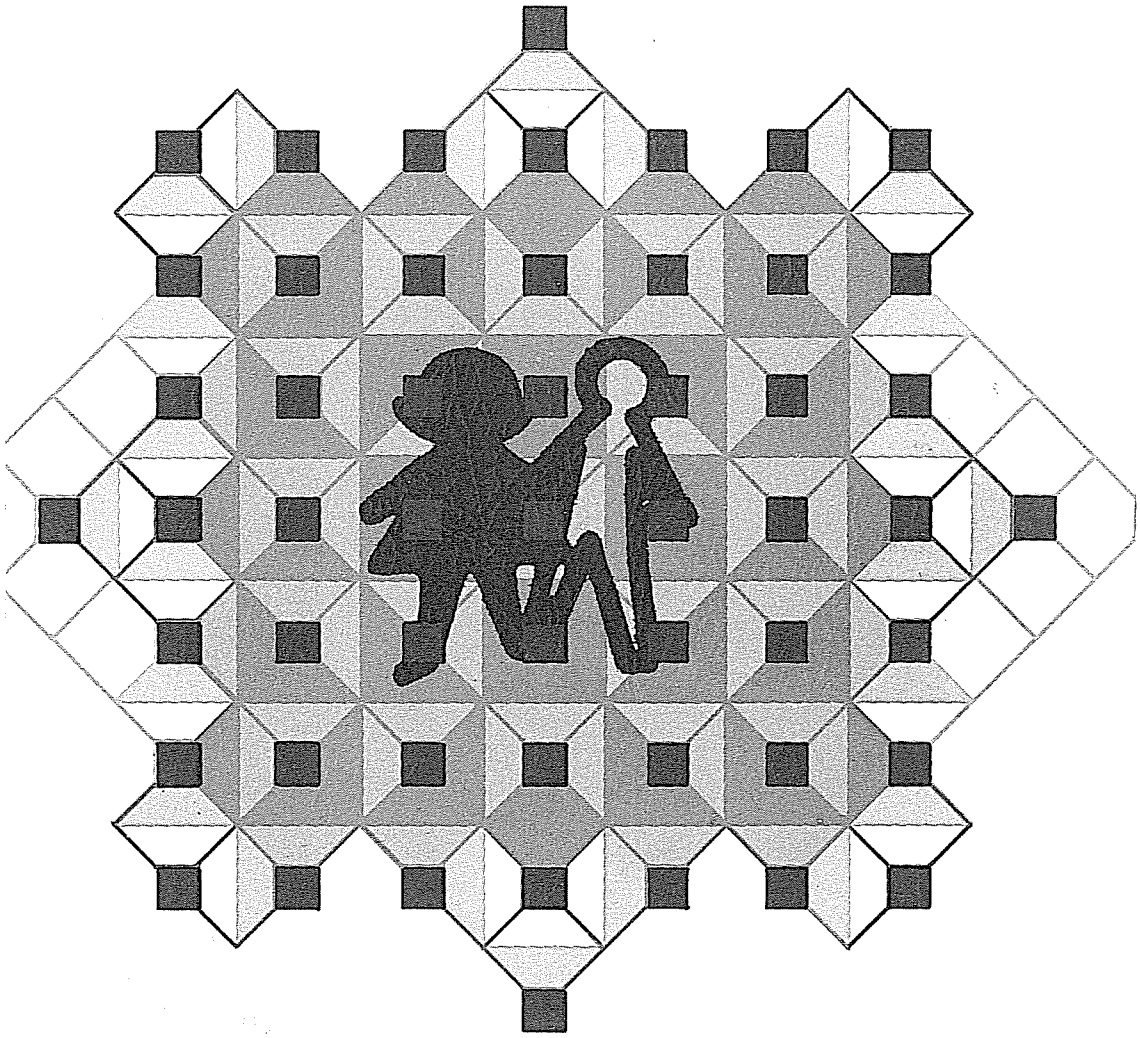
ontario mathematics gazette

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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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A SPECIAL ISSUE

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EDITORIAL
A Special Issue

Good Grief
a Pun



Getting a journal to "bed" is not unlike rounding up my sons for bed by a specific time. An attempt at such a parallelism would have seemed ludicrous a year ago. "Guys, its time for bed; reply . . . answer . . . reply . . . Writers, the deadline for submissions is near; correspondence . . . reply . . . correspondence . . .," and wouldn't you know it somehow the deadline was more or less achieved. Perhaps one should have more faith in human nature . . .

Sitting here finally, relaxing and reflecting by the pool with my G and T reminds me of how good a shower felt after a hard fought football game. The protest of each bruised or pulled muscle reminding you of the exerted efforts; and now in a seemingly similar way my brain tells me that putting a magazine together was also a demanding experience but without reservation a rewarding one.

As promised this issue of the magazine deals with the issue of special education which is also highlighted in the cover as well. To address this topic as broadly as possible I have put together a wide selection of articles, from local writers to individuals who have presented their thoughts and ideas at the 7th Annual Conference on Diagnostic and Prescriptive Mathematics in Vancouver, B.C., including a thought provoking interview with a very well known educator, Dr. Lola May. The flavour may tend toward elementary school mathematics, but keep in mind that much of the difficulties students have in mathematics at any level generally stems from lack of appropriate experiences in the fundamentals, (sounds like a tip on golf instruction). You should find that the strategies utilized in diagnosing student errors are by and large universal and only require modifications in content.

Why a focus on special education for this issue? One can come up with a number of reasons; but most specifically it is largely the behind the scenes activity at the Ministry of Education dealing with this topic that lead me to believe that we would have to face the problem of special education quite seriously in the near future.

Well, as luck would have it, the future is upon us. Bill 82 was introduced in Parliament on May 23, 1980. The news release and excerpts from the Bill appear on page 5 to set the stage for the theme of the journal and what lies ahead for us as a profession. The implications of the Bill will affect all of us and we no longer can ignore it.

In her book "A Guide to the Diagnostic Teaching of Arithmetic," Reisman states, "Clearly, diagnostic teaching strategies can enhance the teacher's effectiveness with all students. The matching of current level of performance and the material to be learned is certainly sensible for any child". This seems like a sensible statement under current educational needs. Recent studies have indicated that there is an alarming increase in the number of students, particularly boys, requiring special education. This seems rather odd relative to the insights that educators have achieved over the past few years in setting realistic goals and models of teaching strategies. It may perhaps be as Lola May suggests in the taped interview that ".... teachers are too reluctant to keep (and work with) borderline cases in the regular classroom because teachers don't have the background to cope with the situations".

I'm not suggesting that we all go out and enroll in classes to upgrade ourselves in the area of special education immediately. But what I would like to suggest is that it might not be a bad idea to find out more about it. Perhaps the student who appears to be slow may not be slow at all; he may just require a slightly different set of experiences than what most children require for concept attainment. The feedback that I have received from teachers who have taken additional courses in special education is that they found some of the techniques suggested for the "special" child work better than traditional strategies for the "normal" child. If this is the case, do we not owe it to ourselves as well as the children we teach to get better acquainted with new techniques in teaching? Besides, if the Ministry is willing to foot the bill to outfit our schools with up to date materials should we not take advantage of it?

I have enjoyed putting this issue together. Its up to you to make it an even bigger "issue". The

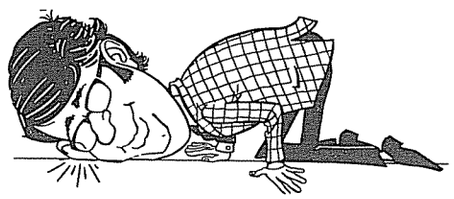
opportunity is there; let's make the best use of it. Let's use the Gazette as a means to disseminate ideas for improving mathematics for the special child. And lastly, I would appreciate your reaction to this issue.



Sam Pravica



P.S.
Support your guest editor. Calculate on sending your experiences on calculators or mini computers in an article to Ralph Connelly at Brock University. I'm sure he would be most pleased to receive your submissions.



WHAT'S NEW AT THE GAZETTE

Bob Smith

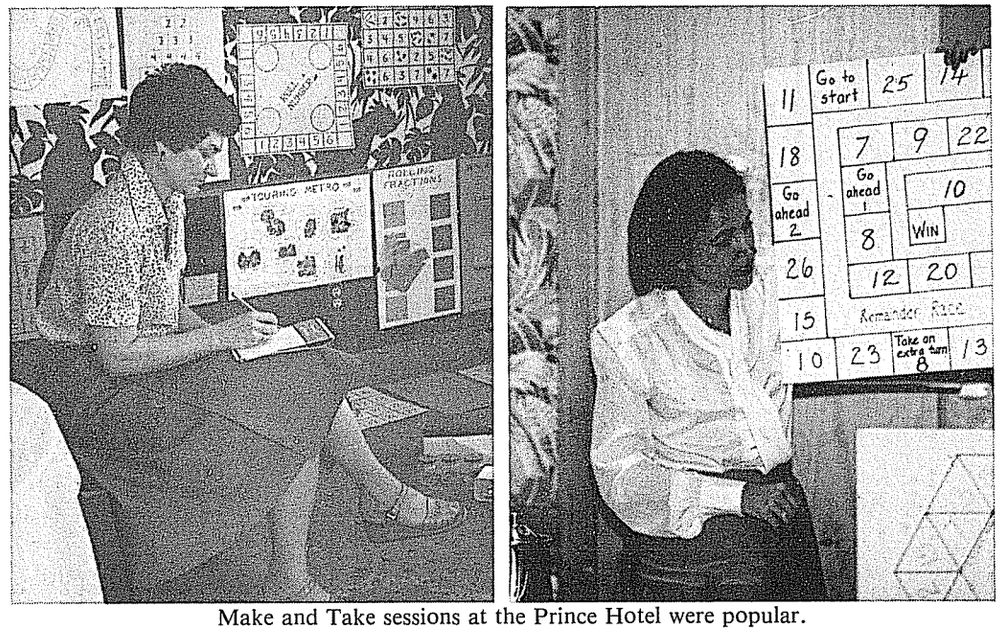
The President, the Executive and Council of O.A.M.E./A.O.E.M. are very concerned about the finances of the Association. As publications (O.M.G. and Abacus) spend approximately half of the budget, your editors were offered guidelines. The one which affects you and me directly is "Each issue (O.M.G.) will not exceed 52 pages". This would result in a saving. At the same time it will mean cutting out of either some regular features, a number of articles, or a bit of both. Another answer to the problem of decreasing finances is to increase our membership. To encourage this, the council decided there will be no Membership Committee this year. Rather, every member of council is attempting to obtain 3 new members. O.A.M.E./A.O.E.M. needs new members. Each of you should be a membership committee. Sign up a new member this term.

As this will be my last year as editor of O.M.G., the Executive has appointed two guest editors. For this issue, our guest editor is Professor Sam Pravica. Sam has been a Mathematics Consultant with the Halton Board of Education and is presently teaching in both the elementary and secondary panels at the Faculty of Education, The University of Western Ontario. For the December issue, our guest editor is Professor Ralph Connelly. Ralph has taught Middle School mathematics in New York State, Elementary Mathematics Methods at Memorial University as well as serving as a consultant for K-6 programs in Newfoundland. He is presently teaching the Elementary Mathematics Methods and graduate courses at Brock University. Do read his interesting and practical article titled "Crossing the Chasm From the Clinic to the Classroom". As we already have some articles about the calculator, he will be using the calculator as a mini-theme for the December issue. Do send him your articles. May both issues go well. We look forward to reading them!

We will miss the services of Bob Stevenson and Duncan Isaksson this year as both are leaving the editorial board. They have served long and well. We welcome back Bill Higginson, after his sabbatical and look forward to his new column on research.

As we continue to try to improve your journal, to make it useful to all the Mathematics Teachers, I am pleased to welcome Maurice Poirier to the editorial board. He has been the editor of COMA Bulletin and the main help in our search for French articles. At present he teachers Gr. 9-13 at ecole secondaire Garneau, with the Carleton Board of Education. Along with all the organizations he belongs to and writes for, most recently he has been a course writer with the CARE project. He would welcome original French articles or French articles suitable for reprint.

I cannot finish this column without a mention about the excellent Spring Conference hosted by SEYMA and so well organized by co-chairpersons Shirley McIntyre and Paul Sherk. It was our first time to be back in a hotel and I must say the Prince Hotel was luxurious. Next year's conference brings us to the Canterbury Inn, Sarnia. As in Chaucer's time, make up a group of friends to enjoy the journey with, and plan to be at the Canterbury Inn next year. See you there.



Make and Take sessions at the Prince Hotel were popular.

ERA ENDS AT QUEEN'S

John Coleman Steps down as Head of the Department of Mathematics and Statistics

On June 30, 1980 A.J. Coleman stepped down as Head of the Department of Mathematics and Statistics at Queen's University. This was a position he had held continuously since coming to Queen's in 1960. A graduate of Toronto and Princeton, Dr. Coleman is well known internationally for his work on the applications of Group Theory.

During the period of his headship he was a leading figure in the Canadian mathematical community, being President of the Canadian Mathematical Congress (as it was then called; 1973-75), Director of the Science Council's Study of the Mathematical Sciences in Canada (1973-76) and Chairman of the Canadian Mathematics Education Study Group (1978-80). From 1974 to 1978 he was, as well, Chairman of the Exchange Commission of the International Mathematical Union. In this latter capacity he was instrumental in establishing mechanisms for supporting mathematics in several third world countries.

To most readers of the Gazette he is perhaps best known as the general editor of the widely used series of secondary school textbooks published by Gage. An enthusiastic teacher himself, John Coleman has been a long time supporter of elementary and secondary school teachers in the province of Ontario. He was for some years, prior to the founding of OAME, Chairman of the Ontario Mathematics Commission. When OAME was founded there was considerable doubt about the organization's ability to support this journal financially. It was a note of John Coleman's generosity and commitment to our organization that he volunteered to be personally responsible for the finances of the **Gazette** should it prove necessary.

In the end our organization did not have to fall back on this kind offer. It is, however, remembered with gratitude and the editors feel certain that the membership as a whole would join them in wishing John a most enjoyable sabbatical leave during the 1980-81 academic year and a satisfying return to full-time teaching in 1981.

Editor's Note: Sorry I couldn't obtain a picture.

LETTERS TO THE EDITOR

Ernestown Secondary School,
Box 100,
Odessa, Ontario
K0H 2H0
1st May, 1980

The Editor,
Ontario Mathematics Gazette.

Dear Sir:

In issue #3, March 1980, J.V. Mussen has written a letter in which he altered the third equation in table 1 of T.J. Racey's article entitled "The Use of Trigonometry to Solve Equations". The alteration is itself incorrect. Relativity theory can be very tricky especially when one is trying to distinguish between the Lorentz transformation equations and the relative measurements on time intervals and lengths . . .

Editor's Note: The University referee in the discussion suggested both T.J. Racey and J.V. Mussen are correct. It depends on how you define your variables . . . it may be just a question of your point of view.

P.S.
Incidentally, the clearest exposition on the special theory of relativity that has ever been written by anybody was written in 1924 by a high school teacher in collaboration with his senior students. It is entitled **Readable Relativity.** The author was Clement V. Durell. It was originally published by G. Bell and Sons Ltd. (London) and has since been issued as a paperback in the Harper Torchbook Science Library series. Copies should be in every high school library.

Yours Sincerely,
E.P. Bettery-Pratt
Ernestown Secondary School
(classroom teacher)

337 Whitmore Drive,
Waterloo, Ontario
March 27, 1980

The Editor,
The Ontario Mathematics Gazette.

Dear Sir:

I would like to comment further on some issues arising out of the recent letters of Roland Gray and myself. First, however, let me emphasize that, although the issues were sparked by a small part of Prof. Gray's most valuable article, my disagreements are really with the M.M.M. (Modern Mathematics Movement) - or, rather, with certain aspects of it. Indeed Prof. Gray's article was written in the context of certain widely accepted practices, and it was not inappropriate for him to abide by them in view of the aim of his article. It is much easier for me to be a maverick, since I am only making one or two points.

Past criticisms of the M.M.M. have produced a very curious result. Many years ago I sounded off on the "negative x" terminology. The curious thing was that this was introduced by the M.M.M. **in order to** improve understanding by means of increased precision (with less ambiguity) of language and greater riguor - to make clearer certain distinctions that had hitherto been blurred. When it was pointed out (and not just by me) that, in fact, the new treatment resulted in **greater ambiguity** and **more** serious errors in understanding (e.g. believing in the existence of negative complex numbers) the response, very strangely, was that the **critics** were trying too hard to be fussy and rigourous! (I believe this type of response is sometimes known as the "double bind", or "double whammy" - it is useful in argumentation in that the recipient can't possibly win).

A similar situation has developed (quite apart from Roland Gray's article and letter) with regard to other matters, such as the explicit use of the concept of naming. This is, of course, a basic concept, and must be dealt with, either directly and explicitly, or indirectly. The point I am trying to make is that **the M.M.M.** (or schools of thought within it) has tried too hard to be precise and rigourous in this connection and has only confused matters (partly because it hasn't been as regourous as is thought). Thus, I **agree with** Prof. Gray's view that "the precision and rigor can develop later", and **am criticizing the M.M.M. for this very reason.**

It may be possible to distill a principle out of all this. The matter of understanding, rigour and precision is clearly of central importance but, equally clearly, subject to backfiring. Rigour and intuition are not opposites - nor are rigour and an approach via simple everyday language. It may be that the most "rigourous" approach to a topic other than a very elaborate logical exercise is highly intuitive, and that the region in between should be avoided.

Designing maxims can be rather fun. For the present situation, what about the following?

"The more you try, explicitly, to be rigourous, the more regourous you have to be."

Congratulations for a fine publication.

Sincerely,
Ralph Staal

BILL 82

by Dr. Betty Stephenson

Editor's Note:
Rather than include the lengthy albeit informative presentation of Bill 82 to Parliament by the Minister, I felt it more appropriate to print the News Release by Gordon Bergman. The News Release deals specifically with the aspects of the bill that affect us directly as a profession. If you wish a copy of the Minister's presentation of the Bill to Parliament please write and I will send you a copy.
P.S.
The Thursday, August 14, 1980 (T3) printing of the Globe and Mail has an interesting article on Special Education and Bill 82. What's your reaction?

Statement by The Honourable Bette Stephenson, M.D.
Minister of Education
Minister of Colleges and Universities
on the Introduction of a Bill
Regarding the Provision of Special Education
The Legislature, May 23, 1980

Mr. Speaker:

Today I am introducing for debate in the Legislature of Ontario a Bill that will have the effect of making the publicly-supported school system legally responsible for the education of all Ontario students.

Education Act, Sect. 2(y)
Require elementary school boards to implement procedures for early identification of the learning abilities and needs of pupils and may provide guidelines in accordance with which such procedures shall be implemented.

NEWS Release

Ontario Ministry of Education / Ministry of Colleges and Universities

Education Minister Bette Stephenson last week introduced a bill in the Legislature that will make school boards responsible for providing special education programs and services for all exceptional children of school age. This includes children with intellectual, communication, physical and behavioural exceptionalities and those with multi-handicaps.

The program will start this year as a pilot project involving 19 school boards. The pilot project will determine the most appropriate planning approaches which will then be used by all boards, assisted by teams from the Ministry, to fully implement Special Education programs by 1985. Phasing-in of the program to all boards will begin in September of 1981.

Dr. Stephenson said that the Ministry will increase grants for new special education programs and services by $75 million dollars by 1984-85, assuring school boards of continuing financial support for such programs. The province will spend an additional $7.5 million in grants this year to speed payment to all boards involved in the provision of new programs and services. The Ministry will also spend $500,000 to provide direct assessment and professional development services to boards to assist them to implement special education programs. During the next four years, as the program is phased-in for all boards, Ontario will allocate an additional $67 million in support of special education programs and services.

"The government is fully committed to continuing the provision of extra funds for special education to ensure that all reasonable needs are met and that no undue burden is placed on the local taxpayers," Dr. Stephenson said.

The Bill, introduced by Dr. Stephenson, will also extend to the Roman Catholic Separate School System the right to operate schools for the mentally retarded.

The more intelligent the creature, the more sensitive it is to suffering.
- Ernest Choquette, La Terre (Librairie Beauchemin Limitee)

DIFFICULTIES AND ERRORS IN COMPUTATION

A report and commentary of recent assessment studies in the U.S.A., Canada and England.

Douglas H. Crawford, McArthur Hall, Queen's University

Introduction:
Concern with basic skills has been mounting in recent years. In particular, parents and the public generally claim that children do not learn their tables adequately and cannot compute satisfactorily. Following the lead of the National Assessment of Educational Progress (N.A.E.P.) in the U.S.A., over 30 states have developed some form of on-going assessment programs, some like New York and Florida going further and instituting minimal competency requirements for high school graduation.

Similar concern has been developing in Canada and England, and in both countries formal assessment programs have been set up. In Canada, both British Columbia and Alberta have conducted assessments recently, while the Assessment of Performance Unit (A.P.U.) created by the Government in England conducted a school mathematics assessment in 1978, the results of which have only recently been released.

This article reports in condensed form a study of seven recent assessment projects in school mathematics - two from the U.S.A. (N.A.E.P. I 1972-731 N.A.E.P. II 1977-78) one from Canada (B.C. 1977) and four from England (the A.P.U. study, two other national survey tests at ages 10 and 16 respectively, and a research study at Brunel University).

The focus of the review was computational skills with whole numbers, common fractions and decimals. Levels of performance and common errors were studied, and hence those difficulties and errors which occured in several or all projects identified. A survey of recent studies and thinking on computational skills was also made, and on the basis of this and the assessment analysis, some recommendations are made for areas of priority in improving such skills.

Some Illustrative Results from the Analysis

Available data on the achievement in computational skills of the student groups tested by each assessment project were studied and summarised in tabular form. Study of these tables then revealed which difficulties and errors were most prevalent. Examples of the major areas of difficulty and error are presented below.

1. **Whole Numbers**

a) **Basic Facts**
Grade 4

$$4\overline{)28} = 6$$

Correct 70%
This error 3%

b) **Place Value**
Age 10

0	6	2	9	9

What is the next number this meter will show?

Write it here | | | | | |

Correct 41%
07300 13%
1731010 7%

c) **Reversal Error in Subtraction**

$$\begin{array}{r} 39 \\ -16 \\ \hline 23 \end{array}$$

	Age 9	Age 13
Correct	55%	89%
Reversal	18%	0

d) **Using the Wrong Operation** (and Reversal!)

Grade 4

$$\begin{array}{r} 185 \\ +412 \\ \hline 373 \end{array}$$

Correct 92%
This error 2%

e) **Regrouping Errors** (Borrowing; Carrying)

Age 10. Multiply

$$\begin{array}{r} 392 \\ \times 7 \\ \hline \\ \hline \end{array}$$

Correct 52%
2771 (41 for 14) 30%
2734 (forget)

f) **Inadequacy of Algorithm** (Basic facts, Place value)

Grade 8 Divide: $45\overline{)1232}$

27 Rem. 17 (corr.) 70%
25 Rem. 7 9%
29 Rem. 27 9%
207 Rem. 17 4%

g) **Effect of Verbal Context**

Age 11 Use of "What number is 5 times as big as 20?" lowered facility by 25% from "Multiply: 5 x 20"

II. **Common Fractions**

a) **Adding Both Numerator and Denominator**

$\frac{3}{4} + \frac{1}{2} =$

	Age 9	Age 13	Age 17
5/4 (correct)	2	35%	67%
4/6 or 2/3		34%	15%

b) **Confusion of Rules and Procedures**

Multiply: 3/4 x 5/7

	Grade 8	Grade 12
15/28	82%	87%
21/20	6%	5%
20/21	6%	5%
8/11	1%	1%

c) **Effect of Words vs. Symbols (Level of Abstraction)**

Age 11 How many halves are there in 2½?

$2\frac{1}{2} \div \frac{1}{2}$ Correct 70%

33%

III. **Decimals**

a) **Effect of Horizontal Format**

```
  .35
+1.4
```

	Age 9	Age 13
Correct	18%	

.35 + 1.4 Correct 1% 52%

.35 ⁵ 1.4 Errors with answers containing digits 4 and 9. 50% 27%

b) **Lack of Understanding of Place Value**

Multiply: .15 x .45

	Grade 8	Grade 12
0.0675 (correct)	63%	78%
6.75	16%	11%
0.675	13%	8%
67.5	3%	1%

A Brief Survey of Related Literature

Ginsburg (1977) claims that children's errors result from organized strategies. He argues that for many children arithmetic as an activity is isolated from their ordinary concerns, and that there is a large gap between their informal (and concrete) and their formal (and abstract) knowledge. The drop in performance when a dollars and cents addition is replaced by the identical addition of numbers alone illustrates well this claim.

Ashlock (1976) has made a powerful contribution to the study of errors in his highly successful book **Error Patterns in Computation.** In it, he cites major error categories such as wrong operations, obvious computational error, and defective algorithm. These are very broad categories, yet useful, none the less. He also points the reader to the very thorough study of errors by Lankford, who conducted diagnostic interviews with 176 Grade 7 students distributed across the U.S.A.

Bright (1978) summarized the results of the National Longitudinal Study of Mathematical Achievement (N.L.S.M.A.), the 1st N.A.E.P. Assessment, and state assessments from Connecticut, Michigan and North Carolina in relation to computational skills. He examined in turn:

1) addition and subtraction of whole numbers,
2) multiplication and division of whole numbers, and
3) fractions and decimals.

He was able to conclude that performance improves across grades, that its level decreases with increasingly complex items, and that performance tends to stabilize, but that this requires several years of instruction. Finally, he argued:

1) that discussion must be of **specific skills** in order to be meaningful,
2) that is must occur in a well-defined context which includes consideration of the place of pocket calculators, and
3) that is must be based on a wealth of data obtained over many years.

Suydam (1978) has reviewed recent research related to the concept of fractions in a very comprehensive paper. Among her main conclusions are:

1) that many students have little conceptual understanding and computational skill with fractions,
2) that this is at a far lower level than with whole numbers, and
3) that instruction has put a premature emphasis on learning algorithms.

"The fraction concept is complex, made up of a variety of components or sub-components and attributes." Learning about fractions is the beginning of a long journey towards understanding rational numbers as the work done by Kieren (1978) clearly indicates.

More recently, Radatz (1979) has argued the need for developing cognitive models of the causes of errors, and produced an information-processing classification of them by consideration of the mechanisms used in obtaining, processing and reproducing the information contained in mathematical tasks. Among his categories are errors due to:

a) language difficulties;
b) difficulties in obtaining spatial information;
c) deficient mastery of prerequisite skills, facts and concepts, and
d) the application of irrelevant rules or strategies.

Finally, a new approach to assessment recently reported by Ekenstam and Nilsson from Sweden (1980) seems very promising. The approach aims to reveal difficult steps in the learning processes of selected topics by starting with a sophisticated question and constructing related questions formed from it in sequence by removing each final step or detail from the previous question. For example, given the question

$$\frac{3x - 2}{2} = \frac{x}{3}$$

a sequence of problems can be constructed corresponding to each step in a particular solution.

$$\begin{aligned} 3(3x - 2) &= 2x \\ 9x - 6 &= 2x \\ 7x - 6 &= 0 \\ 7x &= 0 \\ 7x &= 6 \end{aligned}$$

is such a sequence.

Discussion and Implications of the Findings

Analysis of the performance levels, difficulties and errors of the projects studied confirms generally the conclusions of Bright and Suydam. Performance does improve with grade level, does decrease with increasing complexity and abstraction, and does tend to stabilize after several years of instruction. Computation with fractions reveals poor understanding of the fraction concept, which implies that algorithms are often learnt before adequate conceptual understanding has been achieved.

Other areas of concern identified by the study are the importance of basic facts and understanding of place value. One or the other or both are needed in many contexts such as simple calculation, the carrying out of algorithms, and in work with fractions, decimals and measurement. The place played by the calculator in reducing computational complexity is still being determined. Scrutiny of its use in classrooms and what the effects are is essential in the next decade.

Most of the commonly occuring errors are of a conceptual nature, i.e., they represent a mechanical or rule-bound approach to computation. Pupils need to be taught the meaning of concepts and procedures more thoroughly before formal procedures are carried out. Radatz's four categories of errors can be useful diagnostic tools enabling us to identify what type of error is occuring, while Ekenstam and Nilsson's hierarchical approach to analyzing errors may help to pinpoint where understanding fails in a complex task.

Conclusion:
At both the research and teacher education levels, the awareness of areas of weak achievement and understanding is essential. We must seek continually to improve and refine our knowledge of teaching and learning inadequacies in the area of computation and communicate the remedies we find more effectively to teachers and students.

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"CROSSING THE CHASM FROM THE CLINIC TO THE CLASSROOM"



Ralph Connelly, Brock University

Taken from a presentation at the 7th Annual Conference of the Research Council for Diagnostic & Prescriptive Mathematics, Vancouver, B.C.)

Perhaps calling the gap (or perceived gap) from clinical work to classroom implication a **chasm** is a bit extreme - they're not really all that far apart (how does nimbly navigating the narrow gorge sound?)

At any rate, diagnosis in mathematics has been receiving a great deal of attention lately. Diagnostic clinics are springing up all over the place. (The Winter, 1980 RCDPM newsletter, Vol. 4, #3 supplies a host of information on 25 such clinics). Those fortunate enough to have attended RCDPM Conferences over the last 6 years have been treated to a barrage of information relative to pupil difficulties in mathematics. There is no doubt that the

clinical setting, with a (frequently) one to one testing situation conducted by skilled diagnosticians is our single best source of insight into how pupils learn (or mis-learn) mathematical concepts.

Yet with all the activity, with all the information we have being generated, many classroom teachers seem to feel that there is no way of applying this information to their classroom. ("You don't know **my** pupils, **my** school, **my** principal, **my** board, etc.) The main purpose of this article is to try to give a few suggestions as to how this application might be accomplished. Having conceded that individual diagnosis is best, we should recognize that the suggested ideas on classroom diagnosis should only be used when individual diagnosis is not possible (which, I contend, is **most** of the time for many of us).

First, evidence from the clinic seems to be pointing to one fact that really sets the whole attitudinal and procedural stage for what follows:

> Most pupil errors in computation are **systematic** (i.e. the pupil applies an algorithm erroneously **consistently).**

If the teachers did nothing more than accept that statement, with its corresponding implications for diagnostic procedures, what a great step it would be!

No longer would "Do it again, Johnny, but this time **take your time, be careful** and **think!" be the standard for remediation.**

In other words, as a first step, try to become familiar with some of the common error patterns that do exist. It's impossible to know them all (since pupils are inventing new ones all the time), but there are many "popular" ones that we can learn to recognize; Ashlock's Error Patterns in Computation[2] remains a **monument** in this area. Let's take just a moment to look at a few examples:

An "easy" one: (failure to regroup)

$$\begin{array}{r} 47 \\ +\,28 \\ \hline 615 \end{array} \qquad \begin{array}{r} 26 \\ +\,57 \\ \hline 713 \end{array}$$

Very "common" why? (adds regrouped no. **then** multiplies)

$$\begin{array}{r} {\scriptstyle 3}\,34 \\ \times\ 7 \\ \hline 358 \end{array} \qquad \begin{array}{r} {\scriptstyle 1}\,26 \\ \times\ 3 \\ \hline 98 \end{array}$$

How about division? (right to left, like other operations)

$$\begin{array}{r} 44 \\ 2\overline{)\,88} \\ 8 \\ \hline 8 \end{array} \qquad \begin{array}{r} 14 \\ 4\overline{)\,164} \\ 4 \\ \hline 4 \end{array} \qquad \begin{array}{r} 67 \\ 3\overline{)\,228} \\ 18 \\ \hline 18 \end{array} \qquad \begin{array}{r} 39 \\ 5\overline{)\,465} \\ 15 \\ \hline 15 \end{array}$$

A real "toughie" (greater no. ^{3} lesser; ignore remainder)

$$\begin{array}{r} 233 \\ 2\overline{)\,176} \end{array} \qquad \begin{array}{r} 221 \\ 4\overline{)\,824} \end{array} \qquad \begin{array}{r} 231 \\ 3\overline{)\,713} \end{array} \qquad \begin{array}{r} 213 \\ 3\overline{)\,639} \end{array}$$

O.K. experts, a quick look at fractions; we all know 1/4 + 2/3 = 3/7, but how about (misuse of)

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \qquad \frac{1}{4}+\frac{2}{3}=\frac{6}{7}+\frac{4}{7}=\frac{10}{7} \qquad \frac{2}{3}+\frac{1}{2}=\frac{2}{5}+\frac{6}{5}=\frac{8}{5}$$

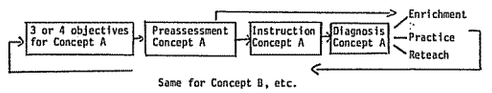
(Regrouping "10")

$$\begin{array}{r} 12\,\frac{1}{8} = 11\,\frac{11}{8} \\ -\,5\,\frac{7}{8} = 5\,\frac{7}{8} \\ \hline 6\,\frac{4}{8} = 6\,\frac{1}{2} \end{array}$$

$$\begin{array}{r} 3\,\frac{1}{3} = 3\,\frac{1}{3} = 2\,\frac{12}{6} \\ -\,1\,\frac{1}{2} = 1\,\frac{3}{6} = 1\,\frac{3}{6} \\ \hline 1\,\frac{9}{6} = 2\,\frac{3}{6} = 2\,\frac{1}{2} \end{array}$$

Okay, step one completed and we're all now experts at recognizing error patterns - So What?

Forgive me for going theoretical, **just for a moment.** A few conferences ago Bob Underhill suggested a model for classroom diagnosis - I'm not much for models, but I've tried this one and it's pretty good (not ideal, mind you, but it works).



To apply this model is not difficult, but it makes many assumptions; that objectives for the concept are available, some method of preassessment is available, some method of diagnosis is available. Let's deal with these separately to show that the task is not overwhelming.

Sources of Objectives:
Board guidelines, Textbook scope and sequence charts (available **free** from publishers), KSU Math Team Math Checklist, other "personal checklists" - many other available (ask around).

Preassessment:
Likely make your own (Innskeep, 1978 NCTM Yearbook[3]) provides many suggestions for how to do this and for diagnosis particularly, but some overall diagnostic tests are available which can be used effectively with "whole class" to preassess a unit's (semester's, year's) work in advance. A list of a few available diagnostic instruments in math is available from the author.

Diagnosis:
Clearly, same test could be used, but I wouldn't suggest this (Again, I would mention Innskeep as a great source of guidance in making your own test). Rather than formal diagnostic testing time, though, I would suggest a different use of homework exercises and homework marking time. Particularly at

the jr. high level, and often at the 4-6 level as well, homework is frequently marked by just putting a "✓" in the corner and occasionally "X"ing out wrong answers. What I would suggest is a simplified "diagnostic" approach to homework assignments and checking:

1. Select exercises carefully to reflect specified objectives ("the odd-numbered exercises on the bottom of page 62" do not guarantee the meeting of this condition). Yes, teacher-made sheets **would** be more effective here, but even a more cautious selection of exercises from p. 62 might suit your needs.

2. It is vital to stress an orderly arrangement of work, and that all work be shown.

3. Look for error **patterns,** not just errors. Even if this means a detailed look at only ¼ of the papers and a cursory glance at the others, over the course of a week's homework you'll have done **some** diagnosis of each pupil's work. This is **far** from ideal, but it's a start! Besides, we all have some pupils whom we know will not be having difficulties (fewer than we'd like, but we have some!) and we can therefore devote more of our limited time to those we expect might be having more difficulties.

4. Write it down **somewhere.** Don't trust your memory to remind you that "I must work with Mary on her division this week". If a checklist for each child (which is what I would strongly suggest) seems unreasonable or unwieldy, how about a folder for each child where when a problem is diagnosed, you write RETURN of this "offending" homework and after the pupil has been made aware of his errors, the paper goes into his file folder (if desired it can be removed when the problem is remediated).

This "homework diagnosis" only deals with the abstract level of Underhill's model, but again, it's a start!

We can also get diagnostic information for the weekly (or whatever interval) quizzes we construct. Because they tend to be more cumulative than our assignments, we can use them to get an idea of where in the sequence of a concept the pupil is having difficulty and the nature of the difficulty. This information can be recorded at the same time the quiz score is being recorded. To effectively use quizzes for this purpose, the same **4** steps as for homework need to be followed. Note that the **time** for constructing and grading the quizzes is **not** significantly greater, it's just that the **focus** of these procedures is altered from what is common practice.

If you're willing to use "off-the-cuff" diagnostic information, a surprisingly fertile source is the math activity corner of your classroom. This may seem like a very unlikely place to organize diagnostic procedures, but some of my best insights into pupil difficulties have come from this source. Nowhere is a pupil more free with his "potential algorithmic shortcuts" than in the heat of a math game competition. To use your activity corner for this purpose, any of your game boards will do. Number the problem cards (this makes it easy to provide a key for pupil use, also) and again concentrate on **sequence** and **objectives** when designing the cards. When pupils are using the game, have them number the problems as they do them. Then just have them leave the sheets when they're finished. "In your copious free time" look through the papers - you'll likely find some interesting results (NOTE: a course in hieroglyphics helps this process). The main reason I like this source, "slipshod" as it is, is that it may have certain attitudinal benefits. That is, when a pupil feels he is being "tested" diagnostically or otherwise, there is sometimes a tendency for the pupil to withdraw; to simply not do the problems, sometimes leading to the diagnosis that he doesn't have any idea about the concept at all. The "game" setting removes that atmosphere and might lead to a clearer indication of the pupil's difficulties.

I haven't touched much on remediation in this article. I am not so naive as to beleive that just getting more diagnostic information from techniques like those mentioned will alleviate pupil difficulties to any great degree. Clearly, remediation in the classroom setting presents an equal, if not greater, challenge than does diagnosis. Many of the sources I've listed in the handout (e.g. Ashlock; Pincus) do discuss remediation also, and I'm confident more ideas in this regard will be forthcoming. Also, I do believe we often get insight into pupils' thinking patterns from the diagnostic information we gather, and that in many cases these insights help us to adjust our teaching strategies so that others are not led into the same erroneous thought patterns, and again (I say for the last time!) "That's a start!"

I hope some of the ideas I have presented will prove useful to you, and that together we can make at least a little progress toward "closing the chasm".

Footnotes:

1. Irons, J. "Programs in progress" in **RCDPM Newsletter** Winter, 1980 Vol. 4, #3.

2. Ashlock, R. **Error patterns in computation.** Columbus, O. Charles Merrill Publishing, 1976.

3. Innskeep, J. "Diagnosing computational difficulty in the classroom" in 1978 NCTM Yearbook, **Developing computational skills,** pp. 163-176.

Editor's Note:
Ralph has an interesting annotated bibliography he would be happy to send you. Write to him!

MODELLING ADDITION & SUBTRACTION WITH REGROUPING

By Barbara Berman and Fredda Friederwitzer
Rutgers, The State University of New Jersey

Introduction
Place value is the foundation upon which much elementary school mathematics is built. The child's ability to understand and apply basic operations with whole numbers and decimals frequently depends upon his/her place value concepts.

The child should have internalized the concepts basic to addition and to subtraction before the teacher initiates specific place value activities. However, the lack of basic fact mastery by the child should not, in itself, prevent the teacher from introducing higher arithmetical operations. Students should be engaged in reinforcement and drill experiences leading toward mastery of basic facts concurrently with the progressive development of new concepts.

Another important consideration for any formal instructional program is the child's learning style. Since most teachers do not have the luxury of working on a one-to-one basis, it is advisable for them to use a variety of teaching modes in order to meet the diverse learning styles of their students (i.e. verbal, visual, tactile, auditory). This is even more essential when one considers that teachers frequently employ one dominant modality in their classrooms and teach the way they themselves learn (Dunn and Dunn, 1979). Unfortunately, children are not always placed with a teacher whose style of learning is similar to theirs. The ultimate result may be the failure of a child requiring visual or kinesthetic stimulation who is expected to learn from a verbally-oriented teacher.

Diagnosis
Operations requiring regrouping present many problems for children. Certain error patterns appear frequently in their work. A close look at these errors can help the teacher understand why they occur. (For a detailed discussion of the diagnosis of error patterns in addition and subtraction, see **Teaching Arithmetic: A Diagnostic Approach,** M. Hopkins, 1977)

Although an understanding of error patterns is often the first step to successful diagnosis, it is possible to diagnose children's errors in other ways as well. Listening to the child describe how an example was done provides valuable insight into thinking processes and arithmetical strategies. The mathematical vocabulary employed by the child may be an important clue to his/her level of understanding and, in itself, may be part of the error process.

The use of manipulative materials can strengthen the diagnostic process. Have the student display the problem with Dienes blocks, squared materials, an abacus or rods, and then work it through. This process enables the teacher to observe the manipulation for each step of the algorism and thus focus on the causes of error. Asking the child to show the problem as a picture is another aid in developing a clear understanding of his/her thinking.

The diagnostic process should not only investigate a child's errors but also attempt to assess areas of strength. Building upon the strengths can become an important part of the overall remedial plan.

Prescription
Prescribing remedial experiences after errors have been diagnosed is not a simple task. Here again, many factors influence the selection of activities and materials required for successful remediation.

It is interesting to note that while systematic errors may be attributable to many causes, a prime factor seems to be the premature transition to paper and pencil activities exclusively when children still require the

active manipulation of materials (Fauth and Jacobs, 1980; Ashlock, 1972). Children must understand how to express a mathematical concept verbally before they are expected to use it in writing (Payne and Rathmell, 1974). They should use the proper language in correct sequence prior to working with materials and then in conjunction with the manipulatives, so that the physical act becomes associated with a mental process which will eventually become internalized as a mathematical concept.

Research indicates that learning is an active process, dependent upon an interplay between the learner and the environment, Piaget, 1971; Engelhardt, 1978. The use of concrete, manipulative materials engages the student in active exploration of mathematical relationships, thus making him/her an involved participant in learning rather than a passive spectator. The use of appropriate materials to model mathematical concepts enables the child to form a mental image of the process which can be recalled at other times when materials are unavailable (Peck, et al, 1980).

A variety of manipulatives are available for modelling addition and subtraction with regrouping. Some of the most commonly used materials are:

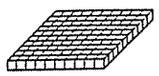
Countable - sticks and rubberbands, counting objects and zip-lock bags;
Structured - bean sticks, squared materials, scored and unscored Dienes materials;
Representational - poker chips and the abacus.

The countable and structured materials lend themselves readily to the development of basic place value concepts because of the obvious relationships among their components.

Representational materials however depend upon the child's memory to establish the relationships among their components. For example, if poker chips are used, there is no demonstrable relationship or order among blues, reds or whites. The order and relationships exist by definition only and, should the definition be forgotten or confused, the materials will be of no help. On the other hand, the use of countable or structured materials precludes this kind of error. The child can readily see that a 1 hundred bundle or flat is not equivalent to 10 units and that an intermediate exchange from hundreds to tens is required.

CHART 1
PLACE VALUE CHART

Flat	Long	Unit
Hundred	Ten	One





This example illustrates the idea that manipulatives are not, in themselves, a panacea. The authors feel there is a hierarchy of materials, which proceeds from the countable to the structured and then to the representational. Teachers must take cognizance of this hierarchy when selecting materials for remedial uses. The developmental level and the age of students needing remediation may affect their ability and/or willingness to use materials. In order to prescribe an effective remedial program therefore, the teacher must be familiar with the strengths and limitations of each material and must consider carefully what material is to be used with which child.

In the authors' view, structured materials (scored and/or unscored) are suitable for most students. For this reason, these materials will be discussed at length. The base-ten blocks are excellent tools for developing the mental imagery required for the regrouping process. (Base-ten structured materials can be found commercially as Cuisenaire "Power of Ten", multi-base arithmetic blocks, metric cubes or squared materials).

A vital prerequisite for successful work with the base-ten blocks (or any other manipulatives) is that children understand the relationships inherent within the materials, (i.e. 10 units = 1 long and 10 longs = 1 flat) and can demonstrate as well as verbalize these. It is important as a next step for students to be able to match the materials with a given numeral and vice versa.

Students are now ready to engage in manipulative activities for addition and subtraction. They should have many experiences with both operations without doing any writing at all. Their manipulation of the materials and their verbal descriptions of the activity provide the teacher with insight into their conceptual development.

When children are able to complete complex examples using materials without hesitation, they are ready to write the **answer only** for each example they work out. This involves understanding the value the manipulatives have and translating the knowledge into a

written numeral.

At this point, the written algorisms for addition and subtraction may be introduced.

$$\begin{array}{r} {\scriptstyle 1} \\ 53 \\ +\ 29 \\ \hline 82 \end{array} \qquad\qquad \begin{array}{r} {\scriptstyle 4} \\ 53 \\ -\ 29 \\ \hline 24 \end{array}$$

Each manipulation with the base-ten blocks by the student should be recorded by the teacher only. This process should be repeated as frequently as needed, with the students manipulating the materials and the teacher recording. When the students are able to correctly verbalize the procedure, they are ready to do their own recording.

Initially, the recording (i.e. writing the algorism) should be done in conjunction with the materials. As a next step, the children should be encouraged to develop visual imagery of the manipulative process (i.e. close their eyes and try to picture how they would manipulate the materials to solve the problem). The students should then use the materials to check their visual memory and record the algorism as well. The ultimate goal is for the child to complete the algorism without reliance on the manipulatives. However, it is not the teacher's decision but that of the child to "give up" the use of materials. This will occur when he/she feels comfortable without them. It is important to note that the materials should be available at all times so that students can use them if and when they need to.

Summary

The procedures described may appear to be time-consuming and not specifically remedial in nature. However, one must remember that remediation is not a simple process of correcting just the obvious error. More often than not, the glaring computational error is merely symptomctic of more basic, deep-seated misconceptions. The goal of remedial activities is not to bandage the wound but to heal the infection.

The use of manipulative materials, as prescribed in this paper, enables the child to rethink and relearn basic place value concepts. These procedures have been classroom-tested and provide success in eliminating common regrouping errors.

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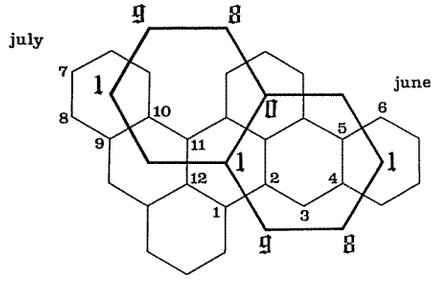
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A STUDY OF SPECIAL EDUCATION IN THE BOROUGH OF YORK

Background

The controversy which has arisen in recent years concerning special education provided an opportunity for study. The process of selecting children for special education in the Borough of York was a concern. The child is identified by the classroom teacher and her recommendation goes via the principal to the special education department. The child is then interviewed and tested by the itinerant school psychologist. If the psychologist recommends special class placement, the recommendation goes to the admission board consisting of the special education team for the Borough of York. Finally it is the parents' privilege to accept or reject this recommendation. Children whose parents accept the recommendation go into the special education stream; those whose parents reject the recommendation go into the regular stream. The purpose of this study was to compare the academic achievement in mathematics and self concept of students presently assigned to special education classes with those of similar ability presently in a regular class.

It is interesting to note the philosophy behind the formation of special classes. It is assumed that due to a lower pupil-teacher ratio, pupils receive more individual attention from a teacher with special education qualification. This treatment is assumed to improve the student's self concept, increase his self confidence and indeed mould a better product. Special education students were expected to improve academically as well as socially.

Statement of the Study

The objective was to compare the mathematical achievement and self concept of pupils presently attending special education classes with a group of students who were recommended to be placed in such classes but for some reason, such as parental choice, remained in the regular classroom.

Subjects

A group of 26 girls and boys presently in special education classes in the Borough of York system was chosen. The second group was chosen from the regular classrooms and paired according to age, I.Q. and sex.

Procedure

Individual tests were administered in

mathematics and self concept. Each pupil was withdrawn from his or her classroom and the tests were administered in the following manner: a brief introductory discussion always preceded the testing which proceeded in the same sequence - mathematics and self concept.

Measures

(a) **Mathematics Test** was a wide range criterion referenced test which included criteria such as shape, size, number and operations.

(b) **The Self Concept Test** was subdivided into four categories: family, peer group, school and general.

Results

The independent variables were sex, I.Q. and age. The dependent variables were the two test totals and the subdivisions of the self concept test. The pairs of names were arranged in order of age beginning with the youngest.

The findings were tested by two t-tests for matched pairs. With regard to each scale, the t-test was concerned with the question "Is the obtained difference in scores of students in special and regular classes more consistent than could be obtained on a chance basis?"

Figure I shows the results of the t-tests:

Significance of Difference Between Group Means

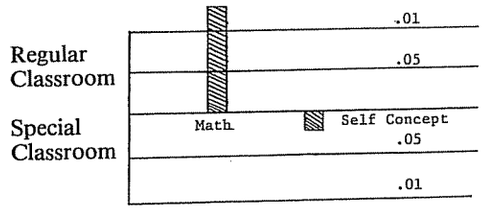


Fig. I

The higher scores of children in **regular** classes are significantly different from the scores of children in the **special** classes in mathematics. The difference in overall self concept was not significant. However the children in special classes were more likely to answer questions about **school matters** in a manner which suggested a **more favourable self concept.**

Summary of Results

All the students in this study had been recommended by the Borough of York Special Education Department for special class placement. However only half the students in the study had accepted special class placement, while the other half was in the regular class placement.

The null hypothesis stated that no difference would be found between the scores of the regular group and those of the special group. The rejection of the null hypothesis with regard to mathematics implies that in mathematics students in the regular stream make greater progress than their opposites in the special stream. The rejection of the null hypothesis with regard to self concept sub scale on school implies that students in special classes are happier about school. (Ed. Note -Apparently separate t-tests were run on the sub-scales of the self-concept test, and the "school" sub-scale was 'significant'. This is not mentioned specifically in the results). Thus it would seem that in mathematics what is supposed to be happening to special class students is not really happening. However life in the special stream is comparatively easy and teachers do not expect too much from the students.

Educational Implications

The basic assumption underlying the formation of special education classes is that higher academic attainment and social adjustment will be achieved under these conditions rather than in the regular class environment. This study refutes this assumption. It also indicates that there is no significant difference in social adjustment between the two groups with the exception of the self concept sub section on schools in which the special education pupils show more favourable self concepts.

The results of this study imply a need to modify the organization of the special education system. Regarding attitude to school the regular classroom teacher could perhaps learn from the classroom environment of the special education teacher. One must consider the feelings of special education students in the general stream - always finding themselves at the bottom of the class - usually a failure. An alternative arrangement would be a withdrawal program whereby students in special classes be placed in regular classrooms and withdrawn on a regular basis by a specialized teacher, thus taking the best from the two different settings.



RUMINATING ON A CALCULUS PROBLEM

E.J. Barbeau, University College, University of Toronto

If we put too much emphasis on technique and require students to work through reams of exercises, they may fail to appreciate the content of the problems they are asked to solve or to understand exactly what a given method is capable of delivering. For this reason, it is worth pausing over a standard textbook problem, asking what sort of answer one might expect, examining alternative approaches and judging which might be best. This is a good idea for students of all abilities. The slow student can catch his breath; the average student will find himself with a firmer footing; the good student, who is all too ready to gobble exercises down whole, will learn to be much more critical about his solutions and to seek out efficiency and elegance.

Earlier,* I showed how this might be done with Grade 10 problems. Here I would like to take a standard Grade 13 calculus problem, and using not only my own ideas but also those of some teachers I have discussed this with shine a light on various facets of it.

This is the problem: A fisherman wants to cross a stream 200 feet wide in his rowboat and to reach a place on the far bank of the river 500 feet away from the point directly opposite his launching point. If he can row at 3 feet per second and walk at 5 feet per second, find his shortest time to reach the desired place.

1. Hidden assumptions. Draw a diagram to illustrate the situation. Is there any current? What is the shape of the river? Where is the fisherman to begin with? Where is his destination? What are the possible paths? Must the fisherman begin by rowing?

2. Casing the situation. Draw a diagram in which the river consists of parallel straight lines, S is the initial position of the man, P the point directly opposite S on the far bank and D his destination. Realize that there are many paths: he could walk around on one side, row around for a while, walk some more, etc. Because we are looking for minimum time of travel, we can eliminate most of these since there will be a better path. Arrive at certain principles: the path should consist of straight line segments; he should never move away from the bank; he should have only one stint in the river. Conclude that it is enough to consider paths whereby the man rows from S to a landing point L situated between P and D and then walks directly to D. How many degrees of freedom are there in choosing this path? (One: once L is chosen, the path is determined. Thus we have a one-variable situation.) What would be the best path if land and water speeds were equal? If land speed were much quicker than water speed? What would be expected in the case at hand? Look at some numerical data? How long would the journey take if L = P? L = D? L is some other point. Is there a trade-off between the shorter distance and longer time taken when the route is largely in the water? This is an important part of the analysis, as the student gets some sense of what the answer might be and how he is going to set up the function to be minimized.

3. Choosing a variable and setting up a function. Having decided we have a one-variable problem, what should the independent variable be? The distance LP? SL? LD? The angle PSL? LSD? PLS? DLS? Something else? What is the dependent variable? (What are we trying to minimize?) What expression relates them? When setting up a function, it is extremely important to specify the domain of definition? It alerts the student to the possibility of an end-

point minimum. Students who fail to do this should be heavily docked. How far can we go in analyzing the function without using calculus? What specific information do we want to get from the calculus?

4. First possibility. Let x be the distance in feet. Let T(x) be the time in seconds when L is chosen. Then

$$T(x) = \frac{\sqrt{200^2 + x^2}}{3} + \frac{500 - x}{5} \quad (0 \le x \le 500).$$

Plot the graphs of each of the two expressions on the right side; explain the graphs verbally. What happens when you sum the graphs? Can we minimize T without calculus? (If your bright students get a little smug about this type of problem, ask them to do this. It makes them think a little.) Suppose the minimum value assumed by T is K. If we try to solve T(x) = K, we are led to the quadratic equation in x?

$$16x^2 - 90(K - 100)x + 25(200^2 - 9(K - 100)^2) = 0.$$

There will be a real solution if and only if the discriminant is real. The condition for this is $(K - 100)^2(900 \cdot 25) \ge 8000^2$, or $|K - 100| \ge 160/3$. (Discuss how much simplification, multiplying out is appropriate.) Since T(x) is at least the time taken for the man to row straight across the river (200/3 sec.), $K - 100 \ge 160/3$ or $K \ge 460/3$. Thus T(x) = 460/3 is solvable with coincident roots and T(x) = K is not solvable for $140/3 < K < 460/3$. We find that T(150) = 460/3.

Use of the calculus demands that we determine the first derivative:

$$T'(x) = \frac{5x - 3\sqrt{200^2 + x^2}}{15\sqrt{200^2 + x^2}}$$

$$= \frac{16(x - 150)(x + 150)}{15\sqrt{200^2 + x^2}\,(5x + 3\sqrt{200^2 + x^2})}.$$

Note the manipulations on the first derivative to get it into a form from which information can be extracted; in particular, note that here it is appropriate to rationalize the numerator. The solution is not complete without some argument that indeed we do have a minimum when the first derivative vanishes at x = 150. (The first derivative also vanishes at -150; why do we ignore this?) We have a number of arguments available. Those with strong stomachs can try the second derivative test. However, we took the trouble to do a little work on T'(x) which enables us to read off at once that $T'(x) \le 0$ for $0 \le x \le 150$ and $T'(x) \ge 0$ for $150 \le x \le 500$. Thus T(x) decreases to a minimum at x = 150 and increases thereafter. An alternative would have us check for a minimum by first principles:

$$T(x) \ge T(150) \iff \frac{\sqrt{200^2 + x^2}}{3} + \frac{500 - x}{5} \ge \frac{460}{3}$$

$$\iff 25(200^2 + x^2) \ge (800 + 3x)^2 \iff (x - 150)^2 \ge 0.$$

Since the last inequality always holds, $T(x) \ge T(150)$ on the domain.

Suppose we had determined that T'(150) = 0 and that T' did not vanish elsewhere for $0 \le x \le 500$, but we did not know the sign of T' elsewhere. Since T' is continuous, by the Intermediate Value Theorem, we at least know that T' does not change sign on the open intervals (0, 150) and (150, 500). Thus there are four possibilities for T according as it increases or decreases on each of the two intervals. Computing T(0) and T(500) will help us determine which is correct. Once again, we have avoided the second derivative. There could be classroom discussion the best method to identify the character of an extremum; any of the methods covered here could be best or worst for a given problem.

5. Second possibility. If θ is the angle PSL, we are led to minimizing the function $100\left(\frac{2\sec\theta}{3} + \frac{5 - 2\tan\theta}{5}\right)$ on the domain: $0 \le \theta \le \arctan(5/2)$. This can be handled in a similar way to the first possibility. Now compare the two approaches. Which is better? Why? Is there any way it could be judged in advance which is likely to be better?

In connection with this, I would like to pass on the inspired comment by Brian Dorrepaal relating this problem to Snell's

Law in the theory of light refraction. A ray of light passes from a point A in medium I to a point B in medium II. Its path is such as to minimize travel time. If its velocity in medium I is v_1 and in medium II is v_2, and if it leaves medium I at an angle θ_1 to the normal of the interface and enters medium II at an angle of θ_2, then, by Snell's Law, $\sin\theta_1/\sin\theta_2 = v_1/v_2$. In the fisherman problem, medium I is the river, medium II the land, the interface is the far bank, $v_1 = 3$, $v_2 = 5$, A = S, B = D, $\theta_2 = \frac{1}{2}\pi$. Hence $\sin\theta_1 = 3/5$ for the angle minimizing travel time.

6. Third possibility. Instead of one variable, we could use two which are linked. If x and y are the distances PL and SL respectively, we find that we have to minimize $\frac{500 - x}{5} + \frac{y}{3}$ subject to the constraints $0 \le x \le 500$, $200 \le y \le 100\sqrt{29}$, $y^2 - x^2 = 200^2$. Thus, we could regard y implicitly as a function of x, using the connecting equation to get an expression for y', which can then be used to minimize the objective function. What are the advantages of this method over the first possibility? Notice how we can evade getting tied up with radicals.

However, this possibility can be looked at in another way which is reminiscent of a linear programming problem. We have to minimize

$$\frac{500 - x}{5} + \frac{y}{3} = c \qquad (*)$$

subject to the condition that (x,y) is on the hyperbola $y^2 - x^2 = 200^2$. (*) is the equation of a straight line of slope 3/5, and we are looking for the smallest c for which this line cuts the hyperbola in the positive quadrant. For this c, the line will be tangent to the hyperbola (why?). To handle the situation avoiding calculus, we use $(\overset{*}{x})$ and the equation of the hyperbola to get a quadratic equation in x; the requisite c can be found from the condition for equal roots. Using calculus, find at what point the hyperbola has a tangent of slope 3/5, and find that value of c which makes the line (*) go through this point. I am grateful to Cyril Lewis for this idea.

7. Finishing off. The final stage, often neglected by students, is to give, in plain English, a statement of their conclusion in the context of the problem. This should be in the form of an instruction to the fisherman as to what he must do. A heavy penalty should be imposed on the student who does not translate his solution out of its mathematical garb; if the penalty is not paid now in marks, it will be paid later in a much more severe fashion.

Conclusion. Of course such an extended treatment cannot be applied to every problem. But it should be used sufficiently often to demonstrate that the value of any mathematical tool increases with the flexibility with which it can be employed. The student who is aware of options and can exercise judgment will often find himself in a position where he can take shortcuts with confidence, avoid tedious and error-inducing computations, and monitor his solution at every step, so that he can pick up on anything which does not look quite right.

The better students should be encouraged to go further. How does the location of the optimum position of L depend on the width of the river, or on the length of PD? The answer should be obtained both intuitively and mathematically. Suppose the river is a metres wide, the destination b meters downstream, the water velocity p metres per second and the land velocity q metres per second. What conditions on a, b, p, q will ensure that the optimum landing position is directly opposite the starting position? at the destination? What other practical situations lead to the same type of problem? For example, we could be laying a pipeline or building a highway across two sorts of terrain for which the construction costs per mile differ. What happens if the destination is on the same side of the river, but the man has to pick up his wife from the opposite bank? The possibilities are endless.

* Ontario Mathematics Gazette, Vol. 18, No. 3 (March 1980), p. 51.

ARITHMETIC ERRORS: REFINING TEACHER OBSERVATION

by Dan G. Bachor, University of Victoria



Increasingly the teacher is expected to provide extra assistance for the child with special needs within the context of the regular classroom setting. In order to meet this mandate the mathematics teacher will need the assistance of testing and teaching tools that are relatively straightforward.

Scriven (1967) suggested that there must be a closer match between the goals of instruction, what is being taught, and what is being tested. Given this match is successful, another important consideration may be how the student is being taught and how the student is being tested. Although the how of teaching is extremely important, only the latter issue of testing will be discussed here. In order to meet Scriven's concern over what to test to ensure a match to what is taught, the how of testing must be considered. One suggestion to assist in accomplishing this match is for the mathematics teacher to employ a systematic observation system of examining student work.

Currently, a teacher, when evaluating the arithmetic errors of special needs children, is likely to conclude that failing children do not know their basic math facts. While in some general sense this conclusion is correct, the student has obtained an incorrect answer, it is somewhat misleading. Stating that the student does not have mastery over basic facts is making the inference that s/he does not understand a large class of material. In fact, the student may be making a single algorithmic error that results in mistakes, which are only observed in certain types of problems. For example, several students have difficulty with zero. They will apply the rule that zero from any number is zero or N-0=0. If zero does not occur in the problem the student will respond correctly, so s/he does know basic facts but not how to subtract zero. The teacher's observation system is not fine grain enough to identify what help the student needs in terms of corrective programming in mathematics.

Bachor (1979a) proposes an observation system that asks teachers to look carefully at student errors that are found in classroom assignments. The basic shift required in observing student performance is to change from looking only at final answers to carefully evaluating how the answers were calculated. This is not to suggest that procedure is more important than the final answer. The goal of the student is to obtain a correct final answer; however, the diagnostic-oriented mathematics teacher must be interested in the strategy the student used to obtain the answer.

Bachor (1979b) suggests that there are five steps involved in completing a systematic observation of student performance, which he termed error analysis. These steps are:

1. Lists strengths and weaknesses in several assignments completed by the same child over several days. The purpose of this first step is to form some hypothesizes about the type of error being made by the student. If possible, it is best to watch the student complete some of the assignments and have him/her describe what they are doing. Watching the child and talking with him/her may help you isolate the incorrect procedures.

2. Prepare a series of items which test the theory that you have concerning the child's error. Just because you think you know the way a child is solving a problem doesn't mean you have isolated the strategy the child is actually employing. There is more than one way to arrive at the same incorrect answer; try to set up your problems to eliminate the number of strategies that could have been used to solve any problem.

3. Once you have clearly identified the error pattern being employed by the student, prepare a teaching objective based on what you have identified. This step is very important in that testing is often not related back into the teaching process. If testing does not result in more effective teaching the testing probably should not have been completed in the first place.

4. Design some instructional technique to correct the identified errors, which will meet the teaching objective.

5. Complete a follow-up error enalysis to check if the teaching technique was successful.

An example of how to use error analysis will illustrate the nature of the observation system and how error analysis may be translated into teaching objectives.

Darcy was having difficulty subtracting. After

watching Darcy work on several problems and collecting his classroom work samples the following hypothesis was suggested: When given a problem in which there are two digits in the minuend and one in the subtrahend where borrowing is not necessary; Darcy subtracted the subtrahend from both digits in the minuend. For instance, 89 - 3 = 56. Several problems were then designed to confirm the suggested hypothesis and Darcy answered all these problems in the same manner. Three examples demonstrate his consistency in following the same procedure: 75 - 5 = 20, 45 - 2 = 23 and 22 - 2 = 00. These examples were selected from test items that were prepared to the theory about Darcy's subtraction technique (step 2). It is interesting to note that Darcy only made a subtraction error when the number of digits in the subtrahend and the minuend were uneven. He would correctly answer problems like: 13 - 12, or 44 - 13, or 123 - 111. However, as soon as there were an uneven number of digits he would apply his subtraction rule: Use the digit in the subtrahend or the closest digit to the left hand side in the subtrahend and take it away from all the digits in the minuend. Thus, 123 - 13 is solved by first taking 3 away from 3 and arriving at 0. Next Darcy would take 1 away from 2 and write 1. Finally, he would say, I take away 1 and write 0.

Once this specific algorithmic error had been pinpointed, a teaching objective and a teaching strategy were prepared that resulted in Darcy being able to determine when and when not to subtract. After a period of two weeks, a follow-up error analysis was completed in which it was confirmed that the corrective teaching strategy had worked.

Error analysis is an observation system which can be used directly by the classroom teacher to pinpoint incorrect strategies. It has the advantage of allowing the teacher to check what the student is doing and whether or not the corrective teaching technique actually worked while maintaining the child with difficulties in mathematics in the regular classroom.

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INTERVIEW

Focus on Special Education with Dr. Lola May



When I initially began attending N.C.T.M. conferences as far back as 1969 the name Lola May seemed to be synonymous with new ideas in elementary school mathematics. There was hardly a mathematics conference that I attended that she was not present at as a workshop leader; a very dynamic speaker, a treat to hear, and an inspiring message to give. I was exceptionally pleased when I discovered she too would be attending the 7th Annual Conference for Diagnostic and Prescriptive Mathematics in Vancouver and agreed to be interviewed then. Picture, if you will, a warm, wet April morning; two huddled in a corner of the lobby of a busy hotel with a tape recorder trying to ignore inquisitive stares of passers by.

S.P. - Sam Pravica
L.M. - Lola May

S.P. I don't remember a conference that I've been at that you haven't given a workshop. You have a way of motivating people and presenting math in a very enjoyable way.

L.M. Thank you; I travel a lot.

S.P. One of the things we are concerned about in Ontario is special education. You might give us an idea of what's happening in your neck of the woods.

L.M. The current focus is on mainstreaming. The most difficult aspect about mainstreaming is adjustment for both the students and the teacher.

The student is generally behind in his/her work relative to the other students. This makes it very difficult for the student if he/she tries to keep up, and difficult for the teacher in trying to match appropriate learning material to the student's developmental stage and sequence. The answer, at least in Winnetka, seems to be in small class sizes. We like a primary class of not more than 21 students and middle grade classes with not more than 24 students.

S.P. Maybe we better define what is meant by mainstreaming, some of our readers may not be familiar with the term.

L.M. Mainstreaming means "special" children spend one hour in special education with somebody else, then integrated with other childen in a "regular" classroom for the rest of the time.

Do you have the problem in Canada, of having more and more children requiring special education?

S.P. Yes we do.

L.M. I don't know what's causing that. I work in this suburb where the average I.Q. is 116. The parents are mostly professional people and the percentage is going up every year.

S.P. Do you think perhaps we may be losing sight of what teaching is all about?

L.M. Yes, I think part of it is that teachers are too quick to give up. They are too quick to say "he ought to be tested because he's not learning". A lot of people have gotten to the point where they want everybody to be average in the room. We allow this to happen because there are a lot more facilities available. We have somebody for every disability it seems, which is great because children are being taken care of, but now we have so many people that we can send students to that I think teachers themselves are not working with some of the borderline cases as we did in the past.

S.P. How do you define special education?

L.M. Well of course special education includes both gifted and the others, that is anyone that isn't average or norm in the classroom, whatever the average is of course. I have trouble with average.

I've been teaching 35 years and as old as I am I think back about my teachers who used to teach everbody the same thing, at the same time, and would stand up in front of the class and say "if you try hard enough, you can get an A in English and Math, an A in Social Studies, A in Science", and I remember sitting there thinking I was good in Math, but there was someone who could write poetry far better than I could and no matter how hard I tried, I could never do as well as they and we were both being rated on that same kind of a standard. I always thought that was strange.

S.P. I'm the same way in spelling. No matter how hard I try, I can't seem to improve very much.

L.M. I was so glad at age 30, when I found out that there was zero correlation between spelling and intelligence.

If they had all of these things when I went to school in Kenosha, Wis., I probably would have ended up in special education because I didn't work according to the norm. Teachers didn't like me because I day-dreamed when classes were boring and I think they thought I didn't understand, because I wouldn't do the mundane things until they practically threw me to the principal. In spite of this I did succeed.

S.P. I enjoyed working with your Super Cube with gifted students. Why is it not used more?

L.M. With Super Cube, some teachers feel they don't know enough mathematics and so they are

frightened of it. There are a lot of teachers, for example, who will say, "I don't let the children use it because I don't know anything about probability, or I never heard of topology". My attitude about that is you don't have to know much about the topic. For these children, if the directions for the activity are clear the children should be given the freedom to talk to each other and draw their own conclusions. The teacher should be there to lend encouragement and have an interest in exploring the topic along with them.

S.P. How would you help the child who has difficulties assimilating math?

L.M. I am convinced that a lot of teachers don't know how to teach mathematics to slow children. You see, you need a different kind of approach. And the different kind of approach is not another ditto sheet. If the first ditto sheet didn't work, then the second and fourth and fifth is not going to work either.

S.P. You're making it worse by reinforcing the misconception.

L.M. Yes some teachers don't have alternative teaching styles, or they don't in some cases even know how to sequence the ideas on paper for someone who is having difficulties. You have to develop the concept piece by piece, and then use the Gestalt and bring it all together.

The skill of teaching is being able to take a concept from a textbook and identify these little parts and put them in some kind of meaningful sequence.

But a lot of teachers just do not have the skill to teach mathematics.

S.P. Who do we blame that on?

L.M. Teacher education. I don't know enough about Canada, but I can say this about the United States, our student teaching in 90% of the places isn't worth anything. We put them in a classroom with somebody who is not a master teacher; somebody who signed up for it because they want an extra body for some extra help. Often the very best teachers won't take student teachers because they know its work if you do it right. But teachers should do it as a professional responsibility. Teachers learn to teach through modelling. That's why we need master teachers to work with student teachers.

S.P. Where would you put your money then? Have we overdone curriculum?

L.M. We should be putting money into good inservice.

I really think we should relook at inservice and do a very good job there. Some educators are doing it, and do a very good job, but many teachers are turned off with inservice because so much of what they have been asked to come to has not been worth their time. What is needed is inservice that is well organized, well thought out, where teachers feel they really learn something. Something they can take with them and use. Most inservice is based on a one shot deal. Teachers don't get a chance to talk about their problems and get alternative suggestions. They don't know how to handle a child's problem so they try and get someone else to do it.

If we can give them the proper skills then we have a chance at winning.

S.P. How much background do you think a teacher should have before he/she is ready to teach special education?

L.M. Well, I don't know the exact number of courses, but I do know that they've got to have a lot of experience teaching children in a regular classroom setting. But they need a lot of work because of the various subject areas that have to be dealt with, the reading area is one thing and the math area is another thing, and you know, both need experience in teaching

decoding skills because these skills are different for each area.

S.P. Do you think a non-experienced teacher with one year of special education courses would be sufficiently prepared to take on a special education class in September?

L.M. That would be terrible! I don't see how you could do that. I would think that you would need at least a year in just special education alone. It would be a miracle to do it in one year along with learning the fundamentals of teaching.

You need the background to do remedial math as you well know, you have to know math very well. And then you have to know a great deal about how children learn.

S.P. Is the psychology of learning being adequately translated into the classroom.

L.M. There's a breakdown there. I think psychologists have sort of been talking to themselves, writing love letters to themselves, and teachers have never understood what they are talking about. I think they've really got to communicate with teachers now and state it in language that we can all understand. We need to know what style of learning is best for the child, and how to detect these different styles of learning. For example, yesterday afternoon I was in a session where the instructor asked us to tell a story but you could not use the letter "N". We each took turns adding a sentence. It started "I had a little dog whose", and I started saying "name" and they all made a fuss. I then asked what are we doing this exercise for? The instructor answered, "Lola, when you were doing it were you listening for all the "N" words or were you seeing them?", and I said, "I was listening for them as I was saying them". Someone else said, "Oh, no, no, the "N" words all came up, I could see "name". The Instructor said, "you see that's one short test of whether you are audio or visual". And you could tell quickly just by asking that, without going through a whole bunch of other things. It would be the first indicator. I thought to myself, isn't that a simple little test.

S.P. We're just scratching the surface. I sat in on a couple of sessions in special education and that's one of the messages that comes through, that we have to make sure that we recognize the learning modality of the individual for best learning. But isn't it true that if auditory is an individual's best mode of learning that somewhere along the line he has to learn to develop the visual mode also?

L.M. I think that is true and you have to make an effort to do it. But I think that what the instructor was trying to say is you pick up which modality is the one they use with more ease and achieve some success with them, and then you try to develop the others.

S.P. Is that message coming across?

L.M. No, I don't think so.

S.P. How do we bridge the gap between what's happening in psychology and classroom practice. For example Piaget was very much misunderstood.

L.M. Yes, very much.

I think the answer is in inservice, done by people who understand the theory and are able to put it in language that teachers can understand.

S.P. School systems make a big to-do about meeting the needs of kids. Do you think that curriculum has gone far enough for teachers to be able to do that?

L.M. No.

The majority of curriculum is still written for the average. There isn't real curriculum for those who deviate from that, I don't think. I think we have to get across to the people outside the school that when you say 'take

care of the needs of the children', that is a very simple statement and its a good statement, but its a very difficult thing to do and we've got to get it across to them how difficult it is for a teacher to meet all the "special" needs. The worst thing we can do to the gifted for example is bore them, because they learn bad habits and they rust. The gifted need time and space and that's so easy to say. However, in a classroom then to have a different time schedule its very difficult to organize and coordinate. Grouping is a very difficult thing to to, to be prepared for the groups and have a knowledge for it, and to have enough strength to be able to do this over and over all day long. You know it's very hard.

S.P. Do teachers allow students to verbalize their thinking enough?

L.M. No, but that's partly, I think because the class sizes are too large. Or they haven't been taught how to do it. You see, I think when you get more than 35 children in a class it becomes very difficult to do reflective teaching. My line when I give a speech is "Dialogue Is An Act Of Love". Five minutes alone with a child is worth a hundred minutes of hot air at the blackboard. But you and I know, just being practical about it, that if someone gives us 35 children its going to be very difficult to spend 5 minutes alone with each child. Reflective teaching only comes out of reflective thinking which is what the child does; out of conversation alone with somebody. Classrooms today have too many people in them to do this kind of thing.

Its listening to the inner voice that enables us to understand the child's level of conceptualization.

Children need to build a rich vocabulary and learn to make decisions. I think a lot of the children that end up in special education haven't developed listening and thinking skills and its because they've been deprived of a home environment that fosters this.

S.P. Do you think school administrators are aware of what is going on, in terms of curriculum development; what kinds of inservice is important?

L.M. My, I have a pessismistic view about it, I'm sorry you asked me, I can't be very positive about that. I've seen very few superintendents who really know very much about curriculum, even principals. It seems that things have changed such that there are many more problems and pressures. The principal is the curriculum leader in his school and of course he should be. But when you go and talk to them about it they don't have any idea of what its about. They don't get into the classroom because they have too many other things to do.

S.P. Many teachers seem to find difficulty moving from one stage of development to another with children and some spend too long in one stage. What can we do to help them come with this problem?

L.M. You know my answer to that is, people who are good people watchers, have no difficulty knowing when to move on. Most teachers are not watching the child, they're watching the curriculum material. Watch the child! The child gives you the clue. You have to take risks. Watch the child, make a change, then if the child is frustrated you can always go back to where you were. I always say to the child, "I'm terribly sorry, I made a mistake, would you forgive me?" "We moved too rapidly and it was my fault", you know the child is always willing to forgive you. Inadvertently the teacher lets the child feel that he has failed, not by saying it directly, but by saying "Oh, my, we can't do that now can we? This is too difficult isn't it? Well let's try something else," and what I've said to you is "you've failed, I thought you could do that and you couldn't do that".

S.P. What is your feeling about

children memorizing number facts in grade 1?

L.M. I think that's terrible. Teachers get so upset when the child doesn't know the facts right away. The problem is that these are memory tasks and at this level memory is one of the weakest areas.

S.P. Particularly in special ed?

L.M. Yes. And here we are trying to force them to do something in the weakest areas. Isn't that amazing?

S.P. That's funny.

L.M. "If you would just try harder."

S.P. But they are trying. As a matter of fact I've noticed that the kids who have learning problems, or something like that, are generally always trying harder.

L.M. Giving everything they have.

S.P. What direction do you think special education should be taking?

L.M. Well, I think that special education should encompass both the gifted and children who are having difficulty. I think we ought to train more teachers.

I met Margaret Mead in an airplane, the year before she died. She was awfully upset about the money going to the Title One program and not enough money was going in to the gifted program. She said, "you know there is nothing wrong with taking care of those that are disadvantaged. One thing people have got to remember is when you study other civilizations, society only goes forward on those that are gifted. The gifted help make a society progress, and when you ignore your gifted then your society stops progressing". I think for awhile, in the United States, they thought it was not democratic to take care of the gifted.

As for children with reading problems, I never quite understood why, when it had been shown that if you had fewer children in first and second grade that you would have fewer reading problems; they then gave you 35-40 pupils per primary class. As a result of this high pupil teacher ratio remedial reading teachers are needed in Junior High School. Why don't you put your money in grade one and two and have fewer that can't read that you have to have to take care of later on?

S.P. That's a good question.

L.M. Isn't that right? See what I mean? Not very good budgeting is it?

S.P. No, its not.

L.M. But education doesn't always make sense does it?

S.P. I think that probably goes back to what we were saying. It seems education is more politically inclined rather than looking at where the priorities really are. They do say we must look at the needs of the child but how about the needs of the teaching profession?



THE MOOF: A LOW STRESS AIDE

by L.B. Hutchings,
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I. BACKGROUND

Low stress algorithms allow slow students or students with specific disabilities to acquire a high level of mastery with relative ease. Low stress algorithms allows students of normal achievement to save a great deal of curriculum time, to enjoy great computational power, and to remain accurate even when higher levels of stress are applied.

The "mechanical" basis of low stress algorithms is the practice of clearly, and concisely recording

every operation. The full record of binary events is readily accessible. This allows critical changes in procedure and pedology, and produces cognitive and affective conditions very different from those usually occuring with conventional procedures. More detailed descriptions are available. (Hutchings, '76, '80)

Low stress whole number procedures exist for addition, subtraction, multiplication, long division and two forms (multiplicative and subtractive) of short division. Addition, subtration and multiplication can be operated vertically or horizontally. Division is only horizontal. There are also three low stress aides. These may be prepared before use if desired, each requires only a 3 x 5 index card for permanence. Or, the relevant portion of each can quickly and easily be generated on any assignment sheet. They are extremely simple.

The first, in curricular order, is the Differential Operations Overt Format or DOOF. This allows any child who can count to quickly find any addition or subtraction fact.

Second is the Multiplicative Operations Overt Format, or MOOF, whose first purpose is to allow easy access to all multiplication and division facts to anyone who can add or count. It has several other important applications.

MOOF consists of 10 columns of the 10 digits. The columns are ordered 0 through 0. Each column contains the same numberal, a digit, repeated 10 times. Low stress notation of the binary sums is made but a cumulative sum is not recorded and neither a line or plus sign are shown.

Third is the POOF, a modified sive of Eratosthenes.

Analogous algorithms exist for the rational domain, and there is a special modeling system, "flower figures", for both domains.

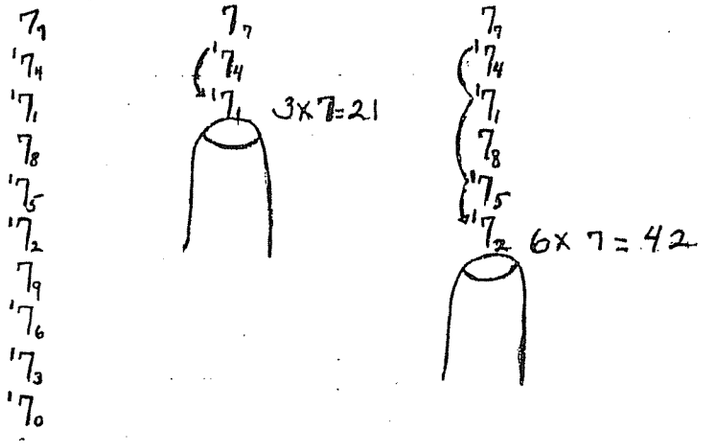
However, in order to understand the MOOF it is necessary to recognize low stress notation and procedure for whole numbers. Regrettably a detailed discussion and description exceeds the scope of this article. However, the most relevant forms can be briefly demonstrated.

II. USE OF THE MOOF

A. The MOOF for basic multiplication facts.

Let's assume that a student has difficulty with the seven times table. The relevent portion of the MOOF is a column of 10 sevens, showing the binary additions low stress notation but with the column-sum line and plus sign omitted. If a student is familiar with low stress notation and is able to count to 18, this arrangement can be quickly generated for any of the numbers 0-9. Once generated, the arrangement provides all multiplication and division facts for the number used.

For example, if the student wanted to know 3 x 7, he would count down 3 sevens, cover the remainder of the column with his finger, and read the sum. Or, if he wished to know 6 x 7, he would count down 6 sevens, cover the remainder of the column, and read the sum.



A significant serendipity is development of the meaning of multiplication as repeated additions. Use of the MOOF tends to be self-correcting. It not only accesses facts but teaches the facts.

Recommended use in elementary school allows a complimentary exposure to flash cards. However, it is suggested that, at secondary level, flash cards be discontinued and MOOF use be allowed discreetly and inconspicuously. The computer-related aspects of low stress techniques should be emphasized when introducing them to secondary populations who, whatever their needs, may resent any attention given to basic facts and operations.

B. The MOOF for basic division facts

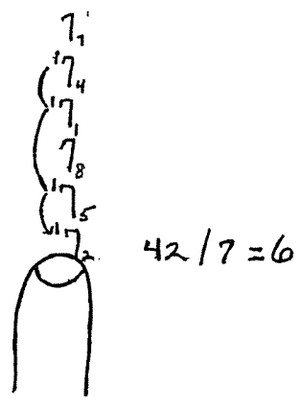
When we found multiplication facts with MOOF, we followed these steps:

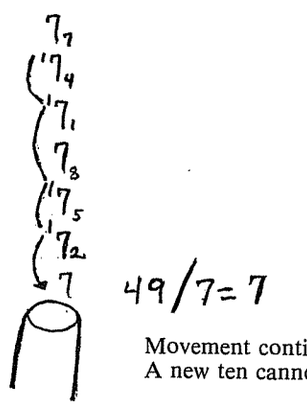
1) Count down the addends
2) Match units
3) Count up the tens

As one of the meanings of division is the reversal of multiplication, it is logical that reversal of these facts will give us division and incidentally, but, usefully show one of the meanings of division.

1) Count down tens
2) Match units
3) Count up the addends

If a student wishes to find the answer to 42/7; he counts down 4 tens, notes that the unit matches, then counts up the addends. Similary if the student wanted to know 49/7; he would count down 4 tens, note that the units did not match, proceed until they did (in this case, one more place) and count up the addend as before.



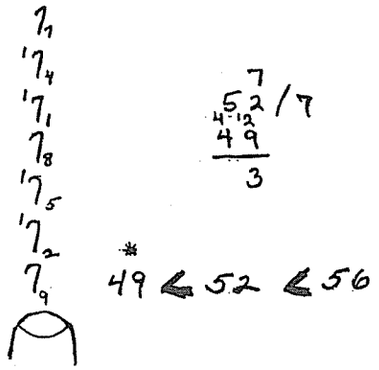


Movement continues until units match. A new ten cannot be passed, of course.

Note: In seeking a match on units, the subsequent ten cannot be passed. If there is no match, the number considered is not part of a basic fact for the number generating that column.

C. ***MOOF for quotient estimation.***

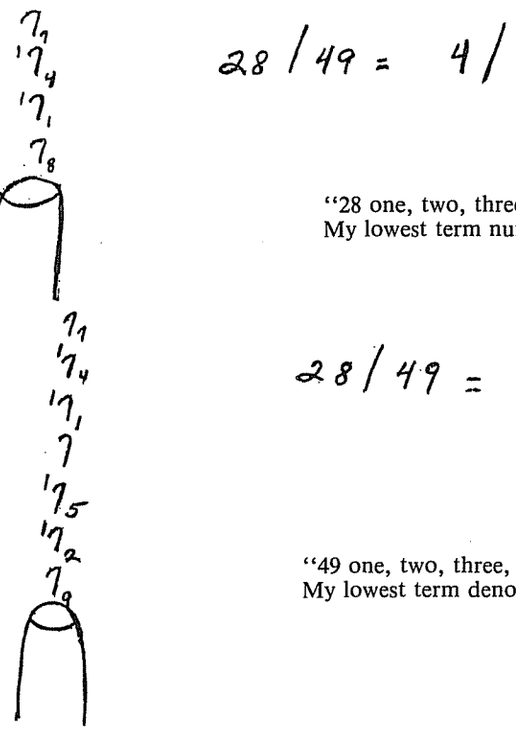
Whether a 2-digit number is a basic fact or not, if it is part of some dividend or partial divident, the MOOF can be used to estimate an appropriate quotient. For example, in this problem, 52 is considered. The student **slides his finger down the MOOF column until he finds the largest sum which is less than the number considered.** 56 is larger than 52, so the child stops after 49, then counts up the addends from that point, as with division facts. The number obtained is the estimated quotient, in this case 7.



This procedure can be used in a technique for quickly factoring three digit numbers, and in both low stress subtractive short division and low stress multiplicative short division. Both short divisions are easily seriated to factor large numbers by one, or two digit numbers. However, as these procedures are applied primarily to advance fractions they are beyond the scope of the present article.

D. *MOOF for lowest terms of simple fractions -- column procedure*

Example: Consider the fraction 28/49. 1) The student looks for a MOOF column which contains both of these numbers as sums. 2) Once found, each is counted out in the manner of a division fact and the numbers obtained are in lowest terms. a) "28; one, two, three, four. My lowest term numerator is four." b) "49; one, two, three, four, five, six, seven. My lowest term denominator is seven. c) So 28/49 = 4/7.



"28 one, two, three, four.
My lowest term numerator is four (4)".

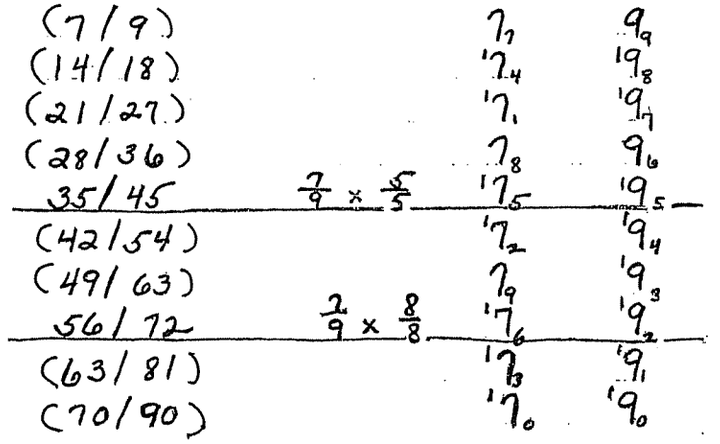
28 / 49 = 4/7

"49 one, two, three, four, five, six, seven.
My lowest term denominator is seven (7)".

E. *MOOF for equivalent terms -- row procedure*

Example: If a student needs to find equivalent terms (both higher and lower) for 42/54, a row is located in which both sums appear. It is not necessary for columns to be adjacent. However, if the student wishes to use a sliced MOOF, so columns can be placed next to each other, that is

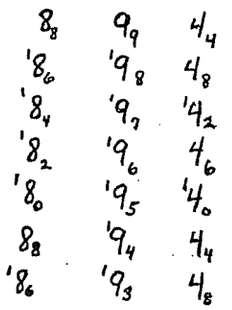
acceptable. In fact, for students with severe reading problems, such placement is desirable. Higher and lower terms are simply read from higher and lower rows of relevent columns.

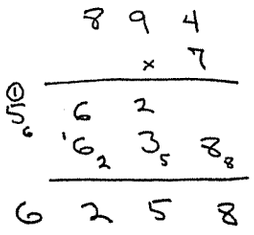


Note: This is a very fast way to find relevent equivalent terms and tends to demonstrate the meaning of equivalent terms; that is, multiplication by various forms of unity. e.g. 5/5 or 8/8

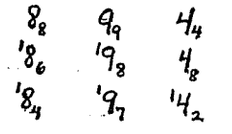
F. ***The sliced MOOF for multiplication.***

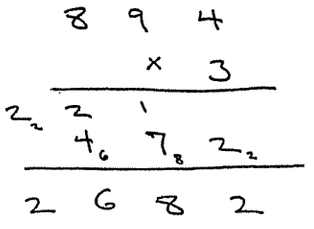
It is possible for a slow student to assist his whole number multiplication by arranging the MOOF columns to form the large factor of any particular problem. Able children sometimes enjoy this as a novelty. The procedure can be either a meaningful aid or a recreation. (It is virtually identical in principle, to Napierian multiplication) Also, a student who has reading problems, or tends to be uncertain of his work, can simplify his work beyond the use of the regular MOOF by arranging the appropriate columns of the sliced MOOF to express the larger factor of the problem. Consider the problem 7 x 894. The student arranges the columns of the sliced MOOF so that each row expresses the larger number of 894. The partial product of low-stress multiplication can be found by counting down to seven, because seven is the other factor, and reading across the sliced MOOF. These columns express the larger factor. The smaller factor, seven, is considered by counting down to the seventh row in this arrangement. The partial sums at the seventh row are the partial products of the multiplication in proper position for multiplication.





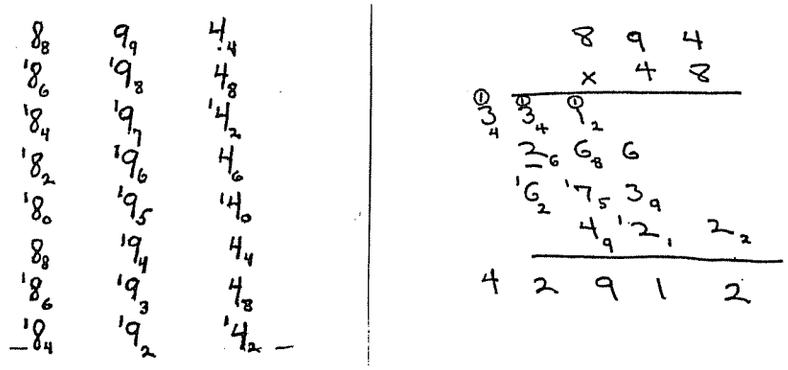
Consider 894 x 3 in the lesser factor.





Now, consider problems with more than one digit - e.g. 894 x 48

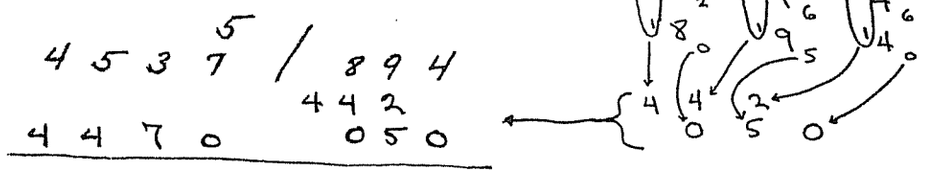
Procedures are exactly as before, except that the second of partial products is off-set one place to the right with respect to the first set of partial products exactly as it would be in unassisted low-stress multiplication.

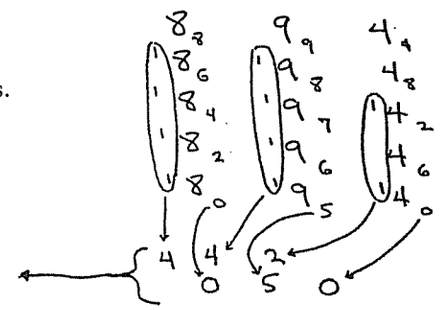


G. ***The sliced MOOF for division - A variation of quotient estimation.***

A slow child or one with a particular disability, especially a reading disability, may aid himself in division by using the previously described quotient estimation combined with the low-stress analogue of Napierian multiplication. 1) Specifically, the student arranges the appropriate columns of the sliced MOOF to express the division problems' divisor. (Figure 10) 2) Then the student slides a ruler or hand down the arranged columns until he finds that set of partial sums which will yield the largest sum which is less than the dividend. 3) When this occurs at the fifth row, with the partial sum 442/050, this set of partial sums is also the set of partial products resulting from the divisors' multiplication by the correctly estimated quotient, and is recorded in the usual manner of low-stress division. 4) The correctly estimated quotient can be found by counting back up the addends. e.g. "One, two, three, four, five. Put five at the top."

The correctly estimated quotient can be obtained by counting up the addends.

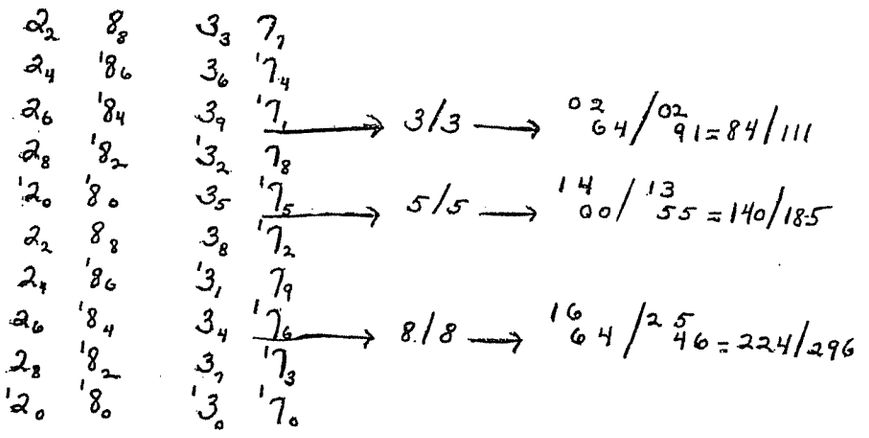




H. ***The sliced MOOF for higher terms - doubled columns.***

The higher terms of simple fractions can be easily obtained by arranging columns of the sliced MOOF expressing the numerator and denominator in correct horizontal sequence. e.g. 28/37

The form of unity raising the term (e.g. 5/5) determines the relevent row. The new expressions of numerator and denominator are copied from this row with correct partial product placement, and then added to show the higher equivalent term as above.



I. *A general comment on large numbers.*

Note that multiples of digits of any reasonable number of places can be considered in any MOOF procedure by cycling around the column **mod** as appropriate. Of course, unlike regular modular procedures, the accumulated tens must be noted.

References

Hutchings, Lloyd B. "Low Stress Algorithms" **Yearbook of the National Council of Teachers of Mathematics.** 1976. Chapter 16.

"Research on Low Stress Algorithms" **Research Monograph of the Council for Diagnostic and Prescriptive Mathematics.** 1980, Chapter 16.

SOLUTIONS TO PROBLEMS IN "PROBLEM SOLVING"

by Charles D. Gallant
St. Francis Xavier, University, N.S.

Editor's Note: These problems appeared in Volume 18, #3, March 1980, pp. 36-39

1. MARTIN GARDNER -- Dice Problem

Three dice are thrown. Someone takes the number on one of these dice, multiplies it by two and adds five. This result is multiplied by five, and the second number on a die added to this result. This answer is multiplied by 10 and the third number added. The person working the trick *(not having seen the dice)* looks at the result of these operations.

By subtracting 250 he can find the three numbers on the dice.

Explain WHY this works!

Solution: Select Appropriate Notation.

Let x, y, and z, denote the respective face values. Then, the instructions given yield the following sequence:

$2x + 5$; $10x + y + 25$; $100x + 10y + z + 250$; $100x + 10y + z$. The latter, of course, is the number xyz. Seeing this result enables the person working the trick to name the three numbers.

2. If you pick 50 telephone numbers and add the last two digits, what sum will turn up most often -- 18, 12, 9, or 0?

Solution: Construct a Table.

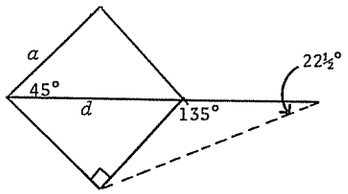
a) Sum = 0 $0 + 0$
   Sum = 18 $9 + 9$
b) Sum = 12 $6 + 6$; $5 + 7$; $7 + 5$; $4 + 8$; $8 + 4$; $9 + 3$; $3 + 9$.
c) Sum = 9 $9 + 0$; $0 + 9$; $1 + 8$, $8 + 1$; $2 + 7$; $7 + 2$; $6 + 3$; $3 + 6$; $5 + 4$; $4 + 5$.

Clearly, a sum of 9 is expected to occur most often.

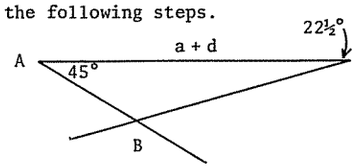
3. Given a segment of length $a+d$, where a is the edge and d the diagonal of a square, construct the square using Euclidean tools, only.

Solution: Work Backwards.

Consider the completed figure.



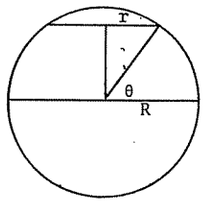
This construction can be obtained by the following steps.



Erect perpendiculars at A and B, etc.

4. At what latitude is the distance around the earth half of what it is at the equator?

Solution: Construct a Model.

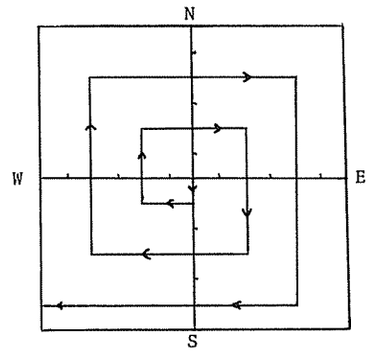


Clearly, $r = R \cos \theta$. Then, the distance sought, $D = 2\pi r = 2\pi R \cos \theta$. Therefore, $2\pi R \cos \theta = \frac{1}{2}(2\pi R)$ and $\theta = 60°$.

5. One day, while hiking in the royal forest, Sir Bumble became lost. He came upon a marker which indicated he was in the exact centre of the forest. He knew that the forest area was a square, 60 meters on a side. Not knowing how to get out, he started walking in the following pattern: 5 meters south; then 10 meters west; then 15 meters north; 20 meters east; 25 meters south; and so on.

How far did Sir Bumble walk before he finally got out of the forest and on which side did he exit?

Solution: Make a Figure.



6. A man and his grandson have the same birthday.

Is it possible that on SIX consecutive birthdays the man's age is an integral multiple of the grandson's age?

Solution: Guess and check.

Man's Age:	61	62	63	64	65	66
Grandson's Age:	1	2	3	4	5	6

7. What is the maximum number of questions required in order to determine my telephone number if each question is answered "Yes" or "No"?

Solution: Make a Simpler Problem.

To determine, in this way, a number between 1 and 2 requires one guess. Continuing in this way, we obtain:

Number Between	Number of Guesses
1 - 2	1
1 - 4	2
1 - 8	3
⋮	⋮
$1 - 2^k$	k

In order, then, to guess a seven digit phone number, i.e., a number between 1 and 999999, we require 20 guesses since $2^{20} = 1048576$.

8. A drawer contains 4 blue and 8 red socks. How many socks must be taken out *(the room is dark)* to ensure one of each color? To ensure a pair? To ensure a red pair?

Solution: Systemize.

Designate the socks as follows: B1, B1, B2, B2, R1, R1, R2, R2, R3, R3, R4, R4.

a) To ensure one sock of each color requires nine socks, since the collection of all 8 red socks would not satisfy the condition, but one additional sock would.

b) For a pair -- Seven.

c) For a red pair -- Nine.

9. An alumnus wishes to leave St. Francis Xavier University sufficient money to ensure that a prize of $100 can be awarded each year at graduation, forever. (Assume 9% interest.)

Solution 1: Guess and Check.

$1211.11 less initial award of $100 leaves $111.11 which will earn interest of $100 during the year.

Solution 2: Draw from Cognitive Background.

The amount of money which must be invested now (called present value) to be worth $100 in one year is $\frac{\$100}{1.09}$, in two years $\frac{100}{(1.09)^2}$, etc., from the formula

$A = P(1+r)^n$. Then Since $S_\infty = \frac{a}{1-r}$, the money which should be invested now is

$$\$100 + \frac{\$100}{1.09} + \frac{\$100}{(1.09)^2} + \ldots = \frac{100}{1 - \frac{1}{1.09}}$$

$$= 1211.11.$$

10. If $a-1$ and $a+1$ are twin primes greater than 10, then, $a^3 - 4a$ is divisible by 120.

Solution 1: Consider a Simpler Problem.

Show $a^3 - 4a$ is divisible by 8, 3, 5.

a) Clearly, $a^3 - 4a = (a-2)a(a+2)$ has each factor even, hence, is divisible by 8.

b) Then $a^3 - 4a = 8\,(k-1)k(k+1)$ which is the product of three consecutive integers, hence, divisible by 3.

c) Since $a - 4a$ is even it must end in zero to be divisible by 5. Consider the following table of terminal digits.

	a	a^3	$4a$	$a^3 - 4a$
1.	0	0	0	0
2.	2	8	8	0
3.	4	4	6	8
4.	6	6	4	2
5.	8	2	2	0

Clearly, cases 3 and 4 cannot occur since, e.g., if a ends in 4, $a+1$ ends in 5 and being greater than 10, could not be a prime.

Solution 2: Draw from a Cognitive Background.

$120 = 5!$ and $a - 4a = (a-2)a(a+2)$

Since $a-1$ and $a+1$ are primes > 10, they are not divisible by the factors of 120, hence we may consider,

$$\frac{(a-2)(a-1)a(a+1)(a+2)}{120} = C_5^{a+2}$$

which is a natural number.

11. Begining with the statement numbered 1, arrange these in a logical order. (Each statement should be on a small separate sheet.)

a) $\sec B \cdot \csc B$

b) $\frac{\sec^2 B}{\tan B}$

c) $\tan B + \frac{1}{\tan B}$

d) $\sec B \left(\sec B \left(\frac{1}{\tan B}\right)\right)$

e) $\sec B \left(\frac{1}{\cos B} \cdot \frac{\cos B}{\sin B}\right)$

f) $\sec B \left(\frac{\sec B}{\tan B}\right)$

#1) $\tan B + \cot B$

g) $\frac{\tan^2 B}{\tan B} + \frac{1}{\tan B}$

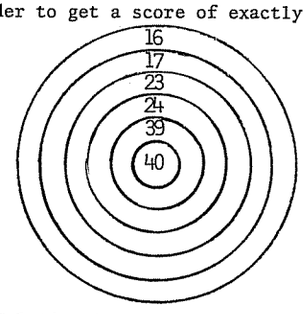
h) $\frac{\tan^2 B + 1}{\tan B}$

i) $\sec B \left(\frac{1}{\sin B}\right)$

Solution: Systemize.

Proper Sequence: #1, c, g, h, b, f, d, e, i, a.

12. What is the minimum number of darts which must be tossed (assuming accuracy) in order to get a score of exactly 100?

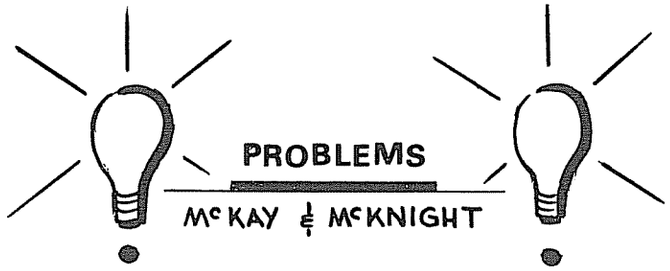


Solution 1: Guess and Check. Six darts; 16 twice and 17 four times.

Solution 2: Systemize.

Clearly, one requires at least 3 darts. The natural inclination is to use the larger numbers. These are successively eliminated leading to the conclusion as above.

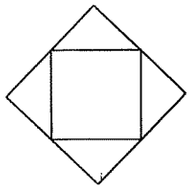






I. UP TO GRADE 10

19.1.1: If the area of the interior square is 6 cm^2 determine the area of the exterior square.



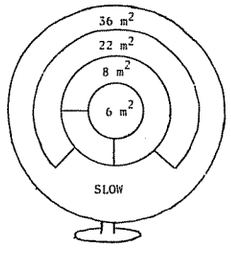
19.1.2: George earned 3/4 as much as Tim earned. George spent an amount equal to 1/2 of Tim's earnings and Tim spent an amount equal to George's earnings. If each saved what he didn't spend, what is the ratio of George's savings to Tim's savings?

19.1.3: Calculate the unknown without the aid of a calculator or computer:

$$\frac{1234567890}{1234567891^2 - (1234567890 \times 1234567892)}$$

19.1.4: Find all the solutions for the equation $|x|^2 + 3|x| - 10 = 0$.

19.1.5: Road signs in Ugah consist of 4 sections as shown, each painted a different colour. It is found that 5 litres of yellow, 3 litres of blue and 1 litre of red paint are just sufficient to comply with this restriction. What must the colour scheme be?



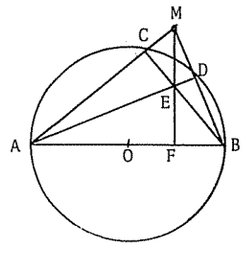
II. UP TO GRADE 13

19.1.6: Find the ordered pair of real numbers (x,y) which satisfies the simultaneous system $8^x = 10y$ and $2^x = 5y$.

19.1.7: Five brothers inherited the family farm, which is 25 km^2. They want to divide the farm into five equal areas, but not into parallel strips. Illustrate how this might be done.

19.1.8: One of a pair of dice is loaded so that the chance of a 1 turning up is 1/5, the other faces being equally likely. Its mate is loaded so that the chance of a 6 turning up is 1/5, the other faces being equally likely. How much does this loading increase the probability of throwing a 7 with the two dice?

19.1.9: If AB is the diameter of the circle as shown here and m is a point outside the circle, prove that $\angle MFA = 90°$.



III. ARE YOU READY FOR A CHALLENGE?

19.1.10: Find a number, half of which is a square, a third of which is a cube and a fifth of which is a fifth power.

C.H.I.L.D. A Service Demonstration Centre

by Joan Simons, Merrill Sitko and Dan G. Bachor

Editor's Note: Project Child's resource office is located in Elborn College at the University of Western Ontario.
Congratulations on your $9,000 grant from May Court Club.

The math class was assembled; the students were receiving their marked papers from the teacher. One girl waited patiently for hers. The papers were returned in alphabetical order. The teacher held the girl's paper back until everyone was seated. She became very tense. The teacher asked for everyone's attention. He held up her work for the class to see. Each question was marked incorrect with large red crosses. He then asked the girl to walk around the class and show everyone her efforts. The girl started to blush and felt very uncomfortable. The class started to giggle. The teacher said to the class, "This is what blonde hair and blue eyes does for you." This is an actual instance of the ridicule that one student had to face. It is indicative of the kind of insensitive probing that was typical of the treatment of children with learning disabilities in the past. Greater awareness of students' individual needs have grown over the years and hopefully this type of situation does not occur these days. The feelings of inadequacy these students experience are hidden today as our expectations of students have become more subtle.

Do you have such students in your classes who seem to frustrate every effort you make when presenting mathematical concepts or processes? How does one define a child with learning disabilities? The generally accepted definition is: "Children with specific learning disabilities exhibit a disorder in one or more of the basic psychological processes involved in understanding or using spoken or written languages. These may be manifested, for example, in disorders of listening, thinking, talking, writing, spelling, or arithmetic. The primary cause of such dysfunction is not due to visual hearing or motor handicaps, mental retardation, emotional disturbance, or environmental disadvantage." In the academic sense, we see the child with learning disabilities as one whose actual performance in academic areas is considerably lower than would be expected by his or her intellectual potential and/or general developmental level. Hence, we are mainly concerned in education about the child who shows a large discrepancy between actual achievement and estimated potential. Personally, we prefer to think of the learning disabled child as the child who **must** be taught according to his or her needs in order to learn. These children have been variously labelled as perceptually handicapped, brain injured, having minimal brain dysfunction, dyslexic, educationally handicapped or impaired, hyperactive, hyperkinetic, specific learning disabled, developmentally aphasic, and "those children". If anything, these children reflect our compulsion for labelling or classifying, and our age-old educational disease, i.e., "hardening of the categories".

How many meet the criteria? Depending on definition, the incidence of learning disabilities has varied from one percent to twenty-five percent with boys being four times more likely than girls to reveal the problem. In a recently completed brief (Communique, Vol. 7, No. 1, Winter 1978), which has been submitted to the Ministry of Education, The Ontario Association for Children with Learning Disabilities (OACLD), present evidence that only 12% of children with learning disabilities are receiving the required educational assistance they need to realize their intellectual potential. This means that in Boards of Education across Ontario, 88% of learning disabled children and adolescents are not receiving the help they may seriously need. This does not even begin to consider the quality of the service which is given to these 12%. Great variations occur from Board to Board with some areas giving good programs. In order to appreciate the number of children and adolescents who require special learning assistance, OACLD used a conservative incidence figure of 5.84% of the school-age population as revealing learning disabilities and discovered that 138,000 children in Ontario were being deprived of the special teaching or services they need to overcome their learning difficulties. In their Brief, OACLD reported that 40 Boards out of a total of 76 revealed no services at all for learning disabled children, with many other Boards reporting long waiting lists for assessment and remediation, and severe budget cutbacks in special education facilities. There especially appeared to be a severe lack of services for students at the junior high and high school level.

Awareness of the needs of the special child, who resides in Ontario, is increasing in the classroom in inservice training and in preservice training. For example, a child, who

is having difficulty achieving in the regular classroom setting, may disturb or disrupt his or her fellow students in the attempt to avoid completing the required academic work. By going, in disgrace, to the principal's office he or she has avoided the embarrassment of demonstrating lack of understanding in front of his or her peers. This method of coping with lack of success does not only happen in the classroom but at home as well. Children with learning disabilities will devise coping techniques to avoid or otherwise circumvent unpleasant tasks at home, like thinking of ways to avoid homework. However, teacher and parent awareness of the needs of the special child is only a partial solution. Being aware does not necessarily result in solution. With the increased awareness of learning problems comes the need for support services to both the regular classroom teacher and the parent.

C.H.I.L.D., (Centre for Helping Individuals with Learning Disabilities), is a resource and referral centre for helping parents, teachers and others who are caring for children who are experiencing learning difficulties. This project is sponsored by the London and Middlesex chapter of the Ontario Association for Children with Learning Disabilities, with funding from Canada Works and the Faculty of Education, The University of Western Ontario. It is housed in the Faculty of Education (Elborn College), The University of Western Ontario, London, Ontario. The centre has been a tremendous success in its first year of operation.

The project has several major goals. The first goal is to continue to help parents and regular classroom teachers become aware of the needs of the L.D. student. This goal is being accomplished by presentations made to parent-teacher groups, radio and television appearances, meetings with community agencies to share materials available at the centre and discuss issues involved with identifying and teaching the L.D. child.

A second goal has been to establish a permanent resource centre which contains curriculum material and research data concerning children with learning disabilities. Staff, with the support of volunteers, will locate and develop material. Parents are supplied with information about L.D. and ways of understanding the education systems, how to locate appropriate help in schools and the community. Ideas on how to help their children with study techniques; games to play with their children, to teach them specific skills are also provided.

Teachers are provided with research information that advises them of recent changes in educational thinking about the learning disabled child. An example of the resources available to C.H.I.L.D. which relate directly to the teaching of mathematics is, Project Math. This teaching aid deals with the difficulty learning disabled students have with verbal problem solving. With poor reading skills experienced by many of these students, the capability of unraveling the complexities of verbal problem solving is beyond them.

Also available is a program on a micro-computer which stimulates children making mistakes in subtraction; "Buggy" helps teachers analyze and understand basic subtraction errors.

Staff have compiled a file on mathematics which contain articles on practical classroom ideas and up-to-date research data relating to mathematics.

Parents are shown how to use some of the math games such as Little Professor and Data Man, which have great value in motivating the children to learn basic math skills. **Deal Me In** by Margie Golick is a popular publication parents are encouraged to use. This shows many ways that a simple pack of playing cards can teach important skills such as thinking skills, numerical sequence, number concepts, computation, visual and auditory memory, etc.

The University of Western Ontario students are available through Student ACLD and C.H.I.L.D. to provide tutoring assistance in the home or school.

CHILD CHAT is a parent group held once a month to allow parents to share concerns related to their children and families. ACLD holds general meetings on different topics related to L.D. from the point of view of education, medical and social aspects of L.D.

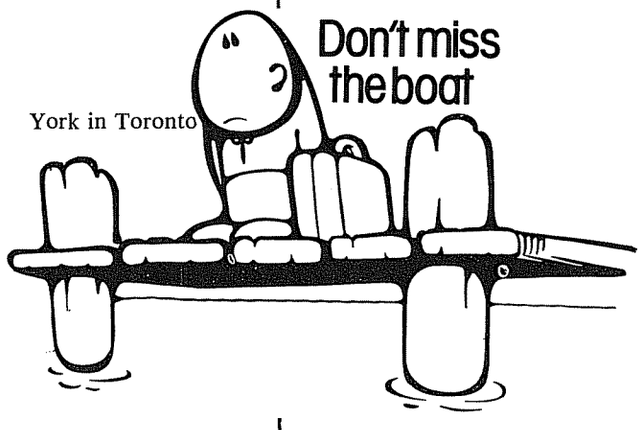
C.H.I.L.D. is a small step taken to help students, teachers, parents and other professionals provide a less threatening environment to allow these children to proceed with their social and educational growth.

Anyone who is interested in learning more about this project should contact the directors, Dr. Merrill Stiko or Dr. Dan Bachor, Faculty of Education (Elborn College), The University of Western Ontario, London, Ontario N6G 1H1. Phone: (519) 679-6023.

COURSES OFFERED IN SPECIAL ED.

The following chart represents a listing of special ed. courses offered at the Universities indicated. The description of courses have been omitted due to content length. Please contact the University concerned for details. You will note that a variety of courses are offered with plenty of room to get involved.

University	Courses Offered	
	SUMMER	FALL/WINTER
Brock; St. Catherines	Special Ed. Part I, Part II, Specialist Learning Problems in the classroom, M.Ed.	Special Ed. Part I, Part II in Hamilton and St. Catherines
Nipissing	Special Ed. Part I, Part II, Specialist	
Ottawa	Special Ed. Part I, Part II, Specialist	Special Ed. Part I, Part II, Specialist
Queens in Kingston	Special Ed. Part I and Specialist	Special Ed. Part II, TTR option in Smith Falls Special Ed. Part II in Peterborough Special Ed. Part I, Kingston
Toronto	Special Ed. Part I, Part II, Specialist	Special Ed. Part I, Part II, Specialist Special Ed. Part I for secondary school teachers in Mississauga.
Western in London	Special Ed. Part I, Part II, Specialist	Special Ed. Part II in Delhi Special Ed. Part I in Kingsbridge Special Ed. Part I, Part II, Specialist in Kitchener/Waterloo Special Ed. Part III in Owen Sound.
York in Toronto		The consulting role in the education of the exceptional student. Individualized education for exceptional students. Introduction to the education of students: -with behavioural exceptionalities -with communication exceptionalities -intellectual exceptionalities



(gifted, show learners, trainable retarded).

Information processing; drama and the exceptional student.

Note: These courses are offered at a variety of centres. Contact the office of continuing education at York University for the centre most convenient for your interests.



DIAGNOSING CHILDREN'S FRACTION CONCEPTS

Carol Novillis Larson
University of Arizona

Many problems that children have with fractions and fractional numbers[1] are based on their lack of adequate fraction concepts. Often times children can model proper fractions with circular regions (pies) and that is the extent of their meaning. What they do seem to have is a set of rules and the "name of the game" is to match up the correct rule with the problem they are working on: "Divide and put the remainder over the bottom number;" or "Cross multiply each part of the fraction by the same number -- or is it add?" There's nothing wrong with these rules if they are based on an understanding of what one is trying to do and the nature of the numbers that one is working with. But for many children a fractional number is a mystery that they approach as two whole numbers, one above the line and one below it. And all of their rules have to do with what to do with these two whole numbers.

When diagnosing a child it is important to test both his/her understanding of fractional numbers as well as his/her ability to demonstrate computational skills. Some children can have a correct method for finding common denominators for adding fractions with unlike denominators but have no real meaning of equivalent fractions. So it is not enough to test computational skills and to only test concepts when a child can't compute with fractional numbers. Ability to compute is not adequate evidence that a child possesses fraction concepts. So if a child needs diagnosis in the area of fractional numbers always begin by testing concepts. And conversely, with children who have been exposed to computational algorithms, test computational skills even when the children demonstrate little meaning. We want all the information that we can get.

The three most common models used in elementary school to bring meaning to fractions are: the area model, the set model and the number line. So in diagnosing children's fraction concepts we want to be sure that they can associate various types of fractions with these three models. By **associate** I mean that: Given one of these models the child can:

1) Name the appropriate fraction term (verbal, level, production).
2) Select from a list of four fractions the appropriate one (symbolic level, recognition).
3) Write the appropriate fraction (symbolic level, production). And I also mean the reverse: Given a fraction the child can:
1) Select from a list of four regions, sets, or number lines the appropriate one (semi-concrete level, recognition).
2) Shade an appropriate region when the region has been separated into n parts or 2n parts, where n is the denominator of the fraction (semi-concrete level, production).
3) Shade an appropriate region where the child has to first separate the region into parts and no hint is given as to the number of parts to make or that they should be equivalent (semi-concrete level, production).
4) Indicate the correct fractional part of a set (concrete or semi-concrete level, production).

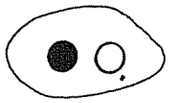
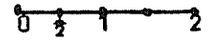
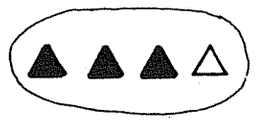
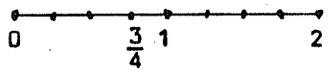
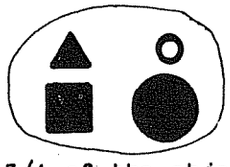
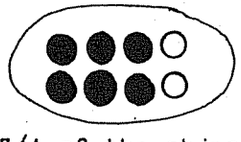
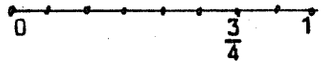
1. **Fraction** is used for a symbol of the form a/b, where a and b are whole numbers and $b \neq 0$. **Fractional number** is used for the number that the symbol a/b represents.

5) Indicate the correct point on a number line (semi-concrete level, production).

The above tasks indicate only some of the types of diagnostic test items that can be used to test children's fraction concepts.

The following chart contains different types of fractions and their related models that can be tested. This chart is not meant to be inclusive, nor is it intended that each model should be tested for each type of fraction. A diagnostician familiar with the many choices can select the models and the fractions to test that are appropriate given the circumstances and the child's responses.

MODELS FOR FRACTIONAL NUMBERS

Area Model	Set Model	Number Line
One-half ($\frac{1}{2}$)		
 $\frac{1}{2}$ is shaded.	 $\frac{1}{2}$ of the objects in the set are shaded.	
Proper Fraction		
Congruent Parts		
 3/4 is shaded.	 3/4 of the objects in the set are shaded.	
Non-congruent Parts		
 3/4 is shaded.	 3/4 of the objects in the set are shaded.	Does not apply.
Number of Parts is a Multiple of Denominator		
 3/4 is shaded.	 3/4 of the objects in the set are shaded.	

Equivalent Fractions

In the preceding region, the child recognizes that both 3/4 and 6/8 indicate the part of the region that is shaded.

$\frac{3}{4} = \frac{6}{8}$

In the preceding set, the child recognizes that both 3/4 and 6/8 indicate the part of the set that is shaded.

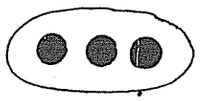
$\frac{3}{4} = \frac{6}{8}$



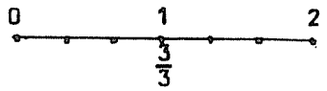
Fractional Names for 1



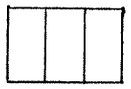
$\frac{3}{3} = 1$



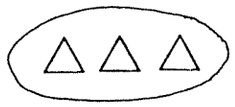
$\frac{3}{3} = 1$



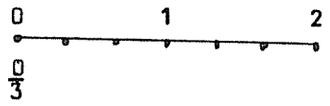
Fractional Names for 0



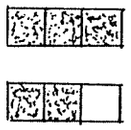
$\frac{0}{3} = 0$



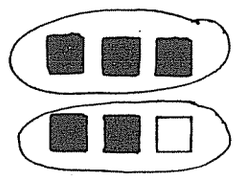
$\frac{0}{3} = 0$



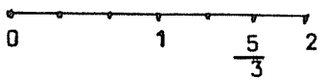
Improper Fractions



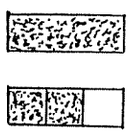
5/3 is shaded.



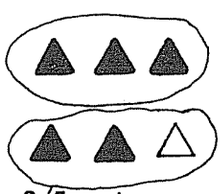
5/3 is shaded.



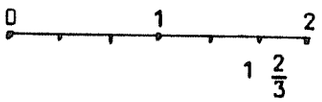
Mixed Numerals



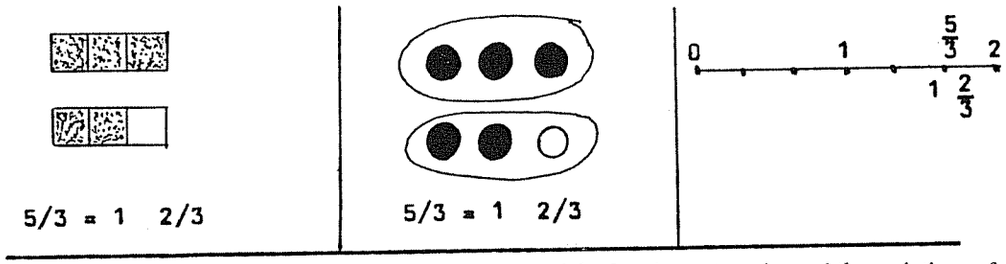
1 2/3 are shaded.



1 2/3 sets are shaded.



Relationship of Improper Fractions and Mixed Numerals



Following is a brief discussion of the reasons for some of the fraction categories and the variations of models that are included in the chart.

One-half or **half** is the first fraction term that children learn. Because of its frequent use in our language it is possible for a child to develop a concept of half and not have a concept of any other fraction. This is the reason for always testing **half** first. It is also the reason we never generalize about a child's fraction concept based on his knowledge of **half.** So knowledge of **half** is a separate category in the chart.

When testing a child's knowledge of other proper fractions it's a good idea to test both **unit** and **multiple fractions.** A **unit fraction** has one in the numerator, e.g. 1/3, 1/4, 1/5, etc. A **multiple fraction** has a numerator greater than one, e.g. 2/3, 3/4, 2/5, 4/5, etc. In the chart only a multiple fraction, 3/4 is illustrated.

Area & Set Models

Models with non-congruent parts also need to be tested. It is a very difficult idea for children to understand that the type of model determines the manner in which the parts are equivalent. In an area model the parts need to be equivalent only in area, they do not need to be congruent. The parts of a set need to be equivalent only in number. The size and shape of the elements of a set have nothing to do with it. If there are eight cars in a restaurant parking lot and someone says half of the cars are foreign cars and half are American cars, would you think, oh, there must be five foreign cars in the parking lot or four? But aren't foreign cars usually smaller than American cars? Size has nothing to do with it. Each car is counted once. So we have two equivalent subset of four cars.

Many children have problem with these ideas. Some children disregard size and shape of parts in all models. Others insist that all parts be congruent for all models. Others insist that all parts be congruent for all models. Sometimes these incorrect rules are abstracted from the very materials that we use in our teaching. If children are only presented with regions separated into congruent parts and with sets that are composed of congruent elements then the child will probably abstract as a defining characteristic of her/his fraction concept that parts must always be congruent. Other children get so involved in the counting process for determining the numerator and the denominator that they completely forget that any kind of equivalent parts are ever necessary.

Number Line Model

The number line is a more difficult model for students than are the area and set models. An important difference between an area model and a number line model is that when using a number line model the child needs to also attend to the scaling. When trying to locate a fraction on a number line of length greater than one, some children disregard the scaling and treat the number line as a unit regardless of its length. So on the following number line they indicate that the point marked by X should be name 1/6 rather than 1/3.



Whenevery a number line of length one is used the number line model is not being completely tested. For this reason it is suggested that children be first tested on a number line of length two or three. Then, only if they are incorrect do we test them on a number line of length one. You will notice that

in the chart most of the number lines are of length two.

Equivalent fractions, improper fractions and mixed numerals are also located on the chart. We need to be sure to test if children can model these as well as proper fractions.

When diagnosing fraction concepts it is important to observe how the child responds to tasks. His/her manner of responding often gives us more information than whether the response was correct or incorrect. For example, many children have been given the following test item:

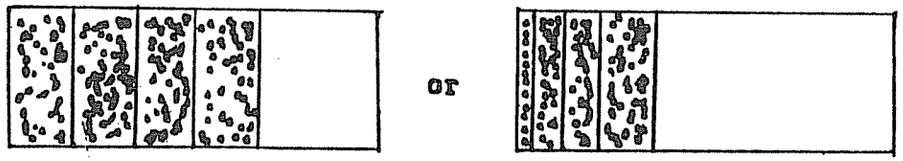
Color 4/4 of the rectangle.



This test item is one in a series of items intended to discover if children are familiar with fractional names for one.

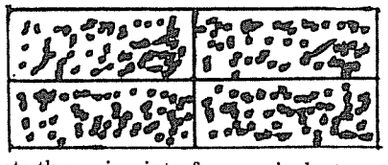
In the past seven years that I have used this item, children have made the following three responses to the above item.

Response 1.



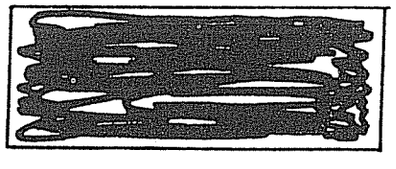
These children make four parts sometimes they are equivalent and sometimes not. But what they have in common is that they don't use the whole rectangle. So we have discovered something about their fraction concept that we weren't even trying to test with this item -- that they don't know that the subdividing of the unit is exhaustive; there can be no extra pieces. We never would have discovered this if all of the regions on the test were correctly separated into equivalent parts and all the children had to do was color in the correct amount.

Response 2.



These children carefully separate the region into four equivalent parts and then color in each part. They demonstrate a great deal of knowledge about fractions but we do not know by this response if they know that 4/4 is a name for one. We might follow-up with the following questions: "How else might you have done this?" and "What would be some other directions that would mean the same thing?"

Response 3.



These children just color in the whole rectangle, usually with quick scribbles. When asked why they respond as they do, these children say: "Well four-fourths is one, so I just colored in the whole thing". These children have demonstrated that they understand that 4/4 = 1.

Both Responses 2 and 3 are correct but they give us different information. Response 2 gives us incomplete information. Because the children could be just following a rule: "Separate the region into the number on the bottom (or the denominator) and color in the number on the top (or the numerator)." They might not even think: "Oh, 4/4 is the whole thing, so I've colored in one rectangle."

The area model, the set model and the number line are just some of the models that can be used when teaching about fractional numbers. They were selected as the focus for this paper because they are the ones used in elementary school mathematics textbooks. In order to help children expand their fraction concept many different types of models should be used. Other effective models are: folding paper, clay, straws and containers of water and sand.

TEACHING PLACE VALUE



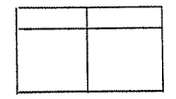
A. Dean Hendrickson, University of Minnesota, Duluth

Understanding of the place value properties of our numeration system develops very slowly and at uneven rates in children. Some of the ideas are grasped quite early and others persist in presenting difficulties well into the third grade and beyond, even with well planned and careful teaching. The most persistent problem seems to be recognizing the number property of grouped material, particularly when groups of different size are presented in the same situation.

Place value ideas are normally given little attention in the schools. The assumption seems to be that if children can count into the decades and if they can memorize manipulations of symbolic forms with 2 digit representations, that they must understand place value. Rarely are any other than pictorial and symbolic models used. Even more rarely is some sequence of instruction developed to present a step-by-step introduction to these ideas. What follows is one sequence of this kind that has worked well with children in grades one, two and three.

The first emphasis should be on the idea of exchanging - of accumulating units to some predetermined size and then exchanging these for some token or fixing them into a standard form as a group and retaining them. Since our positional notation system with symbols has aspects of both of these ideas, both kinds of exchange activity are important.

In such activities, the group size used should vary; different materials should be accumulated by various means to form the groups; and these accumulations should be kept before the child, orally described and symbols gradually attached. Place value mats are very helpful in this.



Shown is a two column mat. Children are given a mat and some material such as UNIFIX or beans and cups and, later, multibase units and longs. Signals such as hand claps, foot stomps and bell rings are used to indicate the number of "ones" to be placed in the right column. A group size is agreed upon - five, for example. This is given a name such as a Quark and that name is used to describe it for the remainder of the activity. As materials accumulate in the right column, whenever a Quark can be made, it must be made and placed in the left column. After each signal to add "ones" the children orally describe the result, i.e., "Two Quarks and three ones." UNIFIX can be linked into a Quark and beans in a cup can be a Quark. Numeral flip cards can be used to attach symbols to the resulting arrangements. One set is used for each column. In the above instance, the oral response would be accompanied by having the numeral cards flipped to show:

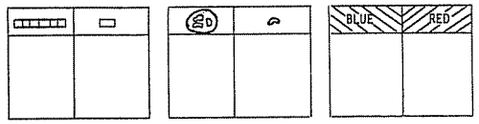
2 3

The next step is to have children record their results on two column sheets. Adding machine tape works well.

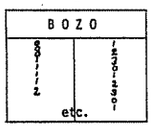
Exchange of a token for a group can be done with chip trading games where the amount added can be determined by a dice, a spinner or other student activated devices. Recording can be introduced in the same way - first manipulating prepared numeral cards, then writing symbols.

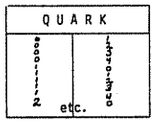
Exchanging in the opposite direction - by decomposing groups to obtain the units and transferring these into the units column and by exchanging tokens of larger groups for smaller groups or units can also be done in this way.

After exchanging in both directions is understood, both addition and subtraction can be incorporated into the activities. It is a good idea to have material in the label sections of the place value mat to reinforce what is in the columns. For example:



As counting results with different size groups are obtained, charts for each size group can be developed and placed around the room, i.e.,

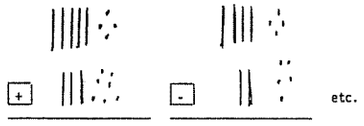




for each grouping size three to nine. (Some children, by using dice and adding machine tape to record, like to run these "computer" charts into 3 and 4 digits).

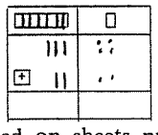
When the group size used is finally ten, probably not until early in second grade for all students, the terms "tens" and "ones" should be used quite consistently. Pictures of the Base Ten Longs and ones can be put into the column headings with rubber stamps.

When integrating place value ideas with computation, base ten materials should be used to arrange situations for addition and subtraction:



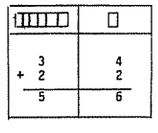
These are set up on work mats as shown below:

(a)



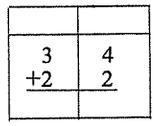
Results are recorded on sheets prepared for two column representations as shown (b):

(b)



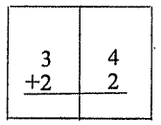
Problems can be made up on (b) sheets and given to students to work by manipulating materials, then recording the results on the sheet with the presented problem. Then, introduce sheets with no pictures to label the columns.

(c)



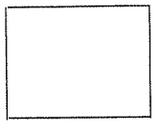
Follow the same procedure of having the children use materials to work the problems. Then introduce sheets with only columns as shown:

(d)



Repeat the same procedures.

Finally, remove the column line so that the usual computation forms are presented. The next step is to introduce the next regrouping size in base ten.



Modify the sheets described earlier into three columns and go through the same sequence of presentations:

The previous activities with grouping UNIFIX links into a lat, stacking cups of beans in a bowl, etc. should be interspersed into work with base then materials -- and with differen size groups!

In introducing Multibase blocks, children should first have free exploration opportunity -to build house, forts or whatever until they are comfortable with them.

Structured activities like building larger pieces out of (1) the same smaller piece and (2) combinations of smaller pieces can be introduced to prepare for the trading and exchange activities.

Insist on the students' making exchanges initially and delay "shortcutting." Set up a "remote" exchange bank to which students must travel if necessary.

One cannot stress strongly enough the following ideas:

1. Grouping and group discussion activities should be done with small group sizes first - before emphasizing ten. This is to reinforce the idea that the grouping size is arbitrary. It also provides for early exchange activity as soon as children "know" five, six or four. It minimizes counting and focuses on exchange as the idea of concern.

2. A variety of materials, modes, formats and game-like situations are needed.

3. Ideas must be developed at the concrete level; symbols must be gradually introduced; interpretations of real situations into symbolic form by the children must precede interpretation of symbolic forms by children.

4. Continual emphasis must be placed on the number property that is retained in a group.

5. Decade counting, backwards counting, counting across decade boundaries and similar oral activities must reinforce the work with materials.

6. Place value must be given major emphasis until about age 10. It seems that not many children are able to think of 34 as three tens and four ones instead of ten, twenty, thirty, thirty-one . . . four, until then.

7. Other kinds of activities present opportunities for reinforcement of place value. For example, children can measure lengths with UNIFIX, count the measurement, then break off links of ten and re-count by tens and ones, recording the result in positional notation. Children can be asked to build rectangles out of Base Ten Flats, Longs and Units, separating the pieces into columns upon completion. They can do whatever exchanges are needed and record the final area in positional notation in tables like the following:

Length	Width	Area
2L + 3U	3L + 4U	7F + 8L + 2U

-or-

Length		Width		Area		
2	3	3	4	7	8	2

RESEARCH COLUMN

by William Higginson, Queen's University

Comment

Some time ago in the context of reviewing a number of publications Bill Higginson made several provocative and uncomplimentary observations about the state of contemporary research in mathematics education. Briefly stated, his position (Volume 16 numbers 1 and 2) was that for a number of reasons the vast majority of the studies carried out in our discipline have little of substance to offer the classroom teacher. In taking this position he was agreeing in essence with the views of our former OAME President, John Egsgard, who made this point several times during his term of office as President of the National Council of Teachers of Mathematics.

At the invitation of the editor, Bob Smith there follows the first of a short series of articles on 'relevant research'. Dr. Higginson notes that his definition of 'research' is a broad one and he invites recommendations for future studies for consideration.

GIFTED STUDENTS IN MATHEMATICS? Are there more of Them than We Think?

Perhaps the most basic assumption which is made about gifted children is that there aren't very many of them. Often this assumption is part of a definition of the group, "the top x% of the ability range" where x is sometimes as small as 1 or 2 and seldom is as large as 10. When viewed from the perspective of different disciplines even the general public seems committed to the idea that mathematics, most of all subjects, is an erea where the majority flounder, some are technically competent and a very small minority are capable of unusually penetrating insights.

It is, therefore, unusual to hear the view proposed that all children are mathematically gifted. The most charitable reaction most mathematics teachers might manage toward such a proposition probably would be that the word gifted was being used in some very unusual sense. More cynically they might suspect that a thesis of this sort could only be the fantasy of an incurable romantic or the theorizing of some ivory tower intellectual unsullied by recent contact with humans under the age of twenty. And so when the leading international journal in mathematics education has a major paper by two classroom teachers (Kent and Hedger, 1980) in which they state "our common experience . . . has guided our

belief that **all** of our pupils are mathematically gifted and that it is our task and sometimes privilege, to make them individually aware of it (p. 150)" it would appear that something out of the ordinary is happening.

In their paper David Kent and Keith Hedger give examples of the work done by their students at the Belper High School in Derbyshire, England. The number and range of the examples they cite, together with the details they give of their methodology is extremely thought provoking. Take, for instance, their remarks about Paul who had arrived at the school with a report which read "below average, but with adequate knowledge of his multiplication tables". Consistent with this Paul scored 46 out of 140 on a diagnostic test soon after entering the school. This mark put him well into the lower half of the incoming class. A little over a year later we find Paul's class doing an introduction to trigonometry via right angled triangles. "The class were given some work to do at home and as an extra Paul was asked if he would like to take a look at an '0' level mensuration question. He was given an old examination paper with the words "you might like to look at Question 1, you should be able to do parts of it". As if to demonstrate his complete mastery Paul did not only Question 1 but also the rest of the paper. In so doing he dismissed a 3D geometry question as if with contempt, he tackled a problem on Linear Law with ease and more. Perhaps the highlights of his weekend endeavour were that he "invented" the formula ½absinC for the area of a triangle and also "invented", for himself with no aid, the 'Cosine rule' (p. 156).

And Paul is by no means unique in his exploits. Mark was described on leaving his middle school at thirteen as "a likeable lad who tries hard, does neat work and is of average ability". A year later he is producing elegant solutions to scholarship paper problems intended for very sophisticated eighteen year olds.

So how does one account for what appears to be a most unusual situation? The critical factor without doubt lies with the individual teachers. They care deeply for mathematics and for adolescents. They share a commitment to a vision of what mathematics might mean to learners. They are sensitive to the human dimension of the classroom teaching and they work very hard in a cooperative fashion. In addition they are exceptions to the general rule that teachers are reluctant to talk or write about their professional experiences. The 'Growing Tall' paper is a particularly comprehensive one. It is, however, only one of many teachers in the Belper school have written (Evans, 1980, Hedger and Kent, 1978; Kent 1978 (a), 1978 (b), 1978 (c), 1978 (d), 1979).

It is the primary purpose of this article to bring these papers to the attention of any secondary school mathematics teacher who would like to think about ways to improve his teaching. (One would hope that this would be all teachers; the journals in question will be available in most Education Faculty libraries if not in Board Resource Centres). As a second stage activity we should follow the example of the Belper staff and begin to share some of our classroom experiences with our peers in a written form. Such documents undoubtedly would not satisfy traditional criteria for research publications (nor would the Belper materials) but they could be of inestimably more value than the twaddle which has constituted an unduly high percentage of the studies carried out under that banner in the past.

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It isn't failing that spells one's downfall; its running away, giving up.
- Michel Greco, *Les Pigeons d' Arlequin* (Les Ecrits du Canada francais)



ENTREVEU AVEC STELLA BARUK

LA LOGIQUE DE NOS ERREURS

Au mois d'avril dernier (1979), Mme Stella Baruk, auteur de Échecs et Maths et de Fabrice ou l'école des mathématiques, effectuait un court séjour au Québec. A cette occasion, Mme Baruk a donné une série de conférences, entre autres dans les principales universités de la province, au cours desquelles elle a exposé ses idées sur l'enseignement actuel des mathématiques. Quinze années d'expérience dans l'enseignement des mathématiques, dont une large partie à des enfants "caractériels" ou "enfants-problèmes", ont conduit Mme Baruk à constater que "l'échec en maths n'est pas l'échec de l'enfant mais celui de l'enseignement". C'est en effet la thèse qu'elle soutient dans Echec et Maths paru en 1973. Mme Baruk fait, dans ses livres, une critique très acerbe de l'enseignement des mathématiques en France, critique que le lecteur québécois peut facilement transposer à l'enseignement dispensé au Québec.

Stella Baruk utilise une métaphore pour le moins surprenante pour nous parler des mathématiques: il est question, en effet, des mathématiques du ciel, des mathématiques de la terre et de celles des limbes! Auquel de ces endroits trouverons-nous la paix éternelle? A vous de choisir! Les mathématiques du ciel sont les mathématiques dont seule une élite peut apprécier les subtilités, et cela après une longue initiation. Le monde céleste des axiomes, des théorèmes et des démonstrations est en effet une source de joie pour certaines personnes, cependant il est malheureusement trop facile pour le commun des mortels de se perdre dans les dédales de la pensée abstraite.

Par contre, les mathématiques de la terre relèvent d'un réalisme profondément enraciné dans la vie de tous les jours. Mais les notions mathématiques fondamentales ne peuvent pratiquement plus être identifiées. C'est ici que l'on devra résoudre le problème existentiel du monsieur qui vend dix balayeuses par jour pendant 25 jours et qui se demande s'il pourra manger le 26ième jour! On se perd dans une concrétisation souvent mal à propos des mathématiques qu'on ne peut plus transcender. Ce "problème concret" détourne l'étudiant du but poursuivi qui est de lui faire assimiler une notion nouvellement apprise.

Le bonheur se trouve donc, selon Mme Baruk, dans les limbes! Celles-ci sont identifiées aux mathématiques de l'erreur, c'est-à-dire qu'un enfant qui fait des erreurs a toujours une bonne raison. Le but de Mme Baruk est de réussir à "reconnaître tous les liens qui ligotent littéralement la pensée d'un enfant et de les dénouer". Par exemple, ce cas rencontré par Mme Baruk: un enfant à qui on faisait répéter: "une application est une relation entre deux ensembles telle que chaque élément du premier ait une image unique dans le deuxième". Toute la classe avait compris, sauf ce petit garçon qui, connaissant la définition de l'application, s'obstinait à dessiner plusieurs flèches au lieu d'une seule. Il répétait: une image unique, et de nouveau de nombreuses flèches apparaissaient. Intriguée, Mme Baruk lui demande: "Qu'est-ce que ça veut dire, pour toi, unique? -- Ca veut dire qu'il est seul... -- Oui et alors? -- Ca veut dire qu'il a toujours plein de cadeaux." Pour cet enfant, unique voulait dire comblé, beaucoup!

Les enseignants doivent donc s'efforcer d'être conscients de leur discours ainsi que des ambiguitées qu'il est susceptible de susciter chez l'étudiant, que ce soit à un niveau simplement phonétique, ou même au niveau de la compréhension du sens des mots employés. Chaque individu a une expérience de vie qui lui est propre et, de ce fait, il lui est tout à fait normal de faire des erreurs, surtout en mathématiques, où les mots utilisés sont souvent

assez éloignés de leur sens premier.

Ainsi, il ne fait nul doute que les mathématiques provoquent les réactions les plus diverses chez les personnes qui entrent en contact un jour ou l'autre avec cette "bête noire": réactions qui peuvent aller d'un certain goût ou même amusement à résoudre des problèmes chez les uns jusqu'à un réel traumatisme chez plusieurs autres, traumatisme qui sera d'autant plus difficile à surmonter que les mathématiques sont part tous, parents, éducateurs et même confrères de classe, considérées comme un instrument de sélection infaillible ou comme le passeport par excellence de la réussite dans la vie. Mme Baruk croit que tout cela est "une formidable mystification qui cache en fait une opération de sélection apparemment inattaquable." En effet, si les bonnes notes correspondent à une bonne compréhension des mathématiques, ce qui est loin d'être une règle absolue, cela prouverait simplement que l'étudiant possède une certaine habileté de l'esprit à jouer avec un certain nombre d'énoncés mettant en cause un certain type de logique intellectuelle. Selon Mme Baruk, c'est un exercice spécialisé qui peut aller très loin dans la complication mais qui, au point de vue de l'enrichissement affectif, intellectuel ou culturel, rest très limité. Les mathématiques ne doivent surtout pas servir d'instrument de sélection qui n'a aucune raison d'être. C'est donc toute une philosophie bien enracinée de l'enseignement des mathématiques que Mme Baruk remet ici en question.

A la suite de toutes ces critiques, ces mises au point sur l'enseignement des mathématiques, Mme Baruk propose une nouvelle approche, une nouvelle façon de les enseigner: dispenser une culture de base, c'est-à-dire un ensemble d'idées sur ce qu'est la méthode mathématique, ce qui la distingue des autres sciences, son histoire... En plus d'élargir les horizons du professeur lui-même, la connaissance des différentes raisons qui ont influencé l'apparition de certaines notions mathématiques, même les plus simples, ne peut qu'être bénéfique à l'apprentissage de l'étudiant et le sécuriser.

Prenons par exemple une notion aussi banale que le "zéro". Même dans des classes pré-universitaires, on retrouve les raisonnements suivants: "trois multiplié par zéro, ça fait trois, parce que zéro c'est rien, alors multiplier par rien, ça fait qu'on n'a rien fait et ça laisse le trois comme ça... Zéro moins trois... Zéro, c'est rien, alors je ne peux pas enlever quelque chose de rien!" Cette litanie du zéro pourrait se prolonger fort longtemps! Pourtant, savoir que le zéro n'est apparu que très tard historiquement dans le langage mathématique et que ce sont des raisonnements comparables à ceux effectués par les étudiants qui furent la cause exacte de l'apparition tardive du zéro, ne pourrait que rassurer l'étudiant sur sa propre faculté de raisonner. Par la même occasion l'enseignant, en prenant conscience du fait que son étudiant suit un cheminement de pensée similaire à celui de la pensée similaire à celui de la pensée des hommes au cours des âges, ne pourrait que constater que les erreurs commises sont tout à fait naturelles.

Danielle Ouellet

All that is gold does not glitter; not all those who wander are lost.
- J.R.R. Tolkien, *The Fellowship of the Ring* (Houghton Mifflin)

Many a person gets a reputation for being energetic, when, in truth, he is merely fidgety.
-Jan McKeithen in Camden County, Ga., *Tribune*

A mind in motion is like a rolling snowball.
- Edward O'Blenis

He who is afraid to ask is ashamed of learning.
- Danish proverb

WHAT'S BASIC? A CONSTRUCTIVIST VIEW*



by Thomas C. O'Brien

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*This article is abbreviated.

Prepared under the support of the National Institute of Education Project on Basic Skills, Choices and Issues. T.C. O'Brien is Professor and Director of the Teachers' Center Project at Southern Illinois University and the author of problem solving materials for pupils, age 11 plus. See **"Solve It"**, (Books 1 - 5). Chicago: Daigger/ETA, 1977.

Introduction. Did you see the film **Casablanca?** Or **Citizen Kane?** Or **The Deerhunter?** Did you go to the movies with a friend? How did your impressions match up with your friend's? Did you see any of the movies more than once? Did you see the same thing the second time around?

My experience -- maybe it's one you can sympathize with -- is that my wife and I almost always come away from a film with differing reactions. "It was about war," she said of **The Deerhunter.** "Or was it about human nature," I said. "It was about friendship," said she. "It was historically inaccurate," said I. Over an ice cream soda we compared our impressions and to this day -- with some points of agreement -- we differ.

We do agree, however, about **Casablanca.** We saw the film for the third time recently. As usual, we agree that we saw a very different film, and we agree further that each of us saw a very different **Casablanca** from the **Casablanca** we remembered.

At each of the movies, each of us paid the same price, each of us spent the same time, and each of us "saw" the same film. Yet it is clear that each of us saw very different films --different even from the same film seen months ago -- and that we will continue to do so in the future.

Constructions. The point of all this has nothing to do with film viewing. The point is commonsensical enough -- people do not merely store what they experience: We are not cameras or Xerox machines. What we all do --what you are doing as you read this right now -- is to interact with the world. We screen reality. We act on it. We interpret it. We modify and re-form it according to the networks of ideas we've already formed.

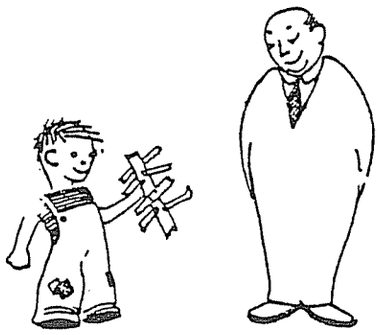
But the things of the world act on us too. When we go to a movie -- **Casablanca,** for example -- the old networks of ideas about government, about Humphrey Bogart, about war, and so forth, alter themselves to incorporate to new information we encourage.

In general, the relationship between people's minds and the world is much like that of a hungry man and a good steak: the hungry man works on the steak, changes it, deforms it, while the steak acts on the man, changes him, enriches his original situation.

Such a view -- a constructivist view -- sees people as active, rather than passive, in dealing with reality. It sees knowledge as a construction, not an accumulation. It sees people not as controlled by "stimuli", but as controllers of the stimuli in the sense that they select and interpret and re-form **the stimuli.** It sees ideas as complex networks growing and alive, not as straight lines. It sees the main intellectual activity of people as organizing and it sees the organizing we do -- starting from the touching and seeing and sucking actions of the newborn through to the elaborate equation-making of the adult physicist -- as tending toward coherence, stability, economy and generalizability. Finally, it sees the mind of man as not satisfied when it has achieved coherence but as questing, as actively searching for and creating novelty to engage, dissonance to conquer, risk to take. The mind is an organism, not a mechanism.

In brief, it's as though we should all -- children especially -- be seen as wearing sandwich boards saying "Under Construction -- Self Employed". (But that's not at all to say that parents and schools have no role to play. They provide the raw materials -- experience -- from which the constructions are built).

Developmental Aspects. When I was five years old, I made a tie rack for my father.



The tie rack I would construct now -- as an adult -- would be very different. Similarly, the mental constructions one makes from age to age are very different. They are different in amount, of course -- one adds experience as one grows older -- but they are different in kind and it is this difference that has interested researchers over the past sixty years.

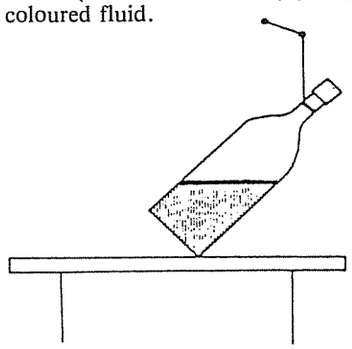
The research is far too lengthy and detailed to repeat here; all that one can do is to hint at its richness.

The research ranges from the actions and co-ordinations of the newborn to the growth of language and of organizing principles (number, time, causality, etc.) in young and school-age children through to the availability and status of thinking and logical abilities in late adulthood.

It generally concerns the question, "What do people make of the world?" How do they organize the things of the world that come in to them? How do they organize the things that are "in" there alrready? And how does this organizing develop? Instead of asking about the effect of the stimulus on the organism, the research asks about the effect of the organism on the stimulus.

The method of research is often surprisingly simple. Much of the research involves careful observation of individuals as they solve problems. How does the infant reach for a rubber duck placed near his right hand when he already holds a rubber frog in his right hand and his left hand is free? (He drops the frog and reaches for the duck with his right hand; up until 6 months or so, infants rarely reach across mid-line). How does the child formulate the rules for a game of marbles? How does the seven year old infer how many candies are hidden if he saw 10 originally, some were taken and now he sees three? What's the four year old's/the seven year old's/the twelve year old's answer to the question, "Why do clouds move?" How does the ten year old/the eighteen year old define "propaganda"?

The results are often particularly striking. In one research, children were shown a drawing of a bottle (or an actual bottle) partly filled with coloured fluid.



Then, an hour later, a week later and six months later, they were asked to draw what they had seen. What commonly happened was that children drew situations that they had not seen and could never have seen because they could not, in fact, exist in the real world.



Findings of this sort (and they are abundant) make it clear that the mind is not a camera, that the mind selects, interprets and reconstructs what comes to it in terms of its existing mental networks. Moreover, the findings show developmental change. In the research cited, 30% of children progressed at six months (without a new look at the bottle), 53% remained the same and "in general, correct drawings after one week remained correct after six months".

The research, sixty years worth in various cultures on a side range of issues with the same general results -- suggests that one's present construction is at once the co-ordination and elaboration of previous constructions and the springboard of new constructions to come. Here is a sketch of some of the main aspects in the flow of events of the development of thinking.

From 0-24 months, roughly, the child constructs and co-ordinates actions -- looking, reaching, grasping, looking and reaching, etc. -- which move from being ends in themselves to being means to ends of purposeful behaviour, begins symbolic and representational thought and action (imitation of people of objects no longer present, symbolic play, drawing and language) and

constructs one of the milestones of intellectual growth, the notion of object permanence. This notion -- that an object exists independent of one's observations or activities -- is one of the fundamental supports of rational life. (Imagine what life would be like if objects were to come and go at random, if objects were thought to lose their existence when we ceased contact with them).

Just as the two year old is different from -- not merely more than -- the newborn, so is the six or seven year old qualitatively and quantitatively different from the two year old. From two to six, the child constructs a wide variety of abilities, the most extraordinary of which is the acquisition, elaboration and co-ordination of language and communication skills enabling him to interact more fully with the world. What an incredibly complex job it is!

But the child still has a long way to go. It is during the school years that still new advances of fundamental importance take place. One of the dominant characteristics of childhood is egocentrism, the view that one's concepts, percepts and point of view are the only ones possible and thus are shared by the world. (Egocentrism is dominant in, but by no means confined to, childhood). Thus the following examples, both from research and from everyday life, should be familiar:

We are driving to a picnic. We are lost. The six year old points to the car in front of us. "Follow him," he says. "He's going to the picnic." "Why?" "Because we are."

Two children are scrapping in the schoolyard. You call them in. Each tells his story. Each sees no sense at all in the other person's story.

On reciprocity and the difficulty of getting "into another person's shoes": "Jane, do you have any sisters?" "Yes, Sue." "Does Sue have any sisters?" "No," or, "Do you have any enemies?" "Yes." "Are you an enemy?" "No."

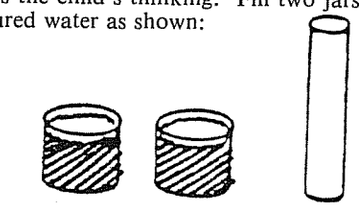
Second grade John writes "John" at the top of all written school assignments despite the fact that there are four kids named John in class. He knows who he is.

Egocentric monologue -- "Conversation" with no listener, no give-and-take, no communication.

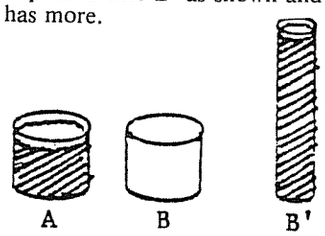
Egocentric dialogue -- "Conversation" with a listener but with referents clear only to the speaker. You telephone your friend and the friend's child answers. "Who is this?" you ask. "It's me." The child knows who he is, so obviously you must know, too.

At an early level it seems as though the child is unable to consider a point of view other than his own. Later, the child is able to admit an alternate point of view -- especially those moderately different from his own -- the child develops this co-ordination.

Co-ordination plays an important role in other affairs as well. A series of classic experiments shows the child's thinking. Fill two jars with coloured water as shown:



Then pour B into **B'** as shown and ask which jar has more.



As with the earlier construction of object invariance, here the child constructs -- over the elementary school years -- the notion of the invariance of **properties** (in this case, volume) of objects. Younger children commonly say that **B'** has more, though they freely admit that when **B'** is poured back into **B'**, the volumes are the same. They tend to be dominated by the looks of things, to center on one dimension (here height) rather than to coordinate demenstions, and to lack a sense of reversibility. How are fruits packaged and displayed in retail markets, by the way? How do advertisers emphasize percept over concept?

Other, more or less informal observations of school age children familiar to parents and teachers are the following:

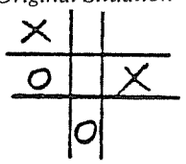
Hold up five fingers. Child counts, "One, two, three, four, five." Hold up the same five fingers and one finger of the other hand and the child starts over again, "one, two, . . ." Five is one issue, six is another.

Young children are often not willing to admit that a person can live in Chicago and in Illinois at the same time, or that a person can be a

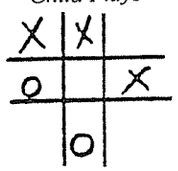
teacher and a mother at the same time.

Playing tick-tack-toe, young children are often incapable of coordinating offense and defense. Further, children are often unable to attend to two alternatives simultaneously. A child with X will often play as shown, ignoring the chance for a double kill.

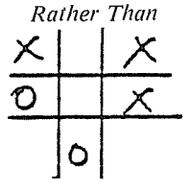
Original Situation



Child Plays



Rather Than



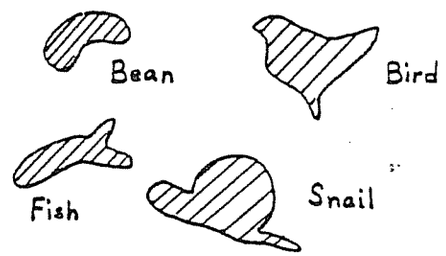
Play "I Am Thinking of a Person", a version of Twenty Questions. At first, children's questions are likely to be unrelated to one another or inconsistent with information already available. In time, children move from atomistic questions, ["Is it Miss Jones?", "Is it Helen Smith?"] to questions involving classification ["Is it a male?"] and which are consistent with previous information.

The period of elementary school coincides with the child's construction of thought operations in which concept triumphs over percept, in which children decenter from their own point of view, and in which coordination of variables and the reversibility of arguments becomes established.

With adolescence, a fourth period begins. This is the period of the **possible.** Here the child becomes increasingly able to deal with all the possible variables and combinations of variables in a given situation ("Here are five differently coloured chemicals; some combination of the chemicals will be colourless. Find it"), are able to construct the all-things-but-one-equal argument of the scientist, and can transcend the immediate here-and-now by creating objects which cannot possibly exist. (In assessing the effects of variables influencing a pendulum, for example -- bob weight, string length, angle of arc, etc. -- the adolescent can effectively create a zero-length kept constant).

It is not the case that thinking stops developing in adolescence. Some researchers are at work on higher levels of development. It **is** the case, however, that many adolescents are firmly rooted in concrete experience or are only at a beginning stage of the period just described. The following problem, adapted here, was fully solved by only 14% of high school students.

Four Island Problem. Pretend that you are going to take a vacation trip to some islands which are new to you. There are four islands in which you are particularly interested: Bean Island, Bird Island, Fish Island and Snail Island.



Your big question is to find out if you can travel between these islands by plane. You ask a travel agent's advice. You first ask if it is possible to travel by plane between Bean and Bird Islands. The travel agent says to you that:

1. People can go by plane between Bean and Fish Islands, but he is not sure if this is a direct route or a route with stopovers.

2. People cannot go by plane between Bird and Snail Islands.

From this information you must figure out if you can go between Fish and Bird Islands. When you ask this, he tells you the two clues again and now adds that he can say for sure the people can go by plane between Bean and Bird Islands, again he does not know whether this is a direct route or a route with stopovers. From this information, figure out if you can go between Fish and Bird Islands. The possible answers are "Yes, for sure", "No, for sure", "I can't tell from the two clues".

Your third and last question is whether you can go between Fish and Snail Islands. Use the information you already have to answer this question. Again, the possible answers are "Yes, for sure", "No, for sure", or, "I can't tell from the three clues".

Memory and Knowledge. An increasingly common view on memory holds that there are several different kinds of memory -- memory in the strict sense (I remember that George Washington was born in 1732.) and memory in the wide sense (I remember that the capacity of

a jar isn't determined merely be the height of the jar.) Additionally, there are different types of retrieval -- recognition ("Didn't I see you at the picnic last week?), recall (Who was the twelfth President of the US?) and reconstruction (31 x 20 is 620 because 30 x 20 is 600 and 31 is one more 20).

Related to these distinctions is the distinction between knowledge in the narrow sense and knowledge in the broad sense. A child can "learn" number facts, for example, by rote -- chant them over and over again -- and the chance is pretty good, in the short term at least, that he can recall them. (Though one of the commonest complaints of teachers is that children return from summer vacation remembering little of what they had learned in June). It's unlikely, however, that he can reconstruct number facts if strict memory fails.

On the other hand, if the child constructs number facts in a relational way (2 + 3 is 5 because 2 + 2 is 4 and 3 is one more than 2, for example) he can reconstruct the 2 + 3 if need be. he can use memory in the wide sense to help him out. Moreover, knowing that 2 + 3 is constructible suggests 1) knowledge is something other than random facts to be stored, 2) it's something one has control over, and 3) old knowledge is capable of being combined to make new knowledge.

Factors Leading to Intellectual Growth. Knowledge in the wide sense is one's personal collection of organized processes which he brings to bear on his environment. How does such knowledge develop? There are four contributing factors -- maturation, experience and activity, social transmission and self-regulation.

Maturation is necessary for development. But maturation alone is clearly not sufficient. Not all eleven year olds (or two year olds or forty year olds, for that matter) are the same in the quality or quantity of their thinking.

Experience is a major factor, especially experience and activity involving manipulation and interaction with objects. (One definition of thinking is "interiorized action"). When a child sorts beads, for example, he is beginning the underpinnings of classifying. When he counts blocks in various configurations, he learns that the number of blocks is independent of a particular configuration.

Social transmission is important as well, especially as it involves give-and-take and genuine exchange of ideas, information and viewpoints. In engaging points of view different from his own, the child chews the steak mentioned earlier and the steak nourishes him.

In general it is self-regulation, where the child -- or the adult -- engages a situation, assimilates it and accommodates his mental networks to it, which is most central to learning. Intellectual development is a form of adaption, and learning in the wide sense is provoked adaptation. It should be clear, by the way, that the match-up (better, moderate mismatch) between the new situation and one's present mental networks is critically important, as is the context in which the new situation occurs. **This** is where teachers and parents play their role, lest children be left to the randoms of everyday life.

Basics. What, then is basic, in the school education of a child? Perhaps it is helpful to distinguish between knowledge in the narrow sense and knowledge in the wide sense and to realize that the two can be mutually supportive. (They can also be mutually indifferent, even destructive).
There are dangers at both extremes. Knowledge in the narrow sense can amount to laundry lists of trivial or atomistic facts or to a surface layer of "right words" with few, if any, underpinnings. And the notion of knowledge in the wide sense can be so airy-fairy that it has no substance, gives no direction to educators, amounts to vague -- though well-intentioned - generalities.

A developmental constructivist would take a wide view of what is basic. He would see self-regulation as basic. That is, children should be prepared to deal with the problems that arise in their world, even though much of their future world is unknown to us at present. Further, they should be self-sustaining. They should not wait for nutriment to come to them; they should be able to find their own nutriment. They should, in general, develop an internal road-map of the world which is coherent and stable and useful and open to new growth.

But this is terribly general. Here's a bit more specificity.

One attempt at redefining basics, perhaps the most ambitious attempt in the history of American education, took place in the curricular revolution of the 1960's. The chief characteristic of the "new curricula" was an emphasis on the structure of the various content areas. This approach, for a variety of reasons -- one being that teachers were rarely encouraged to engage the structure to make it their own, not to say anything of the wide gap between the adult content specialist's idea of

the structure of a discipline and the interests and mental networks of young children and the fact that the "new curricula" reached children largely as knowledge in the narrow sense -- are part of the forgotten past. But structure should not be discarded, lest we have mere bits and tricks.

Robert B. Davis, in a 1967 introduction to the Makison Project, states a position in math education which captures the best of what the "new curricula" tried to do.

> If we devote grade 1 to addition facts up to 10, grade 2 to addition facts up to 100, and so on, we are putting one foot in front of the other, left, right, left, right, . . . This approach is weak in power.
>
> If, instead, we seek those basic mathematical concepts, techniques and attitudes which play important structural roles in the development of the subject, we have a far more powerful approach. . . Cartesian coordinates, introduced (say) at grade 2, give us an ability to relate any arithmetic or algebraic problem to a geometric one, and vice versa. For all the rest of our lives we shall be able to unify algebra and geometry into a single coherent subject. This is **power.**
>
> Once we learn such basic structural concepts as variable, function, mapping and so on, we have a strong structural framework to which **all** of our subsequent mathematical learning can be related. This, again is power! Here we are building cognitive structures that can well serve as foundations for improved structures in the future.

A constructivist is likely to be more concerned with the child's construction of thinking abilities than with the discovery of the structure of content (or with the passive storage of structure, as was often the case). It is suggested that Davis' statement be revised:

If instead, we seek those basic operations, techniques and attitudes which play important and structural roles in the intellectual growth of the child, we have a far more powerful approach. For all the rest of his life the child will be able to continue to build a coherent road-map of reality, one which enables him to construct and test relationships, to infer and to withstand the deductive dominance of percept over concept. This is ***power.*** *Once he learns such basic operations of classifying, ordering, logically multiplying, inferring and so on, he has a strong structural framework from which all of his subsequent knowing can evolve. This, again, is power! Here we are building cognitive structures that can serve as wellsprings of improved and more complex structures in the future.*

In general it is suggested, the goal of education should be to cause intellectual growth to occur, not merely to teach children the facts, rules, procedures, conventions and nomenclature of narrow knowledge.

What are the components of this intellectual growth? Those operations listed above, of course and more generally, an ability to ***organize,*** *to coordinate, to pull into coherence the disparate entities of environment, experience and inference and to regulate between one's internal networks and the demands of external reality. In classroom situations this would call for children to be involved in issues which generate an ability to discern relevant from irrelevant; to distinguish between necessary and plausible and merely possible; to generate, to exhaust, to test and to cancel alternatives; to know, given a situation, what are the necessary conditions, what are the sufficient conditions, and what is the difference between necessary and sufficient; to solve problems and to generate new ones and to inquire and to become self-sustaining in that inquiry.*

There is no space here to be much more specific, save to say that a constructivist approach is by no means at odds with traditional content areas (though it is at odds with the view that knowledge is static stuff), that the approach is widely -- though unofficially -- espoused by American teachers, and that such an approach goes to the heart of what it is that really is basic in the education of children.

What is basic? A summarizing comment:

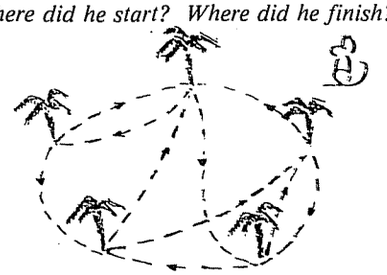
"The main work of the school is surely the fostering and developing of mental life, enabling children to experience more fully and consciously all that life has to offer. This large, overall aim is to be achieved by an infinity of small steps. The material we provide children can seldom be thought of as an end in itself but rather as a means through which effective thinking and feeling are fostered."

APPENDIX

Here are some classroom activities drawn from [or related to] our local work which show a constructivist point of view in action.

1. A monkey went from tree to tree as shown.

Where did he start? Where did he finish?

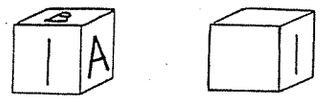


[By Dr. Jerzy Cwirko-Godycki, Warsaw, who used this problem at the Eighth Annual Teachers' Center Residential Course, August, 1979.]

*2. **Number Patterns***

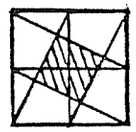
*The subject 'number patterns' also is a rich field for problem solvers. We all know them from psychological tests [What comes next? 1, 3, 5, 7, . . .] In **Solve It** these problems have an extra dimension, caused by the way of presentation [fig. 2], but specially by means of more fundamental questions. This is a good example what mathematics can mean in elementary school.*

3. The right cube is the same as the left cube, but the artist was in a hurry and didn't finish his work. Finish it for him.



*[From T.C. O'Brien, **Solve It**, Chicago: Daigger/ETA, 1977].*

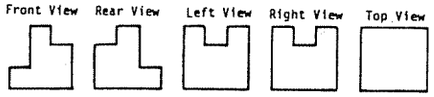
4. Here's a figure drawn on inch-square graph paper. What's the area of the shaded square?



[By David S. Fielker, London, who used this problem at the Seventh Annual Teachers' Center Residential Course, August 1978].

5. One child builds a building from blocks, one block at a time. The building is hidden from a second child. The first child gives a verbal description of his or her actions and the second child's task is to construct an identical building. [The descriptions can also be given in writing].
[From Elizabeth Thomas, East St. Louis, Illinois, included in SEEDBED, Vol. 1, No. 2].

6. These are some nice geometry problems with blocks [fig. 6].



Also on the wiskobas project board attention is given to building blocks. e.g. "de vierkuber" [four cube houses]. On the other hand there are some nice mirror problems [taken from Marion Walter], shadow-geometry, tangrams [the least nice, I think], pegboard and geographical problems. One such geographical problem was a starting point for a lesson during the wiskobas meeting of 1978. Then it was formulated as follows: "Find the direction to Strasbourg" How difficult, but also how motivating such a problem can be -- as well in mathematical-didactical as in organizational sense -- is to be read in the report of this lesson.[3]

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1. Bruner, J. *Beyond the Information Given.* New York: W.W. Norton, 1973. pp. 254-8.

2. Inhelder, B. Memory and Intelligence in the Child. In D. Elkind and J. Flavell (eds.), *Studies in Cognitive Development.* New York: Oxford University Press, 1969. pp. 337-64.

3. Karplus, R. Intellectual Development Beyond Elementary School *School Science and Mathematics,* May 1970, Vol. 70, No. 5, pp. 398-406.

4. Wason, P.C. **et.al.,** *The Psychology of Reasoning: Structure and Content.* Cambridge, Mass: Harvard University Press, 1972.

5. Brearley, M. (ed.), *The Teaching of Young Children.* New York: Schocken Books, 1969, P. 7.

Further Readings for Parents and Teachers

Practical Applications. Here is a brief list of books which teachers have found helpful. Because relatively little work has been done in other areas, they tend to be concerned with mathematics (very broadly defined) and with science.

Brearley, M. (ed.) *The Teaching of Young Children.* New York: Schocken Books, 1969.

Baretta-Lorton, M. *Mathematics Their Way.* Reading, Mass: Addison Wesley Publishing Company, 1976.

Coppele, C., Sigel, I. and Saunders, R. *Educating the Young Thinker.* New York: D. Van Nostrand Company, 1980.

Smock, c. *Mathemagenics Activity Program.* Athens, Ga.: Mathemagenics Activity Program.

Stanfield, J. Maths Adventure. Chicago: Daigger/ETA, 1973.

Background and Research. Here is a brief list of books which teachers have found useful in explaining a constructivist point of view and in detailing the developmental research.

Elkind, d. and Flavell, J. (eds.) *Studies in Cognitive Development.* New York: Oxford University Press, 1969.

Furth, H. *Piaget for Teachers.* Englewood Cliffs: Prentice Hall, Inc., 1970.

Ginsburg, H. *Children's Arithmetic.* New York: D. Van Nostrang Company, 1977.

Sigel, I. and Cocking R. *Cognitive Development from Childhood to Adolescence: A Constructivist Perspective.* New York: Holt, Rinehart and Winston, 1977.



John F. McCartney Scholarship, 1980

Final Selection

Lai Lane Luey
3 Portsmith Rd.,
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York Mills Collegiate Inst.
North York Bd.

W.J. Andrew Fyfe,
24 Strathmore Rd.
St. Catharines, Ont. L2T 2C5
Sir Winston Churchill S.S.
Lincoln Co. Bd.

Bertrand Lo
400 Bloor St. E. No. 95
Mississauga, Ont.
Peel Bd.
Thomas L. Kennedy S.S.

The Winner:
Lai-Lane Luey
Her Teacher: James Corbett

Congratulatons to Miss Luey and to the large number of applicants (116). Such participation can only strengthen the growing appreciation of Math Education in Ontario.

P.S.
Details of the history and procedures for applicants will appear in the December issue.

THE SOCIETY OF DECARTES, 1980

Medalists Are:
J. Bruce White, Windosr Kennedy Collegiate
Walter A. Bany, Brampton Mayfield S.S.



I DIDN'T KNOW THAT

TRICKS — GAMES — PUZZLES

Dr. H. Laurence Ridge, Faculty of Education, University of Toronto

With this issue focusing on special education, the column will be devoted to a type of puzzle participation appropriate for gifted and talented students. This is an example of creative problem solving in which a variety of solutions is possible as well as a neat generalization.

The Sable Island Treasure Hunters had an old yellowed map that gave the following instructions:

"On the island, go to the gallows. From there, walk to the pine tree, measuring the distance. At the pine tree, turn right and walk that same distance to A.

Again from the gallows walk to the oak tree, measuring the distance. At the oak turn left and walk that same distance to B. Join AB. The treasure is at the midpoint of AB.

When the treasure hunters arrive at the island, the trees are still there but the gallows is gone. How could they find the treasure?

This version of the puzzle appeared in the February 1975 issue of **Abacus** (submitted by John Egsgard). More recently it was featured in the August 1979 **Scientific American** (Martin Gardner's column "Mathematical Games") and as problem 400 in the February 1980 **Crux Mathematicorum.**

Initial Investigation

At the outset it helps to find out just how the location of the treasure, T, is related to P and K, the positions of the pine and oak trees, respectively. See how different locations of the gallows, G, effect the location of the treasure, T.

Figures 1 and 2 below show the location of T to be (about) the same for two widely separated locations of the gallows, G1 and G2.

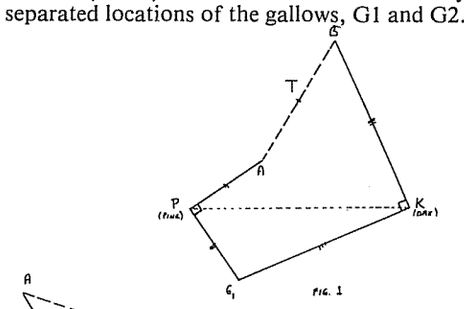


FIG. 1

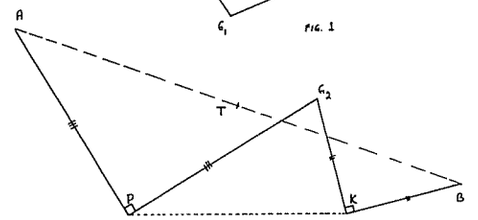


FIG. 2

The Polya technique of investigating extreme cases suggests taking P or K as the starting point (G). Both of these cases show T to be the centre (or centroid) of the square on PK to the left of directed line segment PK (as shown in figures 3 and 4 below).

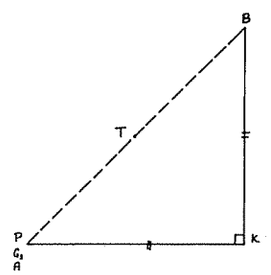


FIG. 3

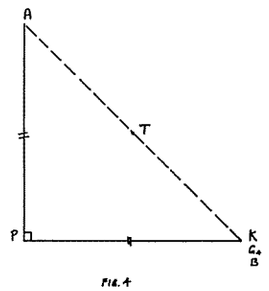


FIG. 4

From this evidence it **appears** that T is always the centroid of this square. The solution of the puzzle would then be:

> Start anywhere and follow the same instructions as if you were starting at the gallows OR start at the pine tree, go half the distance to the oak tree, turn left, proceed the same 'half distance', then dig.

Now all that is left to be done is prove that the position of T is indeed invariant. This is where the variety of solutions applies.

I. BY ANALYTIC AND SYNTHETIC GEOMETRY

Locate points P, K, and G on a coordinate plane (as shown in Fig. 5). To find the coordinates of A and B (and hence, T) draw GG', AA', and BB' perpendicular to the x-axis.

Since $\Delta BB'K \cong \Delta KG'G$ by synthetic geometry (angle-side-angle)

S = DB' = PK - B'K = PK - G'G (by congruency) = K - |y|

t = B'B = G'K (by congruency) = K - x

∴ B has coordinates (K - |y|, k - x)

Similarly, using ΔAPA' and ΔPGG'

u = |y| v = x

∴ A has coordinates (|y|, x).

T is the midpoint of AB and has coordinates

$$\left(\frac{(k - |y|) + |y|}{2}, \frac{(k - x) + x}{2}\right) = \left(\frac{k}{2}, \frac{k}{2}\right) \text{ Q.E.D.}$$

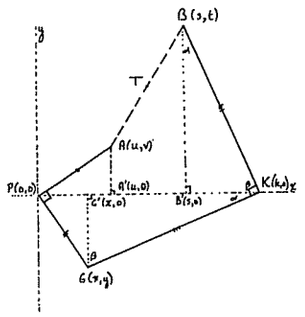


FIG. 5

II. BY VECTORS

As shown in Fig. 6

$\vec{PT} = \frac{1}{2}(\vec{PA} + \vec{PB})$

$\vec{PA} = [-y,x]$ since $\vec{PA} \cdot \vec{GP} = 0$

$\vec{PB} = \vec{PK} + \vec{KB}$

$= [k,0] + [y, k-x]$

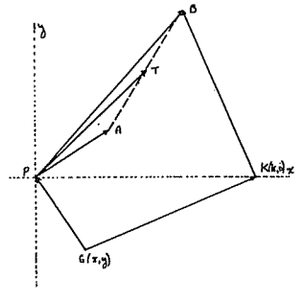


FIG. 6

$= [k + y,\ k - x]$

$\vec{PT} = \frac{1}{2}([-y,x] + [k + y,\ k - x])$

$= \frac{1}{2}[k,k]$

$= [\frac{k}{2}, \frac{k}{2}]$ Q.E.D.

III. BY TRANSFORMATIONS

As shown in Fig. 7

A is the rotation image of G under a rotation of $\frac{\pi}{2}$ about P. For the coordinates of A

$$\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} -y \\ x \end{pmatrix}$$

$\therefore$ A has coordintes $(-y,x)$.

B is the image of G under a rotation of $-\frac{\pi}{2}$ about K. B', P, G' are the $[-k,0]$ translation images of B,K,G respectively. Then B' is the image of G' under a rotation of $-\frac{\pi}{2}$ about P.

For the coordinates of B'

$$\begin{pmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{pmatrix} \begin{pmatrix} x - k \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x - k \\ y \end{pmatrix}$$

$$= \begin{pmatrix} y \\ k - x \end{pmatrix}$$

B is the $[k,0]$ translation image of B'.

$\therefore$ B has coordinates $(k + y,\ k - x)$.

T is found as in solution I or II.

IV. BY COMPLEX VARIABLES

Locate P,K,G on an argand plane as shown in Fig. 8.

$\vec{GP} = \vec{GO} + \vec{OP}$	$\vec{GK} = \vec{GD} + \vec{OK}$
$= -z - 1$	$= -z + 1$
$\vec{PA} = (-i)(-z - 1)^*$	$\vec{KB} = i(-z + 1)$
$\vec{OA} = \vec{OP} + \vec{PA}$	$\vec{OB} = \vec{DK} + \vec{KB}$
$= -1 - i(-z - 1)$	$= 1 + i(-z + 1)$
$= -1 + iz + i$	$= 1 - iz + i$

$\vec{OT} = \frac{1}{2}(\vec{OA} + \vec{OB})$

$= \frac{1}{2}(-1 + iz + i + 1 - iz + i)$

$= i$ Q.E.D.

GENERALIZATION

Rather than turning left and right through $\frac{\pi}{2}$ at the Oak and Pine respectively and proceeding the same distances, turn left

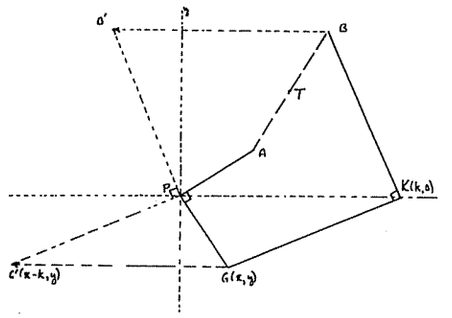


FIG. 7

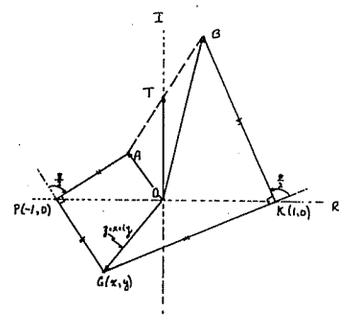


FIG. 8

*Multiplication by i rotates through $\frac{\pi}{2}$ (counterclockwise). (Note that a similar coordinate system could have been used for solutions I, II, and III.)

through an angle θ and right through π - θ (as shown in Fig. 9) and proceed m times as far as the respective distances from the gallows.

Following the pattern in solution IV

$\vec{PA} = me^{-i(\pi-\theta)}(-z-1)$ | $\vec{KB} = me^{i\theta}(-z+1)$**

$\vec{DA} = -1 - me^{-i(\pi-\theta)}(z+1)$ | $\vec{DB} = 1 + me^{i\theta}(-z+1)$

$= -1 + me^{i\theta}(z+1)$

since $e^{-i\pi} = -1$

$\vec{DT} = me^{i\theta}$

The solution to the generalized puzzel would be:

Start at the pine, go half the distance to the oak, turn left through an angle θ and proceed m times the 'half distance', then dig.

Happy treasure Hunting!

Recreationally yours,

H. L. R.

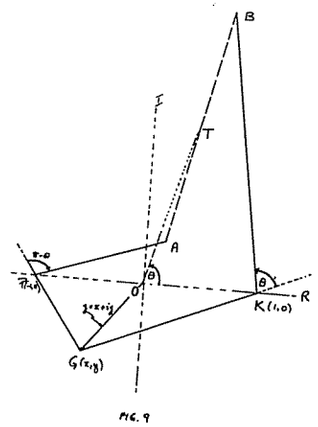


FIG. 9

(m = 1.5, θ = $\frac{5\pi}{12}$)

** Multiplication by $e^{i\theta}$ rotates through an angle of θ (counterclockwise).

BOOK REVIEWS



G. Vervoort, August 1980

After promising our guest editor for this issue to review some books on learning disabilities in mathematics and their remediation I find myself ill at ease about the task ahead. The discomfort is based not on the fact of having no more experience diagnosing or remediating learning difficulties than the typical classroom teacher. The texts reviewed here do not require a background in special education. No, rather the feeling of unease caused by the words itself, **diagnosis** and **remediation,** words which conjure up images of broken bones set and wounds healed so the sun shines again.

Hidden deep, deep inside me there is skepticism about what good the diagnosis will accomplish and what hope there really is for remediation. I do not refer here to such things as diagnostic teacher where the teacher is on constant alert to remove misunderstandings before the accumulated debris blocks the pass. I am thinking of the typical student with learning difficulties at the time he is finally sent to the special education teacher.

Tests and diagnostic charts may reveal for instance that the student never fully grasped the concept of regrouping in subtraction. Does it follow therefore that teaching regrouping will solve the problem? Or will other troubles be waiting in the wings? Will not the same deficiencies that kept the student from understanding regrouping - while his classmates experienced no difficulties - again interfere in the learning of subsequent material? Is there really a cure except for the child who was absent while a crucial lesson was taught? I keep hoping.

Meanwhile the recommended action is being on the alert for a lack of understanding or skills in our classrooms and provide all necessary medication on the spot.

Diagnosing Mathematical Difficulties Robert G. Underhill, A. Edward Uprichard, James W. Heddens. Columbus: Charles E. Merrill Publishing Company. 1980 ISBN 0-675-08195-5.

This book is written as a text for undergraduate and graduate courses to provide a compendium of theoretical and practical models for diagnosing mathematical difficulties. In contrast with the text reviewed below the authors shy away from sweeping generalizations and carefully refer to 'difficulties' rather than disabilities.

Teachers benefit from individual and group diagnostic skills, interview skills and knowledge of available commercial tests. This book addresses all these needs. It also includes a summary of the current research perspectives. The appendices include such useful items as a hierarchy list for multiplication and division, a checklist of mathematical concepts, a mathematical taxonomy, a sample probe task and a sample case study for remediation. The book will prove to be useful not only in college courses but for all teachers. In particular it should be helpful for people involved in writing programs and program loops for Computer Assisted Instruction.

Arithmetic and Learning Disabilites. Guidelines for Identifcation and Remediation. Stanley W. Johnson. Boston: Allyn and Bacon, Inc. 1979 ISBN 0-205-06444-2 $20.95 Can.; ISBN 0-205-06504-X pbk. $10.95 Can.

According to the introduction this book is written for " . . . those who yearn for the key which might unlock the door for children who stand outside, still waiting to know normal learning experience." I fear this book will likely leave teachers and children both standing outside in the cold a while longer. Stripped of all the veneer the author's approach towards learning disabilities in mathematics can be summarized as "find out precisely what it is they can't do, and then make them do it." A sample from the text will illustrate the point.

Curriculum Area: Number Recognition
Learning Disability Type: Spatial Orientation

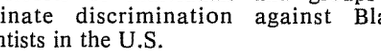
The text then describes in turn a) Problem free behaviour, b) Sample task where deficit child may display difficulty, c) Do it yourself diagnostic activities, d) Non-arithmetic situations where similar behaviours are required, e) Remedial objectives for confirmed problems. All this to conclude that the child at times writes his numerals sideways rather than right side up. Finally all of this is followed by the section Sample remedial activity.

The activities should focus on the child's understanding of spatial concepts. Only with full awareness of what each spatial concept means in terms of real-life situations will the child ever be able to efficiently use the concepts when needed. These opportunities to demonstrate acquired concepts must be presented so that the amount of understanding present does not have to be inferred. Exercises may need to be repeated several times to provide the child with sufficient time to thoroughly learn the spatial concepts involved.

The sample remedial activity requires the teacher to direct the child to circle or indicate all the numbers which are standing up.



This reader ended up feeling he was in a merry-go-round. The diagnostic activity is not always that sharp either. On p. 127 (Counting; Spatial Orientation) it is recommended the teacher poses this problem:

> Ask the child how many of the numbers are standing up.
>
> 

What valid conclusions with respect to spatial orientation does the teacher draw if the child answers 'Three'?

Draw your own conclusion. Mine is that the intelligent sensitive teacher does not need this book. And teachers who lack either of these qualifications ought not to be dealing with learning disabled children regardless how many special education courses they took.

"Black Mathematicians & Their Works"
edited by Virginia K. Newell et al., published by Dorrance & Company - $12.50

This interesting and unique reference work documents the contributions of Black Scholars in the field of Mathematics. It combines reprints of research articles with a comprehensive biographical index, giving data on Black Mathematicians and mathematics educators.

The first Black to be recognized as a mathematician (and astronomer) was Benjamin Banneher, born in 1731; largely self taught, he assisted in the planning of Washington, D.C. The first Black to receive a Ph. D. degree in Pure Mathematics was Elbert F. Cox (Cornell 1925), while it was not until 1949 that Black women received Ph. D.'s - Evelyn Boyd Glanville and Marjorie Lee Browne.

All of the mathematical research papers are beyond the scope of high school students and most high school Math teachers, but the appendices provide an interesting account of the efforts of individuals and groups to eliminate discrimination against Black Scientists in the U.S.

Perhaps this book should be on the library shelves simply to make the point that there are black mathematicians - although for $12.50 (U.S.), it would be a relatively expensive point!

David M. Handley

ONE PROBLEM FROM CRUX MATHEMATICORUM

462. Propose par Hippolyte Charles, Waterloo, Quebec.

Soient A, B, C les angles d'un triangle, Montrer que:

$$\begin{vmatrix} \tan A/2 & \cos A & 1 \\ \tan B/2 & \cos B & 1 \\ \tan C/2 & \cos C & 1 \end{vmatrix} = 0$$

Prove the above identity when A, B, C are the angles of a triangle.

The solution may be found in issue #5 of volume 6 (May 1980). Back issues cost $1, and bound volumes $10. Write to CRUX MATHEMATICORUM, Algonquin College, 200 Lees Avenue, Ottawa, K1S 0C5.

SOLUTIONS TO PROBLEM SET 19.1

19.1.1: The interior square has side $\sqrt{6}$, the exterior square has side x. Using one of the triangles $2x^2 = (\sqrt{6})^2$

$x = \sqrt{3}$.

Then Area (exterior square) $= (2\sqrt{3})^2$ cm^2

$= 12$ cm^2.

19.1.2

	Income	Spending	Saving
George	3x	2x	x
Tim	4x	3x	x

Ration of savings is 1:1.

19.1.3 1234567890 x 1234567892 = (1234567891 - 1) (1234567891 + 1)

$= 1234567891^2 - 1$

Denominator becomes 1 and answer is 1234567890.

19.1.4 (1 x 1 + 5) (1 x 1 - 2) = 0

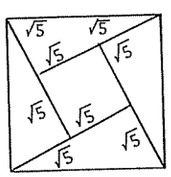
1 x 1 ≠ - 5, 1 x 1 = 2

x = · 2

19.1.5 Each litre covers 72/9 = 8 sq. ft. Mix 2¼ litres (ie 18 sq. ft.) of yellow with a similar amount of blue to make 4½ litres (ie 36 sq. ft.) of green. This leaves just enough blue for the innermost ring. The position of the other colours follow.

19.1.6: Dividing the first equation by the second, $4^x = 2$ and $x = \frac{1}{2}$. Then $2^x = 5y \Rightarrow \sqrt{2} = 5y$ and $y = \frac{\sqrt{2}}{5}$. Thus the the solution is $\left(\frac{1}{2}, \frac{\sqrt{2}}{5}\right)$.

19.1.7:



19.1.8 P(loaded face) = 1/5, P(given unloaded face) = 4/25, P(2 loaded faces together) = 1/25, P(2 unloaded faces adding to 7) = 16/625. There are 5 combinations of 2 unloaded faces adding to 7. - P(2 faces add to 7) = 1/25 + 5 (16/625) = 21/125. P(2 faces add to 7 with normal dices) = 1/6. P(2 faces add to 7) is increased by 21/125 - 1/6 = 1/750.

19.1.9 Let 0 = (0,0), A = (-A,0), B = (A,0) and M = (x,y).

Slope of AM = $\frac{y}{x1 + a}$ and since CB 1 AM, slope of CB = $\frac{(x + a)}{y}$.

This equation of CB is $y = -\frac{[x + a)}{y}(x - a)$.

Slope of MB = y and since AD1 MB, slope of AD = a - x.

Thus equation of AD is $y = \frac{(a - x)}{y}(x + a)$

E is the intersection of CB and AD and so solving

$-\frac{(x + a)}{y}(x - a) = \frac{(a - x)}{y}(x + a)$

we get x = x

Thus E = (x , -) and hence LAFM = 90 degrees. (MF is a vertical line and AF is a horizontal line).

19.1.10: Since the number is divisible by 2, 3 and 5 it must have the form $N = 2^a 3^b 5^c$. Now for half the number to be a square $N = 2^{2k+1} 3^{2\ell} 5^{2m}$. For a third of the number to be a cube $N = 2^{3x} 3^{3y+1} 5^{3z}$. For a fifth of the number to be a fifth power $N = 2^{5p} 3^{5q} 5^{5r+1}$. The smallest values satisfying all these conditions are a = 15, b = 10, c = 6. $\therefore 2^{15} 3^{10} 5^6 = 30233088000000$ is the smallest such number.

TWO APPLICATIONS OF THE PIGEONHOLE PRINCIPLE

R. Honsberger, Faculty of Mathematics, The University of Waterloo

Editor's Note: R. Honsberger used some of this material when he filled in for M. Gardiner in his column in **Scientific American** recently.

The simple-minded claim - if 5 shirts are distributed over 4 drawers, then some drawer will wind up with at least 2 shirts - exemplifies one of the most powerful and frequently used notions in mathematics. It is an example of the socalled pigeonhole principle, which, in slightly more general form, may be stated as follows:

if km + 1 objects are distributed among n boxes, some box will contain at least k + 1 of the objects.

In a still more general form, it makes the obvious claim that not every case in an investigation can be below average and not every case can be above average:

if km + 1 objects are distributed among n boxes, $m \geq 2$, then some box must contain at least k + 1 objects and some box contain not more than k objects.

The great usefulness of this commonplace idea is hinted at in the following two applications.

(i) Problem 1

Observe in the following arrangement of the integers from 1 to 10 that there is an increasing subsequence of length four from left to right (any of 1, 2, 7, 10; 1, 2, 6, 9; 1, 2, 7, 9; etc. will do).

4 1 8 2 7 6 10 9 3 5

Prove that, no matter how the integers from 1 to 10 might be arranged in a row, there will always occur either an increasing subsequence of length at least 4 or a decreasing subsequence of length at least 4.

Solution

For each integer k in the row, let us determine the "coordinates" (a, b), where a = the length of the longest increasing subsequence that begins at k and b = the length of the longest decreasing subsequence that begins at k (counting k itself in each case as the first member of the subsequence). In the given example, the coordinates for the first integer, namely the 4, are (3, 2) and those of the 1 are (4, 1).

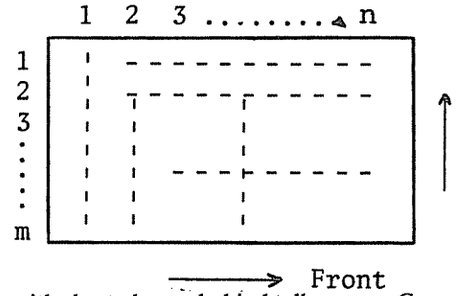
Now, if any of the coordinates is as great as 4, the conclusion follows. Suppose, then, that none of the 10 pairs (a, b) contains a number exceeding 3. This means that each pair (a, b) is one of the 9 pairs it is possible to form by using only the numbers 1, 2, and 3. By the pigeonhole principle, some two of the 10 pairs (a, b) must be the same. Suppose for definiteness that the coordinates (a, b) determined for the integer k occur again for the integer m farther down the row.

row: . . . . . k . . . . . m . . . . . .
(a, b) (a, b)

Clearly k and m are not equal; and if k<m, then the first coordinate of m would be at least a 1, not just a, and if k>m, then the second coordinate of k would be at least b + 1, not just b. These contradictions show that some a or b must be at least 4, and the solution is complete. (Of course, the property can be generalized to a row of $m^2 + 1$ integers with a subsequence of length n).

(ii) **Problem 2**

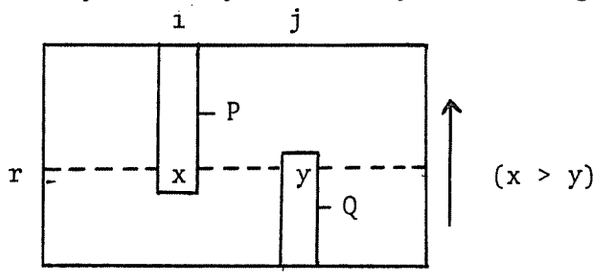
Suppose a marching band, consisting of mn musicians in m rows and n columns, has the members in each row arranged from left to right in non-decreasing order of height (that is, they tend to get taller as you go along the rows to the right). As far as the columns are concerned, the heights



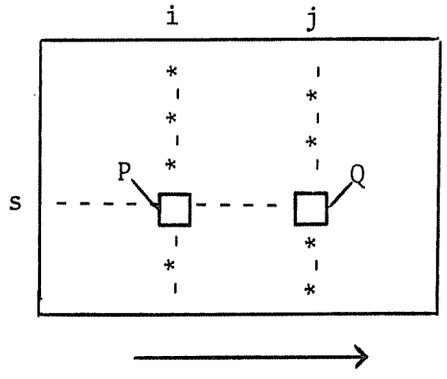
might still be all jumbled up, with short players behind taller ones. Consequently, suppose that the people in each column are similarly arranged from front to back in non-decreasing order of height (putting the shorter ones nearer the front). Of course, it is only to be expected that this will foul up the non-decreasing arrangements of the rows. Prove that this is, in fact, not the case, but that after doing the columns, the rows are still in non-decreasing order from left to right.

Solution

We proceed indirectly. Suppose that in some row r, we have a taller person x to the left of a person y. Suppose x is in column i and y is in column j. Since we have just finished doing the columns, we know that everybody behind x in the segment P of column i is at least as tall as x, and that everybody ahead of y in the segment Q of column j is no taller than y. Since x is taller than y, we have everybody in P taller than everybody in Q.



Now let us reverse the shuffling procedure to the halfway point where the rows have been done but not the columns; in other words, let us unshuffle the columns. In doint so, the players inP wind up



Halfway:

rows done, columns not.

being sprinkled who-knows-where over column i and those in Q to various places in column j. We can think of the m rows as boxes (1, 2, . . . m) into which the players of P and Q are slotted (while remaining in their proper columns, of course). But the number of people in P and Q is clearly m + 1 (together these segments contain enough people to fill one column and have exactly one person left over). By the pigeonhole principle, then, some two of them must wind up in the same row s. Clearly these two could not both come from P or both from Q. Therefore, in some row S, we have in column i a taller player from P ahead of a shorter person from Q in column j, in contradiction of the non-decreasing arrangement of the rows. The conclusion follows.

PLUS + + +
AN INFORMATION COLUMN



Ed Barbeau

80.5 **Pre-College Statistics for the '80's**

John Del Grande, mathematics co-ordinator for North York, organized a committee of Ontario educators in order to develop guidelines and materials in statistics which could be used experimentally in Canadian secondary schools. Financed by the Samual Beatty Fund, the group of 13 has produced two publications:

(a) Pre-College Statistics for the 80's (Skeleton K-13 curriculum and topics for Grades 9-12).

(b) Statistics for Grade 13 (author Neil Walsh, North York) These are copyrighted, but, while not available for commercial use, may be reproduced for experimental classroom use. For further information, write to Mr. J.J. Del Grande, Mathematics Co-ordinator, North York Board of Education, 5050 Yonge Street, Willowdale, Ont. M2N 5N8

80.6 **Call for Competition Information**

The Canadian Mathematical Society Education Committee is anxious to compile a list of all school mathematical competitions held in Canada, information which it hopes to make available across the country. Please send to Prof. E.R. Williams, Department of Mathematics, Memorial University, St. John's, Nfld. AlC 5S7 the following information: (a) name of competition, (b) sponsor and local organizer, (c) region of competition (d) regulations. Include, if possible, recent question papers.

80.7 **1001 Problems, for High School Students**

The Canadian Mathematical Society has just published the fourth book in its series of problems for high school students. This volume contains 100 problems, prepared by E. Barbeau, M. Klamkin and W. Moser and can be obtained for $2.50 each from the Society's office at 577 King Edward, Ottawa, Ont. K1N 6N5.

80.8 **The role of applications in the Undergraduate Mathematics Curriculum**

An ad hoc committee on applied mathematics training chaired by Prof. Peter Hilton has come up with recommendations for a change in thrust to more applied experiences in university programmes. The discussion of "themes" - attitudes, unification of the mathematical sciences, curriculum, applications -- experience programs, societal aspects - makes this a useful handbook for school teachers concerned with curriculum. The 26 page booklet, entitled as in the heading, can be obtained from the Office of Mathematical Sciences, U.S. National Research Council, 2101 Constitution Avenue, Washington, D.C. 20418.

80.9 **The Samuel Beatty Essay Contest**

Two students were awarded first position in the 1980 Beatty Essay contest:

John Chew, Grade 12, University of Toronto Schools
"Limits to Area in the Plane",
Kevin Jardine, Grade 12, West Humber, C.I., Etobicoke, Ont.
"Surface Theory".

Each will receive $100. A third prize of $35 goes to Steve Meszaros, Grade 12, Blakelock H.S., Oakville, Ont.

"Newton vs. Einstein: Comparing the Newtonian
Universe and the Universe of Special Relativity".

This contest, sponsored by the Samuel Beatty Fund, is open to all Canadian school students. Information for the 1981 contest will be available in October from E.J. Barbeau, B201, University College, University of Toronto, Toronto M5S 1A1.



PRESIDENT'S MESSAGE

by Lorna J. Morrow

As the new school year begins, and the work starts again in earnest, the activities of OAME/AOEM also pick up -- via Executive and Council Meetings, Committee work, and Chapter Meetings and Conferences. All of these have been and will continue to be successful only because of the time and energy devoted to them by our many hard-working members.

Before looking ahead to future activities, I would like to look back briefly to thank three of these people for their efforts over the past year. The Annual Conference at the Prince Hotel, May 9 to 11, was a great success primarily due to the two co-chairmen, Shirley McIntyre and Paul Sherk. The able leadership given the organization over the last year by Alex Norrie as President, also deserves a special mention.

For the future, my greatest concern, as it should be for all members, is the financial state of OAME/AOEM. As a volunteer organization our income depends primarily on our fee structure. While fees have been increased slightly over the last few years, expenses seem to be increasing more rapidly. This year will thus be one of re-examing expenditures and practices that have become almost traditional -- for example, the size of Council, the number of Council meetings, the size and number of our publications -- with an attempt to pare costs where possible without affecting quality.

This is not to say that we can expect great changes in the organization immediately but we must be prepared to examine and perhaps alter some of our established practices over the next few years.

Because I find this outlook a rather gloomy one, I'd also like to encourage the members to work towards a more cheerful goal as well. I think we are becoming an organization with something to offer the mathematics teachers of this province, and there are two special groups toward whom I would like to direct our work in the coming year.

There are thousands of elementary school teachers out there looking for help in teaching Math, looking for new ideas, looking for support. We can help via our Annual Conferences, Chapter Conferences, and greater emphasis on things-to-do-tomorroe in the publications. I'd like the elementary teachers of the province to know that there is such an organization as OAME/AOEM to help and support them. You as individual members can help by telling your fellow staff members, talking to your feeder schools, sharing your copies of the Gazette and Abacus, and encouraging membership in the local chapter organizations.

Il y a beacoup de professeurs de classes francais dans cette province. Je veux vous dire que l'association fera de son possible pour vour aider. J'espere que l'AOEM sera vraiment une association bilangue pendant les annees a venir.

I'm looking forward to a good year and hope you are, too. May your classes be small, supplies be plentiful, principals be cooperative, superintendents be supportive, and may you all live happily ever after.

O.A.M.E./A.O.E.M. COUNCIL 1980-81

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D. Alexander, J. Routledge, D. McPhail, L. Morrow, G. Knill

CHAPTER AFFILIATIONS

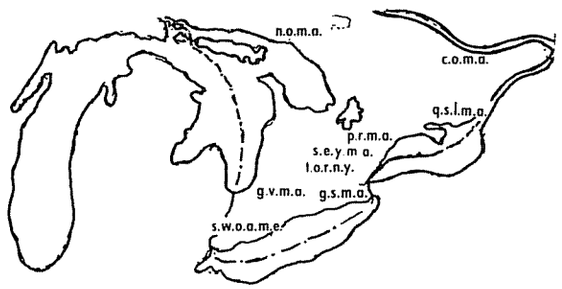
C.O.M.A.	-Carleton Ottawa Math Assoc.	Armstrong, Bruce
G.S.M.A.	-Golden Section Math Assoc.	D'Angelo, Julian
G.V.M.A.	-Grand Valley Math Assoc.	Robinson, Kenneth
N.O.M.A.	-Northern Ontario Math Assoc.	Sandblom, Michael
P.R.M.A.	-Pine Ridge Math Assoc.	La France, Paul
Q.S.L.M.A.	-Quinte St. Lawrence Math Assoc.	Harvey, Elaine
S.E.Y.M.A.	-Scarborough East York Math Assoc.	Sherk, Paul
S.W.O.A.M.E.	-South Western Ontario Assoc. for Math Education	MacEwan, Neil
T.O.R.N.Y.	-Toronto North York Math Assoc.	

Fourth International Congress on Mathematical Education
Berkeley, California U.S.A.
10-16 August 1980



Editor's Note: It would be nice to have what was "cooking" at the conference. Please write and quench our appetite for news.

CHAPTER ITEMS



The Carleton-Ottawa Mathematics Association held their annual meeting and program at the University of Ottawa on April 17, 1980. Surrounded by a superb view of Ottawa, outstanding hot and cold snacks and the comfortable lounge area atop the Colonel By Engineering building, the meeting helped launch the new cooperative education program in Mathematics and Computer Science being offered by the University of Ottawa. The program included presentations by Mr. Charles Monk, one of the most inspired speakers to ever look back on a career in education from the vantage point of retirement, and Dr. J. Oren, the Chairman of the Computing Science Department at the University of Ottawa who spoke on computerized simulation.

At the meeting special tribute was paid to two distinguished members of COMA, Dr. Duff Butterill and Mlle. Roz Seguin, both retiring after lengthy and rewarding careers in education.

COMA continues very active work in the following areas:
Crux Mathematicorium
Mathematics Examining Board
Charitable status
Program
New constitution

COMA and Algonquin College sponsored a voluntary grade 12 exam in mathematics this spring, written by students entering mathematics dependent careers. Approximately 350 students participated in this exam, all of whom will receive information concerning their preparedness for future study in mathematics. This pilot project is expected to lead into an Examining Board sponsored grade 13 voluntary exam for the spring of 1981.

The membership of COMA is now in the process of voting by ballot concerning COMA's possible registration as a charity. This request is also being studied by the Department of National Revenue.

The new executive for 1980 - 81 has been selected and congratulations are in order to the new chairman, Peter Ullrich.

GOLDEN SECTION REPORT

June 1979 - general meeting. It was decided that the chapter would function as two subdivisions, one centered at St. Catharines and the other at Hamilton. Paul Lessard and Carl Riehm were elected Directors for the Hamilton subdivision and Prof. Muller and Ray Marshall for the St. Catharines subdivision.

Oct. 2, 1979 - An evening of workshops at Brock University. There were 9 sessions available and approximately 90 attended.

Nov. 9, 1979 - Mathematics mini conference at McMaster Univesity. There were four workshops available and approximately 90 attended.

Although other workshops had been planned for June 1980, they were cancelled due to the large

number of conferences and related experiences available around this time period.

Oct. 3, 1980 - Tentative plans for another series of workshops at McMaster University.

PINE RIDGE MATHEMATICS ASSOCIATION

PRMA is planning a dinner meeting/workshop for early November.

Bill Jeffers, head of Mathematics at Central Collegiate in Oshawa, who has been a board consultant in Mathematics, workshop leader, chairman of the OAME convention held at Trent University a year ago, and former representative for PRMA is leaving the teaching profession for the position of Technical Supervisor, Nuclear Division, of Ontario Hydro.

QUINTE-ST. LAWRENCE

The Quinte-St. Lawrence Mathematics Association held its annual meeting on April 23, 1980, at Queen Elizabeth C.V.I., Kingston.

The before-supper sessions featured David Cosh who led a session on the "Rotagram", a simple instrument developed by the DIME project at the University of Stirling, Great Britain; Ron Ripley, who led a workshop on Fractions and Decimals; Norman Rice who answered the question, "How Do Pocket Calculators Calculate?" and Hugh Allen who led a "hands-on" session on Flow Charts.

The evening sessions included one by Paul Sherk on trigonometry entitled "Where Do All Those Names Come From?" and another by David Handley on math history topics such as trisecting an angle and amicable numbers.

Hugh Allen was recognized at the meeting for his outstanding contribution to QSLMA since its inception, having served continuously on the executive in various capacities until now.

The newly elected executive is as follows:

President - David Borrowman, Syndenham H.S.
Vice-President - Norah Wilson, Rideau D.H.S.
Secretary-Treasurer - Peter Battey-Pratt, Ernestown H.S.
Area Representatives - Peter Roulston, Western Area
Eric Stulberg, Central Area
John Nightingale, Eastern Area
Alan Olsen, Northern Area

Dr. John Coleman took a few minutes at the end of the meeting to speak with some of the secondary mathematics people regarding a meeting of mathematics and physics professors of both the University of Toronto and Queen's University, and recognized that this was the last QSLMA meeting which he would attend in the capacity of head of the Mathematics Department of Queen's University as he is stepping down from that position at the end of this school year. The encouragement and support that he has offered to QSLMA was recognized and appreciation for his efforts was expressed.

SEYMA

During 1979/80 a fall conference was held, catering to levels K-8. The speakers were very good, but

the conference was not well attended. SEYMA also hosted the OAME/AOEM 7th annual conference at the Prince hotel in Toronto. There is a concern over the declining interest shown in SEYMA. A meeting is scheduled for September, 1980 to consider the future of the chapter.

FUTURE MEETINGS

Local
O.A.M.E./A.O.E.M.
Executive - Sept. 12/80
Council - Oct. 18/80
Executive - Nov. 14/80
Council - Feb. 14/80
Executive - April 3/80

International
O.A.M.E./A.O.E.M. Leadership
Oct. 22-25; 1980 Toronto
N.C.T.M. Oct. 23-25; 1980 Regina
O.A.M.E./A.O.E.M. Annual May 1981 Sarnia
N.C.T.M. April 22-23, 1981 St. Louis
N.C.T.M. April 14-17, 1982 Toronto