic. Having learned how to find the sum of a Geometric Series, we can proceed with our problem.

$$\frac{60}{(1+i)^3} + \frac{60}{(1+i)^2} + \frac{60}{(1+i)} = \frac{60}{(1+i)^3}\left[\frac{(1+i)^3-1}{(1+i)-1}\right]$$

Thus we have $\frac{60}{(1+i)^3}\left[\frac{(1+i)^3-1}{(1+i)-1}\right] = 150$

which reduces to $5i^3 + 13i^2 + 9i - 1 = 0$

Note that this is the same equation as outlined in Solution 6, so we can solve as before.

Solution 8

Solution 8 is really equivalent to solutions 6 and 7, but interpolates actuarial tables for the present value of an annuity. The present value of an annuity of 1 per annum for n interest periods at rate 1 per period is represented by the symbol $A\,\overline{n|}\,i$.

Let the interest rate per month be i.

$60\ A\,\overline{3|}\,i = 150$

$A\,\overline{3|}\,i = 2.5$

From tables

$A\,\overline{3|}\,.0975 = 2.49784$

$A\,\overline{3|}\,.09625 = 2.50336$

By interpolation, the value of i to give $A\,\overline{3|}\,i = 2.50$ is .0970 which checks with the results in Solutions 6 and 7. This last method gives one reason for introducing annuity tables. If these are not available, $A\,\overline{3|}\,.0975$ and $A\,\overline{3|}\,.09625$ can be calculated from the formula for the sum of a geometric series.

The students should now be asked to go to the newspaper and check the interest rate shown in advertisements. Here is an example taken from a local paper.

No Down Payment - Used 2 Door Hardtop

Lots of Extras, Mileage 28,377. Sale Price $2,748

$95.61 per month - 36 equal payments of $95.61

Annual Rate 11.78%.

Problem - Check the rate quoted.

Instalment Price = 36x95.61 = 3441.96

Cash Price = 2748.00

Interest 693.96

Method 1 - Use formulas from Solutions 3 or 4.

$$\text{Rate} = \frac{24C}{A(n+1)} = \frac{24\text{x}693.96}{2748\text{x}37} = .1638$$

Rate = 16.38% per annum.

Method 2 - Direct Ratio Formula (Consumer's Affairs)

$$\text{Rate} = \frac{6mD}{3P(n+1)+D(n-1)}$$

$$= \frac{6\text{x}12\text{x}693.96}{(3\text{x}2748\text{x}37)+(693.96\text{x}35)} = .1517$$

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BRIEF TO MINISTER OF EDUCATION

Copies of a brief on the qualifications and preparation of Teachers of Mathematics (K-8) are available from M. MacGregor, 247 Bright Street, Sarnia, N7T 4E9. This brief was prepared from comments from OAME members and Faculties of Education who were sent a preliminary draft. The brief has now been approved by the OAME Council and sent to the Minister.

REVIEW OF SENIOR DIVISION CURRICULUM

The Council of OAME invites responses to the report of the Committee published in the September, 1975 Gazette. They should be directed to Neil Williamson, 2329 Bridletown Circle, Agincourt, Ontario.