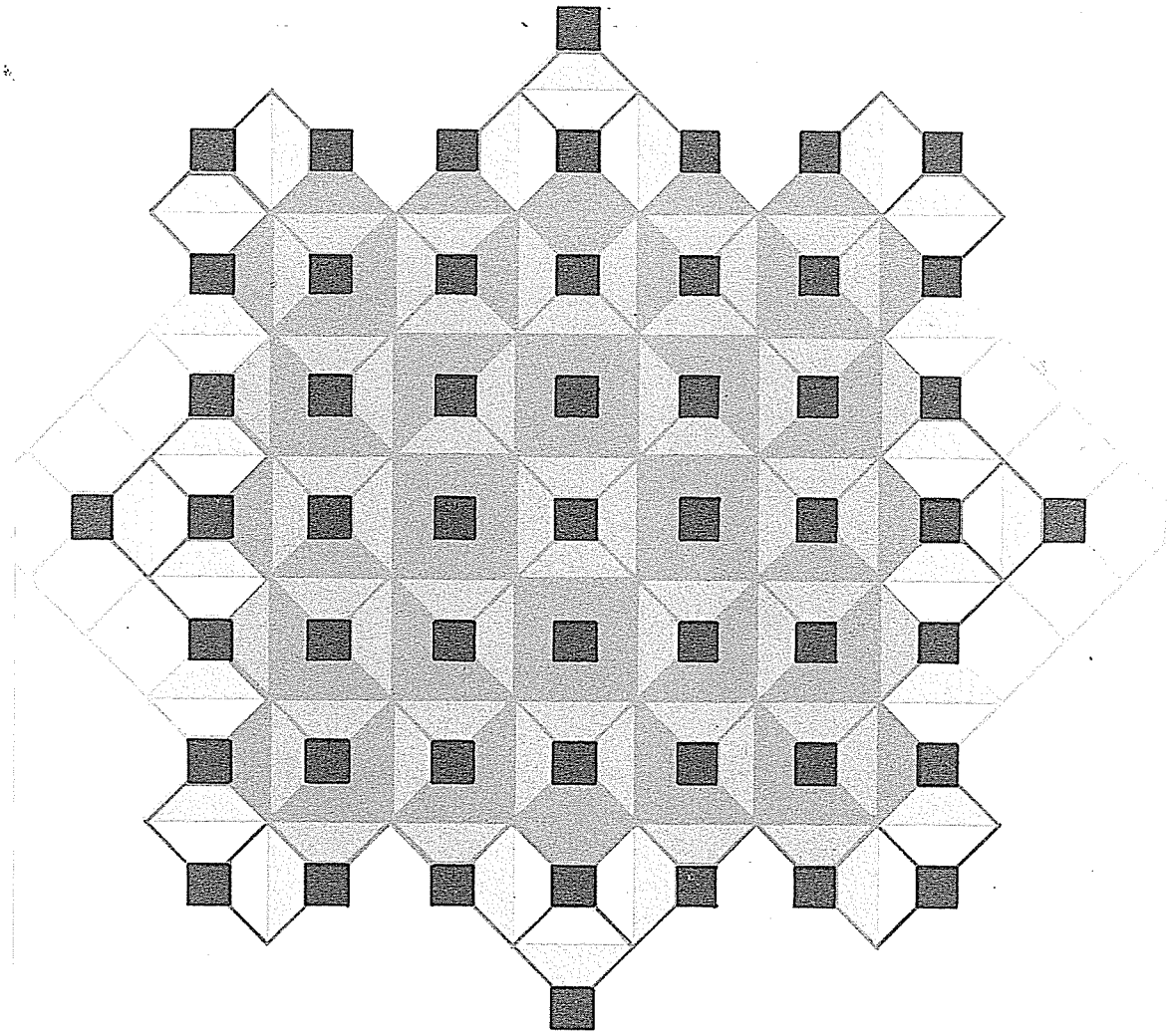
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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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EDITORIAL

It is exciting to be a part of new developments. The Intermediate Division Mathematics guidelines are being formulated. There is a whole process of change going on that we should be aware of. In the last editorial I gave an overview of the changes. Local committees are busy formulating their views, not only about the goals, but also about the results of their field testing, and in some cases about the rationale. The Validation Committee is awaiting the results from which it will compose recommendations. These recommendations must be implemented by a reconstituted writing team. So much for the **goals** of Mathematics for the Intermediate Years and a booklet on the **rationale and strategies** of Intermediate Division Mathematics.

Let us look for a moment to the implementation of such a program. The guidelines will be introduced into the classrooms gradually over the next several years. How effectively depends on us, the teachers. Be of good will. I well remember as a young teacher in the late forties being asked to go to the mathematics revision meeting, meetings encouraged by Colonel Watson of the Curriculum branch where I met with local secondary school teachers of mathematics. A dialogue had begun. Curriculum change, the exchange of ideas between two panels of teachers had started. Today we have Math Co-ordinators whose work is to view the curriculum as a whole from K to 13 and who are encouraging the teachers of the Intermediate Division to work together. This very week, plans are being made by the Faculties of Education to ensure help will be available for those Boards of Education who request it. Undoubtedly, there will be Support Documents issued in the years ahead to ease the way. Yes, there is a need for professional development. The model to be used might well follow the input, workshop, implement-in-the-classroom, evaluate style of other successful inservice education projects. With confidence gained, we teachers will meet the challenge, even if our first attempts are slow and cautious, yet we will succeed in the long run.

Included in this issue you will find three articles which point the way for new endeavour at the intermediate level. I refer to Clare Zeller's article on Flow charting, Doug Henderson's article on Data Analysis and the Motion Goemetry article by Dave Boyle and Ton Griffiths. All should be required reading. At the intermediate level there has been a renewed emphasis placed on problem solving skills, particularly through "real life" examples which might be heeded at all levels. For this reason, your March 1980 Gazette will have as its theme Problem Solving. Do not let this call for papers go unheeded. We will need your contribution.

Hope to see you at the Sixth Annual Conference in May at Trent University. This time, we will enjoy the hospitality of Pine Ridge Chapter and the arrangements being made for us by Bill Jeffers and company. I am looking forward to the jog along the canal with Bill Higginson and Hugh Allen. Do join us.

Bob Smith

LETTERS TO THE EDITOR

Chatham College
Box 316
Pittsburgh, Pa. 15232

Dear Prof. Smith:

I recently attended a conference on Math anxiety where I obtained a bibliography list. Included on this list was the article by Peter Hilton entitled "Some Thoughts on Math Anxiety" which appeared in **Ontario Mathematics Gazette** 1978.

My problem is I cannot find your publication at any of the area libraries and do need the article for a thesis I am writing. I was hoping you could send me a copy of the article. I would be happy to pay for any duplicating costs.

Thank you for your co-operation.

Sincerely,
Jeannette Chakeres

Editors Note: It appears O.M.G. is getting known for the quality of articles carried. Several persons have shown an interest in Peter Hilton's article. Look for Sheila Tobias' new book **Math Anxiety: What It Is, Why You Have It, and What You Can Do About It**, to be published by W.W. Norton, New York.

201 Holmwood Ave.
Ottawa, Ontario
Jan. 9, 1979

Dear Bob,

I have had a letter from one of COMA's members, R. Vaillancourt suggesting that our chapter might well be designated COMA/AMOC, due to its bilingual character.

Christine Hubbard

P.S. Congratulations on another terrific Gazette.

Editors Note: This has now been done.

Faculty of Education,
Althouse College,
London, Ontario
Feb. 1, 1979

Dear Bob:

I was pleased that my comments regarding J.G. Griffiths' letter to you in the December '78 issue of the Gazette drew responses from some of my colleagues. I was also pleased that their reactions were positive, but, they felt that I gave the impression that few shared my concerns.

I am aware that others share my views. The problem is perhaps that there seems to be a general reluctance to bring them into the open because these comments may not be as yet in "vogue".

It seems to me, however, that educators are beginning to question the present curriculum and are becoming more vocal, and, it is perhaps time that the "chaff" be discarded and an attempt be made to find the "kernel" of what may be termed a sound Mathematics education.

My belief is that this can be achieved if educators would be willing to debate their beliefs which would, however, require that they voice them. And what better medium then the Gazette?

Yours sincerely,
Sam Pravica

Editors Note: Great idea! O.M.G. would welcome the dialogue on what comprises a sound Mathematics education.

St. John Fisher School
Forest, Ontario

Dear Prof. Smith:

I attended your last leadership conference and enjoyed it very much. In particular I found the workshops on transformational geometry and the use of the computer immensely helpful. Many of the Monday-morning specials were passed on to my staff and were much appreciated. My congratulations and thanks to the committee which did such an excellent job.

Ray Piché

ANNOUNCEMENT OF THE EIGHTH ANNUAL ELEMENTARY LEADERSHIP SEMINAR

The Ontario Association for Mathematics Education are sponsoring this eighth Annual Leadership Seminar at the Ascot Inn, just north of the Toronto Airport, on Oct. 17-20, 1979.

The fee is $120.00 per participant, which covers registration, lodging and food. In late February, announcements were sent to all Ontario Directors of Education. We suggest if you are interested and have not seen reference re the seminar that you either contact your local Board of Education office or write direct to the OAME office, 247 Bright St., Sarnia, N7T 4E9. Space is limited and we need to know your intention to attend prior to school closing in June.

Mr. D. McPhail, program co-ordinator is looking forward to have the following speakers - Mr. William Blake, Mr. Paul Doucette, Mr. Charles Fader, Mr. D. Green, Dr. W. Innard, Mr. G. Knill, Mr. A. Norrie, and Mr. R. Robinson.

In addition there will be the following group sessions led by Olive Fullerton, Irene Gauvin, Dr. Lukasevich, Roy Murray, and Peter Ullrich.

Mr. John Storey, Ministry of Education, Toronto, hopefully will be a special guest speaker.

AFTER VALIDATION - WHAT THEN?

Sam Pravica, Faculty of Education, The University of Western Ontario

One might get the impression that little is happening behind the scenes while the field trials of the Intermediate Mathematics Guidelines are in progress. Not so! The implementation time line for the revised curriculum guidelines set by the Ministry is such that planning ahead is necessary.

As a member of the Validation Committee, I find the work ahead interesting and challenging. For many, both teachers and committee members, a very real concern, which the title of this article suggests, is "After validation - what then?"

There are two major areas that require attention, (i) inservice education for practising teachers and, (ii) preservice education for new teachers.

The question that is being asked by many is, "How will the Ministry "educate" teachers on the contents of the new guidelines?" Many of us can remember when teams of resource people coopted by the Ministry provided us with the "New Math" input in the early '60's. I am sure that we can concede that the dollar "crunch" has preempted any thoughts in this direction. The members of the Validation Committe are aware of the problem and are in the process of identifying alternatives in implementation strategy. For example the members of the Coordinators' Association have been kept up to date with the validation procedures through their representative on the Validation Committee. This has given boards having a consulting staff sufficient information to begin developing strategies for program development in their local jurisdictions. However, let us keep in mind that program development is just the beginning. It contains only the potential of a sound education. Without adequate inservice in conjunction with dissemination of these new programs little actual translation will take place in the classroom, the heart of the reason for the guidelines.

The curriculum suggests a deemphasis of the abstract and an emphasis toward the practical. Perhaps a parallel should be attempted with teachers when locally produced materials are disseminated.

But what of the boards that do not have resource personnel? Can the Ministry provide the human resources that will provide these boards with the needed expertise to develop their programs and initiate inservice? Indirectly, yes. In his report on declining enrolment, R.B. Jackson indicates

that the Commission received the recommendations that:

> "Faculties of Education should direct their priorities more aggressively to inservice education. Their expert input as researchers, curriculum specialists, project co-ordinators and as professional development facilitators would be welcome."
>
> "Teacher preparation institutes should be called upon to a much greater extent to fulfill existing needs in professional development."

By the time this issue of the Gazette goes to print, representatives of the Faculties and Colleges of Education in Ontario will have met to come to grips with this problem. The Committee will attempt to outline procedures by which boards may request and gain assistance in program development and inservice. This service has always been available to boards requiring the expertise of faculty members. Unfortunately, many boards have not made use of it, perhaps because they were not aware that it was accessible to them.

This Committee will also be reviewing the preservice programs that are presently being offered at the intermediate level at the various institutions. It comes at a most opportune time, for, with the implementation of the new Ontario Teacher's Qualification Regulations, requiring that teachers select an area of concentration, primary-junior, junior-intermediate, intermediate-senior, Faculties of Education are required to make appropriate adjustment in their course offerings. It is hoped, then, that as new teachers become absorbed into the profession over the next few years the input from the institutions relative to the Intermediate Mathematics Guidelines will have a cumulative effect.

In addition to this, the Committee will be looking at ways and means of initiating summer and evening classes for teachers interested in upgrading their knowledge of mathematics to better handle the concepts at the intermediate level.

There is much to be done and a great deal is being done. For those requiring assistance in program development, the expertise is there; the problem is knowing where to find it and how to obtain it. This, the members representing the mathematics departments of the Faculties and Colleges of Education, will attempt to outline and hopefully delineate guidelines for involvement in their respective surrounding areas.

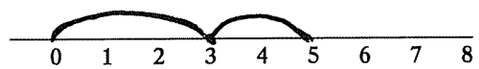
The above description is only a part, albeit an important part, of the types of concerns that face the Validation Committee. The strength of the Committee lies in its representation of the various education groups. Their continued input and ability to come to grips with the problems of revision and implementation will help make the process of validation a more meaningful undertaking.

THE ROLE OF MENTAL IMAGERY INFLUENCES

ELEMENTARY MATHEMATICS TEACHER EDUCATION

Janice L. Flake, Associate Professor, Mathematics Education, Florida State University

A child is expected to:	But can he/she:
1. Solve 5 - 3 = ____	1. Tell a complete story about 5 objects on a table, after which 2 objects are removed from the 5 objects?
2. Solve 2 + 4 = ____	2. Tell a complete story about 2 objects and 4 objects being put together?
3. Show 34 = ____ tens + ____ ones	3. Show how many tens and how many ones when given 34 objects such as sticks? Also if the ones and tens are determined, will

	the child remember 1 ten and 10 ones are the same no matter how the objects are moved around?
4. Understand $7 + (3 + 5) = (7 + 3) + 5$	4. Relate this to concrete materials and "act out" the relationship?
5. Use a ruler to measure lengths	5. Find lengths using one block, or tell the difference between unevenly spaced rulers and evenly spaced rulers?
6. Tell how many grams something weighs	6. Estimate the weight of a given object by picking the object up?
7. Add rational numbers	7. Tell that quarters (e.g.,  ) of equivalent wholes are equivalent?
8. Use the number line for: $3 + 2$ 	8. Relate a "hop" from 0 to 3 followed by a "hop" from 3 to 5 to a "hop" from 0 to 5? Or relate to objects?
9. Solve $3 \times 4 =$ ____	9. Tell how many cookies are needed to give 3 cookies to each of 4 children?
10. Solve $4\overline{)36}$	10. Tell how many sticks each child would get if given 36 sticks (bundled in tens and ones) and told to share them "fairly" among 4 children?

Such questions as these have been shared with inservice teachers in the Tallahassee area. The teachers, in turn, have gone back and examined their own students and found there were many holes in their students' thinking. The results have been quite eye-opening to the teachers. The basic problem found was that the children had very little understanding of the mathematics they were being asked to do.

Mathematics can be thought of as getting right answers, or it can be thought of as developing thinking processes through looking for patterns and relationships. A vehicle for developing such thinking skills is felt by some to be that of **mental imagery** or an image (visual or otherwise) in one's mind of an action, event, or thing. Of particular interest in relation to mathematics learning is the ability to hold an idea in the mind relating to a transformation. For instance, put 5 objects on a table, have the child tell how many objects there are, move the objects around, again ask the child to tell how many objects there are. Observe if the child can answer the question spontaneously or must he/she recount the objects. Has the child been able to hold in his/her mind that five objects will still be five objects no matter how they are moved around? Does the child have a reversibility of thought? See Flake (1978) for additional discussion of mental imagery in the child.

Teachers involved in our inservice classes have become elated to learn of the ideas of mental imagery and its implication to mathematics instruction. Many of these teachers had had the usual undergraduate CUPM mathematics courses, which had been predominantly passive and paper-pencil oriented. They had not learned through "active" methods, nor had they been given any foundation for uses of manipulative aids nor insights into children's thinking patterns. As the teachers grew in these ideas through the inservice classes, their whole concept and attitude toward mathematics changed. (See Flake, 1976). Throughout the inservice courses, three major questions were considered:

1) What are the abstract mathematical ideas we wish to teach the children?
2) What are models for those abstract models?
3) When have the children conceived of the models?

It was felt that the ideas gained by these teachers should become a part of the undergraduate

program for elementary education majors. Thus, a new conceptualization of a teacher education program evolved.

First, a few words about the elementary education majors at Florida State University. A study by Adi (1976) of junior elementary education majors revealed that out of 75 students tested for formal operational thinking, only 12 (or about 16%) of the students tested out at the formal operational stage (which according to the theory, students are suppose to reach at the age of 11 or 12 years old), 26 (or about 35%) of the students tested out at the late concrete operational stage, and 33 (or about 44%) of the students tested out at the early concrete operational stage. Four (or about 5%) showed very little logical thinking. In addition, the Childhood Education Program tested all entering students into the Program with a standardized arithmetic test. In Fall 1977, out of 150 students tested, only 50% were able to achieve scores comparable with the eighth grade mean on the standardized test. (On a personal note, the author has taught elementary education majors in three different states, and has no reason to believe that the Florida State University students are less capable than typical entering elementary education majors).

With the students in mind, the following underlying assumptions have been made:
Prospective teachers need to:

1. Become aware of their own mental images;
2. Build and extend their own images and conceptual understanding of mathematics;
3. Become aware of children's thinking patterns, and learn how to assess them; and
4. Learn how to construct active learning experiences for the children.

Thus, the present sequence of courses has been constructed, and is currently being taught:

1. A Mathematics Course, in which, the students learn by "active" methods and interact with a wide variety of concrete materials and learning experiences. The following content is covered in this course:
   - Classification
   - Seriation
   - Patterns
   - Spatial Relations
   - Measurement (first using arbitrary units, later using the metric system)
   - Whole Numbers
   - Place Value Concepts and Arithmetic Algorithms
   - Rational Numbers
   - Data Collection and Representation

   Students are consciously made aware of their thinking patterns as they are learning the content. Considerable emphasis is made on the shifts from perceiving the world about them to conceiving of it. The main emphasis of the course is to build the students' own conceptual development of mathematics, and to have the students gain an understanding of models for abstract ideas.
2. The second course in the sequence is focused on understanding children's thinking patterns, gaining understanding of when children conceive of models and how to assess it, diagnosis-prescription, an examination of various psychological frameworks for learning, and an over-view of the elementary mathematics curriculum.

   One component of this course involves field experiences with children of various ages and abilities on a one-to-one basis to gain insights into the individual child's thinking patterns. Emphasis is placed on diagnosing the individual children's knowledge and making appropriate prescriptions for the children. Students in the author's classes pursue questions similar to those listed on page 1 with the children, as well as standard Piagetian tasks and "usual" computation skills.

   Attention is given to thinking about Pedagogical Principles for mathematics instruction. Students become involved in organizing their own activity files, gathering and creating activities for each of the areas listed in #1 above, as well as a few additional areas, such as probability, elementary number theory, and geometry.
3. The third course considered designing and impelmenting mathematics instruction. This course is often taught in a Teacher Center site, where the prospective teachers spend a major portion of their time in the public school, working directly with children in classroom settings.

Since we have broken away from the traditional sequencing of having the students take several mathematics courses, followed by a methods course, we continue to give the students additional mathematics learning experiences throughout the sequence of courses.

One interesting phenomenon has occurred. The author insists that her students be able to draw pictures of sticks, Dienes blocks, etc. showing the "actions" for addition, subtraction, multiplication, and division in base 10 as well as other bases. This has turned out to be non-trivial for a number of students. In some extreme cases, some students have been retained for as many as three quarters because of lack of ability to draw the pictures. Finally, when they get so they can do it, they are completely elated and admit that they had not really understood what was going on before. The picture drawing, of course, taps into their own imagery (or lack of it). Many of the students have had number bases in elementary or junior high school, but they admit that they never really understood it, and it was treated strictly at the symbolic level.

Another interesting phenomenon has been occurring at the graduate level at Florida State University. Students from a variety of fields take a graduate level methods course for teaching mathematics at the elementary level. In that course, ideas of mental imagery, discovery learning, uses of manipulative aids, child development as it relates to mathematics learning, etc. are covered primarily through active learning experiences. By the end of the course a number of the students are saying, "Hey, I really feel good about myself, my abilities to learn, and I see why I had problems with math before. But I feel like there is so much more math I need to know." Those people are then asking for a math course taught using the same techniques.

Perhaps a better sequencing for several mathematics education courses for elementary majors is to first focus on processes of learning mathematics, then later press for the learning of the content. It is felt that the sequence we are now using is far more realistic for elementary majors than the usual CUPM courses, but we will undoubtedly continue to seek an even better sequence.

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Adi, Helen. **"The Interaction Between the Intellectual Developmental Levels of College Students and Their Performances on Equation Solving when Different Reversible Processes are Applied."** Ph.D. dissertation. Florida State University, 1976.

Flake, J.L. **"What Makes Mathematical Sense to Children?"** A paper presented at the annual Teacher Corps Conference. Washington, D.C., July, 1976. ERIC SE 021 277.

"What Should Prospective Teachers Know About Mental Imagery?" A paper presented at the Annual Conference of the National Council of Teachers of Mathematics. San Diego, California, April, 1978.

ACHIEVEMENT IN MEASUREMENT AND GEOMETRY IN THE JUNIOR DIVISION

R.S. Smith, Faculty of Education, The University of Western Ontario

In the fall of 1977, the elementary school vice-principals in one of the areas of London undertook a testing program using Grade 4 Mathematics as the subject area. To test all of the mathematics core curriculum was a large task. Test items were written for the various topics including measurement. The refined questions, 140 in all, were printed in 10 booklets so that each pupil met a randomly selected set of 14 questions. Items were multiple choice in nature and the whole thing was computer marked. The results of these tests were then aggregated to give a final score so that item difficulty and per cent correct responses might be obtained. Unfortunately, this computer marked test was inappropriate for Geometry. The committee turned to the report **Measuring Achievement At The Primary and Junior Levels** and found that "Measurement and Geometry objectives are not measured to a great extent at the Junior Division level . . . only a small number of test items, if any, are related to Measurement and Geometry."[1] I was asked to help in this area. It seemed best to use an interview format, to develop a test instrument which would allow manipu-

lation of materials and because of the limited time available to use a random sample of the population only.

Before test items could be designed, it was necessary to specify learning outcomes implied by each objective. Fortunately, some ten years ago, when work first began on a scope and sequence chart for a mathematics program the Measurement and Evaluation Chief, Dr. J. Bramwell was involved. He suggested that the program outline be written with test items in mind. In spite of this, some of the objectives, terminology and phrases which crept in were difficult to translate into test items. If these test items were to measure the knowledge or skill implied by the objective, the question one always had to keep in mind was "What would a child be able to do if the teacher has taught this objective effectively?" Many of the items required the manipulation of polyhedra in place of obtaining information from pictures and the use of measurement instruments rather than the reading of numbers from diagrams. A shoe box full of materials was collected and an interview format was developed. I was interested in what the children did, their actions and stumblings toward an answer as well as the answers they produced. This included the wrong answers as well as the correct ones.

The test instrument was criterion referenced to the London Board of Education Mathematics Core Guidelines. A colleague and I developed test items and trial ran them. The interviews were completed by two other teachers, a colleague and myself. The seventy-nine items were grouped under the four sections, as follows: A. Measurement - linear, perimeter, area, volume.

B. 3-Dimensional Shapes.
C. 2-Dimensional Shapes
D. Symmetry and Congruence

Each of the four sections were set up so that on the left side of the page the materials were listed in brackets under which appeared the question or statement that the tester would use. Directly opposite each question was placed a statement of the objective being measured by that item. The following example is a part of page one.

TEST INSTRUMENT (GRADE FOUR)	INTERVIEW TECHNIQUE CRITERION REFERENCED	
MEASUREMENT		**Computer Card**
Linear		**Yes** **No**
(A line drawn 6.7 cm. A ruler, mm smallest unit Markings start at 0) T. Measure the length of the line. What is the smallest unit? What is the length in mm? Give me your answer in cm using a decimal. Write the measure to the nearest ½ cm.	1. Starts measure at zero 2. States measure in cm & mm 3. Knows mm unit 4. States measure in mm 5. States, using decimal 6. Writes, 6.5 cm.	

The interviews were conducted in Grade 5 classes at the beginning of the grade 5 year but using objectives for Grade 4. For each school the number of Grade 5 children was obtained, and from a random numbers chart, numerals were picked so that a 10% sample would be taken. When the tester arrived at the school the class lists were put together, and the children asked to come to the tester as their number came up. Each interview lasted fifteen to twenty minutes. During some of the interviews, a tape recorder was turned on. Their recorded comments along with written notes proved quite helpful later as the results were reviewed. Because pupils were at various stages in the development of their thinking from concrete to abstract, their talk suggested how far they had developed. The pupils seemed to enjoy the interview but in many instances it seemed they were meeting the various materials for the first time.

What were some of the findings? Keep in mind the purpose of the interview was to measure performance. By chance, a few questions on the V.P.'s pencil and paper matrix sampling test attempted to measure the same objectives and may illustrate the difference in the two kinds of tests. In the following examples T. indicates tester's question from the interview instrument and a

box has been drawn around the question from the pencil and paper test. In each case the % of correst responses is given.

1. T. Write the measure to the nearest ½ cm - item #6 above 41%

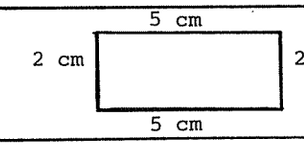
The length of the line segment to the nearest centimetre is _____.

A. 0 cm B. 5 cm C. 7 cm D. 6 cm 86%

2. T. Find the perimeter of this shape. (accepted answer) 38-40 cm. 32%
(see figure 1, no dimensions given to the child)

The perimeter of this figure is:

A. 10 cm B. 7 cm C. 4 cm D. 14 cm

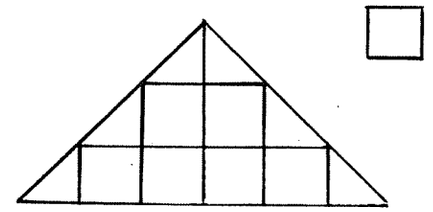


82%

3. T. Find the area of this figure. 1 cm² - acetate grid available 25%
(see figure 2, no dimensions given to child)

The area of the triangle is:

A. 12 cm²
B. 9 cm²
C. 8.5 cm²
D. 6 cm²

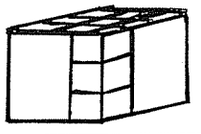


46%

4. T. Find the volume of this structure. (irregular tower of 9 centicubes) 27%

The volume of this cube is:

A. 16 cm³ B. 15 cm³ C. 27 cm³ D. 18 cm³



89%

In the spring of 1978, a second test instrument was designed using the Geometry section of the Mathematics Core Guidelines for Grade 5. The test format was similar in nature and again a group of testers used the same random numbers for a 10% sample of the pupils from the same area schools. For the most part the same pupils were tested. Some of the questions were the same; as the objectives listed in Grade 5 had not changed. Thus some indication of the growth made during that year, really seven months, was noted. For each example, the per cent of responses correct is given.

Comparison of results Gr. 4 and Gr. 5 on same question to same pupils

			% correct	
Topic	**Question**	**Objective**	**Gr. 4**	**Gr. 5**
Linear	Given line 6.7 cm & a ruler	Start measure at zero	59	84
	Measure length of line	States measure correctly	46	49
	Write correctly to nearest ½ cm	Writes measure correctly 6.5 cm	40	38
Perimeter	Show me the perimeter of figure	Knows term perimeter	49	75
	Find perimeter of concave pentagon - a crown	Finds perimeter of concave pentagon	32	37

Area	Show me the area of each figure		Knows term area	56	74
	Colour in standard unit to measure area		Knows area unit of 1 cm^2	10	49
	Find area of convex pentagon a house (using a cm^2 grid overlay)		Finds area of pentagon	31	46
	Show a simpler way of finding area - rectangle		Knows l x w	29	59
Volume	Tell me what volume is		Shows volume concept	15	43
	Show me standard unit used to measure volume		Chooses volume unit 1 cm^3	3	29
	How many cm^3 in this structure?		Counts units of volume	28	80

Symmetry

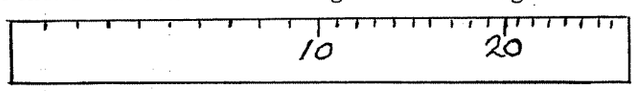
l. of s.	Place the straight edge along a l. of s.	vertical l. of s.	Shows l. of s. for leaf	62	55
		horizontal l. of s.	Shows l. of s. for kite	75	61
		word l. of s.	Shows l. of s. for letter C	63	30
			Shows l. of s. for word WOW	40	22
			Shows no l. of s. for letter F	50	29
			Shows no l. of s. for word BAD	33	19

The Grade 4 questions on symmetry came after the pupils talked about cutting a cube into two identical parts. Gr. 5 did not have this preparatory part. It appears results are susceptible to direct instruction.

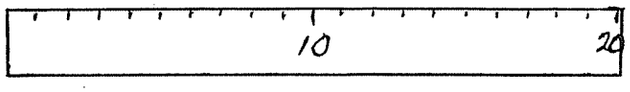
The following are four examples of children's comments and/or work done. These examples illustrate the difficulties encountered in performing the task which was set to measure an objective. The tester's question or statement with pupil responses are set inside a rectangle following the objective in each case.

A. Objective: Knows standard units must be used in linear measurement. Equipment, (2 rulers placed before the pupils. A - the top ruler has 10 mm markings to 10, then 6 mm marking to 20 and 4 mm markings to the end. B - the bottom ruler has regular 1 cm markings.

Top ruler



bottom ruler



T. With which ruler would you measure a line? 62% correct
I then would ask the further question, "Why would you use that ruler?"
Many saw the difference between the rulers and said immediately "because they are the same apart", or "It appears the top ruler has mistakes". Mark said "On the bottom ruler (pointing to the 10) this is one half of the 20". Some pupils got the right answer but for dubious reasons, I would suggest. "Bottom ruler has bigger spaces". "I chose the bottom ruler because it stops at 20 and the other goes past 20. It looks better with the 20 at the very end". "20 is further and may have more cm in between". "It is spread out and easier to look at". Some pupils in their incorrect answer also show dubious thinking. "I chose the top ruler because it has more centimetres. It has 27 cm and the other one has only 20 cm." "The top one is more like a regular ruler. It almost comes to 30 cm". "It has smaller marks so you can round off easier".

B. Objective: Finds the perimeter of a concave pentagon - a crown. (a red bristol board crown shape and a ruler, pencil and paper)

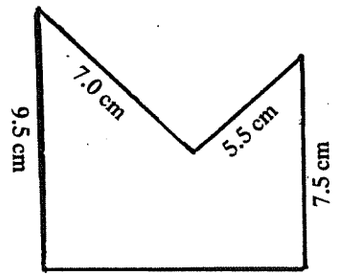


fig. 1

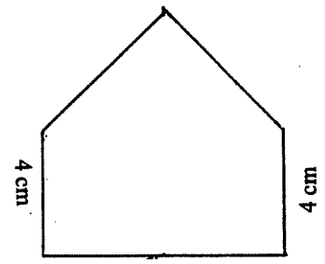


fig. 2

T. Find the perimeter of this figure. 48% correct

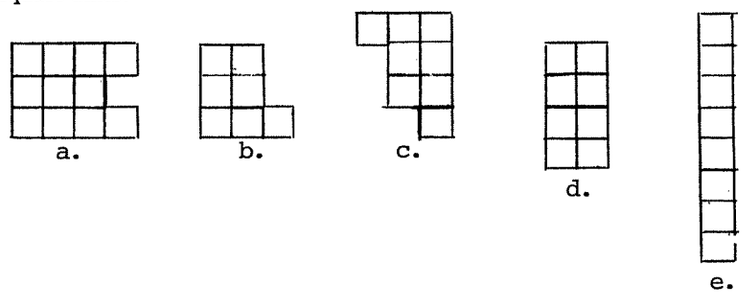
A ruler used, sides recorded and sum obtained. 38-40 cm accepted. Some measured to whole numbers only $7 + 9 + 9 + 7 + 5 = 37$ cm
Some had trouble in listing the lengths, with consequent addition problems.

9	9	5½
9.5	9.4	7½ 9½ 7
9.5	7.6	
5.5	5.8	
7	7	9½

One pupil mixed simple fractions and decimal fractions so created a problem,
e.g. $5½ + 7 + 8.4 + 9.7 + 7½ =$

C. Objective: Recognizes conservation of area.

T. Each of these figures has a number of square units of area. Which one has a different number of square units? 59% correct



Most pupils who got the incorrect answer chose (e), and jumped to that answer. When I questioned them further about why they chose (e), I got the following answers: "the long thin one is different"; "it is straight down"; "it is a longer shape"; "because it is like a ruler". If I pushed further and asked what area meant, they would then go back and count squares.

D. Objective: Estimates volume correctly (centicubes & a small match box).

T. Find the volume of this matchbox. 43% correct

Pupils were allowed to use the cm^3 and place them in the box. As the box was a bit wider and longer (3.3 cm by 4.7 cm but only 1 cm tall) pupils had to see parts of cubes would be needed to fill the box. Most gave 12 cm^3 as the volume - missing this idea. One boy did split the box trying to force the extra centicubes in! We allowed 14-16 cm^3 .

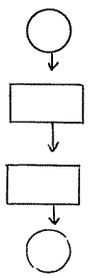
To complete this series of achievement tests for the Junior Division in Measurement and Geometry, I look forward to creating a similar test for Grade 6. The instrument needs refining which will come as teachers become more familiar with this hands-on testing. The vice-principals reported that there was more interest in and more Geometry taught than in former years. It is very interesting work. I have had the opportunity to see and hear what children were doing as opposed to reading the results of a pencil and paper test or of a computer print out. I do believe both have a place.

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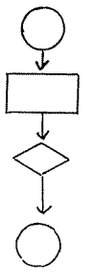
1. **Measuring Achievement At The Primary and Junior Levels,** O.I.S. 1975, p. 213

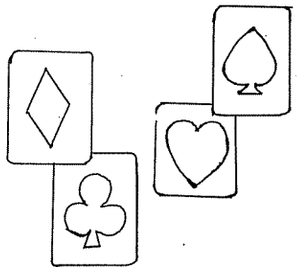
"DO YA WANNA SEE A CARD TRICK?"

C.W. Zeller, Head of Mathematics, Morning Star Secondary School, Mississauga



Phase I: (Yes)
Do you have some cards?
(No)
Doesn't matter shuffle them anyway:
(Fakes a shuffle of imaginary cards)
Finished?
(Yes)
Draw one out:
(Pretends to draw a card)
Got it memorized?
(Yes, sets imaginary deck down --- 7 ♣)
Take its number
(7)
If it is a Jack, call it 11; a Queen, call it 12; a King, call it 13; and Ace, of course 1:
(Right)
Double the number:
($7 \times 2 = 14$)
Add one:
($14 + 1 = 15$)
Multiply by five:
($15 \times 5 = 75$)
Now if it is a club, add five; a diamond, add six;
a heart, add seven; or a spade, add eight;
($75 + 5 = 80$)
Give me the number:
(80)
That means you had the 7 ♣:
(Fantastic!)





The foregoing act with the responses given parenthetically is a powerful introduction to the utility of flow diagrams (charts) as an analytic tool.

Card is the N of S:

$$N \downarrow 2N \downarrow 2N + 1 \downarrow (2N + 1)5 = 10N + 5$$

$$\text{CLUB} = (10N + 5) + 5 = 10N + 10 = 10(N + 1) + 0$$
$$\text{DIAMOND} = (10N + 5) + 6 = 10N + 11 = 10(N + 1) + 1$$
$$\text{HEART} = (10N + 5) + 7) = 10N + 12 = 10(N + 1) + 2$$
$$\text{SPADE} = (10N + 5) + 8 = 10N + 13 = 10(N + 1) + 3$$

Thus, as in the dialogue, the result 80 means immediately a club from the last digit, and the 80/10 - 1 = 7 of clubs in particular. Similarly, an answer of 112 would mean the 110/10 - 1 = 10 of hearts, and the resolution of any three digit number, mnp, is the (mn - 1) of the pth suit from above.

Phase II: This notion can be used further. The same basic idea is used for a game of **Guess Your Pair.** In the initial study of the coordinate system, one frequently confines his attention the ten-square lattice, $L^2_{10} = \{0,1,2,\ldots 9\} X \{0,1,2,\ldots 9\}$. Take $(x,y) \in L^2_{10}$, and proceed through:

i) pick a pair	(x,y) = (7,7)
ii) double your first coordinate	(2x) = (14)
iii) add three	(2x + 3) = (17)
iv) multiply by five	(2x + 3)5 = (10x + 15) = (85)
v) add the second coordinate	(10x + 15 + y) = (85 + 7) = (92)
vi) what did you get?	(92)
vii) your pair was 92 - 15 (done mentally) = 77 i.e. the pair (7,7)	

A generalized flow diagram for a pair (x,y) could be illustrated as follows:

x
↓
kx
↓
kx + p
↓
(kx + p) 10/k
↓
10x + 10p/k
↓
10x + 10p/k + y
↓
A

Then, A = 10x + 10p/k + y, makes A - 10p/k = 10x + y, and thus (x,y) is recovered. It would appear that the choice of k and p must be positive, but otherwise unconstrained.

Phase III: Allow the students to develop **Guess Your Triple** with a flow diagram for selection (x,y,z) as follows:

X
↓
kx
↓
kx + p
↓
(kx + p) 10/k
↓
10x + 10p/k + y
↓
(10x + 10p/k + y)m
↓
10mx + 10mp/k + my + q
↓
(10mx + 10mp/k + my + q)10/m
↓
100x + 10y + 100p/k + 10q/m
↓
100x + 10y + 100p/k + 10q/m + z
↓
A

Thus, A - 100p/k - 10q/m = 100x + 10y + z, and then the triple (x,y,z) is recovered. Again, positive k,m,p, and q seem sensible.

Phave IV: The idea so fascinated one group of students that they worked out a routine for social insurance numbers using the same idea for points from L^9_{10}

Phase V: The reader is invited to communicate with the writer any special cases involving negative k,m,n,p, and q which might be explored.

BAD, BAD ONE HUNDRED

Carol F. Novillis, University of Arizona

Reprinted from the Arizona Teacher of Mathematics, Winter, 1978)

This card game, based on the game Twenty-One, was designed by Anthony Johnson while he was an undergraduate elementary education major at Florida International University in Miami, Florida. It provides practice with adding upper decade facts.

Rules and Play:

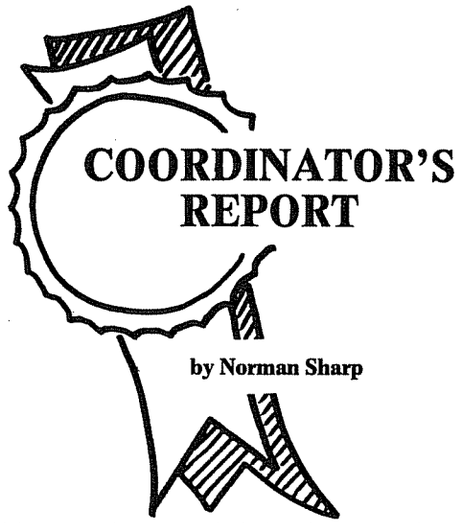
1. For two to four players and using one or two regular decks of cards with the tens and face cards removed from the deck.
2. The dealer shuffles the deck and any player cuts the deck. Two cards are dealt (1 down and 1 up) to each player. The cards have the numerical value presented on the cards with the ace equaling one, except each player secretly chooses one card to represent its face value times 10. This is his tens card.
3. The object of the game is to have two or more cards total 99 or as close to 99 as possible without exceeding 99. After the first two cards are dealt, each player may either pass or be dealt, face up, an additional card. A player may continue to receive additional cards as long as his total is less than 99 or he has not previously passed. If a player has a hand with a total of more than 99, he may be able to stay in the game by changing his tens card so that his new total is less than 100. After everyone has passed, each player identifies his tens card and then his total so that the player with the largest total less than 100 can be declared the winner.
4. The winner of a round scores 10 points. In case of a tie, each person scores 4 points.
5. At the conclusion of each round, the cards are reassembled in a pack and the deal passes clockwise around the table. The winner of the game is the first person to score 100.

Sample

Sample Game:	**Player 1**	**Player 2**
Original Deal	9 3 9 is selected to be the tens card. Total is 93.	8 7 8 is selected to be the tens card. Total is 87.
Next Deal:	8 Total is 101. He's over, so he changes the tens card to 8. New total is 92.	6 Total is 93.
Next deal:	6 Total is 98.	7 Total is 100. He's over, so he changes his tens card to 7. New total is 91.
Next Deal:	PASS	5 Total is 96.
Next Deal:		PASS
Player 1 wins with a total of 98.		

Brain Teasers

1. There are 42 desks in the school room. There are seven vacant seats and two pupils are absent. How many pupils are there on the roll? Answer on pg. 68



During the middle 1960's, large school boards in Ontario began to appoint teachers to the position of coordinator. It appeared that the position of mathematics coordinator was created with the view that courses in mathematics were undergoing changes. Since most of those in the position of coordinator knew each other through their associations in mathematics groups, it was only natural that mathematics coordinators met as a group to discuss common concerns. For several years this group of mathematicians met on an informal basis at the call of any of the individuals, the meeting being chared by John Del Grand, Coordinator for North York Board of Education. Meeting on an informal basis seemed to reflect the way in which the coordinators operated with the teachers and with each other. Lately it has become evident that a formal organization would be required to initiate some of the projects and to tackle some of the responsibilities we felt could be done by a coordinators association.

At a meeting in Hamilton on 1978 01 04, it was decided to prepare a constitution for acceptance by the mathematics coordinators which would indicate the aims and by-laws for the group. On 1978 11 07, a constitution prepared by Michael Silbert, Hamiltoln Board of Education, Warren Sebben, Lincoln County Board of Education, and Alex Norrie, Peel Board of Education, was presented and accepted. The officers for the year are:

President - John Del Grande, North York Board of Education
Vice-President - Michael Silbert, Hamilton Board of Education
Secretary-Treasurer - Norman Sharp, Etobicoke Board of Education

The name of the organization is the Ontario Mathematics Coordinators' Association. Its aims are:

1. To improve the quality of mathematics education in the Province of Ontario.
2. To encourage and stimulate professional growth and development on the part of all members of the mathematics education community in the Province of Ontario.
3. To encourage communication with regard to all aspects of mathematics education amongst all interested parties in the Province of Ontario.

Membership is open to individuals who are working in an educational jurisdiction, have responsibility for the coordination of mathematics programs in all schools within a board of education as defined in the Education Act, and have responsibility for at least three contiguous divisions.

Regular general meetings are scheduled to be held on a bi-monthly basis throughout the school year with the annual general meeting held in May or June each year. Other general meetings or special meetings may be scheduled by a majority decision of the executive steering committee. The quorum for a meeting is two-thirds of the membership; at present there are 14 active members. The executive committee has the usual responsibilities associated with an organization of this type and will always meet prior to a general meeting in order to prepare an agenda. Amendments to our constitution can be made at the general meetings and must be carried by a two-thirds majority vote.

As a group the Association has considered several projects which it may undertake. This year it is concentrating on the Ministry Guideline for the Intermediate Division. In order to look at the broad spectrum of mathematics in the schools, the Coordinators' Association is preparing a rationale for K-13 mathematics for presentation to the Ministry. This will include recent thoughts about "The Formative Years", some positive suggestions for "The Intermediate Guideline" and a look into the future for revision of "Senior Division Mathematics". The five strands upon which the rationale is written are Numeration, Algebra, Geometry, Measurement and Information Processing.

Many boards in Ontario are becoming involved in a testing program to fit into their overall evaluation. The Coordinators see this as their next project. Plans have not been finalized for this endeavour but we feel it will be a way of pooling items and preparing some standardized tests for broad application.

Information regarding the group has been forwarded to the Ontario Association for Mathematics Education, Ontario Coordinators' Association, and to the National Council of Supervisors of Mathematics.

Since this is the first year of formal operations of the Association, there may be several innovative ideas for discussion. We are willing to support any organization which is endeavouring to improve mathematics in Ontario from K - 13.

Special Announcement

1 Day Conference on **Basic Level Mathematics** (H.S.I.) sponsored by O.M.C.A.
Friday May 25, 1979 — To be held in Hamilton
Key Note Speaker: Mr. M.J. McKenna — Topic: Basic Mathematics
Validation Co-ordinator — What Are We Trying To Do?
Where Are We Going From Here?

For further details write: Mathematics Department
City of Hamilton, Ontario
Box 558
Hamilton, Ontario L8N 3L1

Michael Silbert

SOME THOUGHTS ABOUT GEOMETRY

Ian Harris, Faculty of Education, King's College, London
David S. Fielker, Abbey Wood Mathematics Centre

Reprinted from **Mathematics Teaching** September 1974 Number 68

Editors Note: David Fielker has just finished a very long article, based on the final section of his M.Ed. thesis, which will appear in the March issue of "Educational Studies in Mathematics" which sums up his thoughts on Geometry. It includes material from "A Structured Approach to Primary Geometry" which appeared in M.T. No. 63.

We were, in fact, getting towards the end of a weekend conference organized by the London Branch, during which most of us had agreed to work at something geometrical. One idea we had was that too many geometrical topics start in space and end up with numbers. For instance, if one considered the number of intersections possible with n lines one arrived at triangular numbers, and sometimes we felt that anything in elementary combinatorial geometry produced only number sequences.

Instead, we invited our colleagues to start with space and stay in space, or even perhaps start with number and move into space. Most of the weekend, then, was spent with tessellations, "stick" polyhedra, finite projective geometries and paper folding. Such was our effect on the conference room of the hotel in which we were staying that on returning from lunch on the Sunday we had found a middle-aged couple staring through the glass door; as they stood back to let us through they asked, "Do tell us; are you children or inventors?"

We had finished working, and we were talking (which we are sure you will agree is the correct way round). What was geometry about? Why were we teaching it?

True, geometry is a branch of mathematics, in which the same general mathematical activities go on as in other branches: exploration, observation, hypothesis, proof, generalisation, etc. What was particular about geometry? Was it just that it dealt with space?

We began there, and at this point someone was raising the problems of dimension, the relations

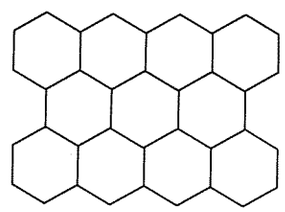
between two-dimensional and three-dimensional space, and the difficulties children often have in transferring from one to the other. But in fact we never did quite decide exactly where to fit this into our subsequent thoughts.

We begin anyway with spatial experiences, and every infant teacher knows that this has been going on long before the children come to school, and she continues the good work. Although this collection of experience is important, perhaps nobody is really doing anything mathematical until they begin the organization of their experiences, and with this organization comes a process of abstraction.

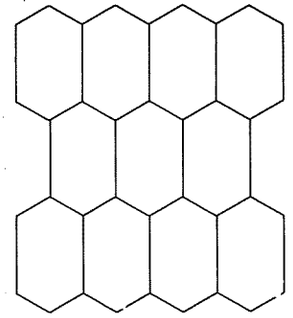
One aspect of the organization is the setting up of a ''scaffolding'' in space, a frame of reference in which space can be described. This may be a coordinate system - the axes of rectangular cartesian geometry, or latitude and longitude on the earth's surface. This seems, however, to be another aspect of geometry that takes us from space to number; the cartesian plane, for instance, transforms points into number pairs and lines into equations, which are relations between sets of numbers.

One way of staying in space is to analyse its structure in terms of algebraic rather than numerical ideas, in other words in terms of sets, relations and operations, and some suggestions for this were made in *A Structural Approach to Primary School Geometry in MT No. 63*. But this algebraisation is more than mere analysis, more even than abstraction. Pip Leaton put his finger on it when he dropped into our conference on the Sunday morning; he quoted Caleb Gattegno who apparently had long ago suggested that whereas algebra took place in the head, geometry was felt ''in the pit of the stomach''. A combination of the two seems to make effective a process of ''internalisation'', so that what begins as only a physical experience eventually becomes something over which we have complete control.

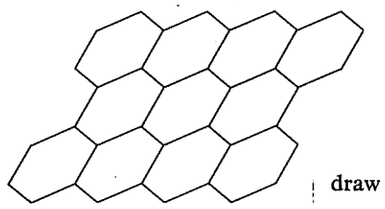
Perhaps an example will help. Let us consider a tessellation of regular hexagons.



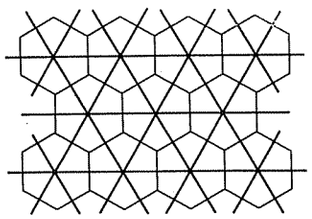
We can take our hexagons and stretch,



shear,



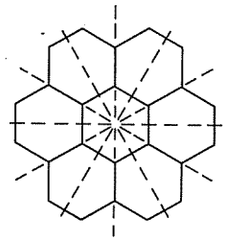
draw the dual tessellation



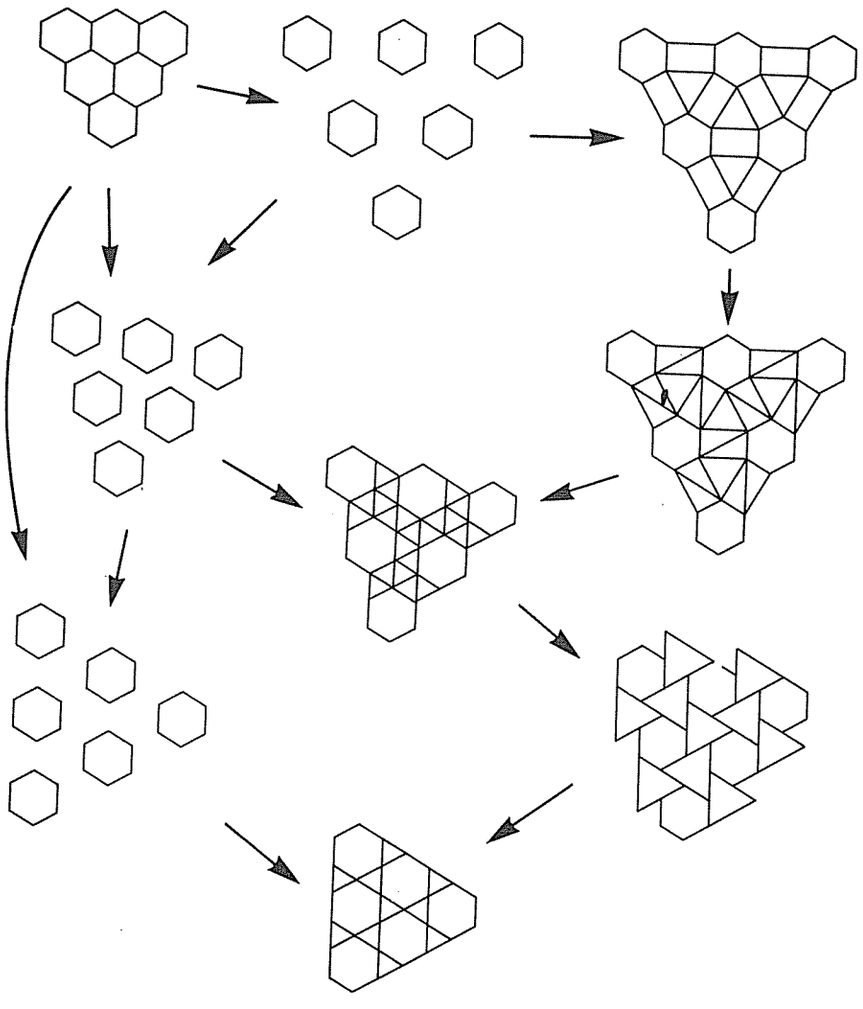
This begins in a purely physical way, passively by observing an array of tiles or a honeyomb, or actively by putting together cardboard or plastic hexagons.

Maybe the idea of a ''scaffolding'' doesn't apply here, but we can move into number rather than space if we consider that regular hexagons tessellate because their angles are 120°, or if we consider Euler's theorem for a plane network which gives a relation between arcs, regions and nodes. Somehow the numbers explain what is happening without letting us ''feel'' it. A purely geometrical analysis would probably involve symmetry. One can look at lines of reflective symmetry (here shown through the centres of one hexagon only) and centres of rotational symmetry.

But we are still concerned with analysis, which in a way is as yet a passive activity. We can be more dynamic, more imaginative, more creative if we manipulate the situation and exert our power over it.



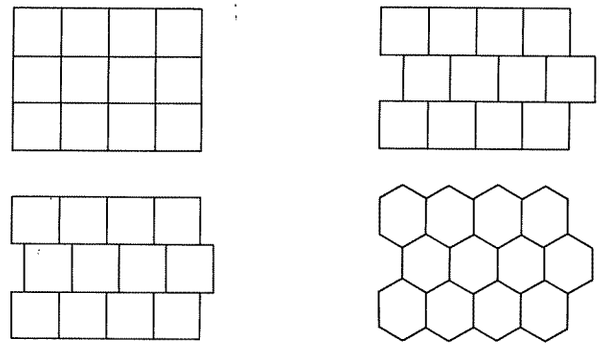
or find other ways of producing different tessellations.



These methods are those of transformation and dissection, two of the ideas discussed in *A Structural Approach*. They do not involve ideas of number, but rely solely on spatial concepts, or ideas of structure applied to spatial concepts. And they help us to internalise the situation, to make it a part of us.

These activities also assist the process of abstraction, which is perhaps what "internalisation" is all about. Transformations and dissections effect change, but in all these changes there are things

which remain the same, invariants of the transformations. What is invariant in all the changes in the abstract structure of the hexagonal array, something deeper than its symmetry, or its topology. Mathematically we can only define certain invariants of subsets of the transformations, since many of them change the symmetry or the topology or some other analysis of the structure. What remains throughout all of them is more of a feeling than a mathematically described invariant. How else could one describe the following series of transformations from squares to hexagons?

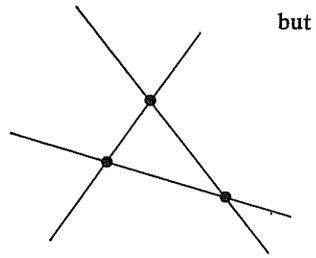


Pernaps the process of abstraction can be better described and also extended if we consider a simpler geometrical object. Take the triangle. Young children meet triangles made of wood, plastic, paper; "solid" triangles; triangles made from rubber bands stretched on geoboards or from geostrips linked together; triangles in pencil on paper, in chalk on blackboards or in print in books; triangular holes. A triangle emerges early as the essence of all these different materials, an equivalence class of concrete objects.

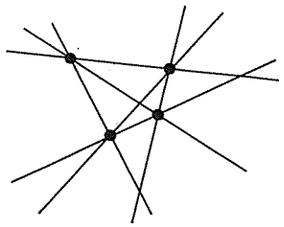
(It is also an equivalence class of shapes. We have met young children who only recognise a triangle if it is equilateral and has a horizontal base, or is isosceles with a horizontal base. But then we have even met teachers who think all hexagons are regular!)

Abstraction is a continual process, and one cannot say that at any point it is complete, only that a certain degree of abstraction has been achieved. So far we have described a fairly full abstraction of the concept of triangle in a Euclidean sense, but there are further possibilities if we make some sort of generalisation, say of the space in which we embody our triangles.

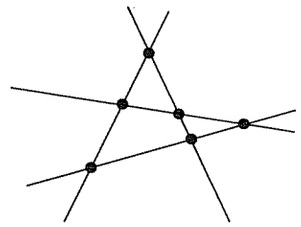
In a projective space, for instance, we distinguish between a triangle and a trilateral. They both look the same,



but a quadrangle (four points, each pair determining a line)



is not the same as a quadrilateral



which has four sides determining six points.

(Only now perhaps do we see the inconsistency of the words we use: triangle, three angles, but quadrilateral, four sides).

On the surface of a sphere triangles look something like plane triangles, but preserve none of the usual angle properties; the angle-sum is more than 180° for instance. In topology, a triangle is three points joined by any three lines, and perhaps in point-set topology it is just an array of three points.



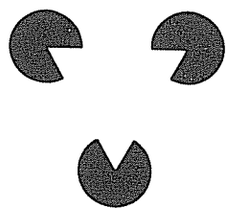
Which reminds us that Benny, aged 2½, fetched the three bottles of milk from the front door and put them on the kitchen floor. "Look," he said. "I made a t'iangle". (So much for Piaget's stage of concrete operations).

One can compare this process of abstraction with what goes on in number. Instead of the concept of triangle, take the concept of 5. This, like the triangle, is an equivalence class, but of sets of concrete objects; it is what all sets of five things have in common. We also generalise this concept by considering its equivalent in other numerical "spaces". We have the ratio 5/1 and the integer + 5, and later the real number 4.999 . . . and the complex number $5 + 0i$. And just as we can consider triangles in finite spaces so we can have numbers in finite arithmetics.

Perhaps there is some sort of pedagogical isomorphism between arithmetic and geometry.

How far can we take the abstraction process? One of us once invited a class of ten-year olds to make as many different triangles as they could, and supplied a collection of appropriate materials. They did very well, even the boy who went away muttering, "I'm going to make a *four*-sided triangle."

We then began a discussion about whether a triangle consisted of just its sides, or whether it included the space inside. And when we cut a triangle out of a piece of paper, were the sides round the triangle, or were they left round the hole? And could you have a triangle without any sides, just consisting of the space inside? . . .



FUTURE MEETINGS

Local

April 4 TORNY Inaugural Meeting
April COMA Meeting
April 24 GVMA Conference
April 25 QSLMA Conference
April 27 SEYMA Conference
May 25 OMCA Conference

Executive Meeting 1979 04 06
Annual Meeting 1979 05 13
Executive Meeting 1979 05 25
Council Meeting 1979 06 02

International

OAME/AOEM 6th Annual, 1979
May 11, 12, 13 Peterborough
NCTM 57th Annual, 1979
April 18-21 Boston
NCTM Calgary Meeting, 1979
October 11-13 Calgary
AAMT 8th Biennial, 1980
January 14-18 Canberra, Australia
OAME/AOEM 7th Annual, 1980
May 8-11 Toronto
NCTM 58th Annual, 1980
April 16-19 Seattle

P.S. England ATM Course at Worcester April 9-12 Diagnostic & Prescriptive Math April 22-24 Tampa.

QUADRILATERALS BY TRANSFORMATION

D. Boyle and T.J. Griffiths, A.B. Lucas Secondary School, London, Ontario

The use of transformations has made possible some new and interesting methods of developing and investigating concepts and properties associated with geometric shapes. An example of this is the development of the named quadrilaterals, such as parallelograms and squares, by applying to points and triangles the transformations of translation, reflection and rotation. These are the congruency transformations, under which figures map into congruent figures.

The major feature of studying geometry by a transformation approach is the visuality of the work and the link with everyday experience. For example, all students will have practical experience with reflections of buildings in water. From this a workable definition of reflection and its properties can easily be obtained.

In this article it is assumed that the students will have had some experience of using transformations in school at a simple level, perhaps using grids or paper folding, but that they have not yet made a formal study. This approach can be used to help bridge the gap, and the degree of formality or rigour can be increased as the study of this topic is developed.

The format of this article is as follows. The specified transformation is first defined and its properties listed as far as they are required in the context. These properties will not be proved, but in every case they are intuitively and observably true. They can all be proved from the definition by using simple methods, such as those used in the traditional Euclidean approach using congruent triangle definitions. The formal justifications would follow for the students at a later stage. The possible shapes which can be obtained by each transformation are then listed, with the methods by which they are constructed and the resulting properties. Some of the properties are inherent in the method of construction, and others are proved.

	R				F		R							
T		A						M	A	T				
											I			
			N	S									N	S
						O						O		

Translation

Definition: A specified translation slides a point to an image point such that the distance and direction moved are the same for any point and its image.
OR - For a given translation the line segments joining points to their images are all equal in length and parallel.
Property - Line segments map into parallel and equal line segments.

Construction

2. points A and B slid by any translation

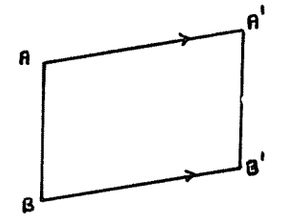


Figure: - Parallelogram
Properties: - opposite sides equal and parallel

Construction
2 points A and B slid by a translation equal in length to AB

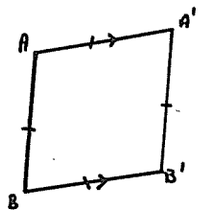


Figure: - Rhombus
Properties: - A rhombus is a parallelogram with adjacent sides equal.

Construction
2 points A and B slid by a translation perpendicular to AB

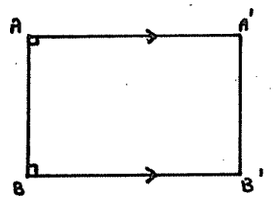


Figure: - Rectangle
Properties: - A rectangle is a parallelogram with right angles

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Construction
2 points A and B slid by a translation perpendicular and equal in length to AB

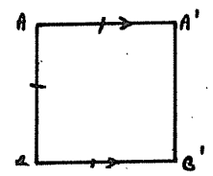
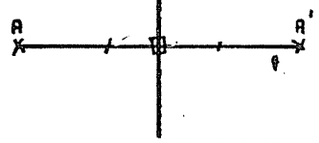


Figure: - Square
Properties: - A square is a rectangle with all sides equal; a rhombus with right angles; a parallelogram with all sides equal and right angles.

L

Reflection



Definition:

If A and A' are mirror imaged in L, then L perpendicularly bisects AA'.

Properties: Line segments and angles map into equal line segments and angles.

Construction with one reflecting line

1 point B on L
1 point A not on L

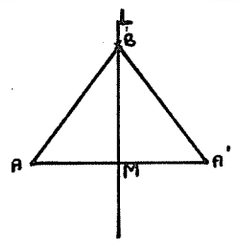


Figure: Isosceles triangle
Properties: -$\angle BAA' = \angle BA'A$
BM bisects AA' perpendicularly.
N.B. The authors acknowledge that this is not a quadrilateral, but it is too neat to leave out.

Construction with one reflecting line
2 points B and C on L
1 point A not on L

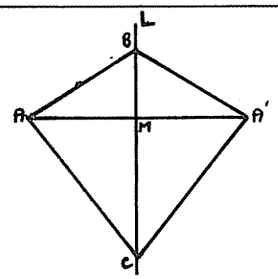


Figure: - Kite
Properties: - BC bisects AA' perpendicularly.
$\angle BAC = \angle BA'C$
$\angle BCA = \angle BCA', \angle CBA = \angle CBA'$

Construction with one reflecting line

2 points B and C on L
1 point A not on L
AB = AC

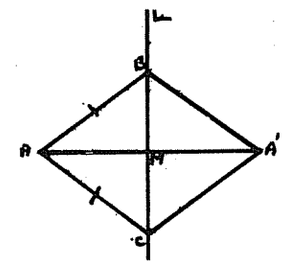


Figure: - Rhombus
Properties: - A rhombus is a kite with all sides equal. Since ABC is an isosceles triangle, M is the mid-point of BC, so the diagonals bisect each other.

Construction with one reflecting line

2 points B and C on L
1 point A not on L
ABC is a right angled isosceles triangle
∠BAC = 90°, AB = AC

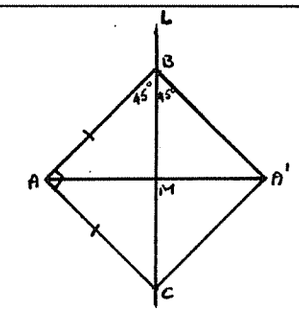


Figure: - Square
Properties: - A square is a rhombus with right angles

Construction with one reflecting line

2 points A and B not on L

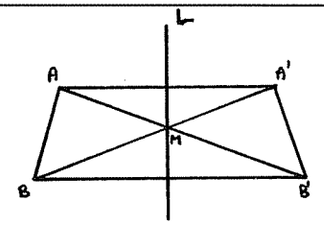


Figure: - Isosceles trapezium
Properties: - ∠A'AB' = ∠A'AB, ∠A'B'B = ∠ABB', since A↔A' and B↔B'
AB'↔A'B
AB' = A'B
also M is its own image and is therefore on L

Two perpendicular reflecting lines

1 point A not on L or L' and the image also reflected.

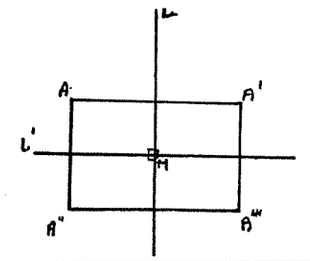


Figure: - Rectangle

Two perpendicular reflecting lines
1 point A on L
1 point B on L'

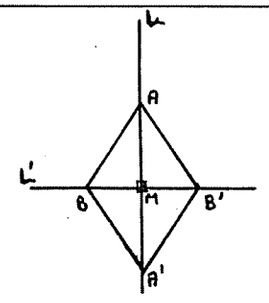


Figure: - Rhombus
Properties: - The diagonals of a rhombus bisect each other perpendicularly.

Two perpendicular reflecting lines

1 point A on L
1 point B on L'
so that AM = BM

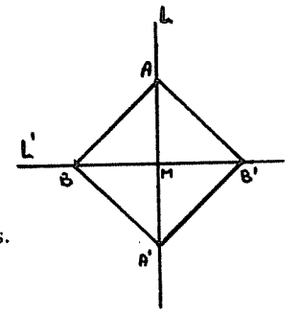


Figure: - Square
Properties: - A square is a rhombus with equal diagonals.

Rotation of 180°

Definition: Each point is rotated about a given centre of rotation by 180°.
Properties: A line segment maps into an image line segment such that the two are equal in length and parallel.

A line drawn to the centre of rotation and its image are collinear.

Construction

Triangle ABC is rotated about the mid-point M of side AC

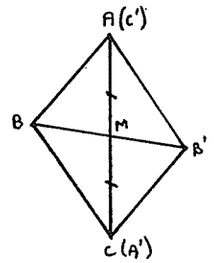


Figure: Parallelogram
Properties: The opposite sides are equal and parallel. The opposite angles are equal. The diagonals bisect each other. This last is proved as an example. B → B' ∴ BM → B'M

so BM = B'M and BMB' is a straight line. M is the mid-point of BB' but M is the mid-point of AC ∴ the diagonals bisect each other.

Construction

Isosceles triangle ABC is rotated about the mid-point M of side BC

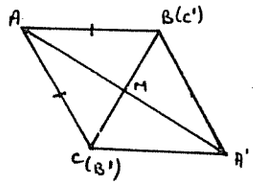


Figure: - Rhombus
Properties: - A rhombus is a parallelogram with all sides equal. Since ABC is an isosceles triangle with M the mid-point of BC, then AM ⊥ BC. ∴ the diagonals are perpendicular.

Construction

Right angled triangle ABC rotated about the mid-point of hypotenuse BC.

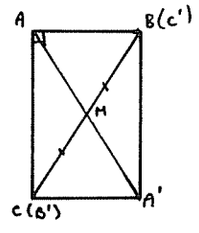
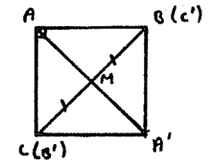


Figure: - Rectangle
Properties: - A rectangle is a parallelogram with right angles.

Construction



Right angled isosceles triangle ABC rotated about the mid-point M of BC.

Figure: - Square
Properties: - All those of a rectangle, parallelogram and a rhombus.

It can be seen in the development of the shapes that this approach also demonstrates how a mathematical topic can be investigated by considering all possibilities in a logical way. The best exmple of this is the devleopment of quadrilaterals through reflection, in which the number of points reflected is increased with the points in all possible positions, and then the number of reflecting lines is increased.

Another concept demonstrated in this approach is that of the inheritence of properties. An example of this is that if a rectangle is obtained by the same transformation as a parallelogram, but with one extra restriction, then the rectangle is a parallelogram and has all the properties of a parallelogram.

It should be noted that not all the named quadrilaterals are obtained by every transformation. For example, parallelograms in general cannot be generated by reflection.

This study can be extended in various directions:

1. The work can be formalised by stating the justifications in proof format.
2. The properties discovered in this development can be used in the study of other geometric figures involving quadrilaterals.
3. A similar approach can be made in developing and investigating regular polygons or even three dimensional figures and polyhedra.
4. The disciplines of architecture, crystallography and art all use the concepts of transformations, and are rich in examples and pictures which can be used as "real life" examples.

A REVIEW TEST PROGRAM

L. Shapiro, W.F. Herman Secondary School, Windsor, Ontario

Problems to Solve:

When this program was started, there were several classroom problems that I wished to solve. Among these were the following:

1. Material studied previously was quickly forgotten if extra work was not given.
2. Enough practice in basic problems was not available in their tests.
3. I wanted to challenge the better students. This was something that I could not do effectively in the classroom.

The Review Test:

In order to solve these problems, I introduced a weekly review test program which was to be written in addition to the usual section tests. The format is as follows:

1. Five problems per week are given to the students in addition to their regular homework. The idea is that they do one extra problem per day. They are responsible for preparing the solutions to these problems. If necessary, they can consult their classmates or teacher for help.

2. On a chosen day of the week, the same day for each review test, the students are asked to write the solutions to either one, two or at most, three of the five review problems that they solved the previous week. The problems are picked at random by cutting a card from a specially

prepared deck of cards. When they are writing the review test, next week's review problems are handed out and the previous week's review test is handed back. The time required to write the review test could be anywhere from 5 to 35 minutes depending on the number and the type of problems picked.

The deck of cards I use is as follows: 30 cards have one numeral on them, a numeral from 1 to 5, with an equal number of each numeral; 10 cards have two numerals on them, a numeral from 1 to 5, all possible ways; 10 cards have three numerals on them, the numeral from 1 to 5, all possible ways. Thus my deck has 50 cards where each numeral has an equal chance of occurring.

Types of Review Problems:

When preparing the review problems, I have the following guide lines:

1. The problems deal with any topic previously studied.
2. The problems are typical test or exam type questions.
3. Most problems stress the basic objectives of the course.
4. Some problems are of a more challenging nature.
5. Some problems deal with more than one topic previously studied.

General Comments

When I started this program, I stressed to the students that the most important part was that they know how to do the problems whether or not they solved the problems themselves. The questions they write as the review test are marked as ordinary test questions. These marks become part of their class mark. For me, review tests are worth between 20% and 30% of their class mark.

I have been using this review test program for over ten years. So far, I have tried it with advanced level classes from Grade 10 upwards to Grade 13. I have used variations of this idea with other classes. The review tests start about the end of September and continue until the end of May. There is a brief stoppage in the tests during exam time. On the average, 23 review tests are written in a school year.

Results From Using the System

After having review tests for many years, I have the following observations.

1. The students seem to have the basic objectives at their fingertips.
2. They get much more practice at basic problems.
3. The better students have an opportunity to attempt some challenging problems.
4. I need less time in the classroom to review for tests and exams
5. It requires a great deal of extra time and effort for the teacher in order to prepare and mark tests.
6. It encourages students to work harder and be better organized.
7. The hard working student is rewarded and the lazy student is penalized.
8. Performance on exams is better.
9. The exam and section test results are very consistent with the review test marks.
10. A teacher has more marks with which to assess the worth of a student.
11. Many of our graduates tell us that the review test program has helped them cope with their University studies.

Conclusions

I feel that this is an excellent aid to supplement the everyday homework assignments. As a result, I would encourage all teachers to adopt a system of this nature.

Sample Review Tests

Math 555 - Week 5

1. Solve for n
   (a) $P(n\ ,\ 5) = 30P(N\ ,\ 3)$ (b) $C(n\ ,\ 4) = 2P(N\ ,\ 3)$

2. (a) How many even 4-digit numbers less than 3000 can be made from the digits 1,2,3,4,5,6 (no repeats)?
   (b) How many words can be made using all the letters of the word ENVELOPE if a vowel is first and last?
3. How many 5-letter words can be made using letters of the word CARRIER?
4. How many committees of 5 can be formed from 7 girls and 6 boys if
   (a) there is no further restriction?
   (b) there are 3 girls on the committee?
   (c) there must be 1 of one sex and 4 of the other on it?
5. How many bridge hands can be formed in which
   (a) there are 8 cards of one suit and 5 of another in it?
   (b) there is at least 2 spades in it?

LES ATTITUDES ET L'APPRENTISSAGE DE LA MATHEMATIQUE

Renée Caron, conseiller pédagogique, C.S. de l'Argile Bleue

Reprinted with permission from **Instantanés mathématiques, APAME, Volume XIV, numéro 3, janvier, 1978**

Note: Ce texte reprend globalement les réflexions que l'auteur livrait aux conseillers pédagogiques en mathématique à l'élémentaire lors de leur session d'étude tenue à Drummondville en octobre dernier.

En voyant ce titre, certains se demanderont: "Pourquoi parler d'attitudes dans l'apprentissage mathématique?" On est en effet peu habitué à considérer les dimensions affectives du comportement de l'élevé dans ce domaine.

Mais pourquoi enseigne-t-on la mathématique ou mieux, pourquoi l'élève fait-il l'apprentissage de la mathématique à l'école?

Pour qu'il sache sa table de multiplication par 8? Pour qu'il puisse résoudre un problème à l'aide de la règle de trois?

C'est pourtant ce qu'on fait depuis longtemps et il ne semble pas que ça nous ait tellement réussi. N'y aurait-il pas quelque chose de beaucoup plus important pour l'enfant que de posséder un ensemble de connaissances et de maîtriser un ensemble d'habiletés à caractère mathématique? Ne devrait-il pas plutôt avoir assumé son expérience mathématique et la percevoir comme un outil que lui permette d'exercer une emprise au moins intellectuelle sur la réalité.

Je voudrais poser ici quelques questions qui m'apparaissent pertinentes à des personnes qui enseignent la mathématique ou qui sont responsables de son enseignement.

- Considérons-nous que l'attitude de la population en général est positive face à la mathématique?
- Croyons-nous que cette attitude devrait être plus positive chez les générations futures?
- Le rôle de la mathématique comme outil d'evolution de l'humanité est-il clair aux yeux des gens?
- L'homme moderne est-il porté à utiliser l'outil mathématique quand il est confronté à un problème? Sent-il seulement que cela pourrait contribuer à une perception plus complète de la réalité?
- La mathématique n'est-elle pas plutôt considérée comme l'apanage d'une élite, les chanceux qui ont la bosse des mathématiques?

Certes on ne veut pas faire de tous les citoyens, des mathématiciens. Cependant, il faudrait que l'apprentissage qu'ils en ont, laisse dans l'esprit de tous, une idée plus exacte que celle qui est généralement répandue, de ce qu'est la mathématique, de ses possibilités et de ses limites.

Une fois, ce voeux formulé, il faut se demander si l'école y peut vraiment quelque chose. Est-il exact de penser qu'en changeant now procédés didactiques et nos attitudes, on pourrait éviter que certaines personnes développent du dégoût à l'égard de la mathématique.

Dans "Crisis in the Classroom", on rapporte le résultat des observations de Dorothy Gardner auprès des enfants étudiant la mathématique, en Angleterre, dans le cadre du Nuffield Mathematics Project. La proportion des enfants qui déclarent préférer les activités mathématiques est plus grande que celle de tout autre matière. On note que ceci est totalement à l'opposé de l'attitude que l'on retrouve habituellement chez les enfants aussi bien en Angleterre qu'aux Etats-Unis.

Mme Gardner a pu noter beaucoup plus qu'une attitude positive face à la mathématique; les enfants font entre autre preuve de confiance, d'initiative et de sens social.

Mais comment se fait-il qu'on n'ait pas réussi jusqu'à maintenant à obtenir ceci dans nos classes ou qu'on ne l'ait obtenu que dans de très faibles proportions. Qui d'entre nous ne rêverait pas d'avoir une classe d'enfants ressemblant à ceux décrits par madame Gardner.

Il semble bien que le simple fait de présenter des connaissances ne soit pas une solution satisfaisante. Il y a une quinzaine d'années, d'ailleurs, on mettait de côté le programme-catalogue. Puis vint le programme-cadre et tous les espoirs qu'il avait fait naître. On parlait d'enseignment individualisé, d'apprentissage personnalisé, lié à la vie, d'intégration, etc.

C'étaient de bien bonnes intentions et nous y avons souscrit avec beaucoup d'ardeur mais on se rend bien compte aujourd'hui que nous sommes encore très loin de l'objectif visé.

Mais quel était-il et quel est-il cet objectif visé? Est-ce qu'il est le même pour tout le monde? Est-ce qu'il est bien clair dans chacun de nos esprits?

On a jusqu'à maintenant, investi le plus gros de nos énergies à répandre et à véhiculer des procédés didactiques que chacun d'entre nous s'enorgueillissait d'avoir mis au point. "Le médium c'est le message" croyait-on avec Mc Luhan.

Mais beaucoup d'entre nous ont eu l'occasion de perdre leurs illusions quand ils ont constaté que les activités qui avaient enchanté les enseignants lors d'une journée pédagogique n'ont jamais eu de suites significatives dans les classes.

En cherchant un peu à comprendre les raisons d'une telle situation on pouvait recueillir des réponses comme: "Je ne vois pas ce que ça peut donner aux élèves." "J'ai essayé de faire cela mais ça prend beaucoup trop de temps". "Après avoir travaillé cette notion pendant deux jours avec eux, il me semble que je n'avais plus d'idées".

Et voilà! A moins de préparer au jour le jour la classe de mathématique pour les enseignants - et alors on manquerait au principe pédagogique de base qui veut qu'on parte des besoins et des intérêts des élèves - il semble bien qu'on n'y arrivera pas de cette façon. Cela semble insuffisant.

Mais qu'a-t-on fait avec les enseignants? A-t-on discuté avec eux de ce qu'on pourrait attendre de l'enfant qui sort de six années d'etudes passées avec eux? Ont-ils eu la passibilité de s'impliquer dans une démarche de définition des objectifs de leur travail?

C'est peut-être à ce point que nous sommes rendus maintenant. Qu'on s'assoie ensemble et qu'on se dise ce qu'on souhaite pour cet enfant qui passera six ans de sa vie à l'élémentaire! Et qu'on ne passe pas trop de temps à discuter sur les connaissances et même sur les habiletés à acquérir. Les discussions sur ces sujets sont la plupart du temps restées stériles, chacun restant sur ces positions.

Certains croient par exemple que les habiletés de calcul sont très importantes alors que d'autres affirment qu'il serait plus réaliste d'initier les élèves a l'utilisation de la calculatrice. Qui a raison? Peut-être que les deux positions sont défendables? La plupart du temps ce que l'on craint de ces positions pédagogiques, ce sont les résultats, les attitudes que ça va provoquer chez l'enfant.

Si l'on s'était entendu préalablement sur ce point des attitudes, peut-être que toutes les autres discussions nous apparaitraient futiles. On serait plus en mesure de faire confiance au collègue ayant des positions différentes des nôtres sachant qu'elles doivent être interprétées à la lumière de ce qu'il attend d'un enfant de l'élémentaire.

On peut en énumérer beaucoup d'autres et de très différents. Les attitudes sont en effet des résultantes de l'expérience globale et personnelle de l'enfant et de ce fait les manifestations seront personnelles et différentes. Il ne sera donc pas aussi facile d'évaluer les attitudes de nos élèves que de mesurer leur performance sur les quatre opérations, par exemple. La plupart du temps, il nous faudra réfléchir sur un ensemble de comportements et sur leurs inter-relations.

Il nous sera sans doute utile alors de nous référer à certaines catégories de comportements lies a des attitudes. Bloom nous en suggère quelques-unes comme l'intérêt, la motivation, **l'anxiété facilitatrice**, la valorisation de soi . . .

Il faut maintenant se demander s'il existe des attitudes qui seront des résultantes spécifiques de l'activité et de l'apprentissage mathématique car tout ce qu'on a dit jusqu'à maintenant, peut tout aussi bien s'appliquer à une réflexion sur l'apprentissage de la langue.

Un groupe de pédagogues du secondaire a réfléchi à cette question. Voici quelques-unes des attitudes qu'ils ont identifiées:

- Une attitude d'appréciation de la mathématique en tant qu'instrument de connaissance et en tant que moyen d'action;
- une attitude d'appréciation de l'utilité des mathématiques dans la vie courante et dans divers domaines;
- une attitude d'appréciation du caractère évolutif et ouvert de la mathématique;
- une attitude d'appréciation de la puissance et de certaines limitations des mathématiques.

Est-ce qu'on peut escompter développer certaines de ces attitudes dès l'élémentaire?

Mais qu'est-ce qu'une attitude? Il semble bien que l'on doive maintenant s'expliquer un peu sur cela aussi.

- ''L'attitude définit une tendance générale d'un individu à agir d'une certaine façon, dans certaines conditions'', nous dit Mager.
- ''Une attitude est un état interne acquis qui influence le choix d'une action personnelle vis-à-vis certaines catégories, de choses, de personnes, ou d'événements'', déclare Gagné.

Il ressort de ces deux définitions que les attitudes d'un individu ne vont pas se toucher du doigt, elles vont se traduire dans des actions et c'est à ces actions qu'il faudra se référer pour évaluer si quelqu'un a une attitude positive ou négative.

Citons ici Mager qui énonce quelques comportements d'un élève qui a une attitude favorable entre autres.

- ''Il en parle de façon favorable''.
- ''Il participe de lui-même à des activités sur le sujet''.
- ''Il participe à autant d'activités qu'il le peut sur ce sujet''.

Il s'agit ici de descriptions quelque peu neutralisées par la généralisation mais dont on peut facilement imaginer ou se rappeler certaines illustrations.

- Les élèves d'une classe de 5e année ont un jour évalué que leur performance en division était insuffisante etont décidé d'un plan pour l'améliorer.
- Des élèves après avoir exploré une notion à l'aie d'un matériel ont apporté le matériel chez-eux pour continuer.

Est-cequ'il n'y en aurait pas d'autres plus spécifiques à l'élémentaire?

La discussion est lancée, des gens y réfléchissent déjà mais une fois qu'on se sera entendu sur des attitudes à développer, il faudra réfléchir sur la manière dont l'élève nous communiquera ses attitudes positives ou négatives.

Cela supposera de la part de chacun d'entre nous une attitude d'écoute face à nos élèves, à leurs besoins et à leurs intérêts et qui sait peut-être en résultera-t-il un boulversement de nos principes pédagogiques et de notre enseignement.

Mais cette fois-ci espérons que chacun d'entre nous saura où il s'en va.

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Editors Note: Let us know if you would like this text translation to be printed in/English. It is available.

MEMBERSHIP NOTICE

The 1978-79 Council authorized that the fiscal year for the Association be May 1st to April 30th of any calendar year. This is to become effective for membership purchased after January 1st, 1979.

With rising costs, the council deliberated long and hard at arriving at a method of not increasing fees but holding the budget balance at the same dollar figure. No longer will there be prepaid postage on the membership application. This, of course, can be eliminated for both the purchase and the member if the membership is bought at the Annual Meeting or a Chapter Meeting.

It has been the policy of OAME to give a member in September an issue of the Abacus and of the Gazette regardless of whether or not he had renewed for the next school year. Since the 1978-79 members purchased their membership under this system, all 1978-79 WILL RECEIVE A SEPTEMBER ABACUS AND GAZETTE. However, all memberships (new or renewal) after January 1/79 will not receive the 1980 September issue of the publications unless they have purchased a membership with an expiry date effective April 30/81. These two items will add several dollars to our budget balance giving the Association opportunities to do more for the mathematics teacher. An application form is printed below and the response last year was excellent. Many teachers who were unable to attend the Annual Meeting in May renewed using this form. We trust the response will be even greater this year.

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INTERVIEW

Taped Talk with Larry Ridge . . . Gerald Rising



Gerald Rising has so become a recognized and well respected authority in Mathematics Education in Ontario through his many well attended presentations at various conferences that we tend to forget that he hails from SUNY at Buffalo. One of his more recent talks was given during MATHEMATICS '78 at York University. It was my pleasure to interview Gerry following his presentation "Use of Calculators in the School Program".

HLR: In many localities in Ontario there is still much controversy as to whether or not calculators of any kind should be used in mathematics programs. It strikes me that some of the thinking which underlies these opinions involves several misconceptions. Could you identify perhaps some of the more flagrant of these.

GRR: Well, I think that's very important and I want to be extremely careful in answering this question. By the way, as a sort of opener, we have dinner table conversations on this subject. My wife is adamant in her belief that the calculator will keep students from learning the tables. My wife joins that large crowd of people that I feel have the attitude towards school, "by God if I had to do it, then they can darn well do it". School, effectively is punishment in other words.

There are a range of concerns. There's the back-to-basics concern, the people who have the belief that the calculator will prevent youngsters from learning what addition is about, what's going on, how to do it, how to do it quickly, and in particular how to do it without a calculator. It's my strong personal belief that the calculator does not and should not get in the way of the basic operations and learning those basic operations. In this regard, I'll make the point once again that I made in the talk. The calculator only applies to a small part of the school program. If you go through textbooks at any grade level, it turns out that the calculator may have a role to play in 10-20% of the curriculum and it should be set aside for the places where it's inappropriate.

Now, I'm not giving research evidence because there is none that's satisfactory to me, but I believe the calculator has an important role. That role can include the learning of basic ideas. I'll give you an example. Suppose we want to teach addition, usually the very first thing beyond counting in the first grade. It seems to me that one way of approaching this is to give students the idea of what addition is about, what it's doing, the idea of accumulation and, in particular the idea of accumulation of costs. You can have students immediately, before there are any basic facts, before they have done tables, or anything like that, add up shopping lists with a calculator. Now they get a very general notion of what addition is about. They can do significant problems with addition. They become comfortable with the idea of addition and now they address the problem of doing it quickly and efficiently, so that they must have the facts in the forefront of their mind. But those facts carry some meaning when they know what addition is about. It's not just '2,3,5' as a kind of license plate number to memorize but its '2 and 3 is 5' as a response to an area of concern that they're generally familiar with. I think that's a very different kind of a role for calculators from the sort of thing that most people foresee. It goes to the very opposite end of the line.

A good friend of mine in our math department took the opportunity to remark to me that I was a traitor to mathematics when I embarked on this calculator experiment*. "Calculators are the devil's invention". That is very close to the phrase that he himself used. By the way, you may not know that when the discoverers or the developers of the proof of the Four Color Theorem made

their presentation to the American Mathematical Society there was only scattered applause so that that reaction to calculator generated kinds of activities is not just isolated in my friend in the mathematics department. I think that's short sighted and I think it may mean that he has made the same kind of misidentification of the kinds of things we are doing with calculators in the schools that the back-to-basics people have made. I see the calculator as playing a positive role and used **only** when it can play a positive role. Certainly if there are areas where it plays a negative role, you should set them aside.

Now it would seem to me that that makes it much tougher for the people who argue against these things to gain ground, but it puts a different kind of responsibility on me and that's to identify accurately those roles and to develop appropriate curriculum materials to make contributions to those roles. Where both of the critics are absolutely right, I submit, is where they direct their criticisms to the currently available materials in the schools.

We do not yet have the right kinds of curricular materials to make appropriate use of calculators. I agree with them whole-heartedly on that. It says "get going", but unfortunately the nature of the resource availability is such that we do not have the kind of sponsorship of development of appropriate activities that we ought to have. That's one of the major problems that we face in Mathematics Education today. For example, the support of the small program that I am working on now at the level of about $50,000.00 a year in no way matches the ocean of support for the kind of mindless activities that the government is underwriting: research of the kind, for example: What happens when you give one class 30 calculators and another class no calculators? What is the difference that's achieved - when there are no curriculum materials to make appropriate use of those calculators? The outcome is predictable. There's no difference except for one interesting item. One of the experiments undertaken by a friend of mine whom I won't name in this one instance, found that the youngsters who did not use the calculators in the experiment, when they were tested afterwards on a test that demanded use of the calculators did not do as well as the youngsters who had had the calculators. I think that's what's called a startling finding in the research group and that's what passes for research in the modern world.**

HLR: I think one of the major thrusts of your talk this morning was that if calculators are to make any sort of inroads into Mathematics Education then they must do so within current curricular topics. Other than that we are looking at a frill type of situation rather than one of a fundamental use in developing concepts. At the present time at the secondary level to what curricular areas do you see calculators in their present state applying particularly well?

GRR: I think you've stated very well my position and let me repeat it because it's so important. If the calculator is thought of as a supplementary device, forget it. We have plenty to do in the schools. We're not doing so well at what we're setting out to do (if we pay attention to Ken Fryer and others, and I think we should). We must find ways of addressing the basic heart of our curriculum in the best possible ways. It's as simple as that. We ought to be looking to the calculator at times when it can make a contribution to that basic curriculum. That's exactly the wort of thing that we are trying to do at Buffalo in this calculator experiment. Now the first thing that that means is that you look at the curriculum and you identify the places where the calculator can contribute. That brings us finally to the question that you ask.

It seems to me that there are a number of these areas that are quickly identified and let me use three of them. The first is graphing. The second is work on logarithms. The third is work on numerical trigonometry. For graphing the calculator gives you a device that allows you, with simple programming, to generate tables of values very quickly and very easily. It allows you to do that kind of point by point graphing that's very important in fixing the basic ideas of graphing in the student's mind and to accomplish a great deal more of that in the same span of time. I think that gives you a tremendous power that we have not had in graphing.

For example, I used to hate drawing the graph of an ellipse because it was just so complicated. You had to square those numbers and take square roots and so on. So I would generally draw ellipses like most of our students today, with maybe four or five points and hope that the symmetries are there that fill in the ellipse. (By the way, I'm certainly not discrediting that. Sketching a graph and using the best and most appropriate points is something that's also part of teaching graphing, but not at the beginning). At the beginning you want your students to go through the labour of plotting the graph point by point and seeing that those points connect up and so on. Well, a programable

calculator gives you a very very nice way of doing that. You can plot 20 or 30 points in the same amount of time that it took me to work out and plot 2 or 3 points, so that you have a basis of rapid plotting of graphs that you did not have before. I think that's a powerful tool and a tool that can contribute. Then the generalizations are made up from those kinds of graphs.

I think that when I went to school as a student, I made the generalizations from almost empty information. I had already been just guessing where the graphs were. It was not a matter of going from carefully plotted graphs to graphs that I could plot by using basic kinds of information. I was just using the basic kinds of information so I did not have that first level working for me.

I made a basic point of the instruction of logarithms and the contribution that calculators can make to that (topic) in my talk and I mentioned there the class session in which students, having studied exponents, were now asked on their own without teacher direction to determine what the 'log' key did - and they did! I was asbsolutely amazed at how quickly they discovered this. They saw that it was a power of ten within about four or five minutes. They were able to record it in the actual-defining exponential equation within ten minutes of the time that they started. These are students who had never met the word in school before and they did it with their own recorded and with us only hovering around in the background, not contributing to the lesson.

It was a startling example of how youngsters could discover on their own merely by testing values with the log key on the calculator. You put 2 in, you get 0.3010 out, you put 3 in, you get .4771 out and so on. Continuing, they came to 10. As soon as they saw an output of 1, they tried 100 and they were immediately getting to the heart of the matter. Well, that's a kind of power that we have not had before and it's the sort of power that we ought to be using in our school program.

You can imagine the sorts of places that you can use the calculator in trigonometry. I'm especially sensitive to trig right now because we are currently developing the work on trigonometry and, of course, the effect here is that the calculator gives you a built-in table of values and it gives it to you in a form that's not just restricted to the 0-90° range but gives you the full range of values for the functions. The result for us: we are almost completely reversing some of the pedogogy here. We are starting with that (full range) and we are noting some of the changes in signs and having the youngsters come up with the regularities.

For example, we will have them early on develop a table of trig functions by 15° up to 450° and then have them look at the regularities, the things that are happening, the places that numbers are repeated and, in particular, the graphs of those functions to see what's happening in trigonometry from that point of view. Then they get to come to some of the structural features that are derived from the basic definitions.

Now I think those are things that involve the calculator in a contributing way to the basic curricuum that we are interested in teaching. Those are the kinds of things that we ought to be prizing out of the curriculum and at the same time prizing out of the calculator, the places where the calculator can appropriately make those contributions. At the same time I think it's equally important that we carefully identify the places where we feel the calculator will not contribute and assure ourselves that we're setting the calculator aside in these topics. We do ourselves a terrible disservice when we suggest that a device that makes a 20% contribution makes anything more than a 20% contribution.

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* Development of Grade 11/12 Curricular Materials for Mathematics with Programmable Calculators supported by the U.S. National Institute of Education.
** Since this interview, Dr. Rising has found another such study (supported by several hundred thousand dollars) that shows the opposite.

Brain Teasers

2. If one mile of fence encloses forty acres, how many acres will two miles of fence enclose?
3. A boy spent one cent on the first day of the month and on each succeeding day spent double the amount of the day preceding. How much more does he spend on the last day than on all the days preceding it?

Answers on pg. 68

SOME LIMITATIONS OF THE STRUCTURE MOVEMENT IN MATHEMATICS EDUCATION: THE MEANINGS OF "WHY"

Stephen I. Brown, State University of New York at Buffalo

If there is any place in the curriculum that one might think that "Why?" is a pretty straightforward question, it should be mathematics. After all, what was the "new mathematics" movement of the 60's all about? We were to give up algorithmic authoritarian thinking as a chief mode of operation in teaching and learning mathematics, and were to be concerned instead with why things worked as we claimed they did. The implication was that **why** is not only important, but pretty straightforward as well.

Sometimes we do find that "why?" is a clear question. Frequently, however, "why?" in mathematics suggests the same kind of ambiguity as in other areas of human endeavor. (What, for example, is the nature of "why" we might offer to a bereaved lover who asks why his friend has left him for someone else? Would a casual explanation be called for, or would some other kind of "why" be appropriate?)

Consider, for example, the following question:

> Given two equilateral triangles, find the length of the side of a third one whose area equals the sum of the other two.

It is a straightforward question. If we represent a side of the first triangle by a and a side of the second by b, then the sum of the two areas can be shown to be:

$$\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} = \frac{\sqrt{3}}{4}(a^2 + b^2)$$

for the area of an equilateral triangle is $\frac{(\text{side})^2 \times \sqrt{3}}{4}$. In order to find the length of the side c of the third triangle, so that its area equals the above we know then that:

$$\frac{c^2\sqrt{3}}{4} = \frac{\sqrt{3}}{4}(a^2 + b^2) .$$

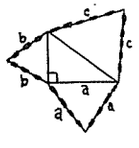
Simplifying we get:

(1) $c^2 = a^2 + b^2$

(2) Thus, $c = \sqrt{a^2 + b^2}$

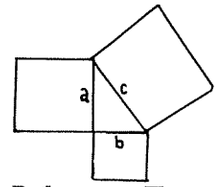
We're done. Ignoring details of calculation, look back at (2). Does it ring a familiar bell? It is in fact an expression of the **Pythagorean Theorem** -- that in a right triangle, if the sides are a and b, and if c is the hypotenuse, then the three lengths are related by (1).

Well now, why do we get the Pythagorean relationship among the sides of three equilateral triangles whose areas are additive? The answer is simple: because our calculations lead us there. But is that answer satisfactory? We would be satisfied with that kind of anwer if someone asked us to find the sum: 1 + 17 + 58. Here, however, we smell something interesting that was **not** captured in our calculation. What the answer suggests is that if we have two triangles that are equilateral with sides a and b then we can find the length of the third side c of an equilateral triangle whose area equals the sum of the other two by the procedure suggested below:



Well **why** should we be able to do that? What is suggested by that simple sketch as a solution to the problem?

If you think about the Pythagorean theorem in totally geometric terms (implying that x^2 is the area of a square whose side equals a) as Euclid himself did, then we can conclude that if you have a right triangle, the sum of the areas of the squares erected on each side equals the area of the square erected on the hypotenuse.



Seen in this light, we observe that the **Pythagorean Theorem** holds not only for squares on the sides of a right triangle, but for equilateral triangles as well! Why? Most of you who recall the Pythagorean Theorem in geometric terms thought of it only as a statement about **squares** on the sides of a right triangle, and **not** equilateral triangles on the sides.

Why? We shall neither excite nor bore you with an answer to the question, but let us suggest that the answer to this "why?" is a deep one and one way of interpreting the answer is that most of us had been carrying around a very limited view of what the Pythagorean Theorem is really capable of asserting.[1]

The fact that the view is limited, however, is not our major point. Rather, we are suggesting that mathematics provides a context within which the most human of all questions -- "why?" -- can be taken **seriously.**

Taking "why?" seriously does **not** mean what it has meant in most existing curriculum however: finding an answer. For the most part the curriculum has focused on having the student demonstrate for any proposition Q that there exists some proposition P so that "P implies Q." But that is a very limited notion of "why" and overlooks the fact that a proof does not always answer the question "Why?" and even if it does in some sense, there are other kinds of "why's" beneath the surface.

Consider a more elementary problem: Find the sum of the first 100 counting numbers. There are many non-plodding ways which we can approach that problem, but one method that is frequently taught is as follows:

(3) Start with: $1+2+3+4+5+\ldots+97+98+99+100$. Now count backwards and arrange so that we have the following:

$$\begin{array}{r} 1 + 2 + 3 + \ldots + 98 + 99 + 100 \\ \underline{100 + 99 + 98 + \ldots + 3 + 2 + 1} \end{array}$$

If we now add vertical pairs, we end up with:

$101+101+101+\ldots+101+101+101+101$

Instead of finding the desired sum we thus have twice the desired sum. How many times do we have 101 as a term in the sum? It is obviously 100 times. But then $100 \bullet 101$ gives us the value of **twice** the sum of numbers from 1 to 100. Since we want the sum only once, the answer is $\frac{1}{2} \bullet 100 \bullet 101 = 5050$.

Fine, again we have proven something, but most people seeing a proof of this for the first time would certainly be left with at least one new kind of **why** question. Why did anyone ever think of re-arranging the original question so as to put it in form (3) above?[2] . That is, there is a difference between understanding that something works and seeing how anyone could ever have come up with an insight that transformed the problem from something "ugly" to something "manageable."

So far we have identified a number of different "whys?". They included:
- "Why," meaning: What logical chain of reasoning would justify a conclusion?
- "Why," meaning: Does this fit into some larger picture?
- "Why," meaning: What might have motivated crucial steps that lead to the conclusion?

We can, however, begin to appreciate other meanings of why -- meanings that have been surpressed in the official curriculum but ones which help direct our thinking about teaching and learning mathematics.

For another dimension to "why," consider the following:

> In the set of natural numbers N = {1,2,3,4,5,. . .}, we define a **prime number** as a number with exactly two different divisors. So, 5 is prime because 1 and 5 are its only divisors. 4 is **not** prime because it has 3 divisors: 1, 2, 4.
> Now instead of focussing on the set of natural number, look at E = {1,2,4,6,8,10,12 ...}
> Using the same **definition** of prime as in N, list the primes in E.

Now there is considerable ambiguity in this problem. Are we asking for the numbers in E that are prime in N (in which case, only 2 will work), or are we asking for those numbers that are prime in E (regardless of what is their status in N)? For the second interpretation, 6 is prime in E because only 1 and 6 divide 6. (Why will neither 2 nor 3 divide 6 in E? If you believe that 2 should divide 6 in E, then you should also believe that 2 will divide 5 in N!)

Once the ambiguity is cleared up, we have many fascinating questions to ask and observations to make. For example, consider

- the number of primes in E
- the search for a formula to generate primes in E
- a method for determining primes in E
- whether or not there is unique factorization of a number into products of primes in E.

You will find some very interesting disparity between answers to analogous questions in N and E. For example, in N you know that every composite can be represented uniquely as a product of primes. Consider 36 in E and show that there are two different prime factorizations.[3]

Yet a different kind of why is suggested by this exploration. We ask: Why do E and N behave differently? Here we are not asking for proofs alone, but for some illumination regarding how the structural differences between N and E account for disparity in conclusions.

There are additional kinds of "whys" that may be difficult to articulate, for they are "whys" that leave us "breathless" no matter how much has been explained -- the "whys" that leave us with a sense of awe and wonder, for they threaten our intuitions about the relationship of the world to mathematical thought. Consider, for example, the following two questions:

- A bug (of no dimensions) decides to travel along the number line from 0 to 1. What is the first number after 0 that it crosses?
- Place a belt around the earth. Now add thirty feet to the length of the belt and again place it around the earth so that the belt and earth share the same center. Consider the rim around the earth left by the longer belt. What could fit through the rim? An ant? A pin? A marble? An aunt? An elephant? (Circumference of earth is around 25,000 miles).

We leave it as an exercise to explore these questions, but urge that after trying to answer them, you attempt to phrase a general "why" question that is captured by your response to both of these problems. It certainly will be a why of a different order of magnitude from others we have explored.

What could be more enlightening than to begin to understand that one reason for our inability to communicate well with ourselves and with others is that we have not very well appreciated the significance of what we are doing when we perform the most human of all acts -- namely asking "why?". While mathematical thinking is not unique in its ability to raise such category questions on the nature of why, for many, because of its distance from what is perceived to be pressing emotional issues, it may provide a less charged analogy in order to begin a search.

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1. Those interested in a resolution of the problem, see, Marion I. Walker and Stephen I. Brown, "Problem Posing and Problem Solving: An Illustration of Their Interdependence," **Mathematics Teacher**, January 1977, p. 4-12.

2. For a detailed discussion of this particular problem as a "peep-hole" through which to approach other humanistic themes, see Stephen I. Brown, "Mathematics and Humanistic Themes" Sum Considerations, **Educational Theory**, Vol. 23, No. 3, 1973, p. 191-194.

3. A forthcoming book is devoted to these questions and many others as well -- in comparing E to N with regard to primes. See, Stephen I. Brown, **Some "Prime" Comparisons**, National Council of Teachers of Mathematics, Reston, Virginia, 1978.

INTRODUCING THE STUDY OF STATISTICS AT THE GRADE NINE AND TEN LEVEL

Doug Henderson, Master Teacher of Mathematics Sudbury Board of Education

When the announcement was made by the Minister of Education that a revision of the mathematics curriculum for the intermediate grades would soon be made, the reaction of teachers in the Sudbury area was generally one of approval. Because of dissatisfaction with the guidelines that first appeared in the mid-sixties, adjustments had already been made both at the Board and school level. The prospect of a new direction in mathematics curriculum with course content more in tune with the likely future needs of the student was a welcome one.

However, within a short time the apparent haste with which the forthcoming guideline was to be prepared began to arouse considerable apprehension among local mathematics teachers. The feeling was that the need for careful planning and extensive discussion before setting new directions in education was being ignored.

In retrospect it is scarcely surprising that the first publication of lists of topics to be included in the new curriculum met with a great deal of resistance and even antagonism. Rightly or wrongly, teachers perceived that they and their students were being used as pawns in what was essentially a political and public relations game.

Amid the controversy surrounding the proposed intermediate guielines several points became clear. Although no one was content with the total guide as it then existed, neither was anyone opposed to every line. As a matter of fact, almost every "new" topic could find at least one champion among local teachers and for some new topics there existed substantial approval.

As a matter of fact, when much of the opposition to new topics in the curriculum was analyzed, the principal objection was often that "it had never been done before". It occurred to us that, rather than deluge the Ministry with opinion of what might or might not be successful, we should choose a topic which we felt should be attempted; and then report on what had or had not been successful.

It was agreed that the philosophical statement in the preliminary version most in tune with our thinking was the desireability of a more practical mathematics curriculum and need for course content whose applicability in the outside world is more apparent. Within this frame of reference the specific topics which we decided to teach were the statistics listed in the guideline.

To accomplish this end each mathematics department head agreed to make room in the grade IX courses of study for about four weeks (40 min/day) for the study of statistics topics. The Board administration agreed to fund a summer writing group of five members for two weeks to prepare teaching materials; and professional activity days in the fall were committed to teacher preparation for teaching statistics. (Our summer group was joined by Neil Truman from the North Shore Board of Education so that we had six persons in all).

Rather than debate the merits of the guideline's approach to statistics, we took the subtopics as listed and attempted to devise the best approaches and student activities that we could find or create to reach the goals set by the Ministry. In combing the literature from Canada the U.S. and Britain, we found very little that suited our purposes. Most "statistics" studies in the texts are either at too low a level, that is they suggest taking endless surveys and drawing graph after graph; or at too high a level and standard deviation is on the second page, hypothesis testing, confidence intervals and correlation co-efficient follow soon after. Although we borrowed some ideas, almost all of the teaching material produced was original.

The following recommendations, and observations then, are based upon experience in 17 secondary schools, with probably the best preparation of both teachers and materials which it is realistic to expect in days of budget restraint.

Grade Nine Advanced Level

(i) Students and teachers enjoy the study of statistics. Students seem to see the potential relevance for themselves and were able to learn the main ideas, with a few exceptions in the unit.

(ii) The ideas associated with organizing data, drawing graphs, must be reviewed even though they are dealt with in the elementary grades.
(iii) The approach to the sampling section should be experimental and activity-oriented. Sampling experiments should be carefully written up and handed in, in the same way as science experiments. The main ideas to get across are the advantages of sampling and some of the dangers in interpreting results of samples. It is not necessary to introduce students to terms like "stratified", "nested" and "cluster" sampling. If these techniques or the equivalent, are suggest by the students, fine. If not, that, too, is fine.
(iv) The only measure of central tendency which should be taught as a core topic is the mean. Distinguishing among mean, median, and mode made little sense to the students and deciding which of the three to use in particular situations was beyond almost all pupils.

Grade Nine General Level

(i) Organizing data, drawing graphs must be retaught rather than reviewed. These activities provide many opportunities for practising work with per cents and decimals.
(ii) As a group, these students did not really appreciate the ideas of sampling and making conclusions from samples. It is recommended that the notion of sampling be postponed to grade X.
(iii) Distinguishing among mean, median and mode is next to futile. However, considerable work can, and should, be done in calculating and interpreting the mean.

Grade X - General Level

(i) The skills of organizing information, drawing and interpreting graphs, and working with the mean must be reviewed.
(ii) Students at this level still have difficulty grasping the need for, and advantages of, sampling. However, they can enjoy and profit from sampling activities. We should keep in mind that we are not trying to prepare the students for the future formal study of statistics, but are trying to impart some general knowledge of how statistical ideas do and will affect their everyday lives. The activities should help pupils better understand such statistically based information as the national unemployment rate and consumer price index.

Grade X - Advanced Level

(i) Some sampling activities and ideas must be reviewed.

To attempt to determine how well the students had assimilated the material, five questions on statistics topics were included on the annual board-wide test in grade IX mathematics. Although these questions do not reflect all of the aims in the unit, the results might give us some indication of what might be expected in terms of student performance.

The next five questions refer to the tally chart below.

A grade nine class had an attendance record for a month as shown in the table.

Days Absent	Number of Students
0	1111
1	~~1111~~ 111
2	~~1111~~ 11
3	111
4	11
5	1

1. How many students were absent for exactly two days?
   (A) 4 (B) 6 (C) 7 (D) 8
   Advanced 78% General 66%
2. How many students were in the class?
   (A) 25 (B) 23 (C) 15 (D) 55
   Advanced 76% General 58%
   Although all students had worked with tally charts, the results of the first two questions show that many do not know how to interpret such a chart if it is prepared by someone else.

3. What was the median number of days absent per student?
   (A) 0 (B) 1 (C) 2 (D) 3
   Advanced 49% General 37%
4. What was the mean number of days absent per student?
   (A) 0.6 (B) 1.92 (C) 1.76 (D) 1.0
   Advanced 42% General 23%
   The general level reached the guessing point on this question.

SOME OVERALL RECOMMENDATIONS REGARDING THE INTRODUCTION OF STATISTICAL TOPICS INTO THE CORE OF THE SECONDARY SCHOOL CURRICULUM

(i) Test results and other problems not withstanding, we agree that statistical topics can and should be part of the core of the mathematics curriculum.

(ii) The general aim should be to acquaint students with the uses of statistics in everyday life and to **prepare** university-bound students for the formal study of the subject. The high school should not try to replace courses in statistics now taught to students studying business, social science or mathematics at university, but should attempt to prepare a basis so that these courses might be more successful.

(iii) Teachers will have to evaluate student progress in this area principally by means other than the traditional paper and pencil test.

(iv) Teacher preparation for teaching statistics is not generally adequate to provide for effective learning beyond the grade nine or ten level. Most teachers, even specialists, have had either one course or no courses in statistics. Those who have had at least one course report the experience in less than glowing terms. Boards, teacher organizations, the Ministry and the universities will have to address themselves to the problems of providing teachers with a more suitable background for the teaching of statistics.

(v) It is urgent that some person or group outline a proposed continuum in statistics so that we may look ahead now at proposals up to grades XII and XIII. Some key questions are:
When and how should we introduce standard deviation and the need to measure dispersion in a distribution? What is the place of probability ideas in the program? To go any farther with sampling ideas, to describe the confidence in a sample result, to test a hypothesis requires some basic probability skills. If these are not to be our course content in the senior division, what is?

One example of the material is given below.

This material is a revision of a document originally prepared in July, 1977 by a Sudbury Board committee whose members were Richard Gauthier, Doug Henderson, Fred Matte, Harold Nelson, Neil Truman and Peter Weygang.

The revision committee members were Doug Henderson, Jim Hill, Gerry Lariviere, Al Rondina, John Savage and Gen Waszczylo.

INTRODUCTION: ORGANIZING DATA

Objective: (i) To introduce students to statistical investigation.
(ii) To establish the need to organize data and introduce students to simple organizational procedures.

Introduction:

Simple In Class Experiment

The problem can be presented to the students in the following manner:

"I'm starting a clothing store specializing in jeans and T-shirts and I have enough money to stock 200 pairs of T-shirts and 200 pairs of jeans. How many of each size should I order? How would you solve the problem? There are 4 sizes of T-shirts, should we buy 50 of each? What sizes of jeans are most popular?

Notes:
(i) Jeans are usually sized by waist in intervals of 1 inch. Have students with a waist size which is an odd number round up to the next even number
(ii) T-shirts are small, medium, large or X-large.
(iii) The store could be the student council if they sell T-shirts or an in-school marketing store.

Lesson Procedures

Day 1:
(i) Explain problem. Establish need for survey.
(ii) Distribute and collect information sheets.
(iii) Write T-shirt data on board in unorganized way. Establish advantage of Frequency table.
(iv) Complete T-shirt Frequency table and graph as homework assignment.

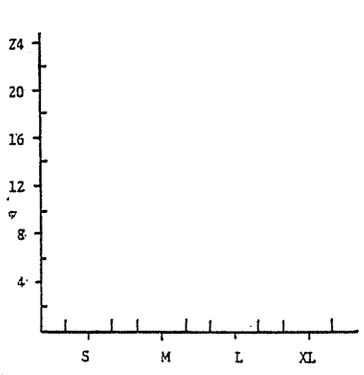
Day 2:
(i) Take up solution to graph (transparency may be quickest).
(ii) Find **approximate** number of each T-shirt size to be ordered.
(iii) Present unorganized data on jean sizes. Have students complete table, graph, number of each size to be ordered.
(iv) Present unorganized data on T-shirt prices. Have students complete table and graph as an assignment.

Day 3:
(i) The data from the other grade nines could be combined to demonstrate to students the need of scaling when drawing graph.
(ii) The results from the other classes might or might not agree with the results of your own class. What is the advantage of having a bigger survey?
(iii) Construct Frequency tables and bar graphs for the larger group. This will introduce a need for choosing a scale. We might re-do the estimates on the different sizes if necessary.

MARKET SURVEY RESULTS

T-shirts

Size	Tally	Numbers
S		
M		
L		
XL		

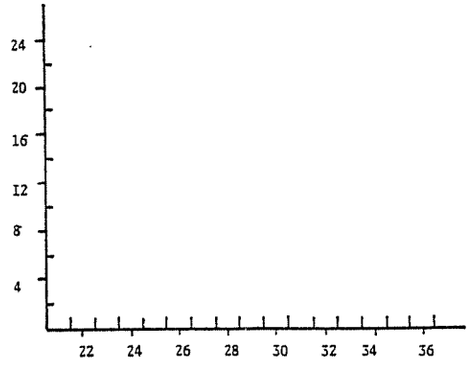


_____ size S, _____ size M,

_____ size L, _____ size XL.

Jeans

Size	Tally	Number
22		
24		
26		
28		
30		
32		
34		
36		



Out of 200 pairs of jeans, I would order:

_____ size 22, _____ size 24, _____ size 26, _____ size 28

_____ size 30, _____ size 32, _____ size 34, _____ size 36.

OPINION POLLS AND STATISTICS

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1. Introduction

Before every major election, the pollsters try to pick the winner; specifically, they try to guess the proportion of the population that will vote for each candidate. Clearly, canvassing all voters would be an impossible task. As the only alternative, pollsters survey a sample of a few thousand in the hope that the sample proportion, denoted by P, will constitute a good estimate of the population proportion, denoted by π. This is a typical example of *statistical inference* or *statistical induction:* the unknown characteristic (π) of a population is inferred from the characteristic (P) of an observed sample taken from the population.

As any pollster will admit, it is an uncertain business. A reasonably good estimate, however, can be obtained to the extent that the sample accurately represents the population of voters. Statistics combined with good common sense can provide that accuracy, and give us an estimate that is customarily written in the form

$$\pi = P \pm \text{a small error.}$$

2. Random Sampling for Estimating Proportions

Here we consider a simple and useful method of taking a sample - the method of random sampling. In random sampling, each member of the population is given an equal chance of being included in the sample, independent of all the other members. For a sample to be random, it cannot be chosen in a sloppy or haphazard way; it must be designed carefully. One possibility is to number all the individuals in the population, and draw the sample by using a chance device such as a bowlful of numbered chips, or a table of *random digits* published in some statistics books (cf. [6, p. 705]). Then we can make the following statement based on statistical theory (cf. [6, p. 223]): If the sampling is random and the sample size n is large, we can state with 95% confidence that π is in the interval (customarily called a 95% *confidence interval*):

$$\pi = P \pm 1.96 \sqrt{\frac{P(1-P)}{n}} \qquad (A)$$

As an example, two days before the Federal Election on July 8, 1974, the Gallup Poll [2] published a survey report based on a sample of size n = 1,000 voters across Canada. They found a proportion P = 43% voting for the Liberals. If we assume the sample was random, then by formula (A),

$$\pi = .43 \pm 1.96 \sqrt{\frac{.43\ (1-.43)}{1{,}000}}$$

$$= .43 \pm .03 .$$

That is, with 95% confidence, the poll predicted that the proportion of Liberals among the whole population of voters would be between 40% and 46%. On election day, the actual proportion turned out to be 43.2%. So the poll prediction was indeed correct (as we would expect it to be, 95 times out of 100).

As another example, the same poll found a proportion P = 16% voting for the N.D.P. Again, if we assume that the sample was random, the 95% confidence interval for the proportion of N.D.P. among the whole population of voters is:

$$\pi = .16 \pm .02.$$

This also correctly bracketed the actual proportion (15%) on election day.

As a final illustration, and interesting example of how T.V. networks make election predictions during the election day is given in [4, p. 137].

Now let us look a little more carefully at (A):

(1) The estimate is not made with certainty; we are only 95% confident. We must concede the

possibility that we are wrong - simply because we were unlucky enough to draw a misleading sample. For example, if a minority of the population is in fact Liberal it is still possible, although unlikely, for us to run into a majority of Liberals in our sample. In such circumstances, the confidence interval (A) would be wrong. Since this sort of bad luck is possible but not likely, we are just 95% confident.

(2) As the sample size n increases, we note that the error allowance in (A) decreases. If Gallup increased the sample to n = 10,000 voters and continued to observe a Liberal proportion of 43%, and 95% confidence interval would become more precise:

$$\pi = .43 \pm .01.$$

This also is intuitively correct: a larger sample contains more information, and hence allows a more precise conclusion. But also notice that, by increasing the sample size 10 times, the error allowance is only reduced by $\sqrt{10}$ times (not 10 times).

(3) Suppose we feel that 95% confidence is not good enough and that instead we want to be 99% sure of our conclusions. If the additional resources for further sampling are not available, then we can increase our confidence only by making a less precise statement. For 99% confidence the formula (A) must have the coefficient 1.96 enlarged to 2.58; this yields the 99% confidence interval:

$$\pi = .43 \pm .04.$$

This is broader and less precise than the 95% confidence interval (A); we must be less precise because we wish to be more certain of being right. In any case, we note that any statistical statement must be prefaced by **some** degree of uncertainty.

3. Why Sampling?

We draw a sample, rather than examine the whole population, for several reasons:

(1) Limited resources. For example, in pre-election polls, neither funds nor time are available to observe the whole population.

(2) Scarcity. Sometimes only a small sample is available. For example, in heredity versus environment controversies, identical twins provide ideal data because they have identical heredity. Yet very few such twins are available.

There are many examples in business. An allegedly more efficient machine may be introduced for testing, with a view to the purchase of additional similar units. The manager of quality control simply cannot wait around to observe the entire population that this machine will produce. Instead, a sample run is observed, and the decision on efficiency is based on an inference from this sample.

(3) Destructive testing. For example, suppose that we wish to know the average life of all the light bulbs produced by a certain factory. It would be absurd to insist on observing the whole population of bulbs until they burn out.

If sampling is required, how should it be done? In statistics, as in business or any other profession, it is essential to distinguish between good luck and good management. For example, suppose that someone bets you $100 at even odds that a fair die will turn up 6. You accept the challenge, but the die happens to turn up 6 after all, and he wins. He has merely overcome his bad management with good luck. Your only defense against this combination is to get him to keep playing the game - with your dice.

If we now return to our original example of pre-election polls, we note that the sample proportion of Liberals may misrepresent the population proportion for either of these reasons. No matter how well-managed and designed our sampling procedure may be, we may be unlucky enough to turn up a Liberal sample from a Conservative population. Formula (A) allows just for such bad luck. From that formula we confirm that the best defense against bad luck is to "keep playing"; by increasing our sample size, we improve the reliability of our estimate.

The other problem is that sampling can be badly mismanaged or biased. For example, in sampling a population of voters, it is a mistake to take their names from a phone book, since poor voters who often cannot afford telephones are badly under-represented.

Other examples of biased samples are easy to find. Informal polls of people on the street often are biased because the interviewer tends to select people who seem civil and well-dressed; a surly worker or harassed mother is overlooked. An MP cannot rely on unsolicited mail as an unbiased

sample of his constituency, since this is a sample of people with strong opinions and includes an inordinate number of cranks and members of pressure groups.

The simplest way to ensure an unbiased sample is to select each voter at random, as described in Section 2. Not only will it be free of bias, but it also will satisfy the assumptions of statistical theory, and allow us to make scientific inferences of the form (A).[1]

In practice, reputable pollsters such as Gallup use sophisticated sampling methods that combine random sampling with some other twist that reduces cost or improves accuracy. One such twist, for example, is *stratified sampling*, where the population is divided into strata. For example, Canada could be geographically stratified into 5 regions: B.C., Prairie provinces, Ontario, Quebec, and the Atlantic provinces. A random sample is taken from each stratum, and the 5 sample proportions are combined to give the national proportion [6, p. 673]. For further examples and methods, see [1, chapter 12] and [5].

4. More Remarks on Random Sampling

Because it is the basic building block of any scientific poll, let us return to simple random sampling. There still remain some interesting subtleties:

(1) In a pre-election poll, ideally the population sampled should be the population of eligible voters who *actually vote on the election day*. Since it would be too late to wait till the end of the election day to sample, the population from which a random sample is taken before the election can only be the population of eligible voters, or at best the population of eligible voters who would *likely* vote. One way to determine the likelihood of voting is for the pollster to ask, "Do you plan to vote?" or the question, "If an election were held today, which party would you vote for?"

(2) If the sample size n is large *relative* to population size N, i.e., the sampling fraction n/N is substantially different from 0, a more accurate confidence interval can be obtained by replacing (A) by the formula

$$\pi = P \pm 1.96 \sqrt{\frac{P(1-P)}{n}} \sqrt{1 - \frac{n}{N}}$$

and if n is small (whether or not N is small), the confidence allowance should be widened by adding $\frac{1}{2}n$.

(3) A recent poll conducted by Gallup in December 1978 and published in the front pages of many newspapers on January 11, 1979, was based on 1,015 personal, in-home interviews. The sample proportions were $P_1 = 38\%$ for the Liberals, $P_2 = 40\%$ for the $P_1C.$, $P_3 = 18\%$ N.D.P. and $P_4 = 6\%$ for other. Assuming that the sampling was random, if (A) is applied to P_1, P_2, P_3 and P_4, we will be *less than* 95% confident that *all four* confidence intervals thus obtained will correctly bracket the population proportions π_1, π_2, π_3, and π_4 *simultaneously*. The confidence could be as low as $(95\%)4 = 81\%$. In order to achieve 95% confidence in all 4 intervals *simultaneously*, the confidence intervals should be wider than those obtained by (A) (cf. [6, p. 509]).

(4) The Gallup poll in (3) is actually more complicated than we suggested. Some voters in the sample could not be reached, some refused to answer, and some could not decide. For a discussion of how to treat this *nonresponse bias*, see [5, p. 294].

(5) Besides the use of proper sampling methods and right statistical formulas, the way a question is phrased for interview in an opinion poll, and the way a survey result is reported, are also very important. An ambiguous question may provide a highly biased estimate and a vague report may be misleading. [1] gives many illuminating examples. Suggestions for questionnaire design, interview technique, and analysis, and reporting of survey results can be found in [5, Chapters 6, 7, 11]. Other factors such as the sponsor's pressure on the pollster, or the pollster's own interest, may also cause bias.

5. Conclusion

Nowadays polls are conducted for many purposes - not only for election forecasts. Some examples are issues of public interests such as Quebec separation or abortion, marketing of new products, and rating of T.V. programs. In reading poll results, keep in mind that they are only *estimates* and subject to sampling errors and possibly bias (which often goes undetected). This is why different pollsters may give quite different predictions before an election. Statistical theory combined with common sense provides a scientific framework for designing and analysing polls.

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Editors Note: There is a very good film to introduce your students to Sampling that I would recommend. It is Sampling and Estimation
Inferential Statistics Pt. 1 & 2
distributed by International Tele-Film Enterprises
47 Densley Ave.,
Toronto, Ontario
M6M 5A8

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1. The material in Sections 1 to 3 is, with modifications, reprinted from [6, p. 3-5 and p. 8-9], with the kind permission of John Wiley and Sons, Inc., 605 Third Avenue, New York,N.Y. 10016.

THE CALENDAR

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The Julian calendar, introduced by Julius Caesar in 45 B.C., replaced a lunar calendar by a solar. The astronomers of the time estimated the solar year to be 365¼ days. To fit this into years it was agreed that 3 years would have 365 days, and the fourth year (leap year have 366 days. The process of adding a day is called intercalation. Of course, the natural place for an intercalation is at the end of the year, which for the Romans was February.

However, the estimate of 365¼ days is a bit long - by 11 minutes 14 seconds. So in the time of Pope Gregory (1582) it was decided to omit the intercalations in all centenary years, except those that are multiples of 400. The Gregorian calendar is in error (vis à vis the vernal equinox) by about one day in 3323 years.

After this brief historical introduction we can take up the first of the two problems which we wish to solve. The problem is this: on which week-day does a given date fall? In 1882, Zeller gave a solution to his problem as follows. Assign a characteristic number to each week-day by calling Sunday 1, Monday 2, . . ., Friday 6 and Saturday 0. (Modular arithmetic with modulus 7!) Let

$$f(q,m,n,J) = q + \left[\frac{26(m+1)}{10}\right] + n + \left[\frac{n}{4}\right] + \left[\frac{J}{4}\right] - 2J \qquad (1)$$

where $[x]$ denotes the integer part of x. Thus $x = [x] + \alpha$ where $[x]$ is an integer and $0 \le \alpha < 1$, so that $[\pi] = 2$, $[-\sqrt{2}] = -2$. Then the remainder left on dividing $f(q,m,n,J)$ by 7 is the characteristic number attached to the week-day of the q^{th} day of the m^{th} month of the year $100J+n$. Of course $0 \le n \le 99$. On applying (1) however, the convention is made that the January and February of any given year are to be regarded respectively as the 13^{th} and 14^{th} month of the preceding year. Hence $3 \le m \le 14$.

Let us test this formula on 6 May 1978. Here $q = 6$, $m = 5$, $n = 78$, $J = 19$.

So $f(6,5,78,19) = 6 + \left[\frac{156}{10}\right] + 78 + \left[\frac{78}{4}\right] + \left[\frac{19}{4}\right] - 38$

$= 6 + 15 + 78 + 19 + 4 - 38$

$= 6 + 78 + 15 + 4 + 19 - 38$

$= 84$

$= 12.7 + 0$; thus the characteristic number is 0. It is Saturday!

Knowledge of elementary properties of divisibility enables one to find which years of this century have Februarys with 5 Saturdays.

To verify that Zeller's method is accurate I recommend that you refer to Rogosinski, which has been the source of most of my knowledge of the calendar from the mathematical viewpoint.

The second and last problem I want to discuss is the determination of themoveable feast — Easter. It was decreed in AD 325 that Easter (Sunday) should be celebrated on the Sunday following the first full moon which occurs on or after the vernal equinox (21 March). Gauss devised a simple numerical rule for determining Easter in a given year. This rule for Easter, 1900 + N (when $0 \leq N \leq 99$) is as follows:

(i) Divide N by 19, 4, 7. Let a,b,c be the respective remainders.

(ii) Divide 19a + 24 by 30. Let d be the remainder.

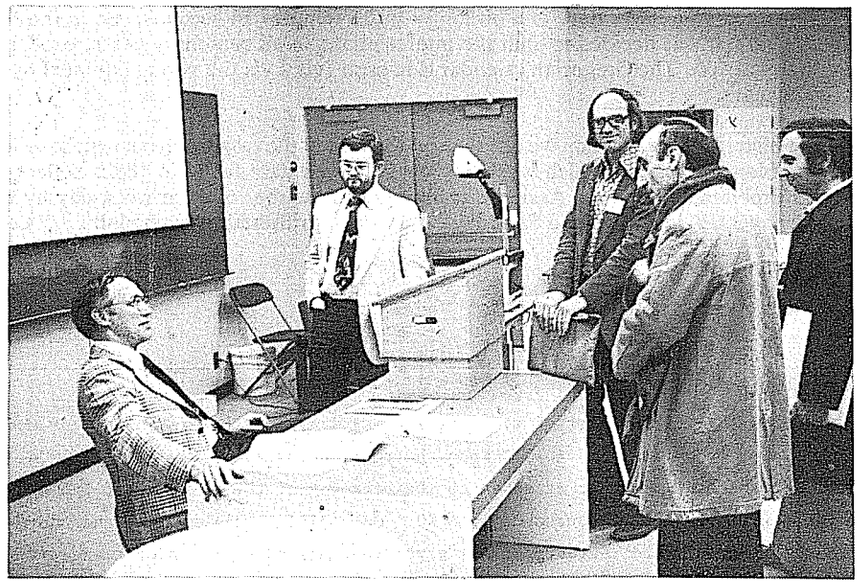
(iii) Divide 2b + 4c + 6d - 4 by 7. Let e be the remainder.

Easter occurs on (22 + d + e) March, if $d + e \leq 9$, and otherwise on (d + e - 9) April. There are two exceptions: If d = 29, e = 6 then Easter occurs on 19 April, and if d = 28, e = 6 then Easter occurs on 18 April.

Problems such as; when in this century does Easter fall on 31 March involve one in solving elementary congruences, and exploring the theory of divisibility. I recommend calendar problems as a source of interesting arithmetical problems.

Reference

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Dave Alexander (F.E.U.T.)

LIMITS AND ROOTS: A QUADRATIC QUESTION

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All calculus students are familiar with the two solutions of the quadratic equation. If $ax^2 + bx + c = 0$, the two solutions are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$. If $a = 0$, these solutions cannot be used since they involve division by 0; in this situation the equation becomes $bx + c = 0$. This has a simple solution $\frac{-c}{b}$.

What happens to the two solutions of the quadratic equation if a is "close to" but not equal to 0? In other words, as a approaches 0 as a limit while b and c are held constant, do the two solutions approach limits as well? The evaluation of the limits of the two solutions follows:

Solution I: $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ (without loss of generality, assume $b > 0$)

$$\lim_{a\to 0} \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \lim_{a\to 0} \frac{(-b - \sqrt{b^2 - 4ac})(-b + \sqrt{b^2 - 4ac})}{2a(-b - \sqrt{b^2 - 4ac})}$$

$$= \lim_{a\to 0} \frac{b^2 - (b^2 - 4ac)}{2a(-b - \sqrt{b^2 - 4ac})}$$

$$= \lim_{a\to 0} \frac{2c}{-b - \sqrt{b^2 - 4ac}}$$

$$= \frac{2c}{-2b} = -\frac{c}{b}.$$

Solution II: $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

$$\lim_{a\to 0} \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \lim_{a\to 0} \frac{(-b - \sqrt{b^2 - 4ac})(-b + \sqrt{b^2 - 4ac})}{(-b + \sqrt{b^2 - 4ac})(2a)}$$

$$= \lim_{a\to 0} \frac{b^2 - (b^2 - 4ac)}{(-b + \sqrt{b^2 - 4ac})(2a)}$$

$$= \lim_{a\to 0} \frac{2c}{-b + \sqrt{b^2 - 4ac}}$$

Since $2c$ is constant and $-b + \sqrt{b^2 - 4ac} \to 0$, the limit approaches infinity.

It is not surprising that as a approaches 0, one of the solutions approaches $-\frac{c}{b}$; this is the value that the solution would be if, in fact, $a = 0$. Many students reason that since only one solution emerges when $a = 0$, both of the limits of the separate solutions of the quadratic equation should approach the same number. This, however, is not the case; one of the solutions approaches infinity as a limit.

This quadratic equation illustrates this pattern of limits: $.00008x^2 - 7.32814x + 4.10182 = 0$. Observe that the value of a is very small relative to that of b and c.

The solutions are: $x = \frac{7.32814 \pm \sqrt{(7.32814)^2 - 4(.00008)(4.10182)}}{2(.00008)}$

$= \frac{7.32814 \pm \sqrt{53.70032}}{.00016}$

$= \frac{7.32814 \pm 7.32805}{.00016}$

$x = .56$ or $x = 91{,}601.$

It is clear that one solution (to two decimal places) is equal to $\frac{4.10182}{7.32814}$ which is $-\frac{c}{b}$. The other solution is extraordinarily large.

What happens to the two solutions of the quadratic equation if b approaches 0 while a and c are held constant?

Solution I: $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$

$$\lim_{b\to 0} \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{0 + \sqrt{0 - 4ac}}{2a} = \frac{\sqrt{-4ac}}{2a} = \frac{\sqrt{-c}}{\sqrt{a}} = \sqrt{\frac{-c}{a}}.$$

Assume that a and c have opposite signs to avoid an imaginary limit; without loss of generality also assume that $c < 0$.

Solution II: $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

$$\lim_{b\to 0} \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{0 - \sqrt{4ac}}{2a} = -\sqrt{\frac{-c}{a}}.$$

These are in fact the two solutions actually resulting from solving the quadratic equation when $b = 0$. ($ax^2 + c = 0 \Rightarrow x = \pm\sqrt{\frac{-c}{a}}$).

What happens to the two solutions of the quadratic equation if c approaches 0 while a and b are held constant?

Solution I: $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ (Assume without loss of generality that $b > 0$.)

$$\lim_{c\to 0} \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{b^2 - 0}}{2a} = \frac{-b + b}{2a} = \frac{0}{2a} = 0.$$

Solution II: $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

$$\lim_{c\to 0} \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - \sqrt{b^2 - 0}}{2a} = \frac{-b - b}{2a} = \frac{-2b}{2a} = \frac{-b}{a}.$$

These are in fact the solutions actually resulting from solving the quadratic equation when $c = 0$. ($ax^2 + bx = 0 \Rightarrow x(ax+b) = 0 \Rightarrow x = 0$ or $x = \frac{-b}{a}$).

If these three limit problems were examined in reverse order, the situation when $a \to 0$ would be even more impressive. When $b \to 0$ or $c \to 0$, the two quadratic solutions simply approach the two solutions which would result if the b and c terms were respectively deleted. A quite different consequence results when $a \to 0$.

This is a good example of the application of the limit concept to a situation familiar to a student before the calculus. The reader is invited to find other such examples.

THE RULE OF 78 - ANOTHER APPROACH

Walker Schofield, Math Head, Oakridge S.S., London

Editors Note: The Dave Campbell article referred to was titled "The Rule of 78 or Math Misused Makes Money" and can be found in **Highlights of the Proceedings of the Fifth Annual Conference.** Both stem from an article in the Sound Off with Gordon Sanderson column published in the local newspaper. Both articles show how a considerable amount of mathematics can be developed from an application, and urge that more emphasis be placed upon the mathematics of money. Are you aware of what "true" interest rate means to your bank. The editors will gladly publish the above mentioned article if you desire it. Let us know.

This article is presented as an additional topic that makes use of the material contained in the article by Dave Campbell. It is most suitable in the Mathematics of Investment course at the year 5 level.

The situation to be considered concerns the case where a borrower decides to repay a loan before the due date. The Ontario Consumer Protection Act outlines the method of calcluating the rebate the borrower is entitled to. For example:

A person borrows \$1730 and is charged \$270 interest for a loan period of 20 months. Repayment is 20 monthly payments of \$100 (starting one month from the date of borrowing). After 8 payments of \$100 have been made, the borrower makes a payment of \$1200 the next month. The problem is to find the interest rebate that the borrower is entitled to.

The Consumer Protection Act states that the actual cost of the loan = $\frac{A}{B}$ × the cost of borrowing where

A = the balance at the beginning + the balance after each payment until the final payment;

B = The sum of the balances that were scheduled in the original agreement.

In the example above, A = \$2000 + 1900 + 1800 + 1700 + 1600 + 1500 + 1400 + 1300 + 1200 + 0 = \$14,400

B = \$2000 + 1900 + 1800 + 1700 + 1600 + ··· + 200 + 100 + 0 = \$21,000

Therefore the actual cost is $\frac{14400}{21000}$ × \$270 = \$185.14.

The rebate is \$270.00 - \$185.14 = \$84.86.

It is obvious that there is a lot of work in writing all the terms down and finding the sums. If there were 150 payments (12½ years) or if the payments were \$165.43 per month the amount of work would increase considerably. Hence the general situation will be considered and a formula developed.

The general problem is when a person borrows \$A and is charged \$i in interest for a total loan of \$P (P = A+i) for N months. The repayment schedule calls for N monthly payments of $\frac{\$P}{N}$. Instead the borrower decides to pay off the loan after making r payments (r < N). We will calculate the cost of borrowing and the rebate.

The calculations that must be made are for A and B as defined previously.

$$A = P + \left[P - \frac{P}{N}\right] + \left[P - \frac{2P}{N}\right] + \left[P - \frac{3P}{N}\right] + \cdots + \left[P - \frac{rP}{N}\right] + 0$$

$$= (r+1)\,P - \frac{P}{N}\,(1+2+3+\cdots+r)$$

$$= (r+1)\,P - \frac{P}{N}\,\frac{(r)(r+1)}{2}$$

$$= (r+1)\,P\left[1 - \frac{r}{2N}\right]$$

$$= (r+1)\,P\left[\frac{2N-r}{2N}\right].$$

$$B = P + \left[P - \frac{P}{N}\right] + \left[P - \frac{2P}{N}\right] + \left[P - \frac{3P}{N}\right] + \cdots + \left[P - \frac{N-2}{N}P\right] + \left[P - \frac{N-1}{N}P\right] + 0$$

$$= NP - \frac{P}{N}[1+2+3+4+\cdots+(N-1)]$$

$$= NP - \frac{P}{N}\,\frac{(N-1)(N)}{2}$$

$$= \frac{2NP - NP + P}{2}$$

$$= \frac{NP + P}{2}$$

Therefore the cost of borrowing is $\dfrac{\frac{(r+1)P(2N-r)}{2N}}{\frac{NP+P}{2}} \times i = \dfrac{(r+1)(2N-r)}{N(N+1)} \times i.$

The rebate is the difference between this amount and i.

To check this formula we use the first example where N = 20, r = 8 and i = \$270. Cost $= \frac{9 \times 32}{20 \times 21} \times \$270 = \$185.14$ as before.

The example that is in the 'Sound Off' column found in the previous article can also be evaluated by this formula. In it N = 150, r = 14 and i = \$13,366 (this figure was rounded to the nearest dollar). Thus the Cost $= \frac{15 \times 286}{150 \times 151} \times \$13,366 = \$2531.57$. This cost is different by 2¢ from the cost shown in the paper, however, this can be explained by the rounding off referred to above.

The formula developed in this article is, of course, not applicable to the situation where the borrower makes double payments some months in order to pay a loan off more quickly. These cases must be dealt with as explained in the first example.

BOOK REVIEW AND RESEARCH

Dr. William Higginson, Queen's University

Case Studies in Science Education: Volume I The Case Reports, Volume II Design Overview and General Findings. R.E. Stake and J.A. Easley (Eds.). Washington, D.C. 20402: U.S. Government Printing Office 1978. (Volume I Stock Number 038000003771 \$7.25; Volume II Stock Number 038000003763 \$6.50. Canadian orders must be paid in U.S. funds and a 25 per cent surcharge must be added).

The U.S. Government Printing office has recently made available the results of a major research study directed by Professors Stake and Easley of the University of Illinois. The Stake/Easley volumes are two of seven publications resulting from the 1977 Survey of Pre-College Science, Mathematics and Social Science sponsored by the National Science Foundation. The objective of the study was to provide the National Science Foundation with a portrayal of current conditions in the K-12 Science classroom, to help make the Foundation's programmes of support for science education consistent with national needs. The project team studied eleven high schools and their respective feeder schools looking in particular at their programmes in science, mathematics and social studies.

Among the observations noted by the codirectors in the preface to their report are: "the textbook usually was seen as the authority on knowledge and the guide to learning. The teacher was seen to be the authority on both social and academic decorum. He or she worked hard to prepare youngsters for tests, subsequent instruction, and the value-orientation of adult life. Though relatively free to depart from district syllabus or community expectation, the teacher seldom exercised either freedom!

The case studies themselves vary considerably in length, format, emphasis and structure. At their

best (Terry Denny in 'River Acres' Texas) they make entertaining and enlightening reading. Others are closer to the turgid prose we have come to expect from research reports. In places there are pockets of flipness which must have made the NSF administrators wince. (The biographical note following the second report of one team member reads "Not a great deal happened in Rob Walker's life between the time of his Pine City Case Study and that of his Boston Case Study. (Hard to believe, considering he spent those four weeks in the cornfields of central Illinois!) ").

The overwhelming impression one is left with after examining these materials is that of an 'embarasse des riches'. The reports smack of authenticity but it is not easy to get a good grip on such a massive amount of information. These reports do, however, put considerable flesh on the rather spare bones of the NACOME report of a few years ago. Constant themes running through the reports are the identification of mathematics classroom activities with vitualistic and authoritarian behavior.

The flavour of the reports can be caught from a few quotations: Three secondary school students give their impressions of the nature of mathematics.

"Putting numbers together in different ways, finding short cuts, learning how to use numbers."

Nothing is hard about Math. I get a B. I don't do good on the big test. Math is doing good work on the tests, big and little.

I'm in level three. That means they won't teach us Algebra. They will only teach us what we have already had. Math is learning over and over what you already know and keep forgetting."

"Mathematics teaching and learning can be said to be much like the children's rhyme, 'When it is good it is very, very good and when it is bad it is horrid' "

In most of our sites we found secondary school mathematics to be just as traditional and work-oriented as learning to compute in the elementary school.The judgement from mathematicians and other visiting site observers was that many of the courses were tedious. Several noted, however, that the students in high school mathematics classes did not appear to find them as dull, uninspiring and irrelevant as the observers did. Teachers and students appeared to accept mathematics as a dry mechanical thing to be done stoically.

NCTM PUBLICATIONS

The National Council of Teachers of mathematics has continued its active publishing policy in the last few months releasing a number of new titles aimed at elementary and secondary school teachers. Among the new releases is the fourth volume of William L. Schaaf's **Bibliography of Recreational Mathematics**. As was the case in the previous three volumes, Schaaf takes a very broad interpretation of the term recreational. Included, for example, are references to articles on topics such as Number Patterns, Polygons and the Pythagorean Theorem. This publication can, therefore, be a valuable tool for any teacher with access to journals such as **The Mathematics Teacher**, **Scientific American** or **Mathematics Teaching**, who wishes to find interesting examples and applications for his classes.

Mathematics History: Activities, Puzzles, Stories and Games is a collection of games and puzzles on historical themes for grades four to six constructed by Merle Mitchell. The format is similar to that made well known by Creative Publications. Adrian L. Hess has written a slim (28 pages) booklet on **Four-Dimensional Geometry: An Introduction.** It is not immediately obvious to whom this booklet might be addressed. The content and method of presentation seems too advanced for students and older readers might be better served by a standard textbook.

If one finds lines such as "Fibonacci numbers surely are a hare-raising experience" and "what's an insect that's not feeling well? (a secant)", funny, then one should immediately order **Mathematics and Humour** by Aggie Vinik et al . **Applications of Secondary School Mathematics** by Bernice Kastner consists of twelve chapters on topics such as Optics and the Geometry of Molecules. Although the applications themselves are frequently of interest the presentations seem generally to be unimaginative.

Easily the most attractive publication of this recent list is Stephen Brown's **Some Prime**

Comparisons, a collection of ideas on number theory. Brown's mathematical taste is first class and he writes well. This book would be a very good enrichment source either as outside reading for a gifted secondary student or as a basis for classroom or math club activities.

BRIEFLY NOTED

The Ontario Educational Communications Authority has a number of first class television/videotape materials in the area of mathematics which seem to be less well known that they deserve to be. Among these programmes are the very sophisticated 'Metric Measure' materials and the highly attractive 'Math Patrol' series. Further details about these programmes can be obtained from:

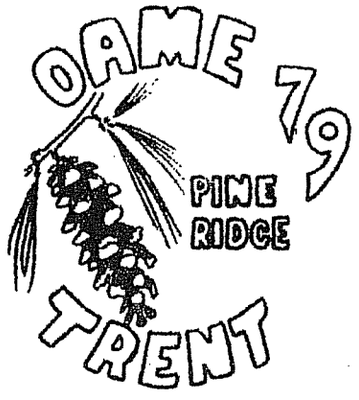
Publications/OECA
Box 200, Station Q
Toronto, Ontario
M4T 2T1
(416) 485-9451

Teachers who have enjoyed the **I Hate Mathematics! Book** will be pleased to know that Marilyn Burns now has two other titles available in the Brown Paper Book series (published by Little, Brown and distributed in Canada by McClelland and Stewart Ltd., 25 Hollinger Road, Toronto). As with the original title these books are full of excellent classroom ideas attractively presented.

One of the interesting spin offs from the women's movement in the United States has been the identification of a phenomenon called Math Anxiety (see the two part article by Peter Hilton in the previous two numbers of this journal). The first two hard cover publications in this area are Sheila Tobias' **Overcoming Math Anxiety** (New York: Norton, 1978 $10.95 in U.S.) and S. Kogelman and J. Warren's **Mind Over Math** (New York: Dial, 1978 $8.95 in U.S.). The latter text takes a more therapy-oriented approach but on the whole the Tobias book would seem to be the superior one.

> "Years ago the revolutionary idea in the teaching of mathematics was that if a student could arrive at the answer by a logical method and correct calculation then he should be given full marks. Now many of these same teachers are saying that the calculation must be done by hand. Pedestrian thinking such as this on the part of the mathematical elite in our school system will surely cause the students to avoid our subject as much as possible. The law may require that they be there in body but their spirits will certainly be elsewhere."

The above statement is typical of the zesty style of material in the **Bit Exchange: What's Going on with Computers in the Classroom** a newsletter produced by Wayne Johnson of Seaway District High School, Box 100 Iroquois, Ontario K0E 1K0. Interested parties can obtain further information from Mr. Johnson.



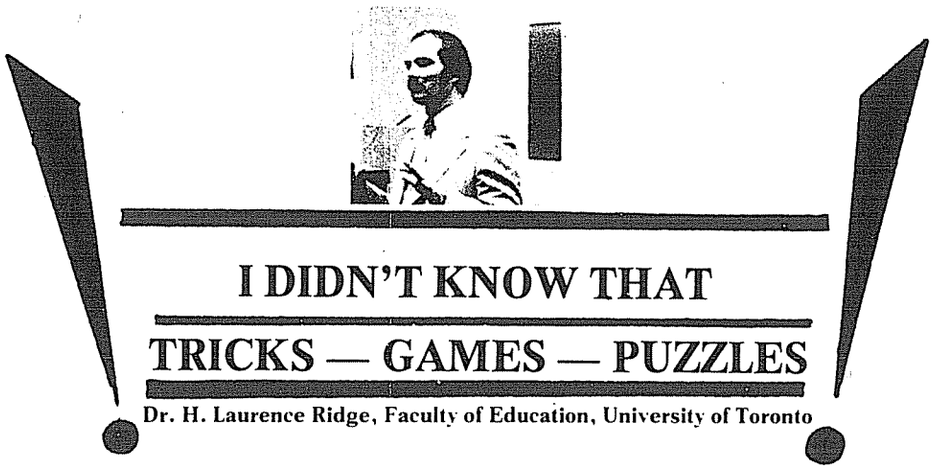
REMINDER

OAME - 1979, May 11-13
Sixth Annual Conference
The Peterborough Conference
by
P.R.M.A. (Pine Ridge)

Brain Teasers

4. On being asked the time John replied, "Add one quarter of the time from noon till now to one half the time from now till noon tomorrow and you will have the exact time." What time was it?

Answer on pg. 68



I DIDN'T KNOW THAT

TRICKS — GAMES — PUZZLES

Dr. H. Laurence Ridge, Faculty of Education, University of Toronto

MULTIPLICATION MEDLEY

This time around we've pre-empted the format of the last few issues in order to react to a most pertinent article by Frances Cacha in the November 1978 issue of the **Arithmetic Teacher**. In "Exploring the Multiplication Table and Beyond", Professor Cacha identifies several patterns in the nine by nine multiplication table. Because such a problem solving activity has such wide applicability from elementary school on up, I would encourage you to turn your classes loose on this one.

I have done precisely that with my Elementary Mathematics Education classes over the past several years and would like to share with you their collective findings to date along with (generalized) explanations (provided by yours truly) suitable for those adept at some algebraic manipulation. The relationships which follow are either distinct from or further developments of those in the Cacha article.

1. **Reducing the multiplication table**
   a. **Easing the load**
   Expose only the 2X and 4X rows. It can be seen readily that the 4X table is simply twice the 2X table. Likewise, the 8X table is twice the 4X table.

   Similarly, the 6X table is just twice the 3X table. Hence 6 X 8 can be found readily by doubling 3 X 8, an application of the 'groupin' (associative) property: $6 \times 8 = (2 \times 3) \times 8 = 2 \times (3 \times 8)$.

   In the 9X table, the sum of the digits of every product is 9. Also, the tens digit (in a 2-digit product) is one less than the other factor. (Eg. $9 \times \mathbf{4} = \mathbf{36}$, $9 \times \mathbf{8} = \mathbf{72}$).

   Other than 7 x 7 (which can be remembered as one of the square numbers) each of the other entries in the 7X table can be derived from its reflection image in the main diagonal (Fig. 1) (Eg. $7 \times 6 = 6 \times 7 = 2 \times (3 \times 7)$).

   The 5X and 10X tables have unique properties which make them easy to remember. Everything else (except 7 X 7) is based upon the 2X and 3X tables. Such a realization can do much to ease the anxiety of 'learning the tables' for many youngsters.

   b. **Equivalent fractions**
   Expose any two rows of the table, say 3X and 5X (Fig. 2). Read the pairs of numbers vertically to see a set of fractions all equivalent to three-fifths. Such patterns can be used to generate many exercises related to equivalent fractions.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	(7)	8	9	10
2	2	4	6	8	10	12	(14)	16	18	20
3	3	6	9	12	15	18	(21)	24	27	30
4	4	8	12	16	20	24	(28)	32	36	40
5	5	10	15	20	25	30	(35)	40	45	50
6	6	12	18	24	30	36	(42)	48	54	60
7	(7)	(14)	(21)	(28)	(35)	(42)	(49)	(56)	(63)	(70)
8	8	16	24	32	40	48	(56)	64	72	80
9	9	18	27	36	45	54	(63)	72	81	90
10	10	20	30	40	50	60	(70)	80	90	100

Fig. 1

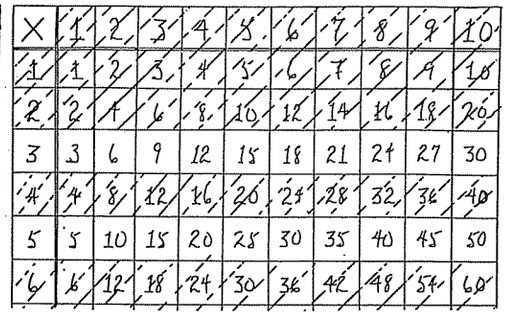


Fig. 2

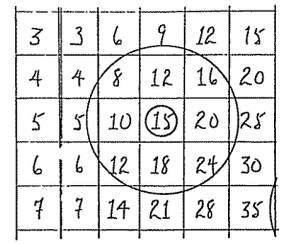


Fig. 3

2. Even and odd

Every odd entry in the table is surrounded by 9 even numbers (Fig. 3). The chart at the right shows why this is always so.

(m-1) (n-1) EVEN X EVEN = EVEN	(m-1)n EVEN X ODD = EVEN	(m-1) (n+1) EVEN X EVEN = EVEN
m(n-1) ODD X EVEN = EVEN	mn ODD X ODD = ODD	m(n+1) ODD X EVEN = EVEN
(m+1) (n-1) EVEN X EVEN = EVEN	(m+1)n EVEN X ODD = EVEN	(m+1) (n+1) EVEN X EVEN = EVEN

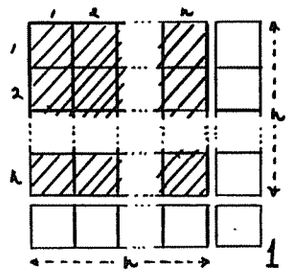


Fig. (i)

3. Diagonals

a. The main diagonal's square numbers and successive odd number differences have a geometric significance (Fig. 4). To each n by n square (of spaces) is added an 'L' (gnomon) made up (as shown as fig. 6) of (n + n + 1) spaces. Algebraically, $n^2 + (n + n + 1)$
$= n^2 + 2n + 1$
$= (n + 1)^2$, the next number on the diagonal.

b. The diagonals parallel to the main diagonal can be based on a rectangular geometric pattern (Fig. 5). To each m x n rectangle (of spaces) is added gnomon made up (as shown as fig. (ii)) of (m + n + 1) spaces.

Algebraically mn + m + n + 1 = (m + 1) (n + 1), the next number on the diagonal. The differences m + n + 1 will be successive odd numbers when (m - n) is even but will be suc cessive even numbers when (m - n) is odd.

Senior level students might try proving this relationship (i) m - n even m - n = 2t for some integer t. m + n + 1 = m + m - 2t + 1
= 2m - 2t + 1
= 2(m - t) + 1 which is odd.
The next difference would be (m + 1) + (n + 1) + 1.
(m + 1) + (n + 1) + 1 = m + n + 3
= 2(m - t) + 3 which is the next greatest odd number after

2(m-t)+1 (ii) m-n odd m-n = 2t-1 for some integer t m+n+1 = m+(m-2t+1) +1
= 2m-2t+2
=2(m-t+1) which is even.

The next difference is (m + 1) + (n + 1) + 1 = m + n + 3
= (2m - 2t + 1) + 3
= 2 (m - t - 1) + 2 which is the next greatest even number after 2(m - t + 1).

Fig. (ii)

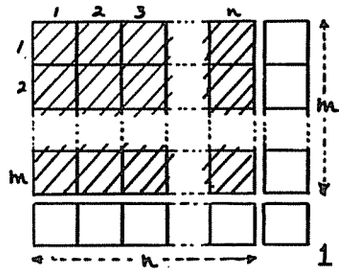
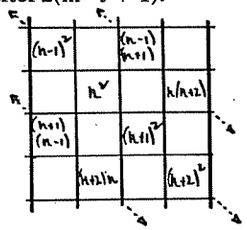
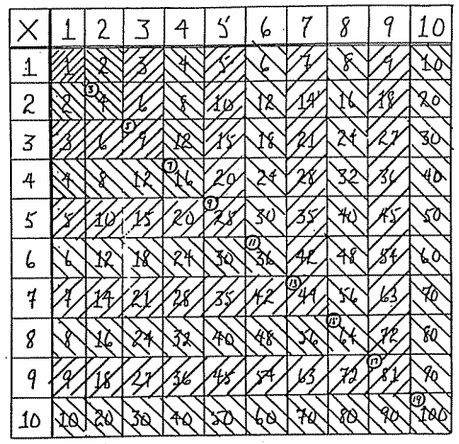


Fig. (iii)



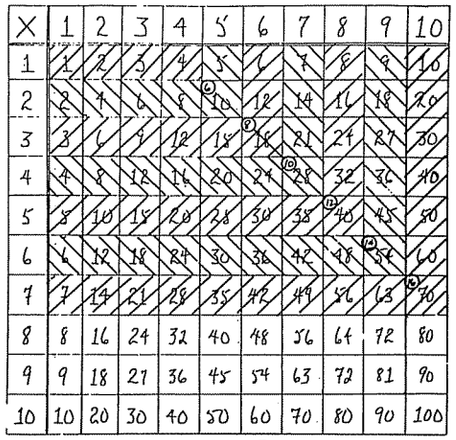
c. Two diagonals parallel to the main diagonal (Fig. 6) consist of numbers each of which is 1 less than a perfect square. The chart (Fig. (iii)), shows the entries corresponding to n^2 and $(n+1)^2$ (in the main diagonal) to be (n+1) (n-1) and (n+2)n respectively. Algebraically, these entries become n^2 - 1 and $(n+1)^2$ - 1 since

$(n + 1)(n - 1) = n^2 - 1$ and $(n + 2)n = (\overline{n+1} + 1)(\overline{n+1} - 1)$
$= (n + 1)^2 - 1.$



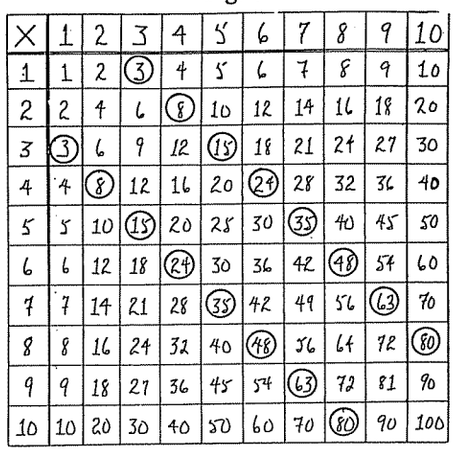
×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Fig. 4



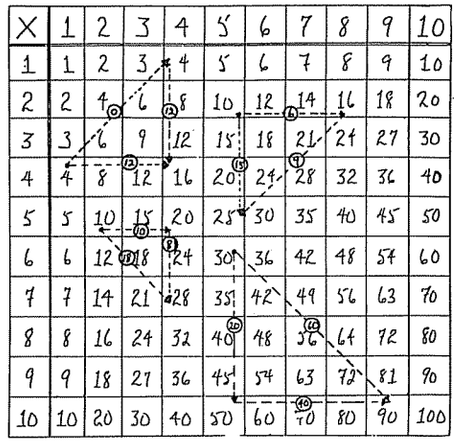
×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Fig. 5



×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

FIg. 6



×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Fig. 7

4. **Digital sums**

The Cacha article displays a table of digital sums* corresponding to the multiplication table. (*27 has a digital sum of 2 + 7 or 9, 48 has a digital sum of 3 since 4 + 8 = 12 but 1 + 2 = 3).

Note that each digital root along the main diagonal is 1, 4, 7, or 9). Which suggests that the digital root of **any** square number is 1, 4, 7 or 9. This is so since any natural number n can be expressed as 9t + k where t and k are also natural numbers but k is less than 9. (Eg. 31 = 9 x 3 + 4). Hence, $n^2 = (9t + k)^2$

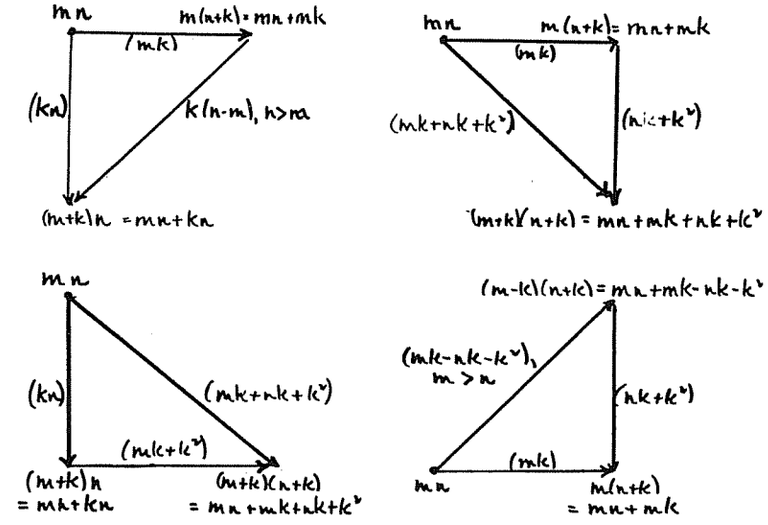
$= 81t^2 + 18tk + k^2$

$= 9(9t^2 + 2tk) + k^2$

whose digital sum is that of k^2 * which we know to be 1, 4, 7, or 9.

5. **Vector addition**

Fig. 7 illustrates a relationship involving rows colums and diagonals which provides a surprising occurrence of vector addition. Every such triangular pattern is represented by one of the following four situations.

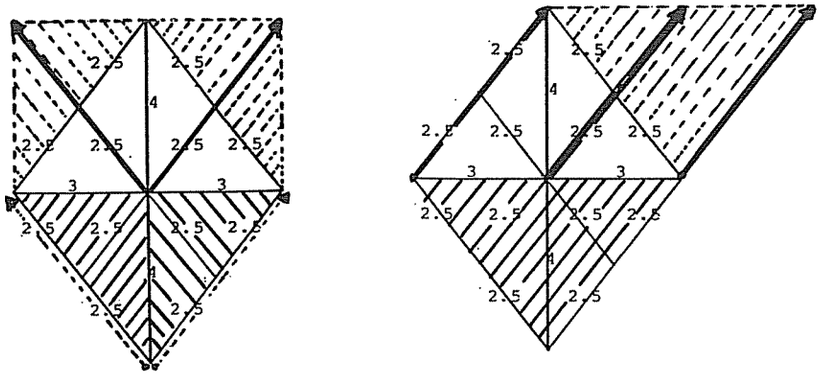


Be sure to let us know about any other patterns which your students come up with.

Solution to Diamond Problem

The completion to Russell Walker's rhyme (and solution to the problem) is twenty-four (cm^2). The result is simplified by translating two of the triangular regions (as shown below) to make a 4 cm by 6 cm rectangle equal in area to the given diamond (parallelogram).

Alternatively, the entire lower half of the figure (traingle) could be translated (as shown at the right) to make a parallelogram with length 6 cm and altitude 4 cm - again, equal in area to the original figure.



*The digital sum of any number is just the remainder after division by 9. See "Getting to the Root of the Problem" in this column in the March 1978 issue).

Recreationally yours,
HLR

TO THE CHAPTERS



I still believe the strength of OAME/AOEM lies in the individual chapters, in individual members who make up the organization. Thus, I challenge each and every member, not the presidents alone, to consider carefully their position on **Basic Mathematical Skills.** To help you in your continued deliberations I am reprinting the testimony of our N.C.T.M. president Shirley Hill, given before a state task force which was preparing recommendations concerning that state's program of testing functional literacy and basic skills. Do read Tom O'Brien's article **Two Essays** which examined the goals of mathematics education at the elementary school level. If you, or a small group of you, or even a chapter are formulating a position paper, perhaps it is worth reiterating the N.C.S.M. position in a summary format. They gave a plausible rationale to suggest that each student needs to be able to:

- solve problems
- apply mathematics to everyday situations
- be alert to the reasonableness of results
- estimate and approximate
- compute with appropriate skills
- use geometry
- measure
- read, interpret and construct tables, charts and graphs
- use mathematics to predice
- understand the role of computers

Mathematics is more than Arithmetic and there is a great need for mathematical literacy. Take time to sort out your thinking and act upon your beliefs.

PROBLEMS

Editors Note: The answers to problems found in the September issue follow this issue's offering. They do take up space. Let us know your feelings. Some have mentioned that they should appear in the same issue, others, that only the difficult solutions be published with the rest being available. The authors do hope your students will continue to submit solutions. Remember to include pupil's name, grade, school and community. Please send to:

Ontario Mathematics Gazette
c/o Mr. J. McKnight
Westdale Secondary School
700 Main St. West
Hamilton, Ontario L8S 3T1

UP TO GRADE 10

17-3-1

A 16 quart radiator is filled with water. Four quarts are removed and replaced with pure antifreeze liquid. Then four quarts of the mixture are removed and replaced with pure antifreeze. This is done a third and fourth time. What part of the final mixture is water?

17-3-2

Some children developed the following coding system for numbers:

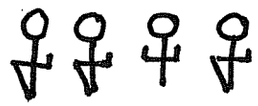

one hand up


both hands down


both hands up

 = 0;  = 1;  = 2;  = 3;  = 4; etc.

What number would be represented by:

 ?

17-3-3

If at a certain instant it is 7 o'clock, what time is it 11,999,999,994 hours later?

17-3-4

When Ernie was as old as Bert is now, Bert's age was half of Ernie's present age. When Bert will be as old as Ernie is now, the sum of their ages will be 99. Find Bert's present age.

17-3-5

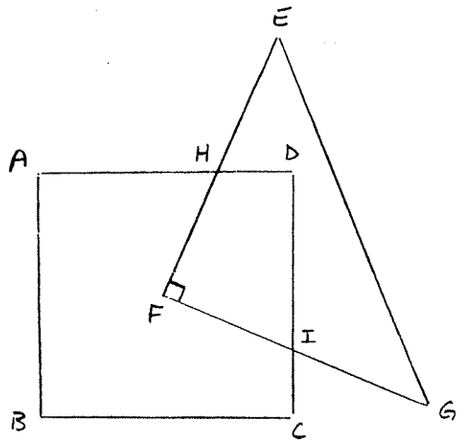
A man was x years old in the year x^2 . How old was he in 1960?

17-3-6

A freight train and an express train travel at constant speeds on straight parallel tracks. It takes 21 seconds for the trains to clear each other when passing in the same direction, but only 6 seconds when passing in opposite directions. Find the ratio of the speed of the freight train to the speed of the express train.

17-3-7

The isosceles right traingle EFG has a vertex at the centre of the square ABCD. Determine the area of the common quadrilateral given BC = 7, FG = 8, HD = 2



17-3-8

Given that $\cos 3\theta = A\cos^3\theta + B\cos\theta$ holds for every real number θ, determine the values of A and B.

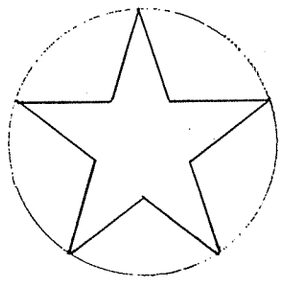
UP TO GRADE 13

CHALLENGE

17-3-9

Find the area of the star if

a) the circle has a radius of 10 units
b) the points of the star are the same distance apart
c) the star is formed by joining each point to the two opposite ones.



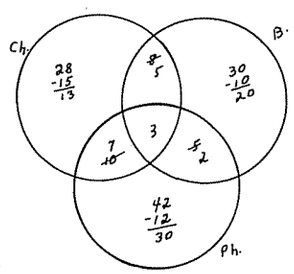
SOLUTIONS

17-1-1 Total games to be played 8 + 16 = 24

75% of games is $\frac{75}{100} \times 24 = 18$

∴ The hockey team must win 18 - 5 or 13 more games.

17-1-2



Total Chem. = 13
Biol. = 20
Phys. = 30
53

∴ The total of all students studying only one science is 63.

17-1-3 No. of rotations of large roller $\frac{1800}{150} = 12$

$\therefore$ inner circumference moves $12 \times 125 = 150$ cm

$\therefore$ rotations of inner cylinder $\frac{1500}{60} = 25$

$\therefore$ the inner cylinder rotates 25 times.

17-1-4 93 people must be eliminated and that will take 93 games.

17-1-5 Let x represent the number of \$1 bills ($x \in I$)

Let y represent the number of \$5 bills ($y \in I$)

$\therefore x + 2(10x) + 5y = 200.$

From the above equation x must be divisible by 5.

If x = 10 we get no solution;

If x = 5

$5 + 2(50) + 5y = 200$ (only possible solution)

$105 + 5y = 200$

$5y = 95$

$y = 19.$

$\therefore$ the cashier paid him five \$1 bills, fifty \$2 bills and nineteen \$5 bills.

17-1-6 Factoring $x^3 + x^2 - 4kx - 4k = 0$

$x^2(x+1) - 4k(x+1) = 0$

or $(x+1)(x^2 - 4k) = 0$

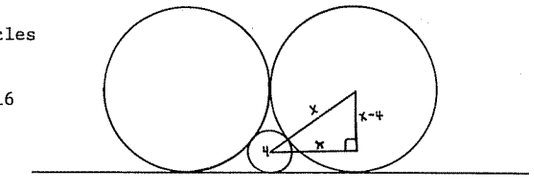
$\therefore$ the roots are $x = 1, \pm 2\sqrt{k}$

$\therefore \pm 2\sqrt{k} = 1$ or $2\sqrt{k} = -2\sqrt{k}$

$k = \frac{1}{4}$ $k = 0$

$\therefore$ k must be $\frac{1}{4}$ or 0.

17-1-7 Let x be radius of 2 equal circles



$(x+4)^2 = x^3 + (x-4)^2$

$x^2 + 8x + 16 = x^2 + x^2 - 8x + 16$

$0 = x^2 - 16x$

$0 = x(x-16)$

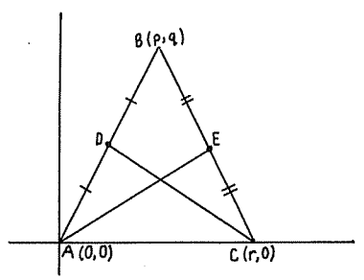
$x = 0$ (inadmissable) or $x = 16$

$\therefore$ the radius of the equal circles is 16.

17-1-8 Given: ΔABC with medians $CD = AE$.

Prove: ΔABC is isosceles.

Proof: Select one vertex A as the origin and construct one side AC along the x-axis. Call the other vertices B(p,q) and C(r,0).



D = mid AB = $(\frac{p}{2}, \frac{q}{2})$ and

E = mid BC = $(\frac{p+r}{2}, \frac{q}{2})$

AE = CD

$$\sqrt{(\tfrac{p+r}{2})^2 + (\tfrac{q}{2})^2} = \sqrt{(\tfrac{p}{2} - r)^2 + (\tfrac{q}{2})^2}$$

$$(\tfrac{p+r}{2})^2 + (\tfrac{q}{2})^2 = (\tfrac{p-2r}{2})^2 + (\tfrac{q}{2})^2$$

$$p^2 + 2pr + r^2 = p^2 - 4pr + 4r^2$$

$$6pr = 3r^2$$

$$6p = 3r$$
$$2p = r.$$

Now $AB = \sqrt{p^2 + q^2}$

and $CB = \sqrt{(p-r)^2 + q^2}$

$$= \sqrt{(p-2p)^2 + q^2}$$

$$= \sqrt{(-p)^2 + q^2}$$

$$= \sqrt{p^2 + q^2}$$

$\therefore$ AB = CB

$\therefore$ ΔABC is isosceles. Q.E.D.

17-1-9 Proceeds from the sale of cattle is x^2 dollars.

Let the cost of the lamb be ℓ dollars.

Let the number of sheep be s.

Since division of the animals is equal, 5 + 1 is even

$\therefore$ 5 is odd

$10s + \ell = x^2$

$\therefore$ ℓ is the units digit of a perfect square and the tens digit must be odd. (s is odd)

Consider the series of perfect squares:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256,..., 576.

All perfect squares with odd tens digits end in a 6.

(i.e., the squares of numbers ending in 4 or 6. This can be proven using $4^2 = 16$ (odd) and $6^2 = 36$ (odd). Only for numbers ending in 4 or 6 do you have an odd carry onto a tens digit which is even before the carry,

e.g., $(10x + 4)^2 = 100x^2 + 80x + 16$

even before carrying; odd carrying

$\therefore$ The units digit, ℓ, must be 6.

$\therefore$ He must be compensated \$2.

A POINT OF VIEW ON THE MINIMAL COMPETENCY MOVEMENT

Shirly A. Hill, President NCTM

This paper was given as testimony before the Florida Task Force on Assessment

I am a mathematics teacher and president of a national organization of mathematics teachers numbering about 47,000 individual members. Thus my remarks will be largely specific to mathematics education, though they may well generalize to other content, and general as to the national picture, though there are implications for this state's basic skills and functional literacy program.

The premises of my position are two:

1) the current demand for attention to basic or essential competencies is a valid and legitimate popular movement, but often misguided and misinformed.
2) the response to public concern comes largely from outside professional education and is essentially political, not always guided by knowledge about students and learning.

Since I believe the "basic skills" movement to be valid in its essential instincts, I think it could represent a welcome challenge and opportunity for education. It challenges us to do something we should always be doing, i.e. examining the goals and objectives of education and trying to achieve societal consensus as to the nature of these goals and the school's role in their achievement. In this sense, the process should be **political** in the best meaning of the word. All segments of society, ideally, participate in policy decisions about the goals of education.

But at the same time, there are serious hazards in what has been in many places a hasty, simplistic response to some very real concerns and problems. The popular notion of accountability, encouraged by the media and supported by representatives in government, is superficially attractive but dangerously simplistic. I believe that in many places, a hurriedly contrived "accountability" measure will create far more problems than it will solve. And the tragedy is that it may miss the greatest opportunity for a popular consensus on the essentials and long-term goals of education. It may leave the deepest problems untouched in order to treat surface symptoms with patchwork remedies.

I do not oppose testing, nor does the organization I represent. We view testing as one, but only one, component of evaluation. I do oppose an almost exclusive reliance on imperfect instruments and an imperfect technology in making decisions so vital to the lives of individual students. To withhold promotion or a diploma on the basis of a single instrument puts far more burden on that tool of evaluation than it is designed to bear.

Furthermore, much of the current minimal competency movement seems to return the role of testing from its task of finding programmatic strengths and weaknesses to that of a screening device for society. This is a philosophical issue of no small consequence in a society that espouses equal educational opportunity and for an educational system that attempts to meet a wide variety of individual needs.

In this regard also, it seems to me that the movement often raises false expectations on the part of the public, and education will once again become the convenient scapegoat when those expectations are thwarted. The testing program is not a panacea; it cannot solve all the educational problems it promises to solve. Achievement cannot be mandated or legislated. If such programs merely placate the public temporarily, we will ultimately face the consequences of our failure to confront the deeper issues of defining our long-term goals.

In my opinion, the hazards of present emphasis on basic skills and minimal competencies revolve primarily around the often narrow, limited, short-sighted view of what is truly basic. Present functional literacy objectives tend to focus on low-level, easily measurable skills and to ignore the thinking processes that may be the most important "basics". They also tend to fragment problem solving processes into trivial bits, which do not sum to the whole - i.e. to the complexity of the interrelationships.

To define the basics as the three Rs is far too restricted an image of the productive citizen - to define the basics of mathematics as the third R is not to understand the nature of mathematics as a useful tool.

There are two dimensions along which the basics of mathematics are being viewed too narrowly, even in the sense of applications or so-called "life skills". One is the tendency to focus on a limited view of what mathematics can provide a worker or a consumer. A recent ad of Mobil Oil speaks to this point. In discussing corporate personnel needs it states that "an individual's ability to deal with abstract problems involving judgment and the ability to reason is often more significant than any technical knowledge."

At the other end of the vocational spectrum, I asked a friend in the distributive industry why he considers mathematics achievement an important characteristic in hiring high school graduates. He said it is not that they are to use particular mathematical tools in a direct obvious way, but he found that they do better on the job because, apparently, of a grasp of patterns and relationships. Mathematics can and does aid development of important processes of reasoning, analyzing, relating, classifying, estimating, inferring, testing, and verifying. But such skills are not developed where an inordinate emphasis is placed on rote or mechanistic learning or application, simply because short-term retention is all that is needed to pass a test.

Current testing programs also take too narrow a view of what it means to be a consumer. The flap over the consumer math results on the national assessment by NAEP concentrated almost exclusively on unit pricing, on the consumer as a buyer. To me, the most essential consumer skill is the ability to deal intelligently with statistical information. This is more than the ability to read information as presented in tables and graphs. The true "basic" for consumers is to be able to draw inferences from quantitative information, to interpret it and to be alert to misinterpretations. We are nowhere more vulnerable as consumers than in the bombardment of statistical data designed to persuade or indoctrinate. There is no area of mathematics more basic and vital to every citizen. This area is barely touched, if at all, on tests.

Another dimension of narrow vision is our short-sighted approach to functional literacy. The basics needed by our students are those which will be essential ten or twenty years from now. The world is not static and the basics are not eternal. We do not have accurate crystal balls, so we must rely on the best predictions by people in the field as to what the trends indicate for future applications.

In mathematics this would surely include literacy about computer and calculator technology. Learning to use a calculator is basic. Learning about the uses and limitations of computers is basic.

This look farther down the road forces us to confront a more difficult issue in mathematics. Right now, everyone's list of minimal competencies includes "appropriate computational skill". So does mine. But we must, sometime soon, examine what "appropriate" means. At present, the computational skill mastery we are testing is dictated primarily by tradition. This ultimately guarantees obsolescence. We need to take a hard look from what might be called a "cost effective" basis at the time required to teach certain techniques in relation to their "real world" uses. For example, a very large portion of the fourth, fifth and even sixth grade is spent on complicated long division algorithm. Yet almost nobody does paper-and-pencil long division in the "real world" any more. Is it worth the time? Time in school is precious. We **need** to confront these difficult decisions, but I believe that present minimal competency programs testing only the **traditional** basics are delaying the process.

Because education is inevitably a futuristic enterprise, it is difficult to pinpoint with confidence the highly specific tools needed at a given future time. It may be that the most valuable and basic ability we can develop is the ability to go on learning. The essentials are those mental processes that characterize a flexible, adaptable problem solver. A college president recently said to his entering freshman class, "When you leave here, people will be far more interested in what you can learn that what you know." If we stress short-term retention of fragmented skills in order to pass tests, we can scarcely expect to develop life-long learners.

There is another misconception about computational skill that calls into serious question the linkage of promotion from grade to grade based solely on performance on a test. Evidence indicates that **mastery** of computation is not necessarily achieved at initial instruction. Skills stabilize generally. Thus, retaining a child in-grade in the early years based on failure to achieve skill mastery may in many cases gain nothing that would not have been achieved anyway.

Another fallacy about mathematics learning is especially widespread and pernicious. It is that the mastery of computational skills is an absolute pre-requisite to other mathematics learning. Even at highest professional levels (though rarely in mathematics education) this fallacious assumption is accepted without question. It represents a misunderstanding of mathematics problem solving and questionable developmental psychology.

In 1975, a National Advisory Committee on Mathematical Education pointed out that many areas of mathematics learning are open to students, even though they have not surmounted all the computational hurdles. The Committee recommended that rather than continue to pound away at skill drill, remedial students get at least a glimpse of other types of mathematical problems solving. It might be productive. It would certainly be humane.

This pre-requisite fallacy may be the most dangerous misconception of all. If we assume that we must focus solely on computations until mastery (as measured by a test) then some students will never get beyond these skills. This may guarantee that **their** skills become obsolete. The line between the disadvantages and advantages will be made more rigid, not relaxed, since those with weaker backgrounds will be trapped in remedial programs, while others have the opportunity to

learn how to adapt to new situations.

This fallacy may mask a new kind of elitism masquerading as equal opportunity for all. Recently, I was reading about the astonishing developments in Artificial Intelligence, in which computer programs are coming closer to being indistinguishable from what we consider "human" thought. It struck me as the supreme irony that while we are on the threshold of programming machines to "reason", we are spending the major part of our educational energies in teaching children mechanistic skills.

TWO ESSAYS

Thomas C. O'Brien, School of Education, Southern Illinois University

Reprinted with permission from **Educational Studies in Mathematics** Vol 7 No. 1 & 2, July, 1976.

Editors Note: Thomas C. O'Brien is professor in the School of Education of Southern Illinois University at Edwardsville and a former North Atlantic Treaty Organization (NATO) Senior Research Fellow in Science. He is the author of five problem books for learners age 9-17. The books, titled "Solve It" are published by Educational Teaching Aids, Chicago.

I

What are the goals of mathematics education at the elementary school level? There seem to be at least two major trends which represent major advances over the widespread "transmission of facts ... to children with a great emphasis on computation".

Robert B. Davis, in an introduction to the Madison Project, states a position which is thoughtfully constructed and which captures the best of what 'modern mathematics, broadly speaking', tried to do.

If we devote grade 1 to addition facts up to 10, grade 2 to addition facts up to 100, and so on, we are putting one foot in front of the other, left, right, left, right, ... This approach is weak in power.

If, instead, we seek those basic mathematical concepts, techniques, and attitude which play important structural roles in the development of the subject, we have a far more powerful approach. Cartesian coordinates, introduced (say) at grade 2, give us an ability to relate any arithmetic or algebraic problem to a geometric one, and vice versa. For all the rest of our lives we shall be able to unify algebra and geometry into a single coherent subject. This is power.

Once we learn such basic structural concepts as viable, function, mapping, and so on, we have a strong structural framework to which all of our subsequent material learning can be related. This, again, is power! Here we are building cognitive structures that can well serve as foundations for improved structures in the future[1] . And in England great emphasis has been placed, in infant and junior schools at last, on the mathematics of the environment. Children count, weigh, measure, and estimate. They work with the environment that is familiar to them, not with the dry symbols and notation and nomenclature of the adult mathematician. Further, they weave their environmental-mathematical studies in with languge, science, art, and (what is called in the U.S.) social studies in what has become to be termed 'the integrated day'.

There may be minor problems with each of the two trends. Of the first, there are questions to be asked: What is the role, in the life of the child or of the adult he will become, of a knowledge of the structural properties of mathematics? What is the role, indeed, of Cartesian coordinates, variable, function, mapping, and so on, in the life of Everyman? What evidence is there, at the elementary school level, of the effect of a knowledge of these things on people's live everyday or intellectual? Of the British approach there is the danger that the environment may become a be-all and end-all of one's investigation and/or that inquiry may rest at the level of description. To be sure, a knowledge of one's environment and an ability to describe same may be necessary parts of one's educational experience, but it is certainly not sufficient. As Morley says, there is a "straight-jacket

... in insisting that the only 'real' mathematics in 'everyday' mathematics''. To do so, he says, is to distort the nature of mathematics and to restrict the range of situations on which to draw for teaching mathematics at age 5-11 or so.[2]

One might wish that Davis' point of view were wedded to British practices. Each position has very great merit and the pooled 'power' might show that the sum is in fact greater than its parts.

But there may be a major problem with each point of view and, perforce, in their conjoining. In each case, one is concerned with the child's knowledge of things - whether it be of mathematical structure or of relationships in the environment - as though those things exist independently of the child.

What, thus, is suggested is a turnabout from those 'outside' things to 'inside' things and their relationship to external reality. What is suggested is a Copernical revolution in mathematics education. For a long time we have been interested in teaching mathematical information - facts and relationships - to children. Allendoerfer has quite rightly suggested that we can do this in better ways, and one solution seems to be to build mathematics curricula in terms of our present and growing knowledge of the ways in which children learn mathematical entities. It may be worthwhile to ask, not what does research on cognitive ability have to contribute to curriculum development in mathematics education, but rather what is the role of mathematics education in the overall intellectual growth of children?

The reader is requested to read Davis' statement once again. It is suggested that Davis' statement be revised as follows: If, instead, we seek those basic operations, techniques, and attitudes which play important structural roles in the intellectual growth of the child, we have a far more powerful approach. For all the rest of his life the child will be able to continue to build a coherent road-map of reality one which enables him to construct and test relationships, to infer, and to withstand the seductive dominance of percept over concept. This is power. Once he learns such basic operations as classifying, ordering, logically multiplying, inferring, and so on, he has a strong structural framework from which all of his subsequent knowing (including, in particular, mathematical knowing) can evolve. This, again, is power! Here we are building cognitive structures that can serve as wellsprings of improved and more complex structures in the future.

In general, it is suggested, the goal of elementary school mathematics education should be to cause intellectual growth to occur, not merely to teach children - or cause them to 'discover' - relationships among entities in the external environment or in the 'body of knowledge', mathematics. The goal of elementary school mathematics should be to encourage the development of cognitive actions, not to ignore them and certainly not to stifle them.

What are the components of this intellectual growth? Those operations listed above, of course, and more generally, an ability to organize, to coordinate, to actively pull into coherence the disparate entities of environment, experience, and inference and to regulate between one's internal constructs and the demands of external reality. In classroom situations this would call for children to be involved in issues which generate an ability to discern relevant from irrelevant; to distinguish between necessary and plausible and merely possible; to generate, to exhaust, to test and to cancel alternatives; to know, given a situation, what are the necessary conditions, what are the sufficient conditions, and what is the difference between necessary and sufficient; to solve problems and to generate new ones, and to inquire and to become self-sustaining in that inquiry.

Mathematics, it has been said, is 'systematic abstract thinking'.[3] It is also, in major part, the study of relationships. Should not the growth of human intellect - of organizing, of constructing and coordinating relationships - be a general and major concern for mathematics education?

II

The "back to basics" movement is a present concern in education. In addressing the issue with teachers, I have tried to distinguish between static knowledge and dynamic knowledge. By static knowledge I mean information stored passively in memory, out of relation with other information (if such a thing is possible). The fact that George Washington was born in 1732 may be an example of static knowledge. So, I suspect, are some aspects of spelling. You obtain it by copying.[4] You either have it or you don't. You cannot reconstruct it. It goes nowhere.

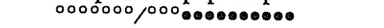
By dynamic knowledge I mean the construction and coordination of relationships. You obtain dynamic knowledge by action, physical or mental. It grows by elaboration. It is susceptible of reconstruction. It generates new knowledge.

Permit, please, an example. One way of getting children to learn number facts is to have them chant or write out, over and over, facts such as $7 + 6 = 13$. This is an example of static knowledge. Such an approach, more widely practiced than we would like to admit dooms the child to random guessing if for some reason he cannot dredge up from memory the single response - 13 - to match the single stimulus $7 + 6$. And the chanting of $7 + 6$ is no help to a new issue, say $7 + 7$. What such an approach does, quite effectively, is to tell children that knowledge is a bunch of unrelated bits and tricks.[5] (Sometimes it is, as when one commits to memory and names of the Capitals of the states but even here one tends to construct mnemonic relationships; more often knowledge involves relationships).

An alternate approach to $7 + 6$ makes the point, I hope. Here is a bead string in which the beads change colors in tens.[6]



Count off 7. Mark your place with a toothpick or a paperclip.



Now go 6 more. What is $7 + 6$? The original problem, $7 + 6$, can be transformed - by the one's action and with the help of the color break-up - to a new problem, $10 + 3$. Now try $7 + 7$. Not only does one have a way of reconstructing $6 + 7$ if memory fails, s/he can construct **new** knowledge from the old!

Inferring is a major instance of dynamic knowledge. By inferring I mean the education of information not directly (e.g. perceptually) available. Suppose you show a child a dozen chips and let him count them. Now ask the child to close his eyes while you cover some of the chips with your hand and stir the other ones about.

In determining the number of hidden chips, the child may guess at random or he may infer - interviews with children produce quite an astonishing range of approaches[8] - the number of chips that are hidden. All these approaches, I suspect, are susceptible of elaboration but rarely is the issues even mentioned in books for teachers or manuals accompanying texts. And so, in school situations, inference takes a back seat to memory.

For older children, three or four or five term series problems provide an arena for inference.[9]

Matt is taller than Jim.
Sue is shorter than Jim.
Al is taller than Bill.
Sue is shorter than Al.
Bill is taller than Jim.

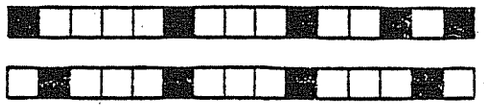
Who is the tallest?

What about children's constructing some problems like these on their own, some indeterminate and others having necessary conclusions? Children can be problem devisers as well as problem solvers.

Finally, here's a pattern in blocks or tiles, or Unifix.



The pattern - white/black - keeps going. What's the color of the sixth block? Black; this is a matter of pereption. What's the color of the ninth? Percption says white. What about the 21st, the 40th, the 1047th? Here the information is not visually available.[10] Now try these, and try them out with chidlren. In each case, determine the color of the 21st, 40th and 1047th block.





The point is that thinking of this sort - involving the construction, testing and application of relationships - is **basic**. To limit children to memory alone is to cheat them and to mislead them, if not to cripple them.

The proponents of the "back to basics" movement are no doubt well-meaning and their message is seductive. That there exists in the hallowed past a wellspring of curriculum to contribute to children's growth is a wish to cherish. The fact of the matter, if we are concerned about the real basics, is that there is little for us to return to. We are in fact, just beginning.

Footnotes

1. Robert B. Davis, **Explorations in Mathematics**, (Palo Alto: Addison-Wesley Publishing Companyh, 1967), p. 9.
2. Arthur Morley **"A New Development in Primary School Mathematics', Mathematics Teaching, 70, March, 1975, pp. 15-17**
3. **John G. Kenney,** 'Conversations on Mathematics Education'**, a collection of informal conversations with leaders in mathematics education which was produced by the author.**
4. **Alfred North Whitehead, in "The Aims of Education" writes, "In training a child to activity of thought, above all things we must beware of what I call 'inert ideas' - that is to say ideas that are merely received into the mind without being utilized, or tested, or thrown into fresh combinations." The original essay was published in 1917. See Alfred North Whitehead, The Aims of Education and Other Essays**, New York: The Free Press, 1957.
5. Erlwanger's dissertation and related articles describe the relation between static curricula and the child's storing of unrelated bits and tricks. See Stanley H Erlwanger, "Case Studies of Children's Conceptions of Mathematics," University of Illinois at Urbana - Champaign, 1974. Also see Stanley H. Erlwanger, "Case Studies of Children's Conceptions of Mathematics," **Journal of Children's Mathematical Behavior**, 1975, Vol. 1, No. 3.
6. This activity was suggested by Hassler Whitney, **Elementary Mathematics Activities, Part A,** Princeton, New Jersey: Institute for Advanced Study, 1974.
7. We have recently completed some videotapes on this issue. One child, after guessing randomly at the number of hidden chips, asked if he could be the one to hide the chips. When the graduate assistant answered correctly and without hesitation, the child (aged 5) said with conviction, "you've got X-ray vision."
8. See Thomas C. O'Brien and June V. Richard, "Interviews to Assess Number Knowledge," **The Arithmetic Teacher**, May, 1971.
9. A rich source of thought-provoking problems such as this, along with commentary and analysis of children's responses, is given in Margaret Donaldson and Donald Witherington, **A Study of Children's Thinking**, London: Tavistock 1963. The problem given here, by the way, has no unique solution.
10. A class of eighth graders, largely unexposed to inferential issues, recently attempted the first problem below by drawing the tiles up to 1047*!*

ON KEEPING MATHEMATICALLY FIT

F.G.B. Muskell, Algonquin College

The teacher of children in kindergarten and the primary grades is responsible for all subjects, secondary school teachers of older students are specialists in a discipline; at university, teachers specialize in some - possibly quite narrow - field of their discipline. All, however, may be faced with the problem of not getting stale - of keeping fit.

Keeping fit means doing exercises, whether physical, spiritual, musical, in language, or in mathematics. This article suggests that keeping exercised in mathematics means attempting to solve problems, and is directed at graduates of mathematics or of mathematics and science, especially those starting to run to fat - mathematically, that is.

Of course, we could and most of us should take some courses after graduation - but not every year for the rest of our professional lives. So a source of problems is if not a necessity, at least a convenience.

Ontario is probably as well off as any place in the world. Professor Ross Honsberger at Waterloo University has given enjoyment and inspiration to a great many teachers of mathematics. It would not be unreasonable for every school library to carry a copy of his "ingenuity" and of one or both of his "gems", published by The Mathematical Association of America (1), (2).

Also from Waterloo comes the Secondary School Mathematics Bulletin with its problems and solutions. It has been going for some fifteen years, and should be on the shelves of Secondary School Libraries (3).

More recently, the Canadian Mathematical Congress (now Society) has embarked upon the publication of 1,001 Problems in High School Mathematics, by three members of the Cong - oops!- Society, one of whom is at the University of Toronto, and one was before going to Alberta at the University of Waterloo (4).

Then there is the Canadian Mathematics Olympiad, and its excellent annual report of results, questions and solution key. The tenth was published in 1978 (5).

From Ottawa comes Crux Mathematicorum with ten problems every month contributed by mathematicians from USA and overseas as well as in Canada. It is devoted especially to problem solving, and is bilingual (6).

Si vous êtes anglophones et vous ne comprenez pas les termes mathématiques français, ou si vous êtes francophones et vous ne comprenez pas les termes mathématiques anglais, vous pouvez obtenir un dictionnaire de terminologie mathematique, lequel a été publié à Montreal - qui n'est pas loin de l'Ontario (7).

If you are already a problem solver but keep your solutions to yourself, perhaps this is the year to share some of them: with your colleagues at hand, or with the readers of the publication that provides you with some of your problems. In any case, a problem editor needs to know how well his selection serves his readership, but he cannot respond to input he does not receive.

No doubt I have omitted books and journals that should have been mentioned. I hope I have; then you can take me to task in letters to the Editor.

F.G.B. Maskell
Algonquin College.

Footnotes

1. Ross Honsberger, **Ingenuity in Mathematics**, 1970, The Mathematical Association of America, Washington, D.C.
2. Ross Honsberger, **Mathematical Gems**, Volume 1, 1973, Volume 2, 1976, The Mathematical Association of America, Washington, D.C.
3. **Ontario Secondary School Mathematics Bulletin**, Faculty of Mathematics, University of Waterloo, Ontario.
4. **1001 Problems in High School Mathematics**, Canadian Mathematical Society, 3421 Drummond, Suite 15, Montreal, Quebec, H3G X7.
5. **Report of the ...th Canadian Mathematics Olympiad**, Canadian Mathematical Society, 3421 Drummond, Suite 15, Montreal, Quebec, H3G 1X7.
6. **Drux Mathematicorum**, Algonquin College, 200 Lees Avenue, Ottawa, Ontario. K1S 0C5.
7. **Dictionnaire français et anglais de terminologie mathématique, French and English Dictionary of Mathematical Vocabulary**, 1970, Librarie Marcel Didier (Canada) Ltée, Montréal, Québec.

Anwers to Brain Teasers

1. 37 pupils on the roll
2. 160 acres.
3. 1 c
4. 9.36 p.m.

ANNOUNCEMENT OF ELEVENTH CANADIAN MATHEMATICAL OLYMPIAD - 1979

Date of Examination

The Eleventh Olympiad will be written on Wednesday May 2, 1979 from 9:00 a.m. to 12:00 noon at schools throughout Canada. The examination papers will be marked in May, and the results will be circulated to schools as soon as possible thereafter.

Date au concours

La XIe Olympiade aura lieu le mercredi, 2 mai 1979, de neuf heures à midi dans les écoloes de tout le Canada. La correction sera effectuée dès le mois de mai et les résultats seront communiqués aux écoles le plus tôt possible.

Conditions of Examination

All examiantion materials, whether submitted for marking or otherwise, are the property of the Canadian Mathematical Congress. Decisions of the examiners are final. Cash prizes and scholarships may be awarded at the discretion of the examiners, who may choose to subdivide or otherwise alter the terms of awards.

Eligibility of Candidates

Candidates must be Canadian citizens or landed immigrants, and must be registered in a high school or its equivalent.

Nomination of Candidates

All candidates of the Olympiad must be nominated by the chairman of a provincial competition or by a school principal. Copies of nomination forms can be obtained from:

The Olympiad Committee
Department of Mathematics & Statistics
Memorial University of Newfoundland
St. John's, Newfoundland A1B 3X6

(The Ontario chairman if Prof. K.D. Fryer, Department of Mathematics, University of Waterloo, Ontario N2L 3G1 and the Ontario quoto of candidates is 50).

CHAPTER ITEMS

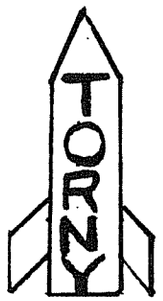
Several projects are reported.

1. Ken Williams of Carleton University with a committee is examining the feasability of an Examination Board.
2. Viktors Linis and committee is examining the three Gr. 13 Math courses.

3. Don Fox is starting a group on Pr. Jr.
4. Dr. J. Poland of Carleton University is organizing a Mathematics Club.
5. Feb. 27th will see a co-operative venture launched as Quinte St. Lawrence & Central Ontario and Eastern Ontario Chapters unite to present a Tri-Chapter Conference. What a good idea! Congratulations to all the organizers.

Glad to see SEYMA now has a Newsletter as well as a fine logo.





Toronto & North Your Chapter

INAUGURAL MEETING
Wednesday April 4, 1979 16:00 - 20:30
Lawrence Park C.I. 125 Chatsworth Dr., Toronto

OTHER MATH COMPETITIONS

1. **Gauss Competiton** - Sponsored by the Grand Valley Math Association. The contest consists of 26 multiple choice questions and is open to students in grades 7 and 8.
2. **Junior Math Contest** - Sponsored by the University of Waterloo. The contest consists of 26 multiple choice questions and is open to students who have not reached their 17th birthday as of December 31, 1977 and have not been enrolled in Secondary School for more than 3 years.
3. **Euclid Math Contest** - Sponsored by the University of Waterloo. The contest consists of 10 multiple choice plus 4 problems for which competitors are expected to write complete solutions. The eligibility requirements are the same as the Junior Math except the year restriction is increased by one.
4. **Descartes Math Competition** - Sponsored by the University of Waterloo. The competition replaces the old Problems Paper which was discontinued in 1967. The Descartes Paper consists of 12 problems of which students are required to answer 10. The eligibility is similar to the Junior Math except the year restriction is increased by 2.
5. **Senior Math Contest** - Sponsored by the Mathematical Association of America et al. It is a multiple choice contest open to all students.
6. **Carleton University Math Competition** - Sponsored by Carleton University. Generally students are given 7 problems with a 2 month time period to complete their solutions. Top performers are then invited to Carleton to write a similar but supervised problems paper. This competition is open to all students with top competitors eligible for an 8 week expenses paid math seminar at the University of Chicago.
7. **McMaster Math Contest** - Sponsored by McMaster University. The contest consists of 3 problems plus one 500 word essay. It is open to all students.



REMINDER

OAME - 1979, May 11-13
Sixth Annual Conference
The Peterborough Conference
by
P.R.M.A. (Pine Ridge)

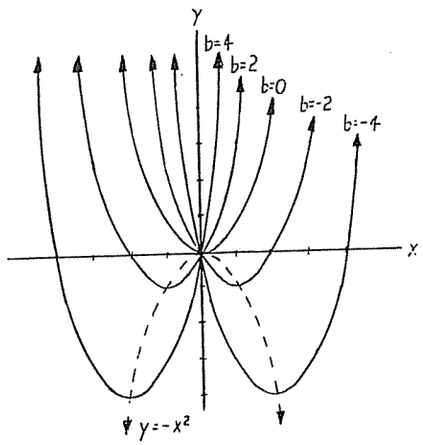
INVESTIGATING THE ROLE OF "B" IN THE FUNCTION $Y = AX^2 + BX + C$

Phillip Harvey

Editors Note: Philip is a Grade 11 student at Loyalist C.V.I. Kingston, taking Gr. 12 Mathematics. Many will know his mother M. Elaine Harvey

Being a grade 12 math student, I was introduced to the parabola and its general equation, $y = ax^2 + bx + c$ early in September. The first things we learned were the effects of a and c on the shape and position of the parabola. I assumed that the next step would be to examine the effects of b on the parabola; so I sketched the graph of $y = x^2 + bx$ for different values of b. I expected at first to find that b moved the parabola horizontally, but instead I found myself with this graph:

Graph of $y = x^2 + bx$ for b = -4, -2, 0, 2, 4



From this graph, it appeared that b did not move the vertex of $y = x^2 + bx$ along the X-axis, but rather moved it along the line $y = -x^2$ with the line of symmetry of $y = x^2 + bx$ being $x = -b/2$.

I realized that, if introduced into the equation $y = x^2 + bx$, a and c would change the shape and position of the parabola which was traced by the vertex of $y = -x^2$, so I adapted the equation to compensate for these changes by adding the constants a and b to the equation $y = -x^2$ to get $y = -ax^2 + c$. This is the general equation for the parabola on which the vertex of $y = ax^2 + bx + c$ is found.

When, in my math class, we omitted observing the effect of b and concentrated on the equation $y = a(x - p)^2 + q$ and the fact that the point (p,q) was the vertex of this parabola, I adapted my equations to find the vertex of $y = ax^2 + bx + c$.

Since the parabola $y = ax^2 + bx + c$ and the parabola $y = -ax^2 + c$ intersect at two points (the vertex of $y = ax^2 + bx + c$ and the vertex of $y = -ax^2 + c$), and the vertex of $y = -ax^2 + c$ is always on the Y-axis (point (0,c) to be exact), therefore, the vertex of $y = ax^2 + bx + c$ is the point of intersection which is not on the Y-axis (assuming that they intersect at more than one point). Therefore, the vertex of $y = ax^2 + bx + c$ is the intersection of

$y = ax^2 + bx + c$ and $y = -ax^2 + c$

$$ax^2 + bx + c = -ax^2 + c$$
$$2ax^2 + bx = 0$$
$$x(2ax + b) = 0$$
$$x = 0 \text{ or } 2ax + b = 0$$
$$x = -b/2a$$

For $x = \frac{-b}{2a}$, $y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$

$$y = \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

$$y = \frac{-b^2}{4a} + c$$

$\therefore$ the vertex of $y = ax^2 + bx + c$ is the point $\left(\frac{-b}{2a}, \frac{-b^2}{4a} + c\right)$

I was curious to find out precisely why the variation of the constant p moved the vertex of $y = a(x - p)^2 + q$ to the line $x = p$ while not changing its elevation. This is why:

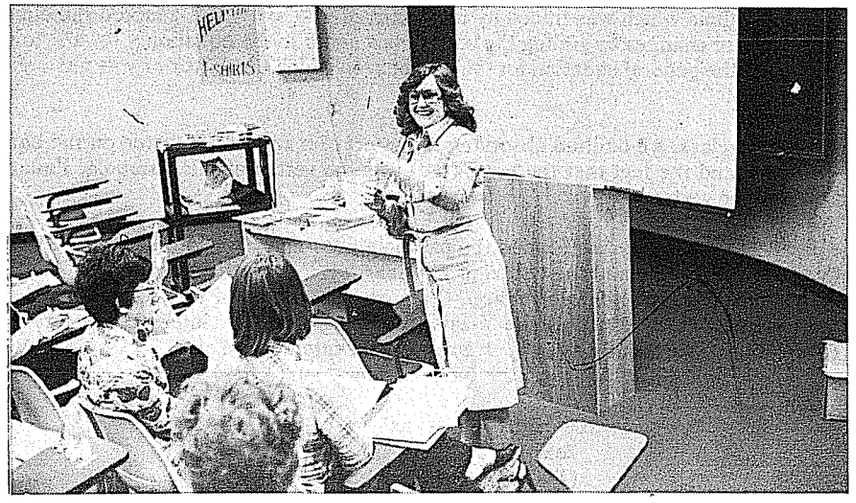
The inclusion of p in $y = ax^2 + q$ to get $y - a(x - p)^2 + q$ not only adds an x coefficient, but also a constant. If we expand $y = a(x - p)^2 + q$ to $y = ax^2 - 2apx + ap^2 + q$, we find that the x coefficient is -2ap. Since the equation for the line of symmetry of $y = ax^2 + bx + c$ is $x = -b/2a$ (the x coordinate of the vertex) and in this case $a = a$ and $b = -2ap$, we get a line of symmetry of $x = 2ap/2a$ or $x = p$ for the equation. This is why the line of symmetry moves to $x = p$, but as the value of b changes, shouldn't the parabola also move down along the parabola $y = -ax^2 + c$? Yes, it should. This is where the constant comes in.

The formula for the elevation of the vertex of parabola $y = ax^2 + bx + c$ is $y = -b^2/4a + c$ (the y coordinates of the vertex), and for $y = ax^2 - 2apx + ap^2 + q$, $a = a$, $b = -2ap$ and $c = ap^2 + q$; so $y = -4a^2p^2/4a + ap^2 + q$ or $y = q$. This shows that p has no effect whatsoever on the elevation of the parabola $y = a(x - p)^2 + q$, because when you added the constant ap^2 to the equation, you brought the vertex of the parabola from the line $y = -ax^2 + c$ to the line $y = q$.

In conclusion, it is easily proven that the vertex of $y = ax^2 + bx + c$ will always lie on the line $y = -ax^2 + c$ by changing the equation $y = ax^2 + bx + c$ into the form $y = a(x - p)^2 + q$ to find the vertex (p,q) and by showing that this will always satisfy the equation $y = -ax^2 + c$. The use of this equation is in no way limited to what I have shown here. It can in fact be used to derive other general equations such as

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \text{ and}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Sharon McPhail (Lambton County)

PREPARATION OF ARTICLES FOR SUBMISSION

It would be appreciated if articles submitted were double-spaced on 8½'' x 11'' paper, with an eye to good format. A clear copy in india ink of figures, tables and diagrams allows for photo reproduction. Some biographical detail of the author is useful. Deadlines for submission of material are **November 1, February 1,** and **August 1.** Editorial correspondence and articles should be submitted to one of the editors listed on the inside cover. Two copies please.

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