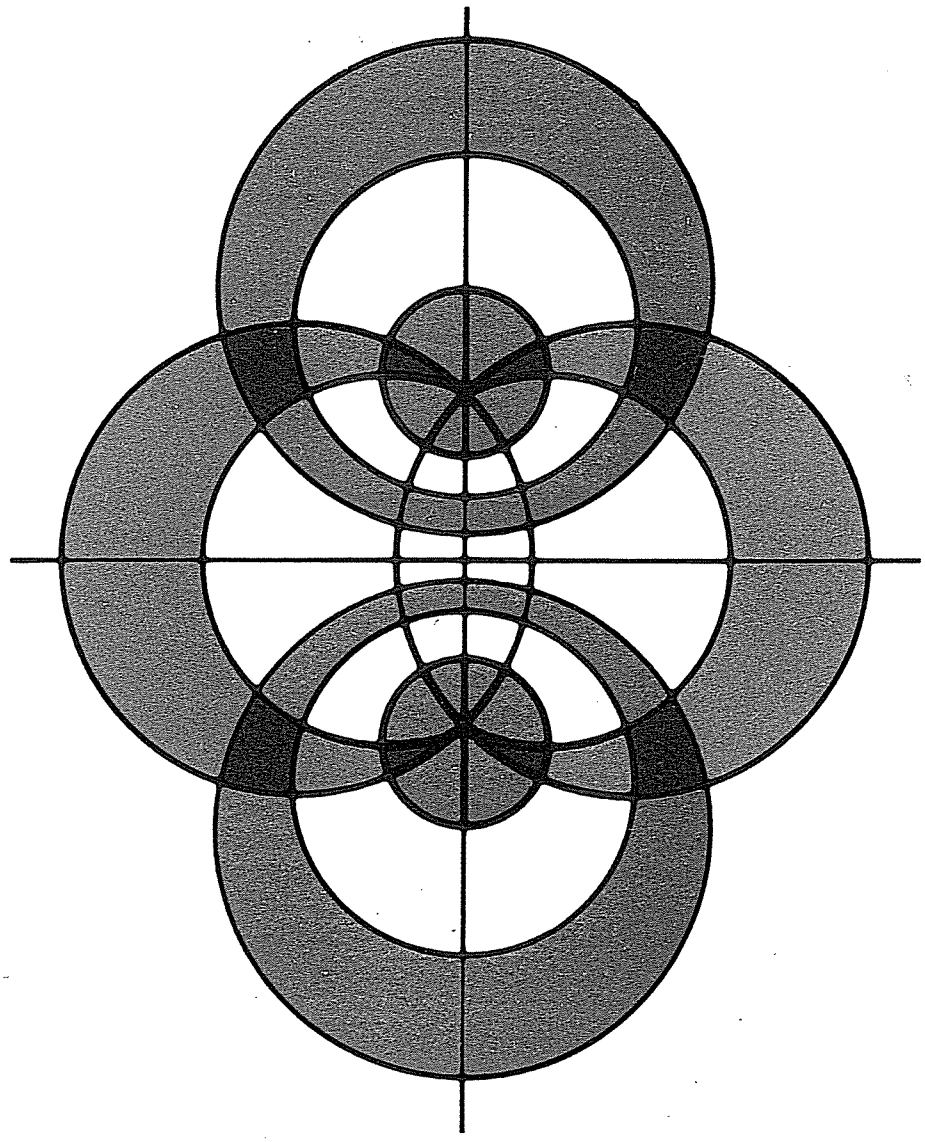
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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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PRESIDENT'S MESSAGE

The 1981-82 school year should prove to be an exciting one, and a busy one. Many of the issues we will have to deal with this year will have a significant effect on mathematics education.

The recommendations of the Secondary Education Review Project will have a real impact on education. The OAME's submission to SERP appears in this issue of *The Gazette.* It is not as detailed as we all would have liked, but we were bound by a very short time line, as were all other groups.

The NCTM Annual Meeting will be held in April in Toronto. The OAME is hosting the conference. This is the first time the conference is being held outside the U.S.A. I urge you to get involved in any way that you can. Let's make this meeting something to remember.

After many growing pains, the Intermediate Mathematics Guideline is beginning its implementation. This, along with the OAIP, will keep us all active for the next few years. I strongly suggest that you share your plans and documents. There are lots of good ideas out there. By co-operating, we can ease the work load for everyone.

Finally, there are microcomputers and calculators. By their nature, they will have the biggest impact on the mathematics curriculum. I think you will see the curriculum change to reflect the role that calculators and computers best play in problem solving. It takes many years of instruction and practice for students to master paper-and-pencil algorithms. There is a difference between understanding the concepts of these operations versus mastering the algorithm. Calculators and computers provide an alternative. They permit us to place an increased emphasis upon higher level skills.

But there are also those that say that by introducing too many "visuals" too early in education we are not developing the "imagination" skill which will eventually affect "creativity".

The use and misuse of microcomputers needs close scrutiny. As I said, since microcomputers will have the biggest impact on mathematics education, it is up to us to make sure they take their proper place in the curriculum.

George Knill

EDITORIAL

With the formation of a new editorial board, it is appropriate to make certain changes to the *Ontario Mathematics Gazette.*

Beginning with this issue of the *Gazette,* articles accepted for publication are placed into one of three sections. The first section contains papers primarily on *Mathematics Education* through all levels. One may find here articles on new teaching techniques, experiences obtained from experimental mathematics classes and innovative pedagogical material. At other times this section may contain studies of strategies of discovery in mathematics and how one best transmits insights obtained from mathematical scholars to our students. The impact of the Secondary Education Review Project (SERP) should be studied here so that the consequences of its possible implementation are well known to mathematics educators throughout Ontario. This is particularly topical in the light of reports of expansion of school mathematics and science programmes in the USSR.

Mathematics as an art form is humanistic, yet scientific and technological in its applications. Many of us studied and still study mathematics for the personal aethetic delight which is experienced by solving problems, understanding new concepts or proving theorems. It is significant that teachers of mathematics prove theorems in Geometrey, Algebra and Trigonometry over and over again for hundreds of years. Why do we not call a theorem that has been accepted by competent scholars a law, such as the "Law" of Pythagoras or the Fundamental "Law" of Calculus? Perhaps, because indoctrination worries us, we seek not only knowledge but complete understanding. We know that through the study of a proof or the analysis of a problem we get closer to the creation of new mathematics and perhaps we prove theorems again and again because it is a ritual where we celebrate the power of pure reason.

The second section is devoted to mathematical expositions. Here we may find survey articles from all areas of mathematics and may see applications of mathematics to the sciences and technologies. It is not intended that this section contains only advanced mathematics, indeed, it is desirable to have many articles expounding aspects of elementary mathematics. In the classroom or study a new perspective or insight on a mathematical concept has been obtained which should be shared through the *Gazette*. In addition one might read in this section historical, cultural and psychological aspects of mathematics.

The *Ontario Mathematics Gazette* is an official publication of OAME and hence should be a link between the many members of the association throughout Ontario and elsewhere. The third section, called *Departments* contains information on events of this and its sister organizations throughout Canada and other countries. Opinion and viewpoints regarding mathematics and mathematics education may be found here. At times interviews with prominent educators are given. In addition, a problem department is included which should be of interest to all students of mathematics. We invite readers to submit problems as well as solutions to problems. Further, there will, intermittently, be book reviews. However, only those books will be considered for review which are deemed to be of special interest to educators and mathematicians. Texts or potential texts for schools and colleges are not considered for review, nor are teaching aids. The reason for this policy should need no further elaboration.

Your opinions are always appreciated but most valued are well written manuscripts for publication which reflect your educational experience, study and mathematical work. Since there are page limitations per issue on the *Gazette* (52 pages!) shorter articles are particularly welcome.

H.P.H.

MATHEMATICS EDUCATION

THE BEGINNER . . . SHOULD NOT BE DISCOURAGED IF . . . HE FINDS THAT HE DOES NOT HAVE THE PREREQUISITES FOR READING THE PREREQUISITES.

Paul Halmos

A STUDY OF MATHEMATICS PROBLEM SOLVING PROCESSES IN ADOLESCENTS

by
Gerald Kulm

This article summarizes a study in progress whose purpose is to produce a comprehensive analysis and synthesis of mathematical problem solving processes used by children in grades 6 to 9. The primary objectives are to draw from this analysis and synthesis the following outcomes: 1) a characterization of problem solving processes used by students in grades 6 to 9, 2) hypotheses about the relationships between problem solving processes and tasks, 3) suggestions for classroom teaching of problem solving, and 4) directions for productive research in mathematical problem solving processes for children in grades 6 to 9.

The topic of problem solving has received considerable interest recently both by mathematics teachers and researchers. Specifically, attention is being focused more and more on the thinking processes that children use in solving mathematical problems. The primary methodology of these studies has been to interview students, asking them to "think aloud", in order to elucidate the cognitive and affective processes, strategies, or heuristics employed while solving mathematical problems.

The one-to-one interview situation has many advantages in studying problem solving. The best performances are often stimulated and it is possible for the researcher to be flexible in order to explore a child's thinking (Hatfield, 1979). A great deal of information is generated about the thinking used by the student, making it possible to consider not only the answer obtained but also the strategies and solution paths that were involved. It is possible to use problem solving tasks which can produce a broad range of processes and which assess the type of thinking actually engaged in while solving school mathematics problems.

This article is based on a paper presented at the Research Council for Diagnostic and Prescriptive Mathematics annual meeting, Vancouver, B.C., April 14, 1980.

Preparation of this paper was supported by the National Science Foundation under Grant No. SED-7920596. Any opinions, findings, conclusions or recommendations expressed are those of the author and do not necessarily reflect the views of the National Science Foundation.

The interview approach to research in problem solving has some limitations. First, it is time-consuming. This makes it difficult to study large numbers of subjects or problem tasks. As a result, researchers often restrict the number or type of tasks, the number of subjects, or both. A second disadvantage is the possible inconsistency across interviews and among interviewers. The subjective judgements made about what the subject is saying or what to infer about problem solving behaviors make reliability a real concern. Methods such as structured interviews or interviewer training may improve the reliability of results, but few studies employ such measures.

The limitations of the interview approach point to the need to combine and synthesize the results obtained by them. The small numbers of subjects in individual studies makes generalization to other populations extremely difficult. The special types or small number of tasks used limits generalizations about processes to other mathematical tasks. The possible lack of reliability for an individual interviewer makes it difficult to conclude that the same process behaviors would be found by other researchers.

METHOD:

The project is proceeding in three major stages: selection, analysis and synthesis.

Selection Stage: The selection of studies for analysis was focused on investigations of mathematical problem solving in which subjects in grades 6 to 9 were of primary interest. This grade level is selected for several reasons. First, in a practical sense, more work has been done with this age group than with others, making it possible to identify a reasonable number of studies. Second, the problem solving abilities of these subjects are in the developing stages, making it possible to study developmental patterns in their acquisition. This age group have not yet studied algebraic or proof-making techniques sufficiently long to develop algorithmic approaches to standard problems. A wider variety of problem solving processes may thus be observed and studied.

The final sample for the analysis consists of audio tapes or transcripts from over 350 subjects in grades 6 through 9 in more than 3,000 problem solving episodes.

Analysis Stage: The major proportion of the time in the proposed project involves analysis of the problem solving data. A coding scheme developed by the Smith group (1977) and described by Branca **et al.** (1979) was modified for use in analysis. The coding technique is well-known by now in mathematical problem solving research. A coding worksheet is used to record the processes observed while listening to the taped protocol. The procedure results in a sequential record of the processes used by the subject.

Reliability of the coding is assessed by having random samples of the data coded by two coders and calculating percentage of agreement. The training is aimed at achieving a 90 percent level of agreement between coders. If agreement drops below 80, retraining is done.

Synthesis Stage: The last four months of the project will be devoted to synthesizing the process and task variable data. Two main categories of independent variables will be used as organizers for this synthesis: 1) problem task structure, and 2) subject variables. The process data will be used in its sequential form to discover and describe strategies and other patterns in which processes occur in systematic forms. Frequency distributions of process data will also be made in order to characterize the overall use or deficiency in use for certain processes. Correlational analyses will be performed to discover relationships among process, subject and task variables.

The relatedness among problem structures will be used to synthesize patterns in processes. Lack of patterns will suggest differences that are due to other task variables (syntax, content, context) subject variables, or other variables unique to the conditions of a particular study.

Developmental changes in process use will be assessed while noting the effects of context or syntax and other intervening variables. Developmental patterns can also be compared for different structures to determine whether the changes are task or structure specific. Finally, developmental patterns will be studied and compared for related problems. Perhaps the processes for working on a special case, for example, develop earlier than those necessary for solving a generalization.

Differences in process employment between boys and girls will be investigated for various problem structures and contexts. Similarly, general or mathematics ability will be used when available to differentiate and determine process patterns.

Some preliminary results of the project will be reported at the St. Louis annual meeting of the

National Council of Teachers of Mathematics. The final report will be completed in November, 1981.

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INDIVIDUALIZED MATH INSTRUCTION FOR THE GRADE 12 GENERAL LEVEL STUDENT

by

Dave Lenchyshyn, Gwyn Jackson and Ray Marshall

The individualized program, which is in its second year at Myer was modelled on a program that was first developed by John Falk at Thornlea Secondary in the early seventies. To help explain the system, the following materials were distributed to those who attended the session.

1. Course of Study (used at Myer)
2. A summary of learning strategies
3. A sample unit (learning materials)
4. Summary of student questionnaire responses

Additional copies may be obtained by writing to the *authors.*

The learning materials consisted of sheets produced on an offset combined with lessons and examples on ditto. Corrections and special remarks appeared on "page 0". By carefully reading the instructions and examples, students soon developed the ability to learn on their own.

After outlining the learning strategies, the classroom arrangements including test row, teacher solution booklets, test sign-up calendars, and storage facilities were described. A short video tape of Dave's class illustrated the process.

The "Course of Study" referred to earlier is used by teachers and students at Myer. The first page describes in detail that students must complete a minimum of seven core units (80% competency on each test) to obtain a passing mark. By completing optional units, final marks between 50 and 100% can be achieved. The 80% competency level for each test is attainable for most students because the unit tests are comparable in format to that of the practice test found at the end of each unit. Students are also required to submit all learning materials properly completed before they are permitted to sign up to take the unit test.

Examinations in November and March have been waived. However, a June *examination based* on core units is available for those who wish it. Students who complete the course (including optional units) prior to June have spares for the remainder of the year.

A unit test consists of test items on 3'' x 5'' index cards. Each card contains one question (which may have several parts) and depending on the unit, a test may contain up to six questions. A master marking scheme with answer key for all units makes marking of tests a relatively routine task. Detailed procedures for testing and re-testing appear in the "Course of Study".

Teaching one class in an individualized basis provides a nice break in routine for the teacher. The approach actually allows the teacher more time to spend with individual students. Increased motivation to complete the course diminishes discipline problems.

To overcome the enormous amount of teacher time required to prepare materials, test cards, etc., the presenters with the help of Sandra Sinclair, who also teaches at Myer, hope to publish these materials, and thus make the program more readily available to other schools. Future plans also call for a similar program to be developed for Grade 11.

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PROBLEM SOLVING: WHY BOTHER?

by
Gordon Nicholls

"I am distressed to see how many people today are afraid of the future and of science together. I believe that these fears are mistaken. They seem to me to misunderstand the methods of science and spring from a gloom about what it has done. We sit under the shadow of the nine o'clock news nursing our sense of doom, and we think ourselves worse off then our forefathers. But a hundred and seventy years ago we were at war for a generation, the working week was eighty hours for children, cholera was more common than flu and the literacy rate in England was only 10% You know all this has been changed . . . We owe this miracle to science. The scientists who have worked it are neither gods nor witch doctors. They have been men: men who had faith in the future, and they used no magic. What they have used is Darwin's method, because that method is science. Science is experiment; science is trying things. It is trying each possible alternative in turn, intelligently and systematically; and throwing away what won't work and accepting what will."(1)

As teachers of mathematics we are part of the scientific community and thus are part of that group which must share the responsibility for the growing public attitude of despair and lack of misunderstanding of science. You may say that as teachers we are not in the foremost cohorts of scientific thought - why blame us? The question of why people despair when faced with nuclear reactors that malfunction or acid rain in their favorite lake is beyond our sphere or influence; however, the presence of large groups of people who do not understand the problem solving process of science is within our domain in the scientific community. The Coleman report(2) went right to the core of our role when it stated that attention to problem solving by the teachers in the elementary and secondary communities was of vital importance.

In the following essay I shall attempt to give a few ideas on how mathematics teachers can make a contribution to the process of problem solving.

"Disgusted with my failure, I went to spend a few days at the seaside and thought of something else. One morning, walking on the bluff the idea came to me, with just the same characteristics of brevity, suddenness and immediate certainty . . .

Most striking at first is this appearance of sudden illumination, a manifest sign of long unconscious prior work. The role of this unconscious work in mathematical invention appears to be incontestable."(3)

The words of Poincaré set forth with elegant precision the experience that many of us have had when we worked on problems in the past. I do not think we can teach this process, but we can share with our students the knowledge that it happens to people who plunge themselves into the dark of the unknown. We can also try to construct our curriculum so that over the years there is a sequence

of opportunities for students to encounter ever more difficult problems and the satisfaction that comes from solving these problems. In the first part of this essay I will try to demonstrate how certain features of the curriculum can be designed to promote problem solving.

In addition to the inspiration in the middle of the night, there is, however, that other approach to problem solving we all know so well - the long tough road of trial, error and memorization. It is frustration on this road that has built up the attitude of fear about problem solving, and it is here that our major contribution can be made. Over the years many teachers of mathematics have constructed aids to help students along this road and many of these aids are both useful and efficient. One of the major tools used is the "Example or Text Book" problem approach. A thorough and well organized teacher can present examples to cover almost every situation. This approach has the very restrictive drawback of a mass of material to memorize. I would like to demonstrate in the later part of this essay how many of the Example problems can be replaced with a set of techniques. If these techniques will work in a variety of situations, then they will replace a large volume of material. I hope to demonstrate to the reader that this is possible.

CURRICULUM DESIGN AND PROBLEM SOLVING

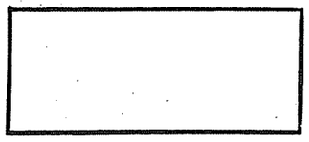
In the design of the high school mathematics curriculum, I would suggest that the following four aspects of problem solving be considered:

1. The importance and use of model building.
2. The significance of the historical approach to the solution of certain problems.
3. The value of the interesting application problem as a motivation for a topic.
4. The need for a set of problems that progress in level of difficulty and can be used as a challenge for both brighter students and average students.

An illustration of how each of these ideas could be used in curriculum construction follows.

Model Building

Here is an approach to the topic of relations that was suggested to me many years ago by A. Wittenberg.[4] The central idea behind the approach is that a two dimensional relation is a model of a physical situation with two variables and there are several different ways to describe the relation, culminating in the algebraic model.



The introductory lesson to the topic begins by giving the students a 40 cm. length of string and posing the following two questions:

How many different rectangles can be constructed? Do all the rectangles have the same perimeter? Ask them to quantify their answer. As the lesson proceeds the results can be recorded in the following ways:

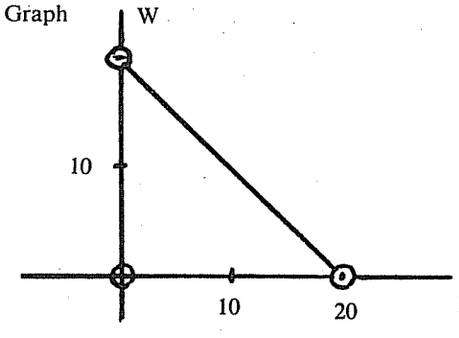
1. In a Table

L	19	18	15	14	12	10	etc.
W	1	2	5	6	8	10	etc.

2. As ordered pairs (L, W)

[19, 1), (18, 2), (15, 5), . . . etc.]

3. Graph



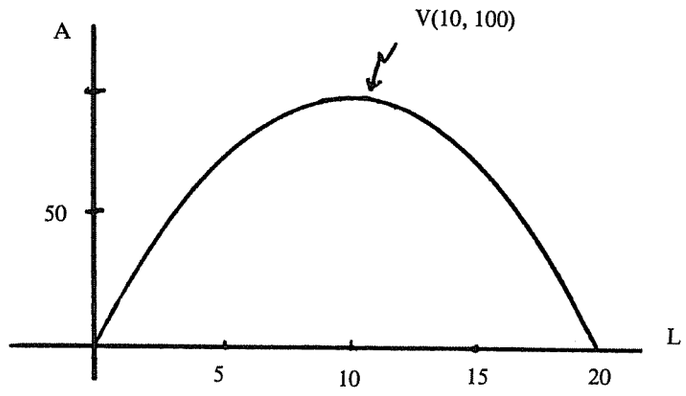
4. **Algebraic Description**

Observation of the table shows the sum of each column is 20.

$\therefore$ L+W = 20

At this point, the student has seen how to build both an algebraic and graphical model of a fairly simple physical situation. The given questions have been answered and the student is now ready to be extended a little by posing the following question:

Do all the rectangles have the same area? My experience with this in a class situation has shown that usually 75 - 80% of the students think that the area will be the same for all the rectangles. To answer the question the same process of chart→ordered pairs→graph is followed and the results come as an interesting surprise. Right at the beginning of the programme on relations it is possible to use the idea of the mathematical model to show the students that their intuition is not always correct. As this unit progresses to develop the traditional work on linear systems and then on to more problem solving the terminology of "model building" should be used and reinforced. This is an ideal place in curriculum design to demonstrate an important process in mathematics. The reader can no doubt think of other parts of the curriculum where the model building idea can be introduced and I leave that to be pursued.



Historical Approaches to Problems

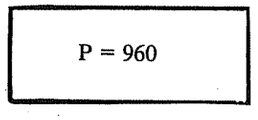
The history of mathematics provides us with many examples of the blind alleys followed and intuitive leaps made by the great minds of the past. I would not suggest that we design the mathematics programme to duplicate all of these historical arabesques; however, I do think that careful study of the past will yield suggestions for some valuable curriculum approaches. There are three very important points in the high school curriculum where this sort of approach could be very helpful: the introduction to Algebra in grade 9, to Analytic Geometry in grade 11, and to Calculus in grade 13. To illustrate how the historical idea could be pursued, I will use the algebra example and leave the other possibilities to the reader.

The literature on the history of mathematics shows that the development of algebra was a response to the need to solve problems that were too complex for ordinary arithmetic approaches. Standard historical texts(5) provide an abundance of examples from Babylonian, Egyptian and Hindu mathematics. The important features to notice from these examples are the processes that were used to solve problems and the long slow evolution to the concept of the variable.(6) With this established as the central thrust behind the development of algebra the next step is to design a curriculum that will provide a reliving of this thinking process. The following suggestion is offered.

Starting with simple arithmetic questions like "I am thinking of a number so that if you double it and add 25 the result is 85. What is the number?", it is possible to have most grade 9 students solve this by a simple trial and error method.

Next, increase the level of difficulty of the problem to the point where the trial and error process must be used in some organized way. The process to be followed is an iterative technique and the values obtained quickly converge on the required number. The question below will illustrate the depth of problem to be considered at the beginning.

Problem



'The length of a rectangle is 20 more than three times the width. If the perimeter is 960, then find the width.'

Solution

Organize the trial solutions in the following way and after each line of calculation draw a conclusion about the calculations made.

Guess	Verify against P = 960	Conclusion
My first guess is width = 10	Length = 30 + 20 = 50 P = 2(10 + 50) = 120	120 is too small ∴ Increase the guess
My 2nd guess is width = 100	Length = 300 + 20 = 320 P = 2(100 + 320) = 840	840 is too small but close ∴ Increase
My 3rd guess is width = 120	Length = 360 + 20 = 380 P = 2(120 + 380) = 1000	1000 is too large ∴ Decrease

After several days of doing problems of this type the students will be psychologically and intellectually prepared to appreciate the significance of the following argument.

Guess
Let the width be w

Verify
Length = 3w + 20
P = 2(4w + 20)

Conclusion
In order to find the correct value for the width the expression 2(4w + 20) = 960, must be true.

Thus the first lesson on algebra involves setting up a problem in equation format. The student will not know how to solve this equation at this stage but the teacher will be providing the motivation for having to learn how to do basic operations in algebra. The curriculum that follows this introduction should be organized in such a way that basic operations are learned quickly and then the introduction to equation solving is presented. Within three weeks of starting algebra, the student should be solving problems of the type illustrated. This is one example of how to construct a part of the curriculum by looking back into the history of our subject. The other suggestions can not be developed here and are left to the reader.

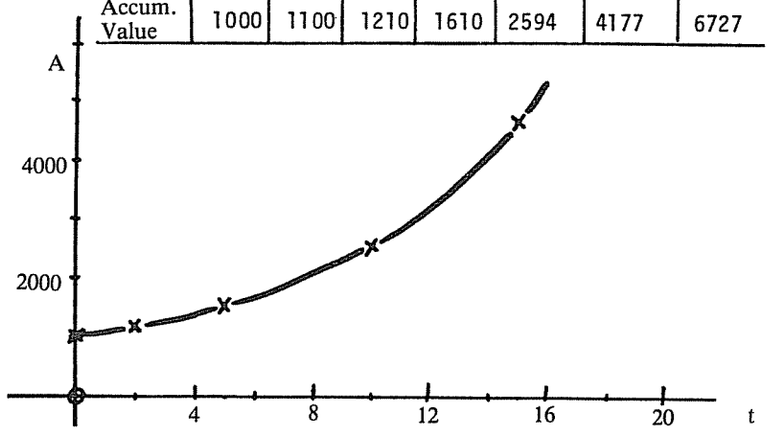
The Interesting Application Problem

The modern uses of mathematics are as diverse in their application as they are in level of difficulty. Fortunately there are many meaningful and interesting examples that are within the scope of the high school programme. Some of these can be used to introduce and motivate a topic. In the design of the curriculum, we should look for interesting applications and see if curriculum topics can be built around them. One very good example that will illustrate the point is the concept of compound interest. This topic is traditionally taught in those courses that have a section on finance and annuities. It is possible however to use this problem to introduce the exponential function.

Present to the students the problem:
'If you invested an amount of $1000 at 10% p.a. compounded annually for t years, then what would your investment be worth after 1, 2, 5, 10, 20 years?'

As the amount are calculated, organize them into table form and then suggest that it would appear to be a functional relationship.

#of years t	0	1	2	5	10	15	20
Accum. Value	1000	1100	1210	1610	2594	4177	6727



The graph follows very easily and with a few trial demonstrations like:
After 2 years $A = 1000(1.1)(.1) + 1000(1.1)$
$= 1000(1.1)^2$

the general formula can be easily developed for t years.
i.e. $A = 1000(1.1)^t$

The stage is now set for development in two directions: the traditional arithmetic of the exponential function topics along with an inclusion of meaningful financial problems throughout the topic will make for a more mathematically interesting experience.

Problem Sets

The appearance of several levels of mathematics contests has helped to provide a source of problems for brighter students. I do not wish to underestimate the importance of the contribution to problem solving by these contests; however, I think a worthy addition to this process would be the publishing of small booklets of graded problems in several areas of mathematics. There should be one booklet for the Intermediate division with problems in arithmetic, algebra and geometry. A second booklet for the senior division should include problems in algebra, analytic geometry, trigonometry and synthetic geometry. A final booklet for the grade 13 level student could contain problems on calculus, algebra, number theory, probability, combinatorics and other topics that could best be determined by university mathematicians. If these booklets were properly prepared, without regard to current fads and jargon, then they could be of use for many years. I would like, in this article, to suggest that the people who have spearheaded the work in preparing the mathematics competitions should be approached to take on this project. It would be a worthy project for the O.A.M.E. to support financially and I hope the executive would give this consideration.

If booklets of this type were available, then teachers throughout the province would be able to incorporate problem solving units into their courses. These units could be assigned to either classes or groups of interested students, provided the problems were properly graded and organized.

TECHNIQUES FOR PROBLEM SOLVING

There are those who approach problem solving in the same way as the young Mozart approached his music and then there are those who always find the going difficult. The vast majority of the students in a secondary school, are in the latter category and they require help in organization and strategies of attack on problems. The techniques that follow are divided into two categories. The first section gives techniques that can be introduced in grade 9 or 10 and should be useful right through to grade 13. The second section offers some techniques that could be introduced at appropriate places in the senior division.

Intermediate Level

Algebra	Geometry
1. Organization Chart	1. Organization Chart
2. Use Specific Cases for Ideas	2. Use Specific Cases for Ideas
3. Build Algebraic Model	3. Diagram Separation
4. Substitute or Convert	4. Add Construction Lines

The chart above lists the techniques that I think are useful in problem solving at the Intermediate Level. Two of the techniques can be used in both algebra and geometry, whereas the other suggestions apply to only one area.

A few examples would help to clarify the meaning of the suggestions.

If the four approaches to geometry in high school are classified under the titles Synthetic, Analytic, Vector and Transformation, and the types of problems that are investigated in each of these areas are studied, then the observer begins to notice a recurrence of several basic problems. For purposes of recall it is helpful to the student to have the body of factual information that he knows organized according to the way he wishes to use this information. Thus the theorems, assumptions and important definitions of synthetic geometry could be organized in chart form to be available for the following types of question:

Prove Angle = Angle	Prove Line = Line	Prove Triangle ≡ Triangle	Prove Line // Line	Prove Area = Area	Prove $\Delta_1 /// \Delta_2$

It is interesting to note that a similar chart for the standard problems in analytic geometry would have the headings:

Prove Angle =Angle	Prove Line =Line	Prove Line //Line	Prove Area =Area	Find Equation of Locus		Intersection of Loci
				By First Principles	By Theorems	

A new component is introduced around locus, but many of the basic problems are the same.

The work with transformation approaches to geometry varies considerably around the province; however, if the basic idea of invariance is considered then problems involving transformations reduce to the question: Which of the following Geometric Properties are Invariant under a Transformation T: $(x, y) \rightarrow (\beta, \psi)$?

Is Collinearity Preserved?	Is Angle =Angle?	Is Line =Line?	Is Triangle ≡Triangle?	Is Line //Line?	Is Area =Area?	Is Triangle ///Triangle	Is Locus Invariant

Here we see a unitying thread that brings the ideas of the previous charts together. As we teach our students a greater volume of material and a variety of techniques, surely we should point out to them that many of the problems we wish to solve are the same or similar and that the solving of the problem becomes a question of which technique to draw upon.

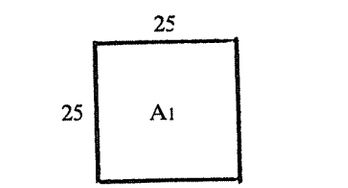
A similar charting approach can be used to organize the theorems and skills of school algebra and it will be left to the reader to create a useful organization.

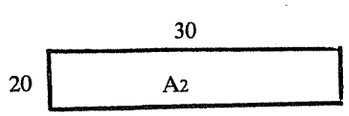
George Polya has written many books on problem solving and teaching.[7, 8] One of his major approaches is the use of the specific case to get the idea. To paraphrase his words:

"If you have a tough problem that you can not solve then create a **similar** but simpler problem. Solve the simple problem and then study the technique you used. Now return to the more difficult problem and you will have a method you can try to follow." An example from algebra will illustrate this approach.

'A fixed length of rope k is used to form a rectangle. Prove that the rectangle with maximum area is the square with sides k/4.'

To understand the problem it often helps to replace k with a number say 100 metres. Thus I am asked to prove that the square with sides 25m has the largest area. ie. $A_1 = 625\ m^2$. Try several other cases to become convinced.

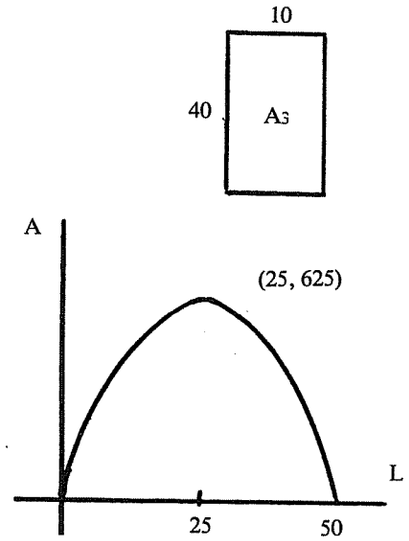




Thus $A_2 = 600\ m^2$
$A_3 = 400\ m^2$

The use of the specific cases would hopefully suggest a chart or a graph of all the data and soon the student is looking at a parabala with vertex (25, 625).

This approach could then be used on the more general problem. For a discussion of many other problems where this approach is



useful I would recommend Polya's **Mathematical Discovery.**(8)

The diagram separation technique and the use of construction lines are so well known that I do not think it necessary to illustrate their use in this article. For the student they should be illustrated at the appropriate point in a geometry programme and then the idea should be retained as part of their problem solving strategy.

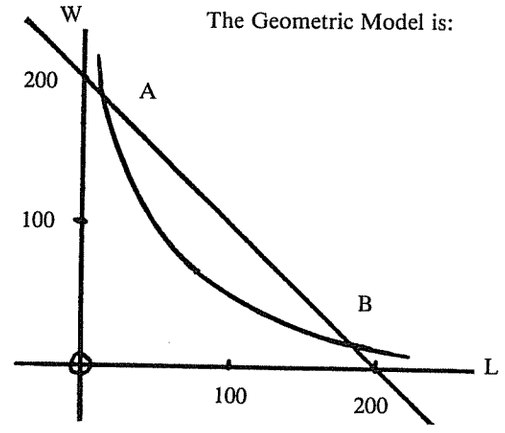
Model building was discussed in the first section and suggestions were given for curriculum change to stress the importance of this idea. Another aspect of model building at this level that is of vital importance is the question: If a problem contains two variables and the model yields two equations then what do we do with them? The first few times that this is encountered it would be of value to use a graph to explain why the algebraic technique of intersection is used to reduce the model to a simpler model.

For example:
'If the perimeter of a rectangle is 400 cm and the area is 9600 cm^2 then find the dimensions.'

The algebraic model for the problem is the pair of equations:

$$L + W = 200$$
$$L \cdot w = 9600$$

The Geometric Model is:



To find the common values for both conditions, it is easy to see on the graph that we require the points A and B. With the graphical model available to explain the concept, it is an easy transition for the students to see that in algebra they solve the pair of equations and thus reduce the algebraic model to a simpler model.

By "Conversion" I refer to the necessity in most problems to change from one mode of expression to another. In the Coleman report, it was noted that one of the major difficulties in problem solving at a more advanced level was the lack of communication between the problem solver and the manager. This process caused difficulty in both directions. Initially the problem solver had trouble in interpreting what was required --- ie. he had difficulty in converting the data and instructions from his manager into a mathematical model. Later difficulty was experienced in explaining the significance of results that had been found. At the grade 10 level the different modes of presentation of information include the English language, numerical tables of values, graphs and algebraic descriptions. Although much time is spent teaching students how to convert information from one mode to another, we do not in general spend enough time on problems that require a conversion from the language format into one or more of the other formats. When problems of a suitable level of difficulty are presented here on a regular basis, the students can be encouraged to ask themselves: "In which format should I try to solve the problem?" There are many problems where the algebraic model is not the best format, but most of the problems presented in texts and on contests stress this style of problem solving. Students should be encouraged to be more flexible in their thinking here and ask themselves the question: "What is the best format for the data?"

Senior Level

Techniques for Problem Solving at the Senior Level

1. Continued Use of Intermediate Level Techniques
2. Use of Algebra on Geometry Problems
3. Transform - Solve - Transform Back
4. Change of Dimension
5. Reverse Logic

As the level of maturity of our students increases we should increase the sophistication of the techniques they can handle. The senior division curriculum includes large sections of work that show the isomorphic relationship between algebra and geometry. The first technique that the student should learn at this level is how to use analytic methods. If successfully mastered this tool can be very useful in a wide variety of situations. It should, however, be pointed out to students that the power of analytic methods on problems dealing with angles is less than satisfactory and thus analytic techniques are not the panacea. One type of problem that is not present in the Analytic Section of most texts, but should be presented at this level, is the problem that is expressed in general geometric terms. The problem solver then has to decide whether to use synthetic or analytic techniques.

For example the following problem can be solved in either format:
'Prove that the lines joining the mid points of adjacent sides of any quadrilateral form a parallelogram.' The solutions are left to the reader.

Once the basic tools of analytic geometry have been mastered then the student is ready for the approach to problem solving that requires the problem solver to transform into a new system, solve the problem and then transform back. For easy reference, I shall refer to this as the T.S.T. technique. Pure Geometry problems that are solved by using analytic methods are examples of this technique in operation. An example that uses the reverse strategy is the following question from the 1977 Descartes compeition.

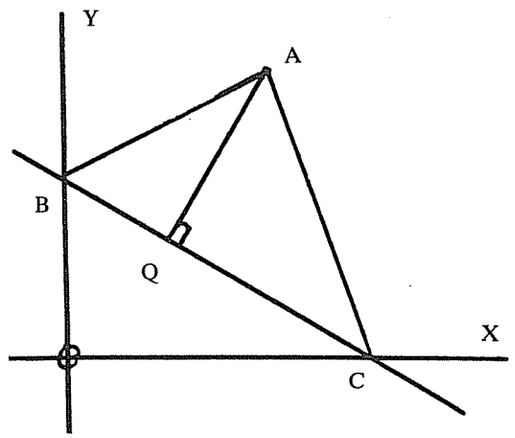
'Find all real points of intersection of the two curves $x^2 - y - 6x + 10 = 0$ and $x + y^2 - 4y = 0$.'

The two curves could be solved by standard procedures, but the solution requires you to factor the quartic $x^4 - 12x^3 + 52x^2 - 95x + 60 = 0$.

An alternate approach is to transform the two parabolae with the transformation T: $(x, y) \rightarrow (x - 4, y - 2)$. The resulting system is $\beta = \psi^2$ and $\psi + 1 = (\beta + 1)^2$. The solution of this system gives the points $(0, 0)$, $(-1, -1)$, $\left(\frac{-6 - 2\sqrt{5}}{4}, \frac{1 + \sqrt{5}}{2}\right)$, $\left(\frac{-6 + 2\sqrt{5}}{4}, \frac{1 - \sqrt{5}}{2}\right)$.
Use the inverse transformation to return to the pre-image position and the problem is solved. At this level of difficulty the student has a very powerful tool, but he will only learn its value by using it on a variety of problems.

The change in dimension technique is one that is used in a very creative way on selected pieces of theory in the senior division work.

For example to prove the perpendicular length theorem in analytic geometry, the standard proof begins with the area of the ΔABC. This two dimensional quantity is later reduced to produce an expression for the one dimensional quantity AQ. Another excellent example of the same technique is in the proof for $\lim_{\theta \rightarrow 0} \frac{\sin\theta}{\theta} = 1$



The theorem begins by changing dimensions and beginning with area. There are other examples in the senior division courses and once again we can identify a technique that could be added to the repertoire of approaches to problems.

The technique of reverse logic is subtle to handle but it is of great use in the hands of skilled problem solver. The multi-step problem built around a logical framework of 4 or more connecting ideas can often be solved by assuming the answer and working backwards. A schema to represent the thinking part of the process of reverse logic is in column one, and the written proof would follow the order of column two. This technique requires considerable practise before it can be used effectively.

Schema for Reverse Logic

Thinking in Reverse w is true			Order of Proof We know f is true, and			
But a	$\Rightarrow$	w	Step 1	f	$\Rightarrow$	d
Also b	$\Rightarrow$	a	2	d	$\Rightarrow$	c
Also c	$\Rightarrow$	b	3	c	$\Rightarrow$	b
Also d	$\Rightarrow$	c	4	b	$\Rightarrow$	a
And f	$\Rightarrow$	d	5	a	$\Rightarrow$	w
where f is a known fact			$\therefore$ w is true			

To conclude, let us remember Euclid's famous quip, "Sire, there is no Royal road to Geometry" and realize that it is applicable to the teaching of problem solving in all areas of mathematics. Although the suggestions offered have been numerous and in some cases sophisticated, we must remember that the skills required for problem solving are demanding and will take work and patience to develop. Perhaps with the slow perculator of TIME some of our students will turn the skills into an art and become problem solvers. But more important, let us hope that our efforts will be rewarded by having a substantial number of our students going on to adulthood with an understanding of the problem solving process, a renewed faith in the capacity of our society to solve the problems of tomorrow and the ability to critically follow the solutions presented to these problems.

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ROOM FOR 'IMPROOFMENT': E.R. Williams' Award-Winning Study of Secondary School Students' Understanding Proof

by

Dr. William Higginson

The idea of proof is a central one in any conception of mathematics. For those who would emphasize the axiomatic and structural features of the discipline it is particularly important. It is not, however, as anyone who has attempted to teach it knows, an easy concept for students to grasp. E.R. Williams, presently Associate Professor of Mathematics at Memorial University of Newfoundland, did research in this area for his doctorate in mathematics education. Working at the University of Alberta under the supervision of Professor A. Olson, Dr. Williams completed a thesis in 1978 entitled "An Investigation of Senior High School Students Understanding of the Nature of Mathematical Proof". At the 1980 Annual meeting of the National Council of Teachers of Mathematics the high quality of this study was recognized when it was announced that Dr. Williams had been selected as co-winner of the First NCTM Dissertation Award.

As subjects for his study Dr. Williams used some two-hundred and fifty grade eleven students in academic streams in ten randomly selected classes from nine different senior high schools in Edmonton. Each student responded in writing to twelve items constructed by the investigator. The student responses were then summarized and categorized.

The fourth of the twelve items runs as follows:

Joe has written a number of interesting equations, some of which are as follows:

$8^2 - 7^2 = 15$ $\quad$ $3^2 - 2^2 = 5$

$5^2 - 4^2 = 9$ $\quad$ $9^2 - 8^2 = 17$

$11^2 - 10^2 = 21$ $\quad$ $14^2 - 13^2 = 27$

From these equations, Joe concludes that for all integers a and b, $a^2 - b^2 = a + b$

QUESTIONS:
(a) Do you think that Joe's conclusion is true _____ or false? _____
(b) If you think that Joe's conclusion is true, then state why.
(c) If you think that Joe's conclusion is false, then state why.
(d) If you think that Joe's conclusion is false, then what do you think might be a correct conclusion?

This was one of the items the students coped with most successfully as almost all of them made a coherent response.

Seventy-five subjects (29.4% of the sample) incorrectly concluded that Joe's argument was valid. Twenty-two (8.6%) rejected Joe's claim but gave no reason to support this conclusion. One hundred and fifty-two of the subjects (59.6%) rejected Joe's claim and gave a reason for doing so. Most of these arguments were based on the production of a counter-example. Very few students made use of the difference of squares identified although one student 'generalized' by claim that $a^2 - b^2 = (a + b)^{a-b}$. ("If the difference between the two numbers is one, then it is true. If the difference is two, then you square the added answer; if true, you cube it and so on.")

The major findings of his study as summarized by the author were:

1. Only those students who were classified as high achievers by their teacher, less than thirty percent of the sample, exhibited any understanding of the meaning of proof in mathematics.

2. Approximately half of the students sampled do not see any need to prove a mathematical proposition which they consider to be intuitively obvious.

3. At least seventy percent of the students sampled do not distinguish between inductive and deductive reasoning and hence do not realize that induction is inadequate to support mathematical generalizations.

4. Approximately eighty percent of the students sampled do not always realize the significance of hypotheses and definitions in mathematical arguments. Approximately sixty percent of the students sampled are unwilling to reason, for the sake of argument, from any hypothesis which they consider

is false.

5. Less than twenty percent of the students sampled understand the method of indirect proof.

6. Almost eighty percent of the students do understand the concept of counter-example.

7. There was no evidence from this study to suggest that high school students understand that a mathematical statement and its contrapositive are logically equivalent or that students realize that a statement and its converse are not logically equivalent.

8. There were no significant differences between the overall level of responses given by females and that given by males.

These are not results that any mathematics educator can find encouraging. It is the merit of this study, however, that it accurately documents the state of the art at this time. Dr. Williams would, I am sure, respond favourably to any request for more information about his research.

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MATHEMATICAL EXPOSITIONS

MATHEMATICS IS THE GATE AND KEY OF THE SCIENCE.

R. Bacon

MATHEMATICS AND GENETICS: AN ELIMINATION PROBLEM

by

David R. Duncan and Bonnie H. Litwiller

Suppose that the presence or absence of a harmful trait is determined for each individual (plant or animal) by the individual's genetic make up. This trait is present in an individual having two recessive genes relative to that trait. If an individual has two dominant genes or one dominant and one recessive gene relative to that trait, then that individual does not display the harmful trait.

Suppose that each individual which displays the harmful trait (having two recessive genes relative to the trait) does not reproduce. This may result from a biological inability to reproduce. It may also result if the harmful trait causes the individual's early demise or if the individual possessing this trait is destroyed as may occur in a plant breeding situation. It is apparent that the fraction of the population possessing this harmful trait should decline in successive generations because of the systematic deletion of the recessive genes which cause it. How rapidly does this decline occur?

Let n be the number of individuals in a given generation. Let x be the number of individuals which are "normal" (not possessing the harmful trait) and y be the number of individuals which are "defective" (possessing the harmful trait). Observe that $x + y = n$. Further, let d be the number of dominant genes in the total population relative to the harmful trait and r be the number of recessive genes in the total population relative to the harmful trait. Since each individual has two genes relative to this trait, note that $d + r = 2n$. Several additional relationships among x, y, d, and r can be derived. An individual is normal if it possesses either two dominant or one dominant and one recessive gene. Assuming that the 2n genes are randomly distributed in a very large population, the probability that a given individ-

ual possesses two dominant genes is

$$\frac{d}{2n} \cdot \frac{d}{2n} = \frac{d^2}{4n^2} .$$

Strictly speaking the first factor,

$$\frac{d}{2n},$$

is the probability that the first gene is dominant, which should cause the probability of the second gene being dominant to be

$$\frac{d - 1}{2n - 1} .$$

The assumption that the population is very large enables us to approximate the second factor as

$$\frac{d}{2n}$$

also.

Similarly the probability that an individual has one dominant and one recessive gene is approximated by

$$\frac{d}{2n} \cdot \frac{r}{2n} + \frac{r}{2n} \cdot \frac{d}{2n} = \frac{2dr}{4n^2} = \frac{dr}{2n^2}.$$

The probability that an individual is normal is thus

$$\frac{d^2}{4n^2} + \frac{dr}{2n^2} = \frac{d^2 + 2dr}{4n^2}.$$

An individual is defective if it possesses two recessive genes. This probability is approximately

$$\frac{r}{2n} \cdot \frac{r}{2n} = \frac{r^2}{4n^2}.$$

As a check, observe that P(normal individual) + P(defective individual)

$$= \frac{d^2 + 2dr}{4n^2} + \frac{r^2}{4n^2} \quad = \frac{d^2 + 2dr + r^2}{4n^2} \quad = \frac{(d + r)^2}{4n^2} = \frac{(2n)^2}{4n^2} = 1.$$

The number of normal individuals is the product of the total number of individuals and the probability that a given individual is normal. Call this Formula I:

$$x = n\left[\frac{d^2 + 2dr}{4n^2}\right] = \frac{d^2 + 2dr}{4n}$$

The number of defective individuals is the product of the total number of individuals and the probability that a given individual is defective. Call this Formula II:

$$y = n\left[\frac{r^2}{4n^2}\right] = \frac{r^2}{4n}.$$

Solving the r in Formula II yields Formula III:

$$y = \frac{r^2}{4n}$$

$$r^2 = 4ny$$

$$r = 2\sqrt{ny}.$$

Since the number of dominant genes is the difference between the total number of genes and the number of defective genes, Formula IV can be derived:

$$d = 2n - 2\sqrt{ny}$$

$$d = 2\sqrt{n}\,(\sqrt{n} - \sqrt{y}).$$

Recall that we wished to investigate the fraction of individuals which are defective and also the decline in this fraction. For the first generation considered, the fraction of the population that is defective is y/n. What fraction of the next generation is also defective? A member of the second generation will be defective if it possesses two recessive genes. What is the probability of this? Not all of the members of the first generation are available for reproduction since none of the defective members reproduce. Therefore only n - y members are able to reproduce. They possess 2(n-y) genes. Originally (first generation), there were

$$r = 2\sqrt{ny}$$

recessive genes in the population; 2y of these recessive genes are unavailable for reproduction. Consequently there are $2\sqrt{ny} - 2y$ recessive genes available for reproduction. The probability that a given individual in the second generation is defective is:

$$\left[\frac{2\sqrt{ny} - 2y}{2(n-y)}\right] \cdot \left[\frac{2\sqrt{ny} - 2y}{2(n-y)}\right] = \frac{(\sqrt{ny} - y)^2}{(n-y)^2} =$$

$$\frac{y(\sqrt{n} - \sqrt{y})^2}{(n-y)^2} = \frac{y(\sqrt{n} - \sqrt{y})^2}{[(\sqrt{n} - \sqrt{y})(\sqrt{n} + \sqrt{y})]^2} = \frac{y}{(\sqrt{n} + \sqrt{y})^2} = \frac{y/n}{\left(\frac{\sqrt{n} + \sqrt{y}}{\sqrt{n}}\right)} = \frac{y/n}{(1 + \sqrt{y/n})^2}$$

Replacing y/n with f (y/n is the fraction of the original population which was defective), the fraction of the second generation which is defective is

$$\frac{y/n}{(1 + \sqrt{y/n})^2} = \frac{f}{(1 + \sqrt{f})^2}.$$

This process would apply to any two consecutive generations; call this Formula V.

Observe that

$$(1 + \sqrt{f})^2 > 1$$

and that consequently

$$\frac{f}{(1 + \sqrt{f})^2} < f.$$

As we had mentioned earlier, the fraction of defectives will decline each generation.

For example, suppose that f = 1/10 or .10; that is 10% of the population of a first generation is defective. The fraction of the second generation which is defective is thus

$$\frac{.10}{(1 + \sqrt{.10})^2} = .05772.$$

Reapplying Formula V to go from generation 2 to generation 3, the fraction of the third generation which is defective is

$$\frac{.05772}{(1 + \sqrt{.05772})^2} = .03752.$$

This new fraction of defectives may be used to determine the fraction of defectives in generation 4. This process may be continued for arbitrarily many generations. Table 1 displays 20 generations beginning with the initial value of f = .10. Also computed are the ratios of the fraction of defective individuals in a generation to the fraction in the preceeding generation.

Table 1

Number of Generation	Fraction of Defective Individuals	Ratios of Consecutive Fractions
1	.10	---
2	.05772	.5772
3	.03752	.6500
4	.02633	.7020
5	.01949	.7402
6	.01501	.7701
7	.01191	.7935
8	.00968	.8128
9	.00803	.8295
10	.00676	.8418
11	.00577	.8536
12	.00499	.8648
13	.00435	.8717
14	.00383	.8805
15	.00340	.8877
16	.00303	.8912
17	.00272	.8977
18	.00246	.9044
19	.00223	.9065
20	.00204	.9148

Initially the eradication of defectives seemed to proceed rapidly. From the first to second generation, the fraction of defectives was reduced over 42%. The rate of eradication slowed continually from that point on. In the twentieth generation there was only a reduction of approximately 8% in the fraction of defectives from the preceeding nineteenth generation.

In terms of absolute numbers, total eradication is very slow in coming. If a constant population of 100,000,000 is assumed in each generation, then the first generation has 10,000,000 defectives while the twentieth generation has 204,000. This is a sizable number!

If the population increases in each generation, assume a 25% increase in each succeeding generation. The twentieth generation would have just over 14,000,000 defective individuals. This is an absolute increase in defective individuals even though the fraction is declining. An agency attempting to care for these defective individuals may perceive that the problem is increasing even though the fraction of defectives continues to decline.

The reader and his/her students are invited to find other situations in science in which mathematical concepts can be employed.

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I DIDN'T KNOW THAT

by
Larry Ridge

With this issue returning to the theme of problem solving I'd like to recount my experiences with the classic 8-people handshake problem which has become so popular as a process problem for audiences not acquainted with formal combinatorics.

> There are 8 people at a party. If everyone shakes hands with each of the others, how many handshakes are there altogether?

This problem came into real prominence through John LeBlanc's article "You Can Teach Problem Solving" in the November 1977 issue of the Arithmetic Teacher. (It is not to be confused with the handshakes problem discussed in the March 1980 edition of this column). The thrust of the LeBlanc piece was that a variety of strategies such as those applicable to process problems could also be exercised in the solution of many textbook problems and hence, that textbook problems did offer more significant problem solving opportunities than had been generally considered.

Through presentation of this problem to groups from Grade 6 to university graduate level, I have gathered a considerable collection of solution strategies and variations which extend the selection presented by LeBlanc. Although you will find it useful to refer to the original article, the following summary includes those strategies for your convenience.

I. DIAGRAM

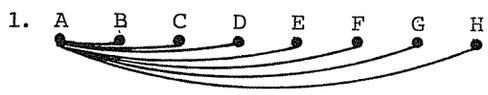


Fig. 1
Each arc represents a handshake, 7 of them.



Fig. 2
Because the handshake between A and B was shown in Fig. 1 it is not repeated in Fig. 2.

When audiences are encouraged to open their minds to many different strategies (even if they know an algorithmic solution) this is one of the first to emerge regardless of the sophistication of the group. (Grade 6 and 7 youngsters enjoyed naming the party-goers: Al, Betty, Chuck, Derek, Eve, Frank, Gerrie, Hannah.) It does provide a systematic counting procedure.

Rarely will this strategy be carried to completion. Rather it serves to generate one of two deductive arguments.

a) Figures 1 and 2 show that A shakes 7 times and B shakes 6 more times.

b) Figures 1 and 2 show that each of A and B shake 7 times. Similarly so would everyone else.

It follows that
C shakes 5 more times
D shakes 4 more times
E shakes 3 more times
F shakes 2 more times
G shakes 1 more time
H shakes no more times.
The total number of shakes is

$7 + 6 + 5 + 4 + 3 + 2 + 1 = 28.$

This would be 8×7 shakes. But this number includes every handshake twice (as shown for A and B in Fig. 2). Hence the total number of handshakes is

$$\frac{8 \times 7}{2} = 28.$$

2.

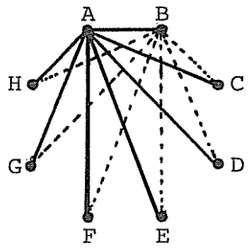


Fig. 3

While this configuration at first appears only as a two dimensional version of Fig. 1 it does offer much more scope in mathematical modelling.
The points representing the people could be considered as abstract points in a plane (no three of which are collinear). The solution then gives the number of line segments which can be drawn on this set of end points. If all the lines are drawn in as in Fig. 4 (which younger students enjoy doing in a range of seven colours) an octagon emerges (regular or otherwise) with all of its diagonals, the number of line segments less the number of sides: $28 - 8 = 20$.

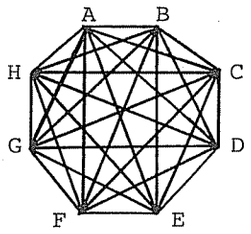


Fig. 4

This latter consideration leads, of course, to consideration of other polygons and the pattern of numbers of diagonals.

3. Diagrams 1 and 2 deal with all 8 of the people. The approach in Fig. 5 takes on the Polya technique of reducing the number of participants and developing a pattern which can be extended to the required number. (Here each line represents a person and points of intersection the handshakes. This is the dual of the situation in diagrams 1 and 2). As each line is added in Fig. 5 it must intersect with each of the previous lines. This creates a number of points of intersection one greater than the number of points created by the previously drawn line.

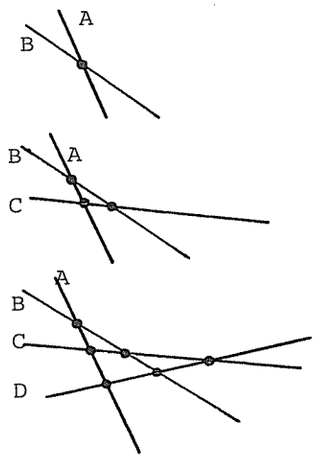


Fig. 5

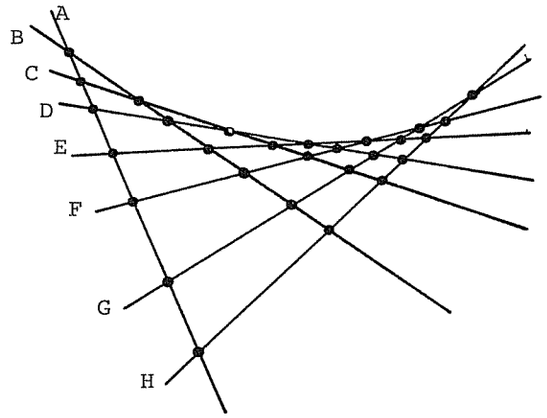


Fig. 6

Deductively, since
2 lines create 1 point
3 lines create (1 + 2) points, and
4 lines create (1 + 2 + 3) points, it appears to follow (and can be shown, Fig. 6) that
8 lines create (1+2+3+4+5+6 +7) points. As a bonus, we also have a solution to the geometric problem involving the maximum number of points of intersection of 8 lines in a plane.

4.

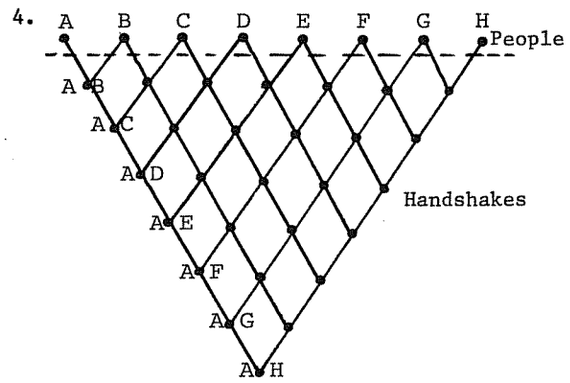


Fig. 7

Figure 7 provides a neat, legible analog of all the handshakes. The top row of dots represents the people. The lines emanating from each of these dots can be thought of as extended hands meeting other extended hands at the points of intersection. The triangular configuration in Fig. 7 shows that the number of handshakes is a triangular number (1, 3, 6, 10, 15, 21, 28, ...) regardless of the number of people involved. It also shows that each of these triangular numbers is a sum of the consecutive natural numbers starting with 1. Another advantage of Fig. 7 is that it can easily be extended for greater numbers of people.

5.

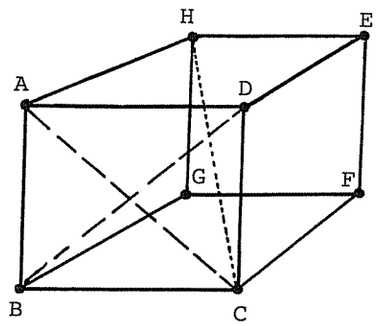


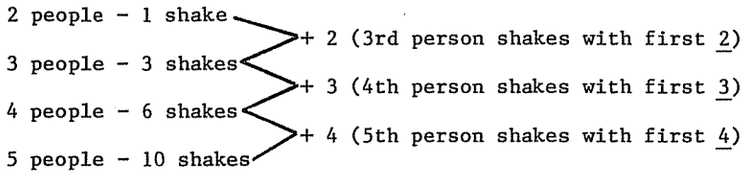
Fig. 8

The 8 people can be thought of as the verticies of a cube (Fig. 8). Each handshake is then one of the sides or diagonals of the cube and conversely.
Number of edges = 12
Number of diagonals or faces (e.g. AC, BD) = 6 × 2 = 12
number of internal diagonals (e.g. HC) = 4; Total = 28.
The disadvantage of this otherwise neat solution is that it is not readily generalizable.

II EXPERIMENT

A group of 8 people can actually go through the handshaking procedure. Difficulty usually arises in keeping track of who has shaken with whom until a system of shaking is devised.

One system again involves the Polya technique of reducing the number of people, identifying and extending a pattern.



By this system each newcomer shakes hands with all those who have already completed their handshakes. This means that the number of shakes increases by the number of people who have previously shaken.

Either the above pattern can be extended to 8 people or the actual shaking should be continued until a pattern becomes evident.

III ORGANIZED LIST

1. List of handshaking activity.

A shakes with B, C, D, E, F, G, H	(7 shakes)
B shakes with C, D, E, F, G, H*	(6 more shakes)
C shakes with D, E, F, G, H	(5 more shakes)
D shakes with E, F, G, H	(4 more shakes)
E shakes with F, G, H	(3 more shakes)
F shakes with G, H	(2 more shakes)
G shakes with H	(1 more shake)

(*B has already shaken with A.)

The above pattern could also have been used as a system in the EXPERIMENTAL Mode. It also relates directly to the approach in DIAGRAM I.

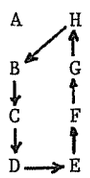
2. Round Robin.

The 8 people could be thought of as competitors in a round robin tournament in which everyone competes with everyone else. The number of competitions or matches would be the same as the number of handshakes.

A standard system for organizing a round robin tournament is shown below.

For round 2 (and subsequent rounds) A stays put while B, C, D, E, F, G, and H rotate as shown at the right. This procedure continues until the round 1 configuration would come up again.

1	2	3	4	5	6	7
A-H	A-G	A-F	A-E	A-D	A-C	A-B
B-G	H-F	G-E	F-D	E-C	D-B	C-H
C-F	B-E	H-D	G-C	F-B	E-H	D-G
D-E	C-D	B-C	H-B	G-H	F-G	E-F



The tournament requires 7 rounds of 4 games each; that is, 28 games.

At first glance this strategy would appear to apply only to an even

number of competitors. How would it apply to 7 players, say?

If there is no player H, then each of the above games involving H would be a bye for the indicated opponent.

1	2	3	4	5	6	7
A-BYE	A-G	A-F	A-E	A-D	A-C	A-B
B-G	BYE-F	G-E	F-D	E-C	D-B	C-BYE
C-F	B-E	BYE-D	G-C	F-B	E-BYE	D-G
D-E	C-D	B-C	BYE-B	G-BYE	F-G	E-F

This time there are still 7 rounds but only 3 games per round for a total of 21, precisely the number obtained by adding the first six natural numbers:

$$1 + 2 + 3 + 4 + 5 + 6 = 21.$$

IV TABLE

1. This is a tabular form of ORGANIZED LIST 1.

PEOPLE	A	B	C	D	E	F	G
SHAKING	(A,B)						
PAIRS	(A,C)	(B,C)					
	(A,D)	(B,D)	(C,D)				
	(A,E)	(B,E)	(C,E)	(D,E)			
	(A,F)	(B,F)	(C,F)	(D,F)	(E,F)		
	(A,G)	(B,G)	(C,G)	(D,G)	(E,G)	(F,G)	
	(A,H)	(B,H)	(C,H)	(D,H)	(E,H)	(F,H)	(G,H)

Note the similarity to a race track tote board for Quinella betting.

The triangular configuration is again evident.

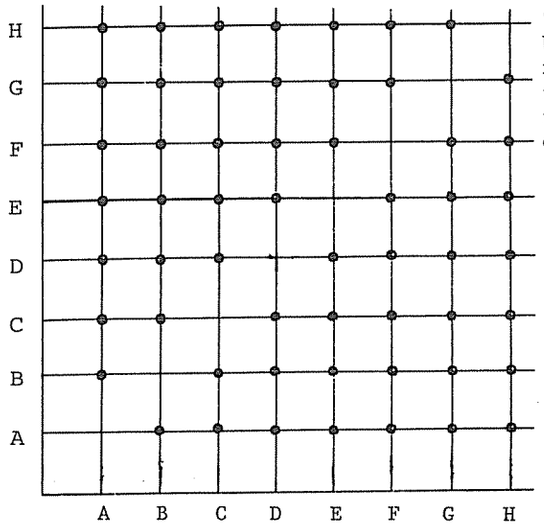
2.

SHAKES WITH	A	B	C	D	E	F	G	H
A	▨							
B		▨						
C			▨					
D				▨				
E					▨			
F						▨		
G							▨	
H								▨

▨ implies "no shake".

Since A shaking with B and B shaking with A creates two open spaces but only 1 shake, the number of shakes is <u>half</u> the number of open spaces; that is, $\frac{1}{2}(8 \times 7)$.

3.



The data in TABLE 2 could be displayed on a coordinate grid as shown at the right. Each dot on the grid represents an open space in TABLE 2.

V DEDUCTION

1. Two deductive arguments have been set out previously in conjunction with DIAGRAM 1.

2. The number of handshakes among 8 people is the number of pairs among 8 distinct things. From combinatorics, this number is

$$\binom{8}{2} = \frac{8!}{2!6!}$$

$$= \frac{8 \times 7}{2}$$

$$= 28.$$

As a matter of interest and transferabality, the textbook problem dealt with by LeBlanc is the following.

> A shopkeeper sold 485 bottles of cola in one day. The cola is packaged in cartons holding 6 bottles. How many cartons of cola were sold?

After attacking the handshakes problem, one group of elementary mathematics education preservice candidates produced nineteen approaches to the problem. May I invite you to use a similar introduction to a multi-pronged attack on textbook problems.

Recreationally yours,
H.L.R.

Faculty of Education
University of Toronto
Toronto, Ontario
M5S 2R7

THE FUNDAMENTAL DIFFICULTY WHICH THE THEORY OF THE INFINITE CONTAINS IS THIS: EITHER AN OUTRIGHT DENIAL OR AN OUTRIGHT ACKNOWLEDGEMENT OF THE BEING OF THE INFINITE LEADS TO MANY IMPOSSIBILITIES.

Aristotle

INFINITE SERIES

by
E.J. Barbeau

1. What is the value of the series?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots ?$$

Surely, it must be zero, for each +1 is cancelled by a −1. But by the same token, it might be 1, for taking apart the first +1, the remaining terms can be divided into pairs which cancel. Notationally:

$$(1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0$$

$$1 + (-1 + 1) + (-1 + 1) + \ldots \quad = 1$$

This is paradoxical. Perhaps the best thing, following Leibniz (1646-1716), is to effect a compromise, and take as the value the average of these two, namely $\frac{1}{2}$. This example is a signal to us that the taking of infinite sums might be a process of a different kind to the taking of finite sums. For example, it appears that the usual rules about bracketing might not hold for infinite sums.

Look at the question of evaluating a finite sum more critically. There are two ways of regarding the process, recursively and operationally. For example, in finding 35 + 21 + 62 + 12, the recursive approach requires us to take the first number, 35, add to it the second, 21, then the third, and so on, so that $35 + 21 + 62 + 12 = \big((35 + 21) + 62\big) + 12\big) = (56 + 62) + 12 = 118 + 12 = 130$. Although it seems as though the result is sensitive to the order in which the numbers are taken, it turns out to be independent of the order. Thus, it makes sense to ignore this aspect and regard summing as an operation which assigns to the set of summands (35, 21, 62, 12) its sum 130. For finite sums, it does not seem worthwhile to distinguish between the two views. However, for an infinite process, consider Zeno's paradox of Achilles and the tortoise:

> "Achilles and the tortoise have a race. The tortoise is allowed a handicap and is permitted to start a certain distance along the course ahead of Achilles. By the time Achilles reaches the starting point of the tortoise, the latter has moved to a second point. When Achilles reaches the second point, the tortoise has moved a little further, and so on, ad infinitum. Thus Achilles can never overtake the tortoise."

Resume of a talk delivered at the O.A.M.E. meeting at Toronto on May 10, 1980.

If we look at the situation as the statement of the paradox invites us to, that is piecemeal, then we have a conundrum indeed. However, if we alter our outlook and regard the race as a whole (not being concerned with a plethora of intermediate cases), we have no problem in convincing ourselves that indeed Achilles will pass the tortoise after a finite amount of time.

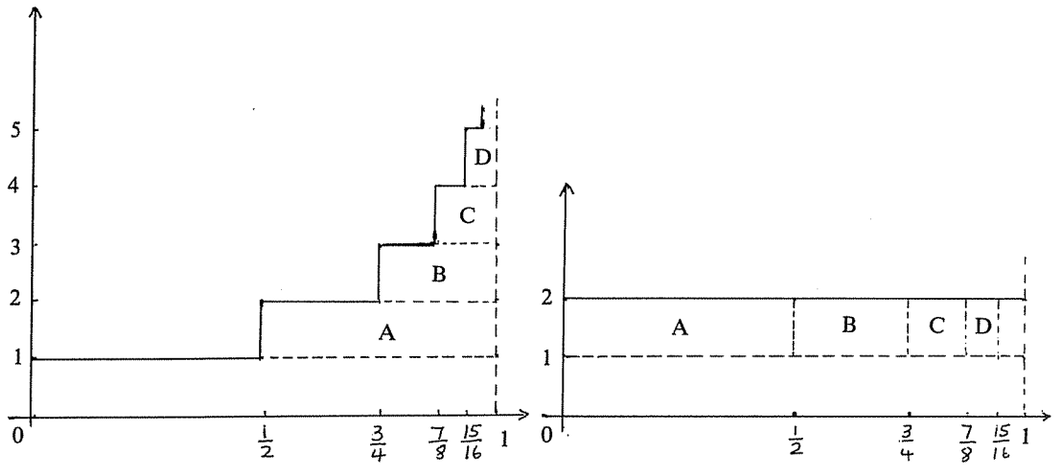
In like manner, mediaeval mathematicians were able to comprehend an infinite number of stages. Consider this proposition of Nicholas Oresme (ca. 1323-1382):

> "A body moving with uniform velocity for half a certain period of time, double the velocity for the next quarter, three times the velocity for the next eighth, and so on, covers the same distance as if it had moved at twice the original velocity over the whole period."

In other terms, if the original velocity is 1, then

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{n}{2^n} + \ldots = 2.$$

The result becomes clear with the aid of two velocity-time graphs illustrating the two motions described; recall that the distance travelled is given by the area under the graph.



Let us turn to the question of assigning a value to the infinite geometric progression: $1 - x + x^2 - x^3 + x^4 - \cdots$. The recursive outlook invites us to consider the result of adding in the terms in turn. When we have added terms to x^n, the partial sum is

$$\frac{1}{1 + x} + (-1)^n \frac{x^n + 1}{1 + x}.$$

Because we would never reach the end, we have to insert an extra ingredient not present in the finite sum case: the notion of limit. We notice that the difference between the partial sum and the quantity $(1 + x)^{-1}$ is an expression, which provided $|x| < 1$, becomes arbitrarily small as n grows. In this way, we can make the usual definition of the sum of an infinite series as the limit (if it exists) of the partial sums. This definition denies a value to any series for which there is no such limit, and we dismiss such series as being divergent.

But hold on! Look at $1 - x + x^2 - x^3 + \cdots$ operationally, as a whole. Multiplying by $1 + x$, we get a big cancellation:

$$(1 + x)(1 - x + x^2 - x^3 + \cdots) = 1 + x - x - x^2 + x^2 - x^3 + x^3 - x^4 + \cdots = 1$$

(this requires a suspension of disbelief about the validity of such an infinite multiplication). The series in a very real sense behaves like a multiplicative inverse of $1 + x$, and nothing has been said about $|x|$ having to be less than 1. From this point of view, we assert the value of $1 - x + x^2 - x^3 + \cdots$ to be $(1 + x)^{-1}$ for any x whatsoever on the grounds that the series behaves like $(1 + x)^{-1}$. Thus, taking $x = 1$, we obtain

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$

while taking $x = -2$ yields

$$1 + 2 + 4 + 8 + 16 + 32 + \cdots = -1.$$

In the eighteenth century, an equation such as the latter caused great consternation among mathematicians. As Leonard Euler (1707-1783) described it, "this is seen by opponents, not undeservedly, to be most absurd, since it is never possible to arrive at a negative sum through the addition of positive numbers." After dismissing the argument that the sum -1 can be accepted for $1 + 2 + 4 + 8 \cdots$ since there are two ways of passing from positive to negative numbers, either through 0 or through infinity, Euler nevertheless finds it valid to assign a value to the series. In his own words:

> "But I think all this wrangling can be easily ended if we should carefully attend to what follows. Whenever in analysis we arrive at an $\cdots$ expression, we customarily convert it into a suitable series on which the subsequent calculation can more easily be performed. $\cdots$ in a calculation it is valid to substitute in place of the infinite series that formula from which the series came. $\cdots$ Since this expression can always be substituted without error for the infinite series, both must have the same value."

However, not too long after this was written (1760), Callet gave an example to suggest that the matter was far from clear. Since

$$\frac{1}{1 + x + x^2} = \frac{1 - x}{1 - x^3} = (1 - x)\ (1 + x^3 + x^6 + \cdots)$$

$$= 1 - x + x^3 - x^4 + x^6 - x^7 + \cdots$$

we can take $x = 1$ to get $1 - 1 + 1 - 1 + 1 - 1 + \cdots = 1/3$. Even worse, for any positive integers m and n, we can obtain from the expansion of $1 + x + \cdots + x^{m-1}$ divided by $1 + x + \cdots + x^{n-1}$ the value m/n for the series $1 - 1 + 1 + \cdots$.

In the face of such anomalies, we might well give up any idea of assigning a value to any divergent series. But these series do actually arise in applications, and in the context of the situation do in fact correspond to some value. Furthermore, it is often possible to capture this value from the series without having recourse to a limit of partial sums. The function

$$u(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt$$

is sufficiently useful as to warrant a table of its values in the Handbook of Physics and Chemistry produced by the Chemical Rubber Publishing Company. We find, for example that $u(5) \doteq 0.00115$. While we can set up a convergent series expansion for this function, it is not useful in practice because it converges very slowly. However, consider the following manipulation (which would make the hair of a rigorous mathematician stand on end):

$$u(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = e^{-x}\int_0^\infty \frac{e^{-v}}{x+v}\,dv = e^{-x}\int_0^\infty \frac{e^{-v}}{x}\,\frac{1}{\left(1+\frac{v}{x}\right)}\,dv$$

$$= \frac{e^{-x}}{x}\int_0^\infty e^{-v}\left(1 - \frac{v}{x} + \frac{v^2}{x^2} - \cdots\right)dv.$$

Since $\int_0^\infty e^{-v}\cdot v^k dv = k!$

we obtain

$$u(x) = \frac{e^{-x}}{x}\left[1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots\right]$$

$$= e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots\right].$$

Accordingly,

$$u(5) = 0.006738(0.2 - 0.04 + 0.016 - 0.0096 + 0.00768$$

$$- 0.00768 + 0.0092160 - \cdots).$$

If we look at the series in brackets, we observe that the terms alternate in sign, and that, after the first six, actually increase in magnitude. This means that the partial sums of the first n terms will fluctuate more and more wildly as n increases, and so will not settle down to any limit. However, before we dismiss the series as meaningless, let us examine the situation more closely.

By way of comparison, turn to a series which is much better behaved. The theory of Fourier series tells us that (with the usual notion of the sum of a series as limit of partial sums)

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots = \begin{cases} -\frac{\pi}{4} & (-\pi < x < 0) \\ 0 & (x = 0) \\ \frac{\pi}{4} & (0 < x < \pi). \end{cases}$$

Substituting $\frac{1}{4}\pi$ for x gives us the Leibniz series:

$$\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots .$$

The partial sums 1.00000, 0.66667, 0.86667, 0.72381, 0.83492, 0.74401, 0.82094, ••• alternately over- and under-estimate the true value (0.78539815) of the series. (To get an intuitive feel, imagine having a graduated cylinder in which you put 1 cc., of fluid, withdraw 1/3 cc., pour in 1/5 cc., withdraw 1/7 cc., continuing indefinitely; what happens to the water level in the long run?) Since the terms of the Leibiniz series tend to zero, the partial sums have a limit, and this limit differs from a given partial sum by no more than the magnitude of the first omitted term. Thus, $\frac{1}{4}\pi$ is within 1/3 of 1, 1/5 of 0.66667, 1/7 of 0.86667, and so on.

In the series for u(5), we might think that the value for the expression in parentheses ought to be within 0.04 of 0.02, 0.016 of 0.2 -0.04, 0.0096 of 0.2 - 0.04 + 0.16, and so on. So, why not look at the partial sum which runs up to the term of smallest absolute value 0.00768? We find that

$$0.006738(0.2 - 0.4 + 0.016 - 0.0096) = 0.001121$$

which turns out to be remarkably close to u(5). If we play the same game for

greater values of x, taking the proper partial sum, we can get even better accuracy in estimating u(x). In fact, it is more practical often to use a divergent series in this way to compute a function than some other series representation which converges.

Extracting the value of even a convergent series can be a practical challenge. While it is true that $\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$, the series on the right side converges so slowly that hundreds of terms are required to get a partial sum agreeing with $\frac{1}{4}\pi$ to even three decimal places. In 1955, Daniel Shanks devised a remarkable method which will give the value of a series, convergent or divergent, to high accuracy, even when only a few initial terms are given. Let $\{s_1, s_2, \ldots, s_n, \ldots\}$ be the sequence of partial sums (s_n the sum of the first n terms) of the series. A new sequence $\{s'_n\}$ is formed using the formula

$$s'_n = \frac{s_{n+1}s_{n-1} - s_n^2}{s_{n+1} + s_{n-1} - 2s_n}.$$

It turns out that in many common situations, if the limit of s_n exists, then $\{s'_n\}$ will converge much more rapidly to the same limit, and if $\{s_n\}$ diverges, $\{s'_n\}$ may converge. This process of going from s_n to s'_n can be iterated as often as desired.

Look what Shanks' transformation does to the partial sums of Leibniz' series for $\frac{1}{4}\pi$:

Original partial sums	First iteration	Second iteration
1.00000		
0.66667	0.79167	
0.86667	0.78333	0.78552
0.72381	0.78631	0.78537
0.83492	0.78492	0.78543
0.74401	0.78568	0.78540
0.82094	0.78522	...
...	...	

The second term of the second iteration gives the values of the series to four decimal places. So in some mysterious way, the first six partial sums contain enough information to determine their limit to this incredible accuracy.

2. All of this seems pretty esoteric. Why should it concern high school teachers and students? Infinite series intervene in the curriculum in the study of geometric progressions and the Bionomial Theorem for fractional indices. Both of these have practical uses, the first in finding the present value of a perpetuity, the second in approximating fractional powers of numbers. Some students in the late high school years have quite a metaphysical outlook, and are fascinated by anything which smacks of the infinite. The question of attaching a value to an infinite series can lead to a good philosophical discussion, from which certain useful points might emerge:

(1) Finding an infinite sum is an essentially different sort of operation than finding a finite sum. Underlying it is an analytic notion (e.g. taking of limits) as well as a purely algebraic notion.

(2) Defining a sum of a series is a matter of choice; different definitions are available, and which is most suitable is often determined by the context.

(3) There is a great need for precision in expressing exactly what is meant by a concept. A student crippled by half-baked ideas is unable to make use of the mathematics he is taught. He must decide on a suitable formulation, know why it should be regarded as preferable, and express it clearly enough to be applicable in practical situations.

3. I have not mentioned the more standard alternative methods of defining the value of a series; this is a substantial branch of analysis known as summability theory, a good exposition of which is contained in the article by C.N. Moore listed below. Another fascinating area is that of convergence tests, means of deciding whether or not the sequence of partial sums possesses a limit. Here one can derive a family of tests of increasing strength, the ratio test, the root test, Raabe's test, Gauss' test, basically from knowledge of the behaviour of geometric progressions. The book by J.M. Hyslop gives a concise coverage of this area. Fourier and factorial series are other areas of interest; factorial series arise in using Newton's interpolation and extrapolation methods. For a treatment, the standard reference is the book on finite differences by L.M. Milne-Thomson. Finally, there are situations in which it is not appropriate to even consider attaching a value to a series, but just to manipulate them formally. The combinatorist and statistician with their generating functions make great play with this approach. For example, if $\binom{n}{r}$ is defined as the coefficient of x^r in the Binomial Expansion of $(1 + x)^n$, then comparing the coefficients of the expansions of the two sides of the equation

$$(1 + x)^{n+1} = (1 + x)^n(1 + x)$$

gives the standard identity $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$, $(1 \leq r \leq n)$.

All of these areas are rich and not inaccessible regions for some of your bright students to explore.

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WHEREVER THERE IS NUMBER THERE IS BEAUTY.

Proculus

DON'T ASK WHAT IT MEANS, BUT RATHER HOW IT IS USED. L. Wittgenstein

MODULAR ARITHMETIC (CONGRUENCES)

by
*Israel Kleiner**

Consider the following problems:

(1) What is the last digit of 2^{1000} ?

(2) What is the smallest prime divisor of $2^{82} + 1$?

(3) Show that 100 ... 01 can never be a perfect square, no matter how many zeros we put in between the extreme 1's.

(4) Determine whether 2561234198751 is divisible by 11, without dividing.

(5) Show that any odd integer raised to the power of any even integer always leaves a remainder of 1 when divided by 8.

(6) Prove that the equation $x^2 - 3y^2 = 2$ has no solutions in integers.

(7) The number 99 ... 9 is clearly not divisible by 2 or 5. We can make it divisible by any number (not containing a factor of 2 or 5) by taking the appropriate number of 9's. Show this.

(8) Show that if (a,b,c) is any integer solution of $x^2 + y^2 = z^2$ then a and b cannot both be odd.

9) $\sqrt{2}$ is irrational - prove.

These problems and many others can readily be solved by what is essentially a notational device for divisibility called "congruence", or "modular arithmetic". It was introduced by the great mathematician Gauss in 1801 in his famous work on number theory entitled "Disquisitiones Arithmeticae". We will introduce the concept via "clock arithmetic", thus: what time will a clock show 4 hours after 10 o'clock? The answer, of course, is 2 o'clock. What we are essentially saying is that 10 + 4 is the same as 2 when dealing with the arithmetic of telling time. To distinguish this notationally from ordinary arithmetic we write $10 + 4 \equiv 2$, where the three horizontal bars "≡" are called "congruence". Moreover, to indicate that we are dealing with clock arithmetic (as there will be other occasions to use the symbol ≡) we add "modulo 12" or simply "mod 12". We thus write $10 + 4 \equiv 2 \pmod{12}$. Similarly, to indiciate that 29 hours after 6 o'clock is 11 o'clock we write $6 + 29 \equiv 11 \pmod{12}$. Note that in each case the difference

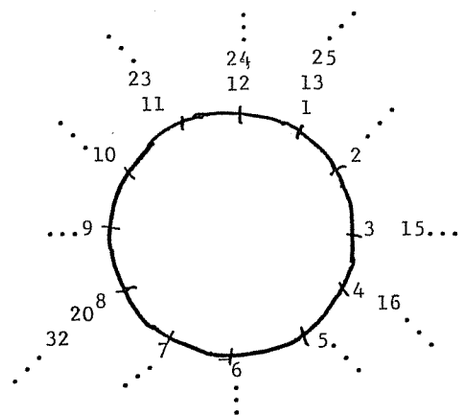


Fig. 1

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* Some of the following material has been presented to mathematics clubs and as "enrichment" material to students from grades seven on. I hope that teachers will find it useful.

between the left hand side and right hand side of the congruence (i.e. (10 + 4) - 2 and (6 + 29) - 11) is divisible by the "modulus" in the brackets, namely 12. (See Fig. 1.)

The same idea can be used in determining, for example, what day of the week it is 16 days after Wednesday. It is clearly Friday, and we can easily settle questions of this type by constructing a "days of the week arithmetic", i.e. an arithmetic modulo 7.[9] We would associate with Sunday the number 1 say, with Monday 2, etc. continuing cyclically in groups of seven. The above problem about the days of the week would then be denoted as $4 + 16 \equiv 6 \pmod 7$, where in this case $(4+16) - 6$ is divisible by 7. Similarly, to find what day of the week it is 183 days after Monday, we would need to find a number x (between 1 and 7) so that $2 + 183 \equiv x \pmod 7$, or so that $185 - x$ is divisible by 7. One readily discovers that such a number is 3 (how can you find that systematically?) so that the desired day is Tuesday. (See Fig. 2)

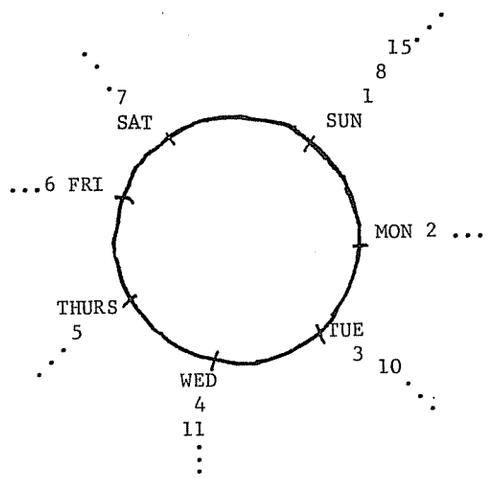


Fig. 2

These are but two "obvious" instances of the very useful concept of congruence which we now define in general. Given two integers a and b (positive, negative or zero) and an integer $m > 1$ we say that a is <u>congruent</u> to b modulo m, and write $a \equiv b \pmod m$ if $m \mid a - b$ (i.e. $a - b$ is divisible by m). This, then, is a "clock" arithmetic with m "hours". The congruence relations "$\equiv$" has many of the same properties as the equality relation "=". Thus, (i) $a \equiv a \pmod m$, (ii) if $a \equiv b \pmod m$ then $b \equiv a \pmod m$, (iii) if $a \equiv b \pmod m$ and $b \equiv c \pmod m$ then $a \equiv c \pmod m$. (Any relation satisfying these three properties is called an <u>equivalence relation</u>, a very important concept in mathematics.) (iv) If $a \equiv b \pmod m$ then $a \pm b \equiv b \pm c \pmod m$ for any integer c. Unlike equality, however, we cannot claim that $\frac{a}{b} \equiv \frac{b}{c} \pmod m$ even if $c \not\equiv 0 \pmod m$. For example, $42 \equiv 18 \pmod 8$ but $\frac{42}{6} \not\equiv \frac{18}{6} \pmod 8$. Try to discover under what conditions one can divide by c.

An equivalent way of saying that $a \equiv b \pmod m$ is to say that a and b leave the same remainder on division by m. (You can prove this by making use of the following important property of intergers: if a and by are integers and $b \neq 0$, then there exist integers q and r such that $a = bq + r$, where $0 \leq r < b$.) A useful and immediate corollary of this formulation is that if $a \equiv b \pmod m$ then $m \mid a$ if and only if $m \mid b$.

We can express divisibility by means of congruence: $a \mid b$ just means $b \equiv 0 \pmod a$. In fact, as noted, congruence represents a symbolic language for divisibility (the basic concept of number theory) - it is a kind of "algebraization" of number theory. This idea gives rise to a new kind of arithmetic for the integers, the so called "modular arithmetic", in which the ordinary relation of equality is replaced by a "weaker" type of "equality" called congruence. (The idea of "equating" or "identifying" objects which are not identical is an important and not uncommon one in mathematics. e.g. In congruence geometry one identifies congruent figures, in similarity geometry one identifies similar figures; in defining the rational numbers one identifies "equivalent" fractions, such as $\frac{4}{6}$ and $\frac{6}{9}$. Note that all these are

examples of equivalence relations.) How powerful that simple idea is we shall see now by solving the problems posed at the beginning of this section. Others will be given at the end of the article.

(1) We first phrase the question in terms of congruences. It is clear that any number is congruent to its last digit modulo 10. Thus, we are trying to find a number x (between 0 and 9) such that $2^{10000} \equiv x \pmod{10}$. Now, $2^4 = 16$ and $16 \equiv 6 \pmod{10}$. Thus, $2^{1000} = (2^4)^{250} \equiv 6^{250} \pmod{10}$. Moreover, $6^2 = 36 \equiv 6 \pmod{10}$ and it becomes clear, in fact, that $6^k \equiv 6 \pmod{10}$ for every positive integer k. Thus $2^{1000} \equiv 6 \pmod{10}$ and the last digit of 2^{1000} is 6.

(2) We see that $2^{82} + 1 \equiv 1 \pmod{2}$ and $2^{82} + 1 \equiv 4^{41} + 1 \equiv 1^{41} + 1 \equiv 2 \pmod{3}$, but $2^{82} + 1 = (2^2)^{41} + 1 = 4^{41} + 1 \equiv (-1)^{41} + 1 = 0 \pmod{5}$. That is, 5 is the smallest prime divisor of $2^{82} + 1$.

(3) Let 100 ... 01 = x (with some given number of zeros). We are asking if we can have $x = a^2$ for some integer a. We first determine what number x is congruent to mod 3. (The reason we choose 3 rather than some other number is because this choice works! - i.e. gives the desired result.) Now, $x = 100 \ldots 01 = 10^n + 0.10^{n-1} + \ldots + 1$, for some n, and since $10 \equiv 1 \pmod{3}$, then also $10^n \equiv 1^n = 1 \pmod{3}$. Thus $x \equiv 1 + 1 = 2 \pmod{3}$. Now, if a is any integer then $a \equiv 0$, 1 or 2 (mod 3). (This is clear if you think about it for a moment, since it says merely that any integer a leaves a remainder of 0,1 or 2 on division by 3.) Thus $a^2 \equiv 0^2$, 1^2 or $2^2 \pmod{3}$ and since $2^2 = 4 \equiv 1 \pmod{3}$ we have that $a^2 \equiv 0$ or $1 \pmod{3}$. But we have shown that $x \equiv 2 \pmod{3}$ and hence we cannot have $x = a^2$ since clearly 0 or 1 cannot be congruent to 2 (mod 3).

(4) We consider, more generally, the problem of divisibility of an integer by 11. Thus, given any positive integer $x = a_n.10^n + a_{n-1}.10^{n-1} + \ldots + a_1.10 + a_0$ (i.e. x is an integer with digits $a_n, a_{n-1}, \ldots, a_0$) we note that $10 \equiv -1 \pmod{11}$, $10^2 \equiv (-1)^2 = 1 \pmod{11}$, $10^3 \equiv (-1)^3 = -1 \pmod{11}$ etc. Thus $x = a_n.10^n + a_{n-1}.10^{n-1} + \ldots + a_1.10 + a_0 \equiv (\pm a_n \mp a_{n-1} \pm a_{n-2} \mp \ldots - a_1 + a_0) \pmod{11}$. i.e. x is congruent to the alternate sum and difference of its digits (mod 11), which implies that x is divisible by 11 if and only if alternate sum and difference of its digits is divisible by 11. In the particular case where x = 2561234198751, we form 2-5+6-1+2-3+4-1+9-8+7-5+1 = 8 and since 8 is not divisible by 11, neither is x.

(5) We wish to show that $(2m + 1)^{2n} \equiv 1 \pmod{8}$. Since any integer is congruent to 0,1,2,...,6 or 7 (mod 8) (cf. problem 3), and since 2m + 1 is odd, it follows that $2m + 1 \equiv 1,3,5$ or 7 (mod 8). Hence $(2m + 1)^{2n} \equiv (1^2)^n$, $(3^2)^n$, $(5^2)^n$ or $(7^2)^n \equiv 1^n$, 1^n, 1^n or $1^n \equiv 1 \pmod{8}$.

(6) If the equation $x^2 - 3y^2 = 2$ had a solution x_0, y_0 say, then $x_0^2 - 3y_0^2 = 2$, or $x_0^2 = 2 + 3y_0^2$. It follows that $x_0^2 \equiv 2 \pmod{3}$. But we have seen in problem 3 above that no integer x_0 can have this property (since either $x_0^2 \equiv 0 \pmod{3}$ or $x_0^2 \equiv 1 \pmod{3}$). Thus no x_0 exists satisfying the equation $x^2 - 3y^2 = 2$. This example is but one instance in an important branch of number theory known as diophantine equations, which deals with solutions in integers of equations with two or more unknowns.

(7) We will show, for example, that 237|99...9 (for the appropriate number

of 9's). The general case will be apparent from this example.

Consider the 238 numbers 10, 10^2, 10^3 ... 10^{238}. At least two of them must leave the same remainder when all are divided by 237 (there can only be 237 distinct remainders), hence these two numbers must be congruent to each other (mod 237). That is, there are positive integers m and n, with $m > n$ say, so that $10^m \equiv 10^n \pmod{237}$, hence $237 \mid 10^m - 10^n = 10^n(10^{m-n} - 1)$. Since 237 and 10^n have no common divisors except 1, it follows that $237 \mid 10^{m-n} - 1$. (This may not be obvious - think about it.) But since $m > n$, $10^{m-n} - 1$ is a sequence of $m - n$ 9's, and we are done.

(8) If m is an even integer, say $m = 2k$, then $m^2 = 4k^2 \equiv 0 \pmod 4$. If m is odd, say $m = 2t + 1$, then $m^2 = 4t^2 + 4t + 1 \equiv 1 \pmod 4$. Thus if a and b are both odd, then a^2, $b^2 \equiv 1 \pmod 4$ and $a^2 + b^2 \equiv 2 \pmod 4$. But $c^2 \equiv 0$ or $1 \pmod 4$, and since $2 \not\equiv 0$ or $1 \pmod 4$, hence $a^2 + b^2 \neq c^2$.

(9) Suppose $\sqrt{2}$ is rational, i.e. $\sqrt{2} = \frac{a}{b}$, where a and b are integers and $b \neq 0$. We can, moreover, assume that a and b have no common factor (by cancelling a common factor from a and b if necessary). Then $2 = \frac{a^2}{b^2}$, or $a^2 = 2b^2$. Now $a^2 \equiv 0$ or $1 \pmod 3$ (see problem (3)) and $2b^2 \equiv 0$ or $2 \pmod 3$, and since $a^2 \equiv 2b^2$ we must have $a^2 \equiv 0 \pmod 3$ and $2b^2 \equiv 0 \pmod 3$ and hence $b^2 \equiv 0 \pmod 3$ (why?). Thus $3 \mid a^2$ and $3 \mid b^2$ which implies $3 \mid a$ and $3 \mid b$ (since 3 is a prime). This contradicts the assumption that a and b have no common factor.

Remark. Although the above problems can be solved in various ad hoc ways, their solution via congruences is conceptual and relatively simple. This is an excellent example of what a "good" notation can do for the solution of a mathematical problem. In fact, it has been said that a good notation goes more than half the way toward the solution of a problem. (The test of a "good" notation is in its suggestiveness and fruitfulness.) Some other examples of a good notation are: (a) The introduction of symbolism in algebra in the 16th and 17th centuries. (Imagine solving equations by having to write out all symbols, relations and operations.) (b) The concept of a matrix, originally introduced (in the 19th century) as a convenient notational device for a system of linear equations, has since "invaded" various areas of mathematics and its applications. (c) The use of the suggestive Leibniz notation for differentiation and integration (still in use today) which simplified considerably the algorithms of the calculus.

Here are some additional problems. They too can be solved using congruences.

(10) Find the remainder on division of (a) 6^{48} by 13 (b) 16! by 17.

(11) (a) If a is any integer and $\bar{a}$ is an integer with the same digits as a but in reverse order, show that $a - \bar{a}$ is divisible by 9. (Note that an integer is divisible by 9 if and only if the sum of its digits is divisible by 9; prove this.)

(b) An integer is called a palindrome if $\bar{a} = a$, (e.g. 33, 252). Show that a four digit palindrome is divisible by 11. Try to generalize.

(12) (a) Show that no square can have its last two digits equal to 79.

(b) If the units digit of a square ends in a 9, what are the possibilities for the tens digit?

(c) Find all possible two digit endings for squares.

(13) General Question: Try to determine which of the results as given in these 12 problems are a consequence of the special properties of the number system we use (i.e. positional, base 10), and which are intrinsic to the integers (i.e. independent of any particular number system used to represent them).

The following references provide additional results and further applications of congruences, as well as other interesting topics in number theory.

1. I.A. Barnett, Elementary Number Theory; Prindle, Weber & Schmidt, 1969.

2. C.S. Ogilvy and J.T. Anderson, Excursions in Number Theory; Oxford University Press, 1966.

3. O. Ore, Number Theory and its History; McGraw Hill, 1948.

4. O. Ore, Invitation to Number Theory; Random House, 1967.

5. H.M. Stark, Introduction to Number Theory; Markham Publishing, 1970.

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MULTIPLICATIVELY YOURS

by

M. Elaine Harvey

Premise: everyone wants affection
Some of the time.
I want it on my terms.
He wants it on his.

If I want it half the time,
And he wants it half the time,
We are likely to coincide
One-quarter of the time.
But if my offering is rejected
My wanting is reduced by half.

If I want it one-quarter the time
And he wants it half the time,
We are likely to coincide
One-eighth of the time.
But if he offers it clingingly,
My wanting is reduced by half.

If I want it one-eighth the time
And he wants it half the time,
We are likely to coincide
One-sixteenth of the time
But if his offering is rejected,
His wanting is reduced by half.

If I want it one-eighth the time
And he wants it one-quarter the time,
We are likely coincide
One-thirtysecond of the time.
But if I offer it clingingly,
His wanting is reduced by half.

If I want it one-eighth of the time
And he wants it one-eighth the time,
We are likely to coincide
One-sixtyfourth of the time.

But I did not begin by wanting it
Half of the six hours
We spend together in a day --
Nor did he.
Half an hour each
Would be generous --
Say, one-twelfth of the time
We spend together for each of us.

We are likely to coincide
One-twothousandthreehundredandfourth
of the time --

Nine seconds per day!

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I TELL THEM THAT IF THEY OCCUPY THEMSELVES WITH THE STUDY OF MATHEMATICS THEY WILL FIND IN IT THE BEST REMEDY AGAINST THE LUSTS OF THE FLESH.

Thomas Mann (Magic Mountain)

DEPARTMENTS

TAKE MATHEMATICS. HOW CAN YOU SHORTEN THE SUBJECT? THAT STERN STRUGGLE WITH THE MULTIPLICATION TABLE, FOR MANY PEOPLE NOT YET ENDED IN VICTORY, HOW CAN YOU MAKE IT LESS? SQUARE ROOT, AS OBDURATE AS A HARDWOOD STUMP IN A PASTURE -- NOTHING BUT YEARS OF EFFORT CAN EXTRACT IT. YOU CAN'T HURRY THE PROCESS.

*****

OR PASS FROM ARITHMETIC TO ALGEBRA: YOU CAN'T SHOULDER YOUR WAY PAST QUADRATIC EQUATIONS OR RIPPLE THROUGH THE BINOMIAL THEOREM. INDEED, THE OTHER WAY; YOU FEET ARE IMPEDED IN THE TANGLED GROWTH, YOUR PACE SLACKENS, YOU SINK AND FALL SOMEWHERE NEAR THE BINOMIAL THEOREM WITH THE CALCULUS IN SIGHT ON THE HORIZON. SO DIED, FOR EACH OF US, STILL BRAVELY FIGHTING, OUR MATHEMATICAL TRAINING; EXCEPT ONLY FOR A SET OF PEOPLE CALLED "MATHEMATICIANS" -- BORN SO, LIKE CROOKS.

Stephen Leacock

PROBLEMS? DID YOU SAY

by
Pat Yip

Notice:
Our problems department is open to all readers. Submissions with or without solutions are solicited. Problems submitted should preferably illustrate some mathematical concept and be of general interest to our readers. Occasionally, problems marked with * will appear; for these, we invite readers to send in solutions. We shall publish the names of the first ten who submit correct solutions. Solutions to problems in this issue will appear in the next issue.

Came across a good one lately? SEND IT IN! (to P. Yip, Dept. of Math. Sciences, McMaster University, Hamilton, Ontario).

20.1.1 **Circles and Such** (submitted by P. Weygang):
With reference to the diagram, prove that,

$$(AB)^2 + (GH)^2 = (EF)^2 + (CD)^2.$$

0 and 0' are the centers of the larger and smaller circles respectively.

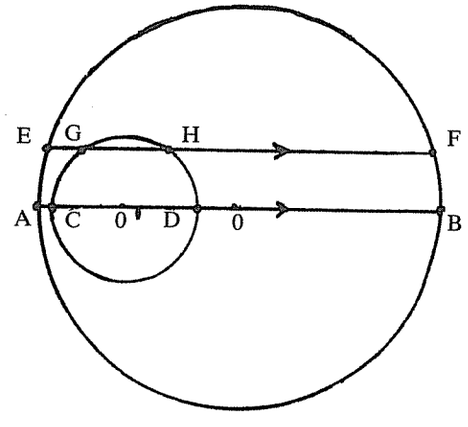


Diagram 1

20.1.2 **The Despised Number** (submitted by H.P. Heinig):
The number 17, Plutarch (40 AD-120AD) wrote, was most despised by the Pythagoreans, not only because it was that day of the month that the death of Osiris[1] occurred, but also because 17 lies between the "square" number 16 and the "rectangular" number 18. They

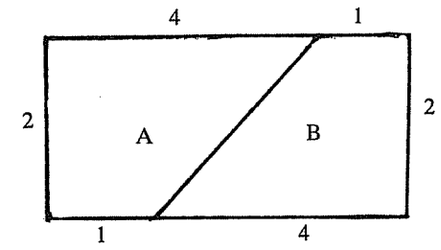
knew that these two numbers are the only natural numbers which are both the area and perimeter of a square and a rectangle (with side lengths a natural number) respectively. Can you show this?

1Osiris was the ancient Egyptian God whose annual death and resurrection personified the self renewing vitality of nature.

20.1.3 **What Heights We Reach:**
It is well-known that in an **equilateral** triangle, the sum of the perpendicular distances from any point **inside** the triangle to the sides is equal to the height of the triangle. Find a similar relation between the sum of the perpendicular distances from any point **inside** a **regular tetrahedron** to the four faces, and the height of the tetrahedron.

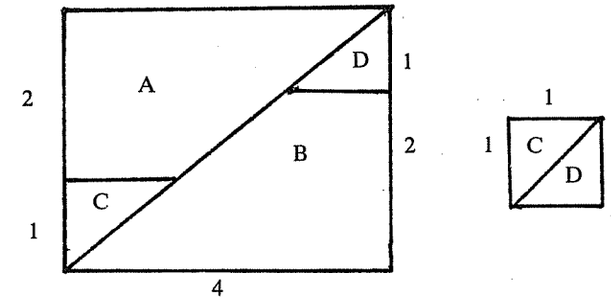
20.1.4 **Where, Oh Where, is the hole?**
To fill in a rectangular hole of 3m x 4m, a carpenter uses two pieces of board; one is 2m x 5m and the other 1m x 1m. These are cut as shown:

Diagram 2



He then re-arranges the four cut pieces thus:
Where is the hole?

Diagram 3



20.1.5 **Roots Without End:**

If $x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$, find x to 2 decimal places.

REPORT ON THE O.A.M.E. SCHOLARSHIP COMPETITION

When the 5th Annual O.A.M.E. Conference was held at McMaster University in May 1978, McMaster made a donation of $3,000 to the running of the conference. As it turned out, these funds

were not needed and the conference organizers decided to return them to the university but in the form of a scholarship.

The O.A.M.E. Scholarship is open to any student of an Ontario secondary school who is entering McMaster University and may be held in addition to any other scholarship. Its value is $600 and will be awarded in the six years from 1980 to 1985. The winner is decided on the basis of an annual mathematical competition held at McMaster. In addition to the scholarship, book prizes are awarded to the top 5 grade 13 students and the top 5 grade 12 students.

The 1980 Competition was written by 113 students on April 26, 1980. A list of the winners and their schools follow.

1. George Tolomiczenko	Etobicoke C.I.
2. John Im	Toronto French School
3. Anand Sohle	T.L. Kennedy S.S., Mississauga
4. Hing-nam Seto	Harbord C.I., Toronto
Philip Chan	Cardinal Newman S.S., Scarborough
6. Sai-kit Eng	Hamilton C.I.
Chang-ki Min	Hamilton C.I.
Isidro Ibasco	Notre Dame C.S., Welland
Woon-kee Chung	East York C.I.
10. Paul Beckwith	Queen Elizabeth Park, Oakville
Tony Hui	Beamsville District S.S.

The 1981 Competition was written by 87 students on March 14, 1981. The winners are as follows:

1. Stephen Worotynec	University of Toronto Schools
2. Yat-Yan Hui	De La Salle College, Toronto
3. Stephen Ko	North Toronto C.I.
4. Wan Loi Wong	Columbia Secondary School, Hamilton
5. Terence Yau	Bloor Collegiate, Toronto
6. Frederick Ying-shu Chan	De La Salle College, Toronto
7. John Robert Maynard	Beamsville District S.S.
8. William Tsoi	Senator O'Connor College School, Don Mills
9. Ward Burchell	Grantham High, St. Catharines
10. Jan Gray	Waterloo Collegiate Inst.

The 1982 Competition will be written at McMaster University on April 24, 1982. Details will be sent to the heads of mathematics departments at all Ontario secondary schools.

The questions from the 1980 and 1981 competitions, together with their solutions, are given below.

1980 COMPETITION PROBLEMS

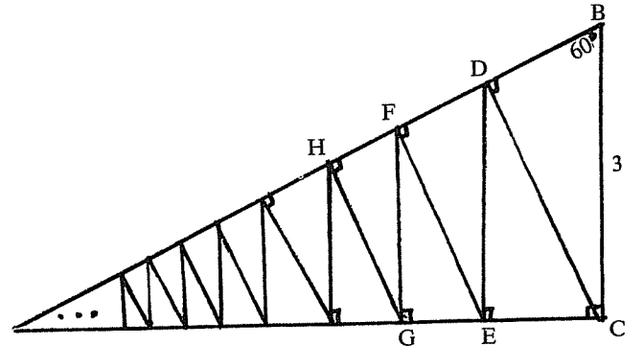
1. Find all values of x such that $\log_5 (x^2 - x) = 2$.

2. Find all values of x such that $\cos 2x = \cos x$.

3. What is the final digit in the number 563^{991} (when multiplied out)?

4. Dan gives Jim as many dollars as Jim has and Sue as many dollars as Sue has. Jim gives Dan and Sue as many dollars as each then has. Sue gives Dan and Jim as many dollars as each then has. Each of them ends up with 16 dollars. How many dollars did each of them have to start with?

5.



A right triangle ABC is given with $\angle B = 60^\circ$ and BC = 3 cm. CD is drawn perpendicular to AB, DE is drawn perpendicular to AC, EF⊥AB, FG⊥AC, and this process is continued indefinitely. Find the total length of all the perpendiculars.

$$BC + CD + DE + EF + FG + \ldots$$

6. Prove that every perfect square can be written in the form 4n or 4n + 1 where n is an integer, i.e., every square, when divided by 4, leaves remainder 0 or 1.

SOLUTIONS

1. $\log_5(x^2 - x) = 2 \iff x^2 - x = 5^2 = 25$

$\iff x^2 - x - 25 = 0 \iff x = (1 \pm \sqrt{101})/2$

2. If $\cos x = \cos 2x = 2\cos^2 x - 1$, then

$2\cos^2 x - \cos x - 1 = (\cos x - 1)(2\cos x + 1) = 0$

so $\cos x = 1$ or $\cos x = -1/2$, which gives $x = 2n\pi$ or

$x = \pm 2\pi/3 + 2n\pi$, where n is an arbitrary integer.

3. The final digits in the successive powers of 563 (or any number ending in 3) are 3, 9, 7, 1, 3, 9, 7, 1, 3, 9, 7, 1, ...
Thus the final digit in 563^n will be 3, 9, 7, or 1, depending on whether the remainder after dividing n by 4 is 1, 2, 3, or 0.

Since $\frac{991}{4} = 247\frac{3}{4}$, the final digit in 563^{991} is 7.

4. **METHOD:** (i) Suppose that initially Dan, Jim, and Sue have x, y, and z dollars. The following chart gives the number of dollars at each stage:

Stage	Dan	Jim	Sue
0	x	y	z
1	x - y - z	2y	2z
2	2x - 2y - 2z	2y - (x - y - z) - 2z = -x + 3y - z	4z
3	4x - 4y - 4z	-2x + 6y - 2z	4z - (2x - 2y - 2z) - (-x + 3y - z) = -x - y + 7z

$$\left.\begin{aligned} 4x - 4y - 4z &= 16 \\ -2x + 6y - 2z &= 16 \\ -x - y + 7z &= 16 \end{aligned}\right\}$$

Solving this system of equations, we get
$x = 26,\ y = 14,\ z = 8$

4. **METHOD** (ii) A few contestants used an ingenious method of working backwards to find the answer without having to solve a system of equations. Remembering that at each stage the total amount of money must be \$48, we can construct the following chart:

Stage	Dan	Jim	Sue
3	16	16	16
2	8	8	32
1	4	28	16
0	26	14	8

Note: Question 4 was first proposed by the great mathematician Leonard Euler in a textbook that he wrote in the 1740's.

5. Referring to the diagram, we have

$$CD = BC \sin 60^\circ = 3 \cdot \frac{\sqrt{3}}{2}$$

$$DE = CD \sin 60^\circ = 3 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = 3\left(\frac{\sqrt{3}}{2}\right)^2$$

$$EF = DE \sin 60^\circ = 3\left(\frac{\sqrt{3}}{2}\right)^2 \cdot \frac{\sqrt{3}}{2} = 3\left(\frac{\sqrt{3}}{2}\right)^3$$

$$BC + CD + DE + EF + FG + \ldots$$

$$= 3 + 3\,\frac{\sqrt{3}}{2} + 3\left(\frac{\sqrt{3}}{2}\right)^2 + 3\left(\frac{\sqrt{3}}{2}\right)^3 + 3\left(\frac{\sqrt{3}}{2}\right)^4 + \ldots$$

$$= \frac{3}{1 - \sqrt{3}/2} = \frac{6}{2 - \sqrt{3}} = 12 + 6\sqrt{3}$$

Here we have used the formula for the sum of a geometric series with $a = 3$, $r = \sqrt{3}/2$.

6. For a perfect square m^2, there are 2 cases. If m is even, then $m = 2k$ (k an integer) so $m^2 = 4k^2$ (and k^2 is an integer). If m is odd, then $m = 2k + 1$ (k an integer) so

$$m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

and $k^2 + k$ is an integer.

1981 COMPETITION PROBLEMS

1. Find all values of x such that

$$\log_{10}\left(x^{\log_{10}x}\right) + \log_{10}(x^2) = 8$$

2. Find all values of x such that $|x + 1| + |x - 2| < 7$.

3. (i) If $x^3 + ax + b$ is divisible by $x - 1$, find the value of $a + b$.

(ii) If $x^3 + ax + b$ is divisible by $(x - 1)^2$, find the value of $a - b$.

4. How many integers are there from one to a million (inclusive) which are either perfect squares or perfect cubes (or both)?

5. In a right triangle, the hypotenuse has length 13 cm and another side has length 12 cm. What is the length of the altitude which is perpendicular to the hypotenuse?

6. There are two tangent lines to the parabola $y = x^2 + x$ which pass through the point $(2, -3)$. At which points do these tangent lines meet the parabola?

7. The Fibonacci sequence F_n (for $n = 1, 2, \ldots$) satisfies

$F_{n+2} = F_{n+1} + F_n$, $F_1 = 1$, $F_2 = 1$. Hence $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$.

Show that $F_5, F_{10}, F_{15}, \cdots, F_{5m}, \cdots$ are each divisible by 5.

For $m \geq 1$ let $G_m = \frac{1}{5} F_{5m}$, so $G_1 = 1$, $G_2 = 11, \cdots$

Find a relationship among G_{m+2}, G_{m+1}, G_m, which is true for every positive integer m.

SOLUTIONS

1. Using the Laws of Logarithms, we have

$$(\log_{10}x)^2 + 2\log_{10}x - 8 = (\log_{10}x + 4)(\log_{10}x - 2) = 0.$$

Thus $\log_{10}x = -4$ or $\log_{10}x = 2$ which gives $x = 10^{-4}$ or 10^2.

2. If $x > 2$, $|x + 1| + |x - 2| = x + 1 + x - 2 = 2x - 1$ which is <7 when $x < 4$. If $-1 \leq x \leq 2$, then $|x + 1| + |x - 2| = x + 1 + 2 - x = 3$ which is always < 7. If $x < -1$, then $|x + 1| + |x - 2| = -x - 1 + 2 - x = -2x + 1$ which is <7 when $x > -3$. Thus the solution of the given inequality is $-3 < x < 4$.

3. (i) If $f(x) = x^3 + ax + b$, then $f(1) = 1 + a + b$. By the Factor Theorem, $f(x)$ is divisible by $x - 1$ if $f(1) = 0$, so $a + b = -1$.

(ii) If $f(x)$ is divisible by $(x - 1)^2$, then we can write $x^3 + ax + b = (x - 1)^2(cx + d)$. Comparing with the left side, we see that $c = 1$ and $d = b$. Therefore

$$x^3 + ax + b = (x - 1)^2(x + b) = x^3 + (b - 2)x^2 + (1 - 2b)x + b$$

The coefficient of x^2 must be 0, so $b - 2 = 0$. Thus $b = 2$ and, from (i), $a = -b - 1 = -3$. Therefore $a - b = -3 - 2 = -5$. [This question could also be done by long division.]

4. The perfect squares $\leq 10^6$ are the squares of the numbers from 1 to 10^3, so there are 1000 squares. The perfect cubes $\leq 10^6$ are the cubes of the numbers from 1 to 10^2, so there are 100 cubes. Included in both of the above sets are the 10 perfect sixth powers $1^6, 2^6, \ldots, 10^6$. Thus there are $1000 + 100 - 10 = 1090$ numbers from 1 to 10^6 which are either squares or cubes.

5. There are many ways to solve this problem (e.g., similar triangles or trigonometry) but the simplest and most elegant is to compute the area in 2 ways. The Pythagorean Theorem gives the length of the third side as 5 cm. So the area is $A = \frac{1}{2} \cdot 5 \cdot 12$. But if the required altitude is h, then we also have $A = \frac{1}{2} \cdot 13 \cdot h$. Thus $\frac{1}{2} \cdot 13 \cdot h = \frac{1}{2} \cdot 5 \cdot 12$

Solve for h: $h = \frac{5 \cdot 12}{13} = \frac{60}{13}$ cm.

6. **METHOD** (i) (using calculus)

Let $P(a, a^2 + a)$ be a point where the tangent from $(2\ -3)$ meets the parabola. Then the slope of the tangent is $m = (a^2 + a + 3)/(a - 2)$

But since $y = f(x) = x^2 + x \Rightarrow f'(x) = 2x + 1$, the slope of this tangent must be $f'(a) = 2a + 1$. Thus $(a^2 + a + 3)/(a - 2) = 2a + 1$. Solve for a.

$a^2 + a + 3 = 2a^2 - 3a - 2 \Rightarrow a^2 - 4a - 5 = (a - 5)(a + 1) = 0$, $a = 5$ or -1.

Therefore the points where the tangent meets the curve are $(5, 30)$ and $(-1, 0)$.

METHOD (ii) (without using calculus)

A tangent line is of the form $y + 3 = m(x - 2)$, where m is the slope.

It meets the parabola when $x^2 + x = m(x - 2) - 3$. Solving this quadratic equation, we get $x = \frac{m - 1 \pm \sqrt{(1 - m)^2 - 4(2m + 3)}}{2}$. For a tangent, there is only one point of intersection, so the discriminant $(1 - m)^2 - 4(2m + 3) = 0$.

This gives $m = 11$ or -1, leading to the same points as in Method (i).

7. (i) $F_5 = 5$ is divisible by 5. Suppose F_{5k} is divisible by 5. Using the given relation $F_{n+2} = F_{n+1} + F_n$ repeatedly, we get

$$F_{n+3} = 2F_{n+1} + F_n$$

$$F_{n+4} = 3F_{n+1} + 2F_n$$

$$F_{n+5} = 5F_{n+1} + 3F_n \quad (1)$$

$$F_{n+6} = 8F_{n+1} + 5F_n \quad (2)$$

Put $n = 5k$ in (1): $F_{5k+5} = 5F_{5k+1} + 3F_{5k}$ (3)

Since F_{5k} is divisible by 5 and $5F_{5k+1}$ is divisible by 5, (3) shows that F_{5k+5} is also divisible by 5. Therefore, by induction, F_{5m} is divisible by 5 for all m.

(ii) Let $G_m = \frac{1}{5} F_{5m}$. Then, using (3), we have

$G_{m+1} = \frac{1}{5} F_{5m+5} = \frac{1}{5} (5F_{5m+1} + 3F_{5m}) = F_{5m+1} + \frac{3}{5} F_{5m}$

Therefore: $G_{m+1} = F_{5m+1} + 3G_m$ (4) Replace m by m+1 in (4):

$$G_{m+2} = F_{5m+6} + 3G_{m+1}$$

But, from (2), $F_{5m+6} = 8F_{5m+1} + 5F_{5m} = 8F_{5m+1} + 25G_m$

so $G_{m+2} = 8F_{5m+1} + 25G_m + 3G_{m+1}$

$= 8(G_{m+1} - 3G_m) + 25G_m + 3G_{m+1}$ (from (4))

$= 11G_{m+1} + G_m$

The desired relation is $G_{m+2} = 11G_{m+1} + G_m$.

James Stewart

REPORT ON THE 1981 CANADIAN MATHEMATICS OLYMPIAD

The 1981 Olympiad was written on Wednesday, May 6th, with 245 students participating from each province was as follows:

Alberta and N.W.T.	24	Nova Scotia	10
British Columbia	39	Ontario	67
Manitoba	11	Prince Edward Island	5
New Brunswick	9	Quebec	62
Newfoundland	7	Saskatchewan	11

205 students were nominated by Provincial Chairman and 40 were nominated by their school principal.

The examination paper is the following:

ANSWER ALL FIVE QUESTIONS
ALL QUESTIONS ARE OF EQUAL VALUE

1. For any real number t, denote by [t] the greatest integer which is less than or equal to t. For example: [8] = 8, [π] = 3 and [-5/2] = -2. Show that the equation
[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345
has no real solution.

2. Given a circle of radius r and a tangent line *l* to the circle through a given point P on the circle. From a variable point R on the circle, a perpendicular RQ is drawn to *l* with Q on *l*. Determine the maximum of the area of triangle PQR.

3. Given a finite collection of lines in a plane P, show that it is possible to draw a arbitrarily large circle in P which does not meet any of them. On the other hand, show that it is possible to arrange an infinite sequence of lines (first line, second line, third line, etc.) in P so that every circle in P meets at least one of the lines. (A point is not considered to be a circle.)

4. P(x) and Q(x) are two polynomials that satisfy the identity P(Q(X)) = Q(P(X)) for all real numbers x. If the equation P(X) = Q(X) has no real solution, show that the equation P(P(X)) = Q(Q(X)) also has no real solution.

5. Eleven theatrical groups participated in a festival. Each day, some of the groups were scheduled to perform while the remaining groups joined the general audience. At the conclusion of the festival, each group had seen, during its days off, at least one performance of every other group. At least how many days did the festival last.

Each examination question is worth ten points. The distribution of marks attained is as follows:

Marks	No. of Students
0 - 9	104
10 - 19	96
20 - 29	32
30 - 39	11
40 - 50	2

Below is the list of the prize winners. There are 16 national prizes and 2 provincial book prizes for each province.

The distribution of national prizes or a provincial basis is as follows: Ontario - 8 prizes; Quebec - 3 prizes; Alberta - 2 prizes; British Columbia, Manitoba and Newfoundland - 1 prize each.

A complete report of the 1981 Olympiad containing a listing of all candidates and their placings together with the problems and solutions will be distributed later this year.

LIST OF NATIONAL PRIZE WINNERS

Placing	Prize	Student's Name	School
First	$1,000.00	ASH, David Walter Thunder Bay, Ont. Teacher: R.G. Garrett Principal: J. Playford	Fort William Collegiate Inst. 512 South Marks St. Thunder Bay, Ont.
Second	$750.00	SMITH, Arthur Paul St. John's, Newfoundland Teacher: Not Available Principal: Not Available	Memorial University of Newfoundland Dept. of Math., M.U.N. St. John's, Newfoundland
Third	$500.00	WEST, Julian Vancouver, B.C. Teacher: J. Parker Principal: A.C.M. Brown	Saint George's School 4175 West 29th Ave. Vancouver, B.C. V6S 1V6

Fourth	$200.00	BARAGAR, Arthur Edmonton, Alberta Teacher: L.J. Pascoe Principal: R.M. Fyshe	Old Scona Academic High 10523, 84 Ave. Edmonton, Alberta T6E 2H5
Fourth	$200.00	BELL, Ian Eric Roxboro, Quebec Teacher: E.W. Fox Principal: (Sister) J. Roberts	Marianopolis College 3880 Cote des Neiges Rd. Montreal, Quebec H3H 1W1
Fourth	$200.00	BOWMAN, John C. Edmonton, Alberta Teacher: L.J. Pascoe Principal: R.M. Fyshe	Old Scona Academic High 10523, 84 Ave. Edmonton, Alberta
Fourth	$200.00	BERNIER, David Ste-Foy, Quebec Teacher: R.D.C. Dionne Principal: H.M.H. Marineau	College Des Jesuites 1150 St. Cyrille Ouest Quebec, Quebec G1S 1V7
Fourth	$200.00	CHEW, John James Toronto, Ont. Teacher: G.A. Fleming Principal: H.D. Gutteridge	University of Toronto Schools 371 Bloor St. W. Toronto, Ont. M5S 2R8
Fifth	$100.00	CAROMICOLI, Adam C. Dundas, Ont. Teacher: J.A. Mattice Principal: R. Begin	Parkside High School Parkside Drive Dundas, Ont.
Fifth	$100.00	DIXON, Alan Geoffrey Nepean, Ont. Teacher: T.A. Fleming Principal: G.M. Ferguson	Confederation High School 1645 Woodroffe Ave. Ottawa, Ont.
Fifth	$100.00	WARLOW, Scott Guelph, Ont. Teacher: R.D. Fleischauer Principal: W.C. Scott	Centennial Collegiate Vocational Inst. 289 College Ave. W. Guelph, Ont. N1G 1S9

THANKS TO ED BARBEAU FOR PROVIDING THIS INFORMATION.

THE SAMUEL BEATTY MATHEMATICS ESSAY CONTEST

The 1981 essays were quite good but did not reach the quality of the best of the first two years' contest. Essays which had enough material to be worked into a prize-winner were deficient in organization, accuracy and thoroughness. Reluctantly, it was decided not to award a first or second prize this year. The third prize was awarded to *Sallianne Deck* of Sarnia Northern Collegiate Institute and Vocational School for her essay *Six numbers: Add any two-Perfect square.*

A number of students obtained "honourable mention". These are:
Helen Earnshaw, Northern C.I. and V.S. Sarnia, for *Leonard Euler*
David Vinke, Glencoe District High School, for *Codes and Ciphers*
Christopher A. Gaspar, Northern C.I. and V.S. Sarnia, for *Map Projections,* and
Lindley Bassarath, Kirkland Lake C. and V.I. for *Mathematical and Biological Population Studies*

A junior prize was awarded to *Stephen Smith* of Royal Orchard School, Thornhill, for *A Study of Pythagorean Triplets* and honorable mention was made of the essay *History of Mechanical Aids to Computation* by *Vivian Varelas,* Grade 10 of Ajax High School.

The *Fourth Samuel Beatty Prize Mathematics Essay Contest* will take place in 1982. The Contest is to encourage students interested in Mathematics to pursue a topic at some length and prepare a

report on it.

Eligibility: Any full time student of an elementary (primary), senior elementary (intermediate junior high) or secondary (senior high) school in Canada may compete.

Prizes: $500 is available for prizes. In the 1979 contest, this was distributed as follows: $200, $100, $80, $60, $40, $20. In 1980 there were three prizes: $100, $100, $35. In awarding prizes, topic, age of contestant and standard of achievement (compared to that of 1979 contest) will be considered.

Topics: A list of topics is available from the organizer. A competitor may use a topic from a previous contest or choose his own topic provided he obtains approval in advance from the Secretary by January 31, 1982. This has been done by a number of students. Request for such approval should be accompanied by a brief summary of what is intended and a reading list.

The essay:
(1) The essay should be 2,000 words long.

(2) The essay may be typed or neatly handwritten in ink, but should be double spaced in either case. 8½'' x 11'' paper or 3-ring note paper should be used.

(3) The essay may be based on reading or on original investigation.

(4) All books used for reference should be listed at the end of the essay.

(5) An essay prepared as a school project is eligible, but should be accompanied by a note from the teacher explaining the nature of the assignment.

(6) An essay which has been submitted or which is about to be submitted for a prize in any other competition outside the competitor's own school is not eligible for this competition.

(7) Teachers may be consulted for guidance, but final preparation of the manuscript should be done entirely by the contestant.

(8) Contestants may seek advice or amplification on any topic from the Secretary. Xerox copies of resource material may be obtained at cost.

Submission of essays:
(1) The deadline for submission of essays is March 31, 1982. It is hoped the winners can be announced by June 1, 1982.

(2) One copy of the essay should be sent in, along with candidate's name, address, telephone number, age last birthday (as of deadline), school, grade and, if applicable, name of teacher advisor. A second copy should be kept by the competitor in case of loss.

(3) Entries and enquiries should be sent to:
Dr. E.J. Barbeau
c/o Department of Mathematics
University of Toronto
Toronto, Ontario M5S 1A1

All entries become the property of the Trustees of the Samuel Beatty Fund.

A book containing the prize-winning essays in the 1979 and 1980 contests has just been published. Copies may be obtained from Dr. Barbeau. For each, please send a cheque for $3.00 payable to ''The University of Toronto'' to cover production and handling costs.

E.C.O.O. and O.A.M.E. DO WE NEED BOTH?

by
Robert E. Drake

E.C.O.O. is the Educational Computing Organization of Ontario. The purpose of E.C.O.O. is to exchange and disseminate information concerning the educational purposes of computers in the

province of Ontario, organize conferences, conduct workshops, and assist policy makers in determining the computing needs of educators.

E.C.O.O. reflects many interests in Ontario education. Administrators, elementary, secondary, and post-secondary teachers are all represented.

Clearly E.C.O.O.'s purposes do not conflict with O.A.M.E.'s but rather complement them. Indeed, E.C.O.O. wishes to help other organizations avoid 'reinventing the wheel'. Many users of microcomputers in the schools are writing programs and these programs for the most part are repeating others' work. For example, my software library includes at least 10 different addition practice programs.

E.C.O.O.'s short history began in September 1979 with a 7 member organizing committee. Its growth has been rapid, from 12 in November 1979 to 200 in January 1980 to 600 in January 1981 to over 1000 at this writing. May 1980 saw E.C.O.O.'s first conference at Harbourfront in Toronto with about 250 attending. A working conference to suggest policy for computers in education in Ontario was held in December 1981. One hundred seventy administrators, teachers, ministry personnel, representatives of O.T.F., O.I.S.E., O.E.C.A., publishers and computer manufacturers were invited, attended and contributed. No ultimate solution was reached but recommendations affecting educational computing were reached. May 1981, E.C.O.O.'s second annual conference with over 800 attending and over 400 turned away. Major names (Papert, Braun, Johnson) in educational computing participated to make the conference an overwhelming success.

E.C.O.O. is most visible through its Newsletter which is published 4 or 5 times per year. It informs teachers of happenings in computing in Ontario primarily, but includes news of interest from elsewhere. The newsletter reflects E.C.O.O. in that is is not aimed at the computer specialist but rather the teacher using computers in the classroom.

Does O.A.M.E. need E.C.O.O.? Yes!

As a body devoted to advancing computing in Ontario education, E.C.O.O. is a resource for teachers using computing. Eventually, E.C.O.O. may be a resource for software evaluation and distribution.

Does E.C.O.O. need O.A.M.E.? Again Yes!

O.A.M.E. and its members contributing to E.C.O.O. will help other Ontario teachers with their problems in computing and mathematics.

We can only complement each other's operation.

Robert E. Drake
President, E.C.O.O.
Brantford Collegiate Institute & Vocational School
120 Brant Ave.
Brantford, Ontario
N3T 3H3
Member: O.A.M.E., G.V.M.A.

COUNCIL COMMENT

by
Bob Robinson

Ed. Note:
In order to keep you informed with regard to the workings of the O.A.M.E. Council, we will include this item in an issue of the *Gazette* following Council Meetings. The Editorial Panel hopes

that you find this section of interest. Your comments are welcomed and should be addressed to:

The Editor
Ontario Mathematics Gazette
c/o Department of Mathematical Sciences
McMaster University
Hamilton, Ontario
L8S 4K1

COMMITTEES

Three committees have been struck by council for the 1981-82 school year. The names of the committees, chairperson and executive liaison are listed.

Computer Committee
Chairperson: William Proctor, 34 Helene Crescent, Waterloo, Ont., N2L 5E5
Executive Liaison: Peter Ullrich

SPECIAL NOTE:

If you have any computer related articles, Board policies, courses of studies, special events or Professional Development activities . . . please send them to the Chairperson listed above.

Membership and Publicity Committee
Chairperson: Eileen Collins, 50 Coral Drive, Hamilton, Ont., L8V 2N6
Executive Liaison: Paul Lessard

Publications Committee
Chairperson: Tom Griffiths, 30 Conifer Court, London, Ont., N6K 2X4
Executive Liaison: David Davidson

LEADERSHIP

An excellent program has been planned for the 10th Annual Leadership Seminar to be held at the Ramada Inn, Mississauga on 1981 10 21-24. More information is found elsewhere in this issue.

N.C.T.M.

The official delegates for O.A.M.E. at the Delegate assembly of the N.C.T.M. in St. Louis in April were Dino Dottori and George Knill. A new Canadain brochure has been prepared by N.C.T.M. Further information pertaining to N.C.T.M. may be obtained from:

Ralph Connelly - N.C.T.M. Representative
11 Berkwood Place
Fonthill, Ont.
L0S 1E0

Plans are well underway for the 60th Annual Meeting of N.C.T.M. in Toronto, 1982 05 14-17. The chairman of local arrangements is Dave Alexander. If you wish to assist, please contact him at:

Faculty of Education
University of Toronto
371 Bloor St. W.
Toronto, Ont.
M5S 2R7

SECONDARY EDUCATION REVIEW PROJECT (SERP)

The president, George Knill, sent a letter to the Ministry by the Ministry's deadline of 1981 06 30 and a copy of this letter will be sent to all councillors.

(Note: A list of councillors is found elsewhere in this issue. Contact a councillor for further information). The councillors were in favour of retaining Grade 13 and not in favour of reducing the high school mathematics significantly. Your comments regarding S.E.R.P. or any submissions are welcomed and should be directed to The Editor, Ontario Mathematics Gazette.

PUBLICATIONS

The new editor of the Ontario Mathematics Gazette, Hans Heinig, was appointed. The names of the other members of the Editorial Panel are listed elsewhere in this issue.

Four issues of **Abacus** are planned for the next school year. The September issue will be prepared by the teachers of Lambton County under the direction of Sharon McPhail.

COMPUTER

The Computer Committee is planning to act as a Resource/Sharing Body to help provide direction for O.A.M.E. members, other educational associations and any teachers involved with computer education. The plans are for this committee to provide some or all of the following - school board reports and activities, courses of study, software and its evaluation, articles, do's and don't's and other Professional Development activities.

FATHER FAUGHT AWARD

The top Ontario school in the 1981 Junior Mathematics Contest and the winner of the Father Faught Award was Loyalist Collegiate and Vocational School from Kingston. Team members were Neale Ginsberg, Jeffery Stroud, Tomo Suzuki, David Hope Simpson and Bobby Hsu. Congratulations!

STATE OF MATHEMATICS IN THE SENIOR GRADES

Revision of the Senior Division will begin in 1982-83. There have been some unavoidable holdups of the recommendations of the Secondary Education Review Project and changes in ministry personnel. The O.A.M.E. should provide input into this important curriculum revision.

SARNIA CONFERENCE

The Sarnia Conference was a huge success and a final report will be presented in the fall.

ONTARIO ASSESSMENT INSTRUMENT POOL (O.A.I.P.)

The O.A.M.E. is involved in a project with O.A.I.P. with regard to Geometry Ideas at the Intermediate Level and also the Junior Mathematics Project.

CHAPTERS

Pine Ridge: Membership is very low but they will continue to administer its scholarship.

C.O.M.A. (Carleton-Ottawa Mathematics Association)

The Annual General Meeting was held on 1981 06 06. Algonquin College's mathematics test for grade 12 was conducted again this year. The Ministry of Education has approached C.O.M.A. to participate in a study of Secondary/Post Secondary Interface. This is one of four studies in the province, all of which are to be completed by 1983 06. The purpose of the study is "to examine and articulate curriculum practices on both sides of the interface, in the hopes of promoting greater understanding and collaboration among educators."

Q.S.L.M.A. (Quinte-St. Lawrence Mathematics Association)

There are more than 100 members of Q.S.L.M.A. These meetings (mini-conferences) were held during 1980-81 in Odessa and Kingston (2). During the 1981-82 school year meetings are planned for October/November, February and April. The executive of Q.S.L.M.A. met on June 17th.

TORNY (Toronto-North York)

An excellent session was held on April 30. It was set around a theme rather than separate one hour sessions. The next meeting is scheduled for 1981 10 30.

N.O.M.A. (Northern Ontario Mathematics Association)

N.O.M.A. covers such a large area of Ontario. The association is planning their next mini-conference for October 3 in North Bay.

O.A.M.E. RECOMMENDATIONS TO S.E.R.P. COMMITTEE

Steering Committee
Secondary Education Review Project
Ministry of Education
Mowat Block, 17th Floor
Queen's Park
Toronto, Ontario
M7A 1L2

164 Hillcrest Avenue
Hamilton, Ontario
L8P 2X4

June 22, 1981

Dear Mr. Chairman:

As the President of the Ontario Association for Mathematics Education, I have been directed by the Council of this organization to submit two recommendations pertaining to the Secondary Education Review Project Discussion Paper. The strength of these recommendations can be appreciated in light of the following background information about the O.A.M.E.

The O.A.M.E. was officially formed eight years ago, however, the roots of the organization go back to 1891 when it was a part of the Ontario Association of Teachers of Mathematics and Physics.

The O.A.M.E. has 1200 members from across the Province. It sponsors two major mathematics conferences each year and publishes two journals, **The Gazette** and **The Abacus. The Gazette** is published three times a year and **The Abacus** four times a year. The Association has seven local chapters which each conduct a minimum of one and a maximum of three mini-conferences each year. The Association also presents three scholarships every year to outstanding Ontario Mathematics students.

The O.A.M.E. Council is composed of forty members, elected by the membership at large. This Council meets three times each year.

At the last council meeting on June 6, 1981, the Secondary Education Review Project Discussion Paper was an agenda item. Due to time constraints, only a few of the Secondary Education Review Project recommendations could be considered.

As a result of these deliberations, the O.A.M.E. Council, on behalf of the membership, make the following recommendations to the Secondary Education Review Project Steering Committee.

1. The O.A.M.E. is strongly in favour of retaining Grade 13.
2. The O.A.M.E. is not in favour of reducing the secondary school mathematics curriculum significantly.

Yours truly,

George Knill
President

CALENDAR OF EVENTS

10TH ANNUAL LEADERSHIP SEMINAR

The highly successful Annual Leadership Seminar is being held again this year at the Ramada Inn on Dixie Road in Mississauga. The dates for this important event are 1981 10 21-24. Another interesting program has been prepared by the committee consisting of Lorna Morrow (North York), "J" Symington (Peel) and Peter Ullrich (Ottawa). The theme of this year's seminar is "Mathematics Across The Curriculum".

The Seminar Administrator this year is again Morley MacGregor who has supervised this excellent meeting in the past.

The purpose is to develop leadership qualities using mathematics topics. Although primarily for elementary teachers, this seminar is of value to principals, secondary school mathematics

department heads and supervisory personnel.

As in the past the number of participants is limited. Letters and applications were sent out to Boards of Education in Ontario in February.

There are a **few** places left . . . so

(a) Inquire to see if your Board is sending anyone
(b) For more information contact:

Morley MacGregor
Leadership Seminar Administrator
O.A.M.E.
247 Bright St.
Sarnia, Ont.
N7T 4E9

10th ANNUAL LEADERSHIP SEMINAR
RAMADA INN, MISSISSAUGA
1981 10 21-24

American Statistical Association - National Council of Teachers of Mathematics: *Leadership Conference, Statistics and Probability in the Classroom;* December 2-6, 1981; 1776 Holiday Inn, Williamsburg, Virginia. For information: Dr. Edna Gross, Division of Teacher Education, 3086 Oliver Hall, Virginia Commonwealth University, Richmond VA 23284.

Association of Mathematics Teachers of New York State: *Annual Mathematics Conference;* October 15-17, 1981; Hotel Syracuse, Syracuse, New York. For information: Phyllis Yudikaitis, 1981 AMTNYS Annual Meeting, 450 Brandywine Avenue, Schenectady, N.Y. 12308.

The *20th Annual Northwest Mathematics Conference* will be held on October 8-10, 1981 at the Portland Marriott Hotel in Portland, Oregon. For information: Tari Querin, 6925 SW Ventura Dr., Portland, Oregon 97223 or Joanne Wilkie, 948 Cumberland SW, Lake Oswego, Oregon, 97034.

The *21st Northwest Mathematics Conference* holds its 21st annual meeting in Vancouver, November 5-7, 1982.

The *First International Conference on Teaching Statistics* will be held August 8-13, 1982; Sheffield, England. For information: The Conference Secretary, ICOTS, The University, Department of Probability and Statistics, Sheffield S374H, England.

MISCELLANEA

Professor A. Rouchier is managing editor of the new research journal *Recherches en Didactique des Mathématiques.* The aim of the journal is to publish articles on studies relating to didactics of mathematics. It publishes articles in French, English and Spanish. Each annual volume includes three issues. Information: A. Rouchier; Université D'Orleans Departement de Mathematiques, 45 Orleans 02 France.

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