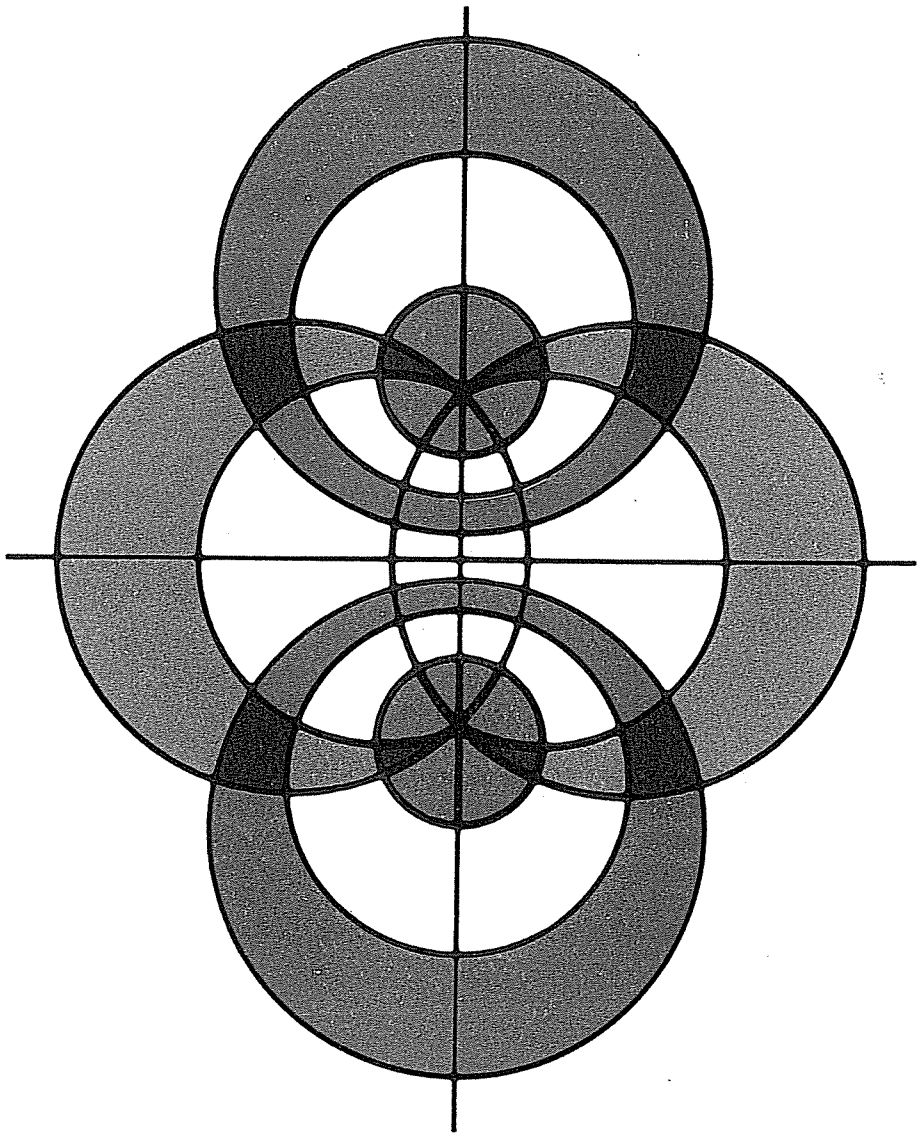
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MATHEMATICS EDUCATION

INDEED, THE AIM OF TEACHING MATHEMATICS SHOULD BE RATHER TO STRENGTHEN THE PUPIL'S FACULTIES, AND TO SUPPLY A METHOD OF REASONING APPLICABLE TO OTHER SUBJECTS, THAN TO FURNISH HIM WITH AN INSTRUMENT FOR SOLVING PRACTICAL PROBLEMS.

Magnus, Philip.

INTRODUCING PROSPECTIVE MATHEMATICS TEACHERS TO ERROR ANALYSIS

by
Dr. Mary Ann DeVincenzo

Analysis of students' errors in mathematics has been stressed as a preferred way of assessing students' mathematical strengths and weaknesses (Denmark, 1976; Englehardt, 1977; Hopkins, 1978; Inskeep, 1978). The National Council of Teachers of Mathematics also recognizes the need for mathematics teachers to be trained in diagnostic and prescriptive mathematics: "Teachers should improve their diagnostic skills and their ability to structure appropriate remediation" (1980, p. 14). It is very important then, that future teachers of mathematics become familiar with specific mathematics errors and the techniques that can be used to identify such errors.

During the Fall 1979 semester, a group of future mathematics teachers were introduced to diagnostic and prescriptive techniques in an undergraduate mathematics course. A particular section of MT 16001, a basic course in arithmetic and algebra required for all undergraduates at our college, was designed as a special section for prospective elementary school teachers. Topics taught in this course included: computations with whole numbers, fractions, decimals, percents, numbers expressed in exponential form, signed numbers, factoring and solutions of linear and quadratic equations. Throughout the entire semester, samples of incorrect written computations were distributed and integrated with some of the topics studied. These errors were cited from various research studies in mathematics and often coincided with the students' errors. Students were required to identify the faulty algorithm that was used in a particular solution and also to hypothesize on possible reasons for the errors. Discussions were then conducted on the class' analysis of the sample of errors and lists of error types were compiled for specific problem types.

The class discussions revealed that the exercises in identifying errors helped the prospective elementary school teachers to obtain the following insights:

1) An error was often unable to be identified from a solution to only one example, e.g., some errors are not revealed with certain number configurations, as in the solution of

$$\begin{array}{r} 24 \\ +32 \\ \hline 56 \end{array}$$

where addition is performed from left to right.

2) Identifying an error was easier when at least three problems of the same type were solved by one student.

3) Correct final answers can sometimes result from the application of a faulty algorithm and this suggests the importance of analyzing the parts of a solution rather than only the final answer.

4) Some incorrect written computations are difficult to identify because different faulty algorithms can result in similar written computations.

5) The reasons for errors are difficult to determine from an analysis of only written computations.

After analyzing errors from written computations, the students observed the limitations of this technique and they recognized the need for an alternative mode of identifying errors. The interview technique, where students are observed as they "think aloud" was presented to the class as another diagnostic technique. Video-taped recordings of interview sessions were available from a recent research study (DeVincenzo, 1980) and were used in class. Many students were better able to understand their own mistakes from several interview sessions which were conducted in class, and they concluded that the interview technique was an excellent means of analyzing errors.

As a way of implementing the knowledge gained from this special section of MT 16001, each

student was asked to observe a student "thinking aloud" in a nearby school, identify an error, obtain information about the error and suggest possible remedial activities. Below are excerpts from some of the students' experiences in the schools:

1) Grade 9 student's computations in a Mathematics Laboratory

$$\begin{array}{r} 8.2 \\ .452 \\ \hline 10.32 \\ .018972 \end{array} \qquad \begin{array}{r} 9.6 \\ .512 \\ \hline 3. \\ 1.3112 \end{array} \qquad \begin{array}{r} 9.6 \\ 5 \\ \hline 14.6 \end{array}$$

Figure 1

The computations in Figure 1 were explained to result from adding corresponding columns and placing the decimal point in the answer in such a way to reflect the total number of decimal places in the problem. This solution technique was used "because there was no addition symbol next to the numbers so I thought I'd use the rule for multiplying decimals." The numbers in the problem were not multiplied because "the multiplication was too time-consuming." After inserting a plus (+) symbol, the three problems were observed to be computed correctly.

This same student proceeded to complete the following exercise:

Circle the larger fraction:

$$\frac{3}{5} \quad \left(\frac{7}{5}\right) \qquad \left(\frac{9}{16}\right) \quad \frac{5}{8} \qquad \left(\frac{12}{15}\right) \quad \frac{1}{2}$$

Figure 2

It was explained that the fraction selected as the larger fraction was the fraction with the larger numerals in the numerator and denominator (see Figure 2). The student teacher explains the motivation for the suggested remedial activities: "I knew the student was familiar with pies because he had earlier expressed the desire to be a chef. I drew a picture of a pie and divided it according to the fractional representations of the problem." Immediately after this demonstration it was reported that the student created his own representations of pies and was able to correct his mistake and properly select the larger fraction in other problems.

2) Grade 4 student's computations in division problems:

$$\begin{array}{r} 539 \\ 55\overline{)2695} \\ \underline{25} \\ 19 \\ \underline{15} \\ 45 \\ \underline{45} \end{array}$$

Figure 3

The solution illustrated in Figure 3 is explained: "We always guess about how many times this [55] will go into this [26]. Five goes into twenty-six five times. ..." At first the student teacher reasoned that the error of using a one-digit divisor was due to the repeated digit in the divisor. However, the solution to the next problem (see Figure 4), revealed the real problem as the student explains the computations:

$$\begin{array}{r} 3699\ R5 \\ 25\overline{)73985} \\ \underline{60} \\ 139 \\ \underline{120} \\ 198 \\ \underline{180} \\ 185 \\ \underline{180} \\ 5 \end{array}$$

Figure 4

"Now twenty-five is around twenty and that goes into seventy, three times and that's sixty. ..." The student teacher concluded that the concept of estimation was the real problem. In the previous problem (see Figure 3), the number 55 over-simplified the choice of 5 as an estimate number: "Both are fives so I use five."

As a way of remediating the error, the student teacher noted that the divisor was a two-digit number and demonstrated where the estimation system can and cannot be used. After this discussion, the student was able to successfully complete three additional division problems with two-digit divisors.

3.) Grade 9 student's computations in a Mathematics Laboratory



Figure 5

The computations illustrated in Figure 5 were repeatedly explained to represent the steps of the long division algorithm: "divide, multiply, subtract and bring down." However, it is observed that these steps are adhered to without placing a zero in the quotient when the division yields a zero quotient at a particular stage of the process. The following remedial activities were demonstrated by the student teacher in the order presented: (a) approximate an answer before doing any computations, (b) check computations using multiplication, (c) reinforce the steps of the long division algorithm: divide, multiply, subtract, bring down, and (d) use bars on the calculations as illustrated in Figure 6.

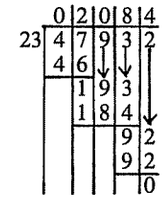


Figure 6

The use of bars on the calculations (see Figure 6), was found to be the most effective and most frequently used remedial activity for this student because "I got a clear way of doing it now which works out."

Integrating diagnostic techniques in a basic mathematics course provided future mathematics teachers with invaluable experiences and insights that helped them better understand and interpret the processes which accompany mathematical computations. It should be noted that approximately one-half, or 15 students, in the college mathematics course, indicated an interest in pursuing a career in education. The other fifty percent of the college students were business, theatre, English and history majors, who decided to enroll in the special math course for prospective teachers, to satisfy a mathematics proficiency requirement at the College. Although only 50% of the college students in the mathematics course were pursuing teacher education, 95% of the class rated the integration of error analysis in a basic mathematics course as superior in effectiveness. The results of this experiment suggest the inclusion of error analysis into mathematics and mathematics method courses.

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THE IDEA THAT APTITUDE FOR MATHEMATICS IS RARER THAN APTITUDE FOR OTHER SUBJECTS IS MERELY AN ILLUSION WHICH IS CAUSED BY BELATED OR NEGLECTED BEGINNERS.

Herbart, J. F.

A SERIES OF CYCLICAL SEQUENCES

by

David R. Duncan & Bonnie H. Litwiller

Teachers are always interested in finding activities which provide for the discovery of patterns. It is an added benefit if these patterns can be found with a calculator. We shall describe and extend several such activities.

Activity I

1. Select a one digit "original" number. For example, select 3.
2. List the consecutive **doubles** of the original number. These are 6, 12, 24, 48, 96, 192, 384, 768, ...
3. Observe the units digits of these doubles. The resulting sequence is 6, 2, 4, 8, 6, 2, 4, 8, ... (a cyclical sequence).

Let us redo Activity I and select 2 as the original number. The consecutive doubles are 4, 8, 16, 32, 64, 128, 256, 512, ... The resulting cyclical sequence is 4, 8, 6, 2, 4, 8, 6, 2, ...

Note that essentially the same cyclical sequence occurs when 2 is selected as the original number as when 3 is selected as the original number except that the sequence begins at a different point.

Table I reports the results of using each of the 10 digits as the original number and applying the doubling process.

Table I

Original Number	Resulting Sequence
0	0, 0, . . .
1	2, 4, 8, 6, 2, 4, 8, 6, . . .
2	4, 8, 6, 2, 4, 8, 6, 2, . . .
3	6, 2, 4, 8, 6, 2, 4, 8, . . .
4	8, 6, 2, 4, 8, 6, 2, 4, . . .
5	0, 0, . . .
6	2, 4, 8, 6, 2, 4, 8, 6, . . .
7	4, 8, 6, 2, 4, 8, 6, 2, . . .
8	6, 2, 4, 8, 6, 2, 4, 8, . . .
9	8, 6, 2, 4, 8, 6, 2, 4, . . .

Observe that 0 and 5 yield the same simple sequence. Also observe that the remaining eight sequences are essentially identical except for different starting points. In fact, **identical** sequences result from using as original numbers 1 and 6, 2 and 7, 3 and 8, 4 and 9. These last identities occur because the pairs of numbers listed are congruent modulo 5, causing their doubles to be congruent modulo 10.

Activity II

1. Select a one digit original number. For example, select 7.
2. List the consecutive **triples** of the original number. These are 21, 63, 189, 567, 1701, 5103, 15309, 45927, ...
3. Observe the units digits of these triples. The resulting sequence is 1, 3, 9, 7, 1, 3, 9, 7, ...

Table II displays the results of using each of the 10 digits as the original number and applying the consecutive tripling process.

Table II

Original Number	Resulting Sequence
0	0, 0, . . .
1	3, 9, 7, 1, 3, 9, 7, 1, . . .
2	6, 8, 4, 2, 6, 8, 4, 2, . . .
3	9, 7, 1, 3, 9, 7, 1, 3, . . .
4	2, 6, 8, 4, 2, 6, 8, 4, . . .
5	5, 5, . . .
6	8, 4, 2, 6, 8, 4, 2, 6, . . .
7	1, 3, 9, 7, 1, 3, 9, 7, . . .
8	4, 2, 6, 8, 4, 2, 6, 8, . . .
9	7, 1, 3, 9, 7, 1, 3, 9, . . .

The resulting patterns of Activity II differ somewhat from those of Activity I. There are two single digit sequences, those of 0 and 5. There are also two basic four digit sequences; one sequence includes only odd numbers while the other sequence includes only even numbers. No two sequences are exactly identical; all sequences begin at different numbers.

The activities may be extended by multiplying selected original numbers by 4, by 5, by 6, by 7, by 8, or by 9. Table III reports the basic cycles that result irrespective of the starting point.

Table III

Multiplier Used	Resulting Basic Sequences	Original Numbers Yielding the Basic Sequence
2	6, 2, 4, 8, 6, 2, 4, 8, . . . 0, 0, . . .	1, 2, 3, 4, 5, 6, 7, 8, 9 0, 5
3	3, 9, 7, 1, 3, 9, 7, 1, . . . 6, 8, 4, 2, 6, 8, 4, 2, . . . 0, 0, . . . 5, 5, . . .	1, 3, 7, 9 2, 4, 6, 8 0 5
4	4, 6, 4, 6, . . . 8, 2, 8, 2, . . . 0, 0, . . .	1, 4, 6, 9 2, 3, 7, 8 0, 5
5	5, 5, . . . 0, 0, . . .	1, 3, 5, 7, 9 0, 2, 4, 6, 8
6	6, 6, . . . 2, 2, . . . 8, 8, . . . 4, 4, . . . 0, 0, . . .	1, 6 2, 7 3, 8 4, 9 0, 5

7	7, 9, 3, 1, 7, 9, 3, 1, . . . 4, 8, 6, 2, 4, 8, 6, 2, . . . 0, 0, . . . 5, 5, . . .	1, 3, 7, 9 2, 4, 6, 8 0 5
8	8, 4, 2, 6, 8, 4, 2, 6, . . . 0, 0, . . .	1, 2, 3, 4, 5, 6, 7, 8, 9 0, 5
9	9, 1, 9, 1, . . . 8, 2, 8, 2, . . . 7, 3, 7, 3, . . . 6, 4, 6, 4, . . . 0, 0, . . . 5, 5, . . .	1, 9 2, 8 3, 7 4, 6 0 5

Ask students to discover why it is not profitable to extend this activity with original numbers greater than 9.

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THE PARTICIPATION IN THE *GENERAL DEVELOPMENT OF THE MENTAL POWERS* WITHOUT SPECIAL REFERENCE TO HIS FUTURE VOCATION MUST BE RECOGNIZED AS THE ESSENTIAL AIM OF MATHEMATICAL INSTRUCTION.

Reidt, F. (1906)

THE URGENT NEED FOR CURRICULUM INTEGRATED MICROCOMPUTER SOFTWARE

by
Don Wiederanders

Sorry folks, we can't wait. We aren't unsympathetic toward software vendors and their problems of copyright infringement. We'd like to be patient until they've solved the problem of how to provide quality, yet affordable software and still make a profit. But we can't wait. We applaud the unselfish efforts of local and regional user groups, but we recognize that most software produced and shared in this way ends up stored randomly on many disks and is accessible only to those who are computer oriented. We know that eventually these programs will be available to the rest of us. But we can't wait.

The word is out. On main street, on the farm, even in the home, everyone knows that the fantastic power of the computer is now available - even to schools. In spite of tight money, funding agencies are buying microcomputers for schools. Overworked teachers are doing a conscientious job of teaching the basics of BASIC to the privileged few who qualify. Every school has its computer groupies. They carry around their box of diskettes, exchange games and programming tips and speak a strange language that is largely unintelligible to the masses. But that's not enough! Effecting the education of ten, twenty, or even fifty percent of our students by the computer just isn't enough. Even if we thought it was, the public knows better. We just can't wait.

If we as teachers are going to realize the potential of the computer in education, its power and versatility are going to have to become available to complement the existing curriculum. Some futurists predict that the computer will, to a large extent, replace the teacher. Others first want to rewrite the curriculum to more fully exploit the power of the new technology. Sorry folks, even if these things were feasible (and we doubt either is feasible, at least in the foreseeable future), we can't wait. A viable first step to integrating the power of the computer into the existing curriculum is available to us now. What is needed are short programs that help us do a better job of meeting the day-to-day needs that confront every classroom teacher. Most of us could use help in presenting "tough" concepts; a tutorial for the student who has been absent; a rerun for the student who "didn't get it"; even enrichment for the student who is "marking time." Such help is, or soon will be, available.

Realizing that we can't wait, the Iowa Microcomputer Curriculum Project (a Title IV-C funded

project) has made a beginning to a solution to the problem:

a) appropriate software
b) keyed to the existing curriculum
c) available to all teachers
d) at a nominal cost

The IMCP's first task was to define relatively small instructional modules (most programs currently available cover too much material). This we've accomplished. The second and much larger task is to accumulate and disseminate appropriate programs for each of these modules. This we've begun.

Recognizing the magnitude of the task for all levels of all parts of the curriculum, the IMCP decided to begin with the curricula for algebra I, algegra II, and geometry. For each course a two dimensional matrix identifies relatively small instructional modules. Figure 1 shows part of the matrix for algebra I.

ALGEBRA I	Concept Development Stages		
Concept Module	Introduction (I)	Skill Building (S)	Application (A)
1 Simplifying numerical expressions (order of operations including parenthesis)	*Some "Introduction" programs may be used as part of the teacher's presentation.*		
2 Variables; evaluation of algebraic expressions and solving simple equations by replacement			
3 Translating word phrases and sentences into the symbols of algebra: a) relatively simple	Disk AI-1 Translating Simple Word Phrases (Introduction) G. Carey	X [1]	
b) more complex	Disk AI-1 Translating Complex Word Phrases (Introduction)	Disk AI-1 Translating Complex Word Phrases (Skill Bui...	

Figure 1

[1]A program is being written or is in the evaluation process for all cells with the "X".

There are 147 cells in the algebra I matrix, 225 in the matrix for algebra II, and 162 cells for geometry. The project currently has programs that are written or are being written for nearly 15% of these cells. These programs are written by teachers - for teachers - to be used with or by their students. Even though the programs that are currently available have gone through the projects extensive evaluation process and have been revised, they aren't perfect! We accept the fact that no program is ever the best it could be. We believe these programs are a good beginning and that each succeeding revision will be better. Our goal is to get good instructional software into the hands of teachers now. We can't wait for perfection. Only the programs that we can use now can help our present students. These programs will also give us a chance to explore the classroom management and instructional dynamics dimensions of integrating the micro into the instructional process.

Phase II of the Iowa Microcomputer Curriculum Project will use the same basic project design to provide microcomputer software to supplement the mathematics curriculum for grades 5-8. Planning has begun for expansion into the physical and social sciences. Though time lines are indefinite, IMCP has made a definite commitment to the total K-12 curriculum. That commitment will be fulfilled as rapidly as human and financial resources make it possible.

At present there are approximately forty authors who have contributed programs for evaluation and distribution through the Iowa Microcomputer Curriculum Project. We need many more!

Currently we are committed to handling only programs written for APPLE II[1] DOS 3.3 and TRS-80[2] Model III. By September, 1982, we hope to be able to accommodate program users and authors of both the Pet[3] (disk) and Atari[4] (disk). Contributing authors receive appropriate authorship credits and other benefits. Further information can be received by writing IMCP or by subscribing to the Micros in Mathematics Quarterly ($3.00 - U.S., $3.50 - Canada) from IMCP, Price Laboratory School, Cedar Falls, Iowa 50613.

[1]TM Apple, Inc., [2]TM Tandy Corp., [3]TM Atari, Inc., [4]TM Commodore Business Machines

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INFLUENCE OF COMPUTERS ON THE SECONDARY SCHOOL MATHEMATICS CURRICULUM

by
Gordon Doctorow

1. Introduction

First calculators and more recently computers have penetrated the traditional classrooms and curricula at all levels of schooling. While this process is uneven, it is nonetheless a general fact of life, certainly in the industrialized countries of the world. And nowhere has the impact been more forceful and direct -- aside from the electronics technology and engineering areas -- than in mathematics instruction/learning.

The subject of this paper is the influence of computers (computing) on the mathematics curricula at the secondary school level based on the published works of the three IFIP world conferences on education and of the 1977 IFIP Working Conference on "Informatics and Mathematics in Secondary School/Impacts and Relationships" (Varna, Bulgaria). The use of the computer in CAI or CMI mode will be dealt with only in passing. Rather we shall concentrate on the ramifications of computing in influencing the topics of the mathematics curriculum, some of the ways in which traditional topics have been changed or new ones added, and some key underlining pedagogical issues posed by the computer.

Traditional Mathematics

The introduction of calculators changed the way in which math problems could be emphasized. With the inexpensive and accurate reckoning capability of the calculator commonly equipped with a good selection of special function keys (trigonometry, exponents), more realistic numbers could be used for problem solving. Given the high speed in which calculations can be performed and checked, more time and effort could be devoted to organizing problem solutions. Furthermore, it has become possible and desirable to introduce numerical techniques for such things as finding the roots of a polynomial.[1] Such techniques have serendipitously led to the introduction of the concepts of algorithm and iteration. Previously abstract vis-a-vis the students' pencil-and-paper experiences, these topics are now being rendered in a concrete proccdural form. The computer, moreover, adds to these effects with a vast repertoire of its own.

For example, the computer is most usually thought of as **the** vehicle for delivering mathematics education. Much courseware has been delivered in the form of mathematical CAI packages as drills or tutorials. Some simulation courseware has also been developed. By and large, the outcome is the presentation of traditional mathematics in a non-traditional and seemingly more effective way (because of the individual feedback provided to the student).

Another step that has been taken in terms of insinuating the computer into the traditional mathematics curriculum has been the introduction of the "electronic blackboard". This refers to using the computer's (usually a microcomputer) graphics capabilities to present and illustrate mathematical ideas on a TV screen or monitor, possibly with color and sound effects and animation as well. This is an instance of the computer used as an audiovisual aid or of computer based instruction. The merit of this approach is that it provides the teacher with a highly interactive medium to illustrate abstract ideas in an interesting and clear way. Besides, the electronic blackboard approach allows the discussion of mathematical ideas to focus on analysis and patterns

inferred from observing the graphic results because the computations are implicitly embedded in the computer.

New Directions

The new emphasis in mathematics education at the secondary school level is on teaching problem solving directly. The focus of instruction is to be on developing heuristic techniques in the manner of the Hungarian Mathematician Polya in his book **How to Solve it.** The new Ontario Ministry of Education intermediate mathematics guidelines tend toward this position. The Ontario Association of Mathematics Educators (OAME) has devoted the entire March, 1981 issue of its journal, the Ontario Mathematics Gazette, to discussions of developing problem solving skills in elementary and secondary school students.

Closely related to problem solving instruction is the study of algorithms. Algorithmic activity is rendered concrete with the use of calculator methods for computations and of computer programming techniques. There is a rich mathematical culture associated with these activities -- one can enumerate flowcharting, iterative and recursive procedures. Boolean operation, defining solutions by the method of step-wise refinement, proving the correctness of algorithms through debugging techniques, etc.

Using the computer, mathematics can become a powerful exploratory activity when used by students to construct mathematical models or to learn from a simulation activity. Presumably, the computer can become the students' personal way of involving themselves in the milieu of mathematics and problem formulation/problem solving activities.

A truly exciting activity that involves a unique use of computers to develop individually tailored problem solving skills is the application of artificial intelligence to student-machine dialogues on a computer (online terminal or microcomputer). This paper will take up some work that has been done in this area.

Changes in Content and Style

To sum up, this paper will focus on (a) changes in secondary school mathematics curriculum content impelled by computer technology and (b) changes in teaching and learning strategies that have been made possible with the application of computers in the classroom.

The latter is a new look at the way in which p ople learn (epistemology). It puts into question the traditional axiomatic-deductive approach to development of math topics in favor of an approach which reflects on the thought processes that go into mathematical activity, i.e. a truly problem-solving, heuristic approach.

It also puts into question the notion of a curriculum prescription as a way of effectively learning mathematics. To quote Seymour Papert in his address to the May 1981 ECOO Conference: "Package knowledge is fundamentally wrong and, therefore, so is the concept of curriculum." Papert declared that categorizing knowledge is a futile enterprise because it (knowledge) grows "beyond anything we can anticipate" and that educators should build on activities that will be continuous with what will be around in 10 years. Pointing out that computers will still be around in 10 years, he argued that the computer's usage in the math curriculum should forcefully demonstrate, **through the use of models,** the real mathematical possibilities in order to inspire and convince students. Turning to individual learning style, he argued that we should not accept the concept that there is only one route through the brain to learn each particular concept but that it was arguably and evidentially more probable that there are many and that through the computer we can "make a mathematics for different personality types". For hyperactive and dynamic children, the computer could provide an active mode of presentation while for "obsessional and quiet types", it could employ a more static form. The point of including Papert in this discussion is that he deals with curriculum in a fundamental way.

Finally, the paper's survey will attempt to shed light on three different models of learning: learning through instructional presentation (CAI drill or tutorial, pre-programmed simulation); learning through placing the student in the role of creator of mathematical ideas (model-building, programming); or learning in an intermediate modality between these two (artificial intelligence applied to problem solving).

2. Changes In Math Topics In The Curriculum

Formative Influences

The advent of computers has had a decisive effect on curriculum thinking if only because the future job market is there. The area of computer science has many close connections with mathematical procedure and methodology. Naturally, suggestions for new curricula have tended to elaborate and emphasize those parts of current mathematics which have a bearing on computing and have annexed new topics which are computer-relevant.[22] In the universities, there has been a considerable shift to topics like structures, linear algebra, Boolean algebra and logic, algorithms, numerical and statistical methods.

Computer Based Teaching

In the area of computer-based teaching of mathematics, world experiences relevant to secondary school mathematics have varied from computer-managed instruction[14], computer-assisted instruction and simulations [15,16,17,18,21] to analysis of mathematical or logical games[23]. In some cases, the content is traditional but in a new format while in others the content is either a reform of previous topics or completely new to the secondary schools.

An approach which is a more cautious encroachment into the traditional mathematics classroom is the use of the computer as an interactive and dynamic audio-visual aid, the "electronic blackboard".[7,26] There are two key arguments for this type of modest and supplementary entry of computers into the classroom: very few mathematics teachers are sufficiently knowledgeable at this stage to feel technically at ease with the computer; it is preferable to make the computer's debut as acceptable as possible from the point of view of these teachers so as to pave the way for a future wide-scale penetration of the computer. This type of change is "teacher-centred" and starts from the existing curriculum. So the design criterion calls for high quality "stand alone" modules. Presumably, the computer is being used as the best possible solution for a particular problem.

As an instructor of teachers from the Netherlands[7] sees it, modular units should allow teachers to switch back and forth from the traditional to the computer and the units must be sufficiently useful to cause the teacher to want to repeat their usage. The conversation between the teacher and the computer has to be very restricted to the actual problem and clear and complete user guides should be provided. As some French researchers[26] see it, the use of a microcomputer like the APPLE with easily viewed television or monitor displays about the classroom should be used to produce accurate graphics to train the students' eyes. Other considerations that they stress are: the quick rendering of complex figures which may be difficult to draw by hand (e.g. polar plots); the use of animation under keyboard control to demonstrate movements of points, progressive drawing, deformations of curves and super-positions of curves; interactivity capabilities so that the students can suggest different variations which can be changed at the keyboard to allow the teacher flexibility; using the computer to provide complex or tedious computational results when the outcome is more important for the lesson than the procedure; building in pleasant aesthetics; building in easy re-programming of the presentations to take into account new insights from the classroom experiences. All these suggestions are directed to inspiring further use of the computers in the classroom.

Examples of topics suggested to conform with the above prescriptions follow.

- Graphing the conic sections illustrating the shapes being traced out, the effects of changing various parameters.
- Demonstrating sums of series through presentation of partial sums and applying this idea to demonstrating limits through calculations produced ad infinitum by the computer.
- Illustrating sums of functions for plotting with the curves shown on the screen.
- Illustrating 3-dimensional figures including transformations.
- Performing statistics experiments using data generated by random numbers demonstrating at the same time the law of large numbers.

Analogies from Data Processing

From the Netherlands comes a rather systematic approach to the problem of formulating mathematics curriculum.[5] Other writers have suggested similar ideas but not as comprehensively.[9] By analyzing intellectual activity associated with data processing power, data processing techniques, concepts of data processing, problem solving and societal concepts of data processing, a set of mathematical prescriptions for each type of activity has been developed.

(a) Data Processing Power

Using the power of the calculator or computer, mathematical instruction should seek exploratory activity, examples of real-life type problems which may use numbers that don't easily "work out", dynamic illustration of concepts through graphics and printouts, and curriculum topics which are useful in developing mathematical models.

(b) Data Processing Techniques

Relevant mathematical topics would centre in on Boolean algebra and logic, numerical techniques, and topology. Algorithmic techniques, debugging strategies, flowcharting, and verification of solutions for optimal results flow out of D.P. techniques. To these can be added conversational programming such as in CAI instruction from the student or to the student.

(c) Concepts of Data Processing

Algorithms as concrete procedures, error estimation and analysis in computations, concepts of variables and sets, manipulation of symbols representing concepts and distinguishing labels from meanings would fall into this category. The study of different number bases (decimal, binary, octal, hexadecimal) could readily fit into a secondary school level topic.

(d) Problem Solving

The idea here would be to teach problem solving strategy as a structured algorithmic approach using step-wise refinement to break down the problem into discrete stages. The structural approach should be used to provide individualized problem solving activities to students by having a computer match the problem to the students comprehension level.

(e) Data Processing and Society

Topics in discussing the moral and social effects of mathematics, for example, in its application to computers with large files that threaten the privacy of individual citizens should be pursued. Furthermore, data processing is tending to revive an interest in the history of number systems, calculation, and computing tools, topics which can be used to enhance and motivate mathematics learning.

Topics For Computer-Oriented Mathematics Curricula

The first curriculum example is taken from a mathematician from the Phillipines[19] whose concern is to devise a math curriculum for a job-oriented stream of students who are headed for computer jobs and for assisting national social development.

- CLASSICAL LOGIC, including flowcharting to introduce the "mathematical way of thinking".
- CLASSICAL SET THEORY combined with flowcharting and writing simple programs which compute sets of numbers.
- NUMBER SYSTEMS, which includes using logic and set theory to define the various number sets (natural numbers, integers, rational numbers, irrational numbers, real numbers, complex numbers), defining the binary operations on these sets, studying algebraic structures (groups, rings, etc.), solving equations to find solution sets out of the domains of particular number sets, working arbitrary binary operations and sets to demonstrate modular arithmetic.
- FUNCTIONS to inculcate the concept of using rules or formulas to establish the relationship between one set of numbers and another set, to define binary operations and compositions with functions, to work with functions which have undefined values to illustrate concepts of the domain and range of a function, and to introduce the ordered-pairs definition as a generalization of firmly established concepts.
- LIMITS, which involves students evaluating values of functions as these values appoach limiting values to demonstrate the pattern of a limit on an intuitive basis.
- DIFFERENTIAL CALCULUS, which uses the computer to explore the limit of a secant to arrive at the concept of tangent at a point.

In the name of making mathematics relevant to the age of informatics, a modern mathematics program designed to encourage mathematicians to orient to informatics is proposed by a writer from Grenoble, France.[22] By the way of illustration the following sample of a secondary school curriculum is offered. This curriculum projection differs considerably from the previous in that the student is at the centre of activity and becomes a creator of mathematics and as an active learner is

led to the normal mode of computer activity.

- Teach sets by allowing students to create their own rather than being given unambiguous groupings which hold little interest for them.

- Once the set is created (in principle responding to a need), the student manipulates it, gives the set a name, gives it some symbolic representations (chosen as the student's own conventions), superimposes a structure which can be represented by Venn diagrams and which embeds the notion of a **file** (closed set).

- Set up exploratory activity by the students in which they find all sorts of things to do with such a set thus leading to the formulation of algorithms by the students themselves (e.g. determining intersections and associativity).

- Use the computer to translate the mathematical algorithms into flowcharts and programs leading to debugging activities.

- Use the computer to demonstrate approximation errors; e.g. ask the computer to calculate 1/17 x 17 and see if the answer is 1.

- Develop the notion of a graph for effective modeling.

The last math curriculum list under consideration comes to us from one of the leading computer scientists in the United States, and in the world for that matter, Anthony Ralston.[29] Although these topics were developed for U.S. university courses designed to bring mathematics education at that level into the mainstream of computer science, their selection has obvious implications for secondary school instruction. What follows is an extrapolation of the high school courses that would be prerequisites for this list or overlap with it.

- Algorithms and their analysis.
- Introductory mathematical logic.
- Limits and summation.
- Mathematical induction.
- The discrete number system.
- Introduction to topology.
- Difference equations and generating functions.
- Discrete probability.
- Functions and recursions.
- Functions, relations and equivalence.
- Algebraic structures -- groups, semi-groups, rings.
- Boolean algebra and set theory.
- Introductory calculus.
- Introductory statistics.

It should be noted that there are many instances in the literature of topics which are included because they relate to computing education but they only step up the level of math required or add to the curriculum while ignoring or underestimating the new cognitive and affective possibilities. There is an important basic curriculum issue here: whether to merely add on or completely restructure. The concept of complete restructuring, however, must be considered from more than the point of view that computer-oriented mathematics is important for future jobs. The computer is an instructional medium in its own right. For the first time in human history, we have a vehicle which can render the concepts of abstract mathematical ideas in a concrete and heuristic form.

3. Changes In The Ways We Do Mathematics

The Methodology of Problem Solving

The 1975 National Assessment of Mathematics (U.S.) concluded that "students in the USA had the ability to calculate but they couldn't apply calculations to problem solving"[4]. Therefore, it seems, a math curriculum devoted to computation has failed to develop problem solving ability. The same is true for the Canadian experience. No wonder that so many knowledgeable math teachers have argued for more use of calculators in the classroom so that computational procedure time could be reduced to allow for more problem-solving and algorithm developing time. As mentioned before, in Ontario this problem is being addressed by the revival of Polya's heuristics and the new emphasis on problem solving at the secondary school level.

It is now being suggested that direct teaching of problem solving is necessary to achieve the necessary transference of skills and method to mathematical problems (and of course other types as well). An article in the March 1981 issue of the **Ontario Mathematics Gazette**[2] argues for teaching problem-solving to students in order to increase their awareness of the process used to solve problems and to get them to memorize a strategy for solving problems. The problem-solving strategy is based on Polya's five-step heuristic: define, explore, plan, do it, look back.

Papert's Use of Computers to Teach Problem Solving

This idea, then, leads to eschewing right-wrong dichotomies as the paradigm of instruction and to embracing the notion of shades of rightness and wrongness which can be corrected with good debugging strategies. In his book **Mindstorms,** Papert takes up this notion in conjunction with the use of the computer as a new "artifact" that provides the means for being able to actually implement the new paradigm.[3] In Papert's epistemology of math learning (fundamentally rooted in Piaget's pedagogy), the emphasis is on induction through exploration and on the student as creator of mathematical idea. Papert summarily dismisses the common belief that people are endowed with bundles of aptitudes; he sees this as symptomatic of the fragmentation and frustration of children's inherently good learning instincts through bad educational practice:

"One of the main lessons learned by most people in math class is a sense of having rigid limitations. They learn a balkanized image of human knowledge which they come to see as a patchwork of territories separated by impassable iron curtains. My challenge is not to the sovereignty of the intellectual territories but to the restrictions imposed on easy movement among them. I do not wish to reduce mathematics to literature or literature to mathematics. But I do want to argue that their respective ways of thinking are not as separate as is usually supposed. And so, I use the image of Mathland - where mathematics would become a natural vocabulary - to develop my idea that the computer presence could bring the humanistic and mathematical/scientific cultures together..."(page 39)

Defining the "mathetic" idea as learning in a general sense in which mathematics is a branch of knowledge linked to other branches in principles and methods, Papert states that current mathematics fails mathematically. He points out that it is an historic adaptation to school classrooms stuck with paper-and-pencil technology with its linear and sequential presentation sequence. Thus topics are selected not on the basis of what is easier to learn but what fits in the best and the result is an "intellectually incoherent set of topics that violates the most elementary mathetic principles of what makes certain material easy to learn and some almost impossible." (page 53)

Papert and his colleagues at MIT have attempted to reconstruct mathematical and mathetic thinking in terms of programming activity by children. Using a simple but powerful command language (LOGO), the students are able to command a mechanical "turtle" to draw geometric shapes or create sounds or colors.[13] The language is suggestive of motions of the body and can be used in a natural way to achieve concepts of algorithmization, debugging, recursive procedures, and finite change. Turtle geometry has in effect become a bridge to Polya through making it possible to learn mathematics by doing it yourself. The success of this work has inspired microcomputer versions of Turtle graphics and similar work in other parts of the world.

In the traditions of Montessori, Dewey, and Piaget, Papert makes the following declaration of what to do:

"Faced with the heritage of school, math education can take two approaches. The traditional approach accepts school as a given entity and struggles to find ways to teach it. Some educators use computers for this purpose. Thus, paradoxically, the most common use of the computer in education has become force-feeding indigestible material left over from the precomputer epoch. In Turtle geometry the computer has a different use. There the computer is used as a mathematically expressive medium, one that frees us to design personally meaningful and intellectually coherent and easily learnable mathematical topics for children. Instead of posing the educational problem as 'how to teach the existing school math,' we pose it as 'reconstructing mathematics,' or more generally, as reconstructing knowledge in such a way that no great effort is needed to teach it." (page 53)

Papert has summarized the computer-based reconstructive approach in his "principles of children's math" (page 54).

-THE CONTINUITY PRINCIPLE. The mathematics must be continuous with the child's well-established personal knowledge from which it can inherit a sense of warmth and value as well as "cognitive" competence.

-THE POWER PRINCIPLE. The mathematics must empower the learner to perform personally

meaningful projects that could not be done without it.

-THE PRINCIPLE OF CULTURAL RESONANCE. The topic must make sense in terms of a larger social context so that transference of learning can take place.

It should be noted that other researchers have developed the same type of learning notions as Papert in, for example, simulations[21], mathematics and art[28], or mathematics as a programming activity. [20,25,27]

One important area of exploration and debate is that of involving students in the task of mathematical modeling. A lot of discussion has taken place regarding the invaluable role that simulations play in immersing the learner in a good learning invironment [6,21] but few more than Papert have come to terms with the student as creators of simulated environments, which is the activity of mathematical modeling. Related to this idea is the concept of students writing teaching programs to teach mathematics to others. The availability of relatively inexpensive microcomputers with good graphics capabilities makes it conceivable that students could engage in such activity thus furthering the concept of individual reconstruction of mathematics.

Artificial Intelligence in Problem Solving

Artificial intelligence is as yet a nascent art in computing activity. It holds out great promise but is held back by gaps in the science of knowledge and of the brain. Nonetheless, some significant small steps have been taken toward a method of using an interactive computer to aid a learner through the stages of problem solving. [18,24] The computer's role is to monitor the student's heuristic processes as (s)he sets out the stages of the solution. The computer can allow for non-optimal solutions by storing in its memory a set of alternate solutions and scanning through the set to see if the student has chosen a true solution. Alternatively, the computer could force the student to proceed only along some pre-conceived optimal path.

Within the modules of the artificial intelligence program would be hierarchies of stages of solution. Thus the computer could detect when a necessary stage has been leaped and provide hints to the student as (s)he progresses. The hinting activity and schema structures resemble Polya's heuristics. In this mode, the computer serves as a personal tutor in the learning of problem solving. Clearly, this type of learning isn't as learner-centred as Papert's but it stands much further in the direction of self-reflection in mathematical activity than the traditional CAI approach.

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A COMPUTERIZED DIAGNOSTIC TEST IN MATHEMATICS

by

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ABTRACT

This paper describes a computerized diagnostic test in mathematics for first year university students. The test is based on Grade 13 or Level 6 mathematics in Ontario. It is designed to diagnose possible areas of weakness in a student's mathematics background. Problems making up the test are classified into four categories: Functions and Relations, Calculus, Vectors and Matrices, and Manipulative Skills. The test package has been implemented on the CYBER Computer at McMaster University, on a pilot basis. A number of first-year engineering students were randomly selected to take the diagnostic, and their subsequent performance in first-year mathematics courses was monitored.

INTRODUCTION

Students entering university have a variety of background in mathematics. This may be due to different highschool curricula, different depths of treatment in different topics, and most important of all, different individual likes and dislikes for the subject matter. The first-year university mathematics curriculum, on the other hand, is usually based on very specific assumptions of highschool background. This is especially true in professional degree programs. However, these assumptions, more often than not, are only vaguely stated as pre-requisites of a particular course. Most first-year students, therefore, are not made aware of their possible deficiencies in preparation until new university material calls for the required background. By then, it may be too late for the student.

The diagnostic test described here provides an early warning to the student with a serious lack of preparation in certain areas. This is done interactively at a computer terminal so that the student obtains the results of the test immediately. His performance in the diagnostic test is kept confidential and he can attempt the test more than once to monitor his own progress. The immediate feedback of his score and its confidentiality are seen as two positive features of this test, one to motivate improvement and the other to remove the stigma of a poor performance among his peers.

We believe that a diagnostic test of this nature, given in the early part of the term will have beneficial effects downstream. To test this, we have compared the performance of this group of students against the rest in their first-year mathematics courses.

DESCRIPTION OF DATABASE AND SAMPLE QUESTION

The collection of problems which form our database is grouped into four categories: Functions and Relations, Calculus, Vectors and Matrices, and Manipulative Skills. Each category is further divided into three levels of difficulty: elementary, intermediate, and advanced.

Typical Grade 12 and Grade 13 textbooks are used as source books for these problems. As well, the Mathematics Department of the Hamilton School Board provided us with some Level 6 Examination papers. We also selected many problems from the MAA High School Mathematics Examinations.

Table I. gives the breakdown of the total number of questions contained in the database.

	Funct. Rel.	Calculus	Vect. Mat.	Man. Skill
Elem.	30	29	29	38
Int.	33	33	27	38
Adv.	29	33	25	27

TABLE I

It must be pointed out that although the classification according to subject matter was not too complicated, the division into levels of difficulty was somewhat subjective. Most problems that require a single mathematical concept and its direct application are classified as elementary. Problems involving either more than one concept and their direct applications, or just a single concept with in-

direct application are classified as intermediate. All other more difficult problems are classified as advance.

Corresponding to each problem, a correct solution, a set of multiple choices and their respective scores are also created and used as part of the database. One major area of weakness in this particular implementation is the lack of graphics capability at the computer terminals. This prevented us from including some good geometry problems which would have been excellent for diagnostic purposes.

We give here a sample question:

GIVEN THE INFINITE EQUATION: X**(X**(X**(.....))) = 2,

FIND THE VALUE OF X.

This is a good example in which certain modifications are necessary for the conventional mathematical notations, due to the display limitations of the terminal. The actual equation is:

$$X^{X^{X^{\cdot^{\cdot^{\cdot}}}}} = 2.$$

The sample question is accompanied by a set of multiple choices:

(1) INFINITY
(2) 2
(3) +-SQRT(SQRT(2))
(4) +-SQRT(2)
(5) NONE OF THE ABOVE

The student is asked to make a choice and enter it into the terminal.

The last component of the database is made up of specific and general diagnostic messages which are selected and displayed to the student, based on the scores in individual categories as well as the overall score. The following is a typical message:

LACK PROFICIENCY IN APPLYING CONCEPTS IN THIS AREA. SOME POSSIBLE AREAS OF WEAKNESS ARE: INVERSE FUNCTION, TRIG. FUNCTIONS, DOMAIN AND RANGE OF FUNCTIONS, SOLUTIONS FOR INTERSECTION OF CURVES, CONCEPT OF TANGENT TO A CURVE, COMPOUND ANGLE FORMULAS.

The message here corresponds to a score of 5 to 9 out of a possible total of 15 in the Function and Relations section.

THE PROGRAM OUTLINE

The program written for the diagnostic test is made up of three basic stages.

(i) Initialization Phase:

When a student logs on at a terminal, his student number, together with the actual time of logging on is used as a 'seed' to generate 12 random numbers. These numbers are used to select the 12 problems making up a test. This scheme, even using our limited database, can generate distinctly different tests for the same student on his second or third try.

(ii) Testing Phase:

This consists of

(a) displaying the question and time allowed for it;
(b) displaying the multiple choices at student's prompt;
(c) accepting student's choice;
(d) returning a score and the correct solution;

(e) displaying diagnostic messages.

(iii) Recording Phase:

A record is made of the attempt of the diagnostic test, consisting of the student number, score for each question, and overall score. A hard-copy of the entire test, plus appropriate diagnostic messages is made available to the student.

The program is written in Fortran V and implemented on McMaster's CYBER machine.

SOME RESULTS AND CONCLUSIONS

150 first year engineering students were randomly selected for the test. They were told the purpose of the test, and urged to take it. It was made clear that this was completely voluntary. Only 52 took the test. It was interesting that all of them completed the diagnostic twice. The comparisons are made on the records of these students.

Table II shows the distribution of the scores.

MARK	PERCENTAGE	
00-39	22	Mean = 51.22
40-49	18	
50-59	30	S.D. = 17.85
60-69	16	
70-79	8	N = 100
80-100	6	

TABLE II

The cutoff average for admission to first year engineering is in the 70's for McMaster, in 1981. These numbers here seem to suggest that the students are not as well prepared mathematically as their Grade 13 marks might indicate. This may be due to many reasons which we shall not dwell on here.

Tables IIIA, IIIB, IVA, IVB contain the distributions for the diagnostic group and for the whole of first year engineering in two mathematics courses, in the mid-year examination.

MARK	PERCENTAGE	
00-39	20.8	
40-49	13.2	Mean = 56
50-59	22.6	
60-69	20.8	S.D. = 19
70-79	9.4	
80-100	11.3	N = 52

TABLE IIIA
Distribution in Calculus mid-year examination
For Diagnosic Group

MARK	PERCENTAGE	
00-39	23.7	
40-49	16.1	Mean = 53.3
50-59	21.8	
60-69	18.5	S. D. = 18.9
70-79	8.4	
80-100	9.4	N = 408

TABLE IIIB
Distribution in Calculus mid-year examination
for all first-year engineering

MARK	PERCENTAGE	
00-39	20.8	
40-49	26.4	Mean = 54.6
50-59	17.0	
60-69	9.4	S.D. = 21.2
70-79	11.3	
80-100	13.2	N = 52

TABLE IVA
Distribution in Eng. Math. I mid-year examination
For Diagnostic Group

MARK	PERCENTAGE	
00-39	27.5	
40-49	21.2	Mean = 50.4
50-59	20.4	
60-69	14.8	S. D. = 19.8
70-79	5.1	
80-100	8.3	N = 400

TABLE IVB
Distribution in Eng. Math. I mid-year examination
for all first-year engineering

The figures in the tables are meant to give only qualitative indications. It is encouraging to see that the diagnostic group performed better, on average, in both mid-year examinations. But, the difference is statistically insignificant. Even this small difference may well be explained by the voluntary nature of the diagnostic test, which may have given us a biased sample. For example, the students who took the test may be better motivated than the average. Only a much wider implementation made mandatory for the selected students can decide whether a diagnostic test of this nature will have significant influence downstream.

As a pilot project, this has pointed out several areas of importance in a diagnostic test. We had no direct follow-up, or remedial assistance for the students aside from some listed textbooks that they might use. Mandatory remedial work would certainly have helped. The fact that only very few of the hard-copies of the tests were actually picked up seems to confirm the desirability of an immediate response, such as we have in this test. There are other aspects of the diagnostic which can do with improvement, such as a wider database, graphics, and even more detailed diagnostic messages based on the answers given. With computerization, these and other changes can certainly be made.

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MATHEMATICAL EXPOSITIONS

THE DISTRIBUTION OF PRIME NUMBERS AMONG THE INTEGERS

by
Israel Kleiner

Introduction

The study of the properties of the positive integers (i.e. number theory) has fascinated both amateurs and mathematicians throughout the ages. The subject is tangible, the results are simple to state and are often suggested quite naturally by numerical evidence. Nevertheless, they are frequently very difficult to prove. "It is just this", said the great 19th century mathematician Gauss, "which gives the higher arithmetic [number

theory] that magical charm which has made it the favourite science of the greatest mathematicians."

An important class of integers are the primes. The primes are important because they are the "building blocks" for all the integers. This means that if we take all possible products of the prime numbers we obtain all the integers without repetitions. The latter is the important part of the above assertion and it means that different products of primes yield different integers. This result, the so called Fundamental Theorem of Number Theory, was already (essentially) known to the Greeks of antiquity. It is not an obvious result. In fact, there are mathematical systems in which it is not true.

Exercise 1. Consider the system of even integers. Show that every member of this system is a product of primes of the system, but that such a representation is not always unique.

The question of which systems analogous to the integers possess the property of unique factorization into primes is an important one in number theory. It gave rise in the 19th century to a new branch of number theory called algebraic number theory, and in particular to such fundamental algebraic concepts as field, ring and ideal. This is a good example of an important feature in the development of mathematics. In an attempt to solve a specific problem one is often forced to introduce and develop new concepts and theories which transcend in their importance the specific problem in question.

Exercise 2. Use the Fundamental Theorem of Number Theory to show that $\sqrt{n}$ is an irrational number for every nonsquare integer n.

In this note we will consider how the primes are distributed among the integers. We ask questions such as these: Is there a pattern that governs the distribution of primes? Is there a formula which describes all the primes or a subset of the primes? Such questions have a long history and are still of very considerable current interest.

General Results

Let us list the first 50 primes:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229.

a) The first question that suggests itself is whether the number of primes is finite or infinite. Of course, we know that there is an infinite number of primes, but this, too, is not an obvious result. (Think about it - why should there be an infinite number of primes? In fact, there are systems which have only finitely many primes.) The proof of the infinitude of primes was given by the Greeks over 2,000 years ago and appears in Euclid's "Elements". (We usually associate the name of Euclid with geometry, but of the thirteen books of his "Elements", three are devoted to number theory.)

b) If we look at the above list of primes we note that there are many pairs of consecutive primes which differ by 2: e.g., 3, 5; 11, 13; 17, 19; 29, 31, ... 107, 109. These are called twin primes. Is there a finite or infinite number of twin primes? The answer to this question is unknown, although it is conjectured that there are infinitely many twin primes. (e.g., there are 1,224 such pairs below 100,000; in 1980 Atkin and Rickert at the University of Illinois found twin primes containing more than 1,000 digits: $1024803780 \times 2^{3424} \pm 1$.) In any case, it was shown by the mathematician V. Brun in the 1930's that there are "not too many" twin primes, in the follow-

ing sense. It is known that the series $\Sigma\frac{1}{p}$, where p ranges over all primes, diverges [1]. Brun showed that if p ranges only over the twin primes the series $\Sigma\frac{1}{p}$ converges [7]. Moreover, the Chinese mathematician Jing-run Chen showed recently that there are infinitely many pairs p, p + 2 such that p is a prime and p + 2 has at most two prime factors.

Exercise 3. Show that the sum of any pair of twin primes is divisible by 12 (except for the pair 3, 5, of course).

Exercise 4. The primes 3, 5, 7 are called triplet primes. Show that there are no other triplet primes (i.e. primes p, q, r such that $q - p = 2$, $r - q = 2$).

c) As we go out into the far reaches of the number sequence the primes get more scarce. For example, the number of primes in the first five blocks of 1,000 is 168, 135, 127, 120, 119 while their number in the last five blocks of 1,000 below 10^7 is 62, 58, 67, 64, 53. The prime 370261, for example, is followed by 111 composite integers. For the hard to impress we set down a sequence of 999,999 consecutive composite integers: $10^6!+2$, $10^6!+3$, $10^6!+4$, ..., $10^6!+10^6$, where $10^6! = 1 \times 2 \times 3 \times 4 \times \ldots \times 10^6$.

Exercise 5. Give an example of 123456789 consecutive composite integers. Generalize.

d) The above examples show that there are arbitrarily large gaps between primes. Nevertheless, consecutive primes are "not too far removed" from one another in the following sense: given any integer n there is always at least one prime between n and 2n. This result, established in the 19th century, is known as Bertrand's Postulate (see [6]).

Exercise 6. Show that there is a prime betwen $\sqrt{n}$ and n for every integer $n > 2$.

Exercise 7. If p_n denotes the n-th prime, show that $p_n < 2^n$ for every integer n.

Exercise 8. Show that there are at least 3 primes having 185 digits each. Generalize.

e) Let us summarize what we have so far. Numerical evidence indicates that the frequency of primes decreases, although there is no regularity in their rate of decrease (e.g., there are 8 primes between 9,991 and 10,090 and 12 primes between 67,471 and 67,570). There seem to be infinitely many primes that are as close together as possible (the twin primes), while there are arbitrarily long sequences of consecutive integers without primes between them (Ex. 5). Still, we will always find a prime "not too far away", after a long sequence of composite integers (Bertrand's Postulate).

This apparent irregularity in the distribution of primes prompted the famous mathematician Euler of the 18th century to say: "Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate."

f) In a sense, Euler's pessimism was unjustified. For, although we find no regularity in the distribution of the primes considered individually, Gauss found regularity in thier distribution when considered collectively. In mathematics one must be able not only to give the correct answers, but to ask the right questions. Gauss tried to count the number of primes in a given

interval instead of looking for a rule that generates successive primes. Put differently, he tried to describe not "how" but "how often" the primes occur in the integers. In technical terms, let $\pi(x)$ denote the number of primes $\leq x$ (x a real number). Gauss tried to find a formula describing the behaviour of the function $\pi(x)$. He conjectured that $\pi(x) \sim \frac{x}{\log x}$ (the log is to the base e; "~" means approximately equal to, in the sense that $\pi(x) / \frac{x}{\log x} \to 1$ as $x \to \infty$.) A look at the evidence considered so far should persuade us that the search for an exact formula for $\pi(x)$ is overambitious. If we write Gauss' conjecture as $\frac{\pi(x)}{x} \sim \frac{1}{\log x}$, it says that the probability of picking a prime from the first x integers is approximately $\frac{1}{\log x}$. Gauss was unable to prove this fundamental result, known as The Prime Number Theorem. It was proved only about 100 years later (in 1896) by very high-powered mathematical methods.

Prime-Producing Formulas

We will now attempt to find formulas which produce primes. We undertake our search for such formulas in several stages. The requirements for the type of formula we seek will be progressively less demanding at each successive stage.

I. We first wish to find a "nice" formula f(n) which will produce all the primes and only primes as n ranges over the positive integers. (A formula f(m,n) with two or more variables will do.)

a) One such formula, albeit a trivial one, is:

$$f(n) = \begin{cases} 2, & \text{if } n \text{ is not a prime} \\ n, & \text{if } n \text{ is a prime} \end{cases}$$

(Although it is difficult to describe what a "nice" formula is, the above is certainly not a "nice" formula.)

b) A more interesting formula is

$$f(x, y) = \frac{y - 1}{2} [|\beta^2 - 1| - (\beta^2 - 1)] + 2, \text{ where } \beta = x(y + 1) - (y! + 1)$$

and $|\ |$ denotes the absolute value. It is not difficult to show that the above function gives all the primes, only primes and, in fact, each odd prime exactly once as x and y range over the positive integers (see [4]). This seems very impressive. On closer scrutiny, however, we note that

$$f(x, y) = \begin{cases} 2 & \text{if } \beta^2 \geq 1 \\ y + 1 & \text{if } \beta^2 = 0 \ (\text{i.e. } \beta = 0 \text{ or } x = \frac{y! + 1}{y + 1}) \end{cases}$$

and the case $f(x, y) = y + 1$ occurs if and only if $y + 1$ is a prime. This follows from an elementray result in number theory known as Wilson's Theorem, which states that an integer n is prime if and only if n divides $(n - 1)! + 1$. This shows that the above formula is essentially a restatement of Wilson's Theorem and thus not very impressive or useful.

Many other functions which represent all the primes and only primes have appeared in the mathematical literature. They are all, however, (to the best of my knowledge) of the type which would not be considered "nice" or useful. They may be restatements of Wilson's Theorem (as in (b)above) or similar characterizations of primes; additions of 1 to itself p-times, p a prime (as in (a) above); formulas for the n-th prime in terms of the first n

primes(!), and so on.

c) As a result of deep work in logic the contemporary mathematician Matijasevich has produced (in 1970) a polynomial which assumes all the primes and only primes for its positive values (it assumes also infinitely many negative nonprime numbers). Since the polynomial is of degree 25 in 26 variables, we shall not produce it here (see [5] for details).

II. We now consider functions which produce only prime values (but not necessarily all the primes).

a) The mathematician Fermat (17th century) thought that $F(n) = 2^{2^n} + 1$ was such a function. It is not. In fact, while $F(0) = 3$, $F(1) = 5$, $F(2) = 17$, $F(3) = 257$, $F(4) = 65537$ are primes, $F(5) = 4294967297 = 641 \times 6700417$ as was shown later by Euler (see [3]). (Fermat obtained many important results in number theory but rarely proved any of them. Most were later proven to be correct - this one is an exception.) No other F(n) (they are called Fermat Numbers) are known to be prime. The largest Fermat Number known to be composite if F(1945) - it has more than 10^{582} digits! But it is not known if, for example, F(17) is prime or composite. It is an open question whether infinitely many of the Fermat Numbers are prime.

An interesting connection between Fermat Numbers and geometry was discovered by Gauss: a regular polygon of n sides can be constructed with straightedge and compass if and only if $n = 2^t \times p_1 \times p_2 \times \ldots \times p_s$, where the p_i are distinct Fermat primes.

b) A formula similar to Fermat's was suggested by his contemporary Mersenne. Mersenne's formula is $M(n) = 2^n - 1$.

Exercise 9. Show that for M(n) to be prime n must be a prime.

We thus need to consider only $M(p) = 2^p - 1$, where p is a prime. Here $M(2) = 3$, $M(3) = 7$, $M(5) = 31$, $M(7) = 127$ are all prime but $M(11) = 2047 = 23 \times 89$. Twenty seven of the M(p)'s (the so called Mersenne Numbers) are known to be prime. (The 27th, $2^{44497} - 1$, has 13,395 digits and was discovered in 1979.) It is not known if M(p) is prime for infinitely many p.

There is a connection between Mersenne Numbers and "perfect numbers", a number n being perfect if the sum of its divisors, excluding n, is equal to n. (e.g., 6, 28 are perfect since $6 = 1 + 2 + 3$, $28 = 1 + 2 + 4 + 7 + 14$.) The result (due in part to Euclid and in part to Euler 2,000 years later) states that an even number n is perfect if and only if $n = 2^{p-1} \times M(p)$, where M(p) is a Mersenne prime.

Exercise 10. Find four perfect numbers (other than 6 and 8).

Exercise 11. Show that if $n = 2^{p-1} \times M(p)$, where M(p) is a prime, then n is perfect. (This is Euclid's contribution. The converse, due to Euler, is more difficult to prove.)

c) A formula producing only primes was found in 1947 by the mathematician W. Mills. He showed that there exists a real number a such that $[a^{3^n}]$ is prime for n = 1, 2, 3, ... (see [2]). (Given any real number x, one defines [x] to be the integer closest to x and $\leq x$.) This is not a very "nice" formula - we do not know what the a is nor what the primes are, but at least we know that such a formula exists.

Exercise 12. Show that no polynomial $f(x) = a_0 + a_1x + \ldots a_nx^n$, a_i integers, will produce only primes (x assumes all positive integral values). (Hint: assume the contrary, and suppose f(m) = p for fixed integers m and p,

with p a prime. Show that f(m + kp) cannot be prime for all k = 1, 2, 3, ...)

III. We are now looking for a function which will produce infinitely many primes (thought not necessarily only primes). Trivial examples are f(n) = n and f(n) = 2n + 1.

Exercise 13. Show that f(n) = 4n + 3 assumes prime values infinitely often as n ranges over the positive integers. (Hint: the proof is similar to the standard proof showing that there exist infinitely many primes.)

a) Effecting a grand generalization of the above exercise, Dirichlet (19th century) showed that if a and b are arbitrary fixed positive integers without common factor, then F(n) = an + b yields an infinite number of primes for n = 1, 2, 3, ... (Note that an + b is just an arithmetic progression with first term = b and common difference = a.) This is a very important result and is difficult to prove. It belongs to the branch of number theory called analytic number theory (i.e. the methods of investigation entail the use of infinite processes).

Exercise 14. Show that there exist an infinite number of primes whose unit digit is 7. Generalize.

Exercise 15. Show that there are infinitely many primes with at least 89 consecutive digits = 0. Generalize.

b) In 1963 Bredihin showed that the function $f(x, y) = x^2 + y^2 + 1$ yields infinitely many primes as x and y range over the positive integers.

IV. Finally we give a number of functions each of which produces only a finite number of primes:

$f(n) = 2n^2 + 29$ is prime for $n \leq 28$

$f(n) = n^2 - n + 41$ is prime for $n \leq 40$

$f(n) = n^2 - 79n + 1601$ is prime for $n \leq 79$

In fact, it has been shown that for any n there is a polynomial f(x) of degree n with integer coefficients which takes prime values for x = 0, 1, 2, ..., n.

The above are examples indicating that "scientific" induction based on the verification of even numerous cases is no substitute for a mathematical proof.

Here is a small sample of the many open questions in the field:

(i) Does $f(n) = n^2 + 1$ yield infinitely many primes? (The conjecture is yes.)
(ii) Do there exist infinitely many primes with 1's as their only digit? (e.g. 11,11111111111111111111111).
(iii) Are there an infinite number of primes which remain prime after every permutation of their digits? (e.g., 13 and 113).
(iv) Is there a prime between n^2 and $(n + 1)^2$ for every integer n?
(v) Is every even integer a sum of two primes? (The conjecture is yes - this is the famous Goldbach Conjecture, made in the 18th century.)

Despite the results we have outlined, I believe it is fitting to conclude with the following quote from H. Weyl, one of the foremost mathematicians of the 20th century, who more than 290 years after Euler echoed the former's sentiments: "... It is most surprising to find that the distribution of primes among all natural numbers is of such a highly irregular and almost mysterious character."

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THE PERSONALITY OF MATHEMATICS

by
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"If Einstein had been born into a primitive tribe which was unable to count beyond three, lifelong application to mathematics probably would not have carried him beyond development of decimal system based on fingers and toes."[1]

What is mathematics? We, in the Western world of the twentieth century, have difficulty imagining a civilization without mathematics. And yet, even for us, the word 'mathematics' escapes easy and precise definition. Mathematics has been regarded as a body of knowledge, as a method, as a language, as a tool, as a set of rules, as an original form of human thought, as an object of satire, as a source of controversy, as a subject for humour, and as an activity which has given some men untold pleasures and driven others to madness. On a slightly more general scale, mathematics is seen to belong simultaneously to the three disciplines -- art, humanity and science.

"Mathematics is an art because it creates forms and patterns of pure thought that exhibit the noblest achievement of the human mind. It has become one of the great humanities because it is a method of expressing, explaining and communicating (and guiding) man's total behaviour. It still reigns as queen of the sciences in its clear rigorous, logical structure, and in doing so serves as an ideal and goal for the perfection of other sciences, as researchers in those fields seek to discover the laws of the physical, biological and social phenomena of our universe."[2]

This 'something', this 'being', that is mathematics has an individuality all its own. With the exception of a few centuries, mankind has consistently concerned itself in one way or another with mathematics. Its usefulness is almost universally recognized. When G. H. Hardy wrote "Mathematics is a useless science", he took care to clarify his remark: "By this I mean that it can contribute directly neither to the exploitation of our fellowmen, nor to their extermination".[3] He would not deny, however, that the influence of mathematics pervades our modern life -- our technology, agriculture, architecture, economics, our artistic enjoyments and even our recreations are subject to its influence. There are few more 'popular' subjects than mathematics from the standpoint that most people have some appreciation of it.

Mathematics is unique among the sciences in the scope of motivations which have generated it. The range of motivations includes those of the primitive tribesman desiring to 'count' his belongings, those of Archimedes manufacturing military weapons for the tyrant of Syracuse to fight the

Romans, and those of Bertrand Russell examining the foundations of logical thought.[4]

One of the most distinguishing characteristics of modern mathematics is its universality. Although this universality is not complete -- the socially backward countries are also those lacking in mathematical activity -- shared among those mathematically active countries is a common language which, once slight variations in symbolism are clarified and the natural language translated can be understood by mathematicians the world over. "Being expressed in the one universally intelligible language as yet devised by human beings, the creations of mathematics are independent of national taste, as those of literature are not."[5] This is not to deny the existence of a distinctly national character in mathematics for the preferences of the French for function theory, the English for applied mathematics, the Germans for foundations, and the Italians for geometry are not denied in positting a universal language. But algebraists from Russia, Germany, and America can understand the same algebra.[5a]

Mathematical development is sometimes free, sometimes tied by rigour, yet attached to mathematics there is a beauty "which guarantees neither its truth nor its utility. But to some it brings the fit of being able to live matchless hours, to others the certainty that mathematics will continue to be cultured for the greatest good of all and for the greatest glory of the human adventure, by men who expect no material profit for themselves."[6]
Mathematics is not, and never can be completed, "for mathematics cannot escape the uncertainty that attaches to any human enterprise."[7] Mathematics is not absolute, unchanging and static. Its greatness endures "not in a stagnant state, but in a state of vitality and effectiveness ... Greek geometry and Greek tragedy have survived through the ages and retained the active power to excite our admiration and increase our happiness today."[8]

Greek geometry is not but one of the prevailing forms of geometry as we know it, and Greek tragedy is but one art from in the current body of literature. But they endure. Sophocles has become a monument of Greek culture embodying the ancient Greek ideals of serenity, self-restraint, and spiritual peace. His plays, 'Oedipus the King' one of the most notable among them, are hailed as great landmarks of balance and harmony, in which form and idea are perfectly united. In this he and his work reflected the culture into which he was born -- the habits, beliefs, values, and traditions of the Greek people of his day.

Through the sciences, the arts, through philosophy, astronomy and technology, mathematics contributes to the development of social, religious, political, and economic aspects of a society. But mathematics is affected as much as it affects, for it is a part of culture and the course it follows is determined by the religious, philosophical, industrial, agricultural and mathematical requirements or characteristics of the given culture.

It would appear that mathematics was born whenever the material necessities of life demanded. The idea of number arose in order that tribesmen could count those things that were important to them, for example, their wives, their children, their flocks. Even the most underdeveloped of agricultural economies required numerical information regarding the changing seasons, and so calendars had their beginnings. When new demands are made by science, agriculture, religion, architecture or politics, new mathematical methods are designed. Primitive man needed numbers, but not geometry. The ancient Egyptians, in order to restore the landmarks which had been destroyed by the periodic floodings of the Nile, invented geometry.

Although mathematics arises to satisfy needs, we cannot explain its development in a given culture strictly in terms of economic needs. Otherwise, how could we answer the question: Why did England, at the time when she was the most industrial and mercantile nation of Europe, refuse to accept the metric system? (Canada is about to take this step.) Perhaps non-economic factors such as national pride, or prejudice against foreign discoveries, provide a partial answer. Both patriotic addiction and prejudice have hindered mathematical progress over the ages. Although the Hindu-Arabic numerals were originally brought to Europe through contacts of Moorish merchants and through scholars who studied in Spanish universities, prejudice greatly hindered their adoption and not until the sixteenth century did they receive standard usage throughout Europe. The Philosophe, Voltaire, popularized Newton's work in France and tried to convince the French that their patriotism to the memory of Descartes was impeding their scientific progress. The English in their strict adherence to Newton, subsequent to the Newton-Leibniz priority dispute, caused similar delay.

If historians were to ascribe outstanding characteristics to the major cultures of past ages there would no doubt be some semblance of agreement that the culture of Babylonia and Egypt was steeped in mysticism and sensuality; the Greeks were preoccupied with ideas and ideals; the Romans

were concerned primarily with politics, military prowess, and conquest; and that Western Europe during 600 A.D. to 1100 A.D. was dominated by theology.[9]

Babylonian and Egyptian mathematics dealt primarily with astronomical and calendar problems. It consisted of a set of rules, derived in part from the earlier Sumerians, to cope with the everyday problems of land measurement, surveying, construction of temples, tombs and other religious buildings. The ends these people pursued in mathematics were largely utilitarian. However, even in ancient Babylonia, some mathematics was pursued for aesthetic reasons alone. "Babylonian creativeness extended to the fields -- astronomy, religion, government and law -- allows us to gain some idea of the cultural level which a society must reach for mathematics to be more than a tool, for mathematics to advance along the lines that have made it the logical, philosophical, and universal subject of modern times."[10]

The period of Greek domination was one which stressed purity, simplicity, beauty, harmony, and perfection. The Greeks, who during this era excelled in almost every field of human intellect, were the first to develop mathematics solely for the sake of interest, and it mirrored their love for speculation and pure reason and their divorce from practicality. The Greeks loved clarity, order and precision. To them, mathematical ideas were purely abstract, and the fundamental notions were those of number and of form. They developed abstraction, generalization, analysis and synthesis of ideas.

Greek mathematics was divided into logistica, the art of calculating, and arithmetica, the abstract theory of numbers. The Greek scholars viewed logistica as beneath the dignity of mathematicians and philosophers. The practical applications, scorned by the scholars and encumbered by an inadequate system of numeration, were therefore assigned to lesser persons, while the scholars devoted their attention to arithmetica, geometry and philosophy.

"The cultural conditions in the Mediterranean area during the flourishing of the Greek culture and for centuries thereafter were such that no systematic development of algebra was made. They left it for the Arabs to preserve and transmit, via Africa and Spain, the Hindu-Arabic mathematics (and the system of enumeration universal now) along with Greek geometry."[11]

Greek mathematics was profoundly influenced by the Pythagorean school of thought according to which truth, beauty, and the good must be sought in unity, in the finite, and in tranquility. Pythagorus believed that "Natural numbers and their ratios rule the universe."[12]

The antidialectical character of Greek geometry, associated with the Pythagorean ideas preaching the exclusion of mathematical infinity, a disgust for the concept of an irrational number, and the superiority of the integer, especially as it evolved into superstitious beliefs and numerology, prevented a development of mathematical analysis.

Mathematics in early China was comprised mainly of computations and solution of algebraic equations, with no geometry comparable to that of the Greeks. Wilder has proposed that the different directions taken by Greek and Chinese mathematics were determined by the primitive cultural conditions existing in those cultures during their prehistorical periods -- "and having received an initial push in a certain direction, the mathematical subcultures no doubt proceeded to a certain extent on their own momentum, operating mainly under evolutionary forces within (in the case of Chinese mathematics) or under both evolutionary and diffusionary forces (from other parts of the general culture; especially philosophy in the case of Greek mathematics.)"[13]

Slavery was a part of the Greek civilization and undoubtedly facilitated the intellectual pursuits of the ruling elite. But slavery existed also in ancient oriental society. Yet the character of scientific development in these civilizations was very different. The decline of slavery in the Graeco-Roman society is oftimes cited as a factor in the decline of their science, but there does not appear to have been a similar relationship in the Oriental society. "Its slavery showed little decline and the political and cultural expansion, contraction and stagnation of oriental forms of society did not necessarily correspond with the developments in slavery or in science."[14] The Graeco-Roman slavery was involved in production and industry, whereas Oriental slaves were engaged for public works and luxury. This distinction may provide a partial explanation. Nonetheless, the institution of slavery per se was not responsible for Greek greatness.

In the mercantile civilization of India there developed commercial arithmetic and a form of geometry to meet their technical needs. The atmosphere in which Hindu mathematics developed did not dictate the preoccupation with logic and rigour borne by the Greeks. Moreover, the Hindus were not subject to the Pythagorean prejudices against irrational numbers, and thus actually succeeded in establishing a pre-algebraic mathematical system better than that had by the Greeks.

As did the philosophy of the day, the religious beliefs and activities influenced the development of mathematics. The prevailing ideology sometimes fostered and sometimes stifled development. Much of early mathematics developed for religious purposes. However, there were also instances when the philosophical and religious attitudes discouraged the study of mathematics. The gradual decay of the Roman slave society caused many of the logically trained minds, particularly Churchmen, to turn to the spiritual rather than to the scientific. Natural sciences and mathematics were neglected. The early Christian leaders scorned pagan learning and prohibited their followers from learning Greek works. Mathematics was especially forbidden since all of the well-known mathematicians of the day, from Ptolemy to Proclus, were heathen.[15] We recall that Hypatia, the first recorded woman mathematician was an anti-Christian martyr.

Ancient Rome contributed almost nothing to mathematics. After the Fall of Rome, Western Europe began a millenium of stagnation overlapping what is sometimes known as the 'Age of Faith'. "From about 400 to 1500 A.D. mathematics reflected the general cultural condition of the times --meagre and barren."[16] The contributions to mathematics that were made were primarily those of Boethius, Alcuin, Gerbert, and Leonard Fibonacci of Pisa. Except during the Dark Ages which spanned the early part of the Age of Faith, ending approximately at the time of the first Crusade, the other cultural elements such as art, music, architecture, literature and philosophy continued to be generated. These conformed to the prevailing theological view of life.

The twelfth, thirteenth and fourteenth centuries witnessed the erosion of the feudal system and emergence of powerful commercial city states among which were Milan, Venice, Florence, and Genoa. The accompanying increased trading and commercial activities, advances in technology, craft, architecture, navigation and military engineering influenced the developing mathematics.

Mathematics, while markedly affected by social and political factors, seems best to develop in quietude. "The first mathematical research institute was founded, not in Athens with the sharp class struggles, but in Alexandria where economic activity was carried out in relatively peaceful surroundings."[17] The century between the end of the Napoleonic wars and the outbreak of World War I was a century more peaceful perhaps than any other century since the Roman peace. Not oddly, this century approximates that known as the 'Golden Age of Mathematics.' With a few exceptions, notable among them Thales - a militant member of his class, Descartes - who enlisted in the dutch army at that time one of the spearheads of the bourgeois revolution, and Galois - who participated actively in the July revolution and shortly afterwards met his death in a duel, most mathematicians sought solitude in which to create.[18]

Political persecution has resulted in the dissolution of cells of mathematical activity, most well known examples perhaps being Gottingen and Warsaw. These disruptions did however allow a diffusion and synthesis of ideas that might not have otherwise occurred. Increased mathematical activity has occasionally accompanied heightened nationalism. Consider for example the period during the Napoleonic wars following the French Revolution, and Germany, after the political disturbances of 1848.

Individuals outside of mathematics have sometimes greatly affected the mathematics. Martin Luther, highly influential in Germany at the time of the Reformation, furthered mathematics by engendering in the people of his day a fuller appreciation. In a tract published in 1524 Luther wrote, "If I had children, they should not only study languages and history, but they should also learn singing and music, together with the whole of mathematics."[19]

"Mathematics does not grow because a Newton, a Riemann, or a Gauss happened to be born at a certain time; rather great mathematicians were made because cultural conditions -- and this includes the mathematical materials -- were conducive to developing them."[20]

"In most sciences one generation tears down what another had built, and what one establishes another undoes. In mathematics alone each generation builds a new story to the old structure."[21] Growth in mathematics has been largely the result of evolution, diffusion, and synthesis and these processes have occurred within the cultural milieu. Euclid's **Elements,** considered a mathematical classic, is a compilation of the works of previous Greek mathematicians. His axiomatic method evolved gradually through the ideas of his Greek predecessors, among whom were Thales, Pythagorus and Eudoxus. Ideas very similar to those of Fermat and Descartes had been held by Nicole Oresme. Even Einstein attributed relativity to centuries before him. Archimedes had employed infinitesimal procedures which could cause him to be considered the originator of integral calculus. He was able to anticipate infinitesimal calculus but could not get his contemporaries to adopt it. Not until Kepler's day was much added to Archimedes' contribution. At that time a number of mathematicians revived the Greek mathematicians' techniques and attempted to improve

them. Cavalieri, Huygens, Wallis, and a host of others who had come within a step of the final synthesis, preceded Newton and Leibniz. Thus we are forced to smile at the superficiality of Pope's verse: *"Nature and Nature's Law lay hid in night, God said, 'Let Newton be' and all was light."*[22] Newton was a man of genius. But men of comparable genius have surely existed at various times and places in history. But the calculus was not discovered in these other times and places because the necessary cultural elements were not present. "There were just as great potential analysts and algebraists during the Greek era as during the age which produced Weierstrass and Kronecker, but among other things the necessary analytic and algebraic elements were lacking in Greek cultures."[23]

Genius cannot function in an intellectual vacuum. It functions only under appropriate cultural stimulation. Our introductory quotation, when preceded by the assertion, "The mathematical genius can only carry on from the point which mathematical knowledge within his culture has already reached"[24] becomes self-evident.

The cultural process is not restricted to one geographical location but goes on rather uniformly over a wide area housing various populations. When the necessary cultural elements are present, mathematical discoveries can be made simultaneously by two or more persons working independently. Consider the following list:

Law of inverse squares: Newton 1666; Halley 1684
Introduction of decimal point; Pitiscus 1608-12; Kepler 1616; Napier 1616-17
Logarithms: Burgi 1620; Napier-Briggs 1614
Calculus: Newton 1671; Leibniz 1676
Principle of Least Squares: Gauss 1809; Legendre 1806
A treatment of vectors without the use of co-ordinate systems: Hamilton 1843; Grassman 1843
Geometry with axiom contradictory to Euclid's parallel axiom: Lobatchevsky 1836-40; Bolyai 1826-33; Gauss 1829[25]

This list is not exhaustive. Yet already there are too many discoveries cited to dismiss them as coincidences.

"When the cultural conditions are right, it (mathematical discovery) will occur without fail."[26] The time was right. If Newton and Leibniz had not been born, someone else would have taken their place, perhaps a bit later, but just as surely. Although one individual might respond better in one culture than another, he is basically a neutral medium, a transmitter of culturally determined ideas.

However, to regard the individual **only** as a constant in the development of mathematics, as the anthropologist L. A. White has suggested, is to overlook an agent in the formation of the personality of mathematics. Mathematical works are to some extent tempered by the personalities of the mathematicians themselves. They are likewise responding to cultural as well as physiological stimuli. No two individuals even within the same culture are identical. They exhibit different traits. Some may find in mathematics that which Bertrand Russell described as "the true spirit of delight, the exaltation, the sense of being more than man which is the touchstone of the highest excellence."[27] To the field of mathematics they bring their personalities moulded sometimes perhaps by the intellectual backgrounds from whence they came.

Many of the greatest philosophers were at the same time outstanding mathematicians: Pythagorus, Pascal, Leibniz. Others including Fermat, Descartes, Huygens and D'Alembert had studied law. Still others had been primed with the knowledge of medicine and theology. Cardano and Jean Bernoulli fall within the former group, and Pascal and Jacques Bernoulli within the latter. At least one man, Lewis Carroll (Charles Lutwidge Dodgson) combined the roles of children's story writer and mathematician. These men did not sever their concerns for mankind and their capacity of human understanding when they donned the intellectual garb of the mathematician.

Mathematicians may be the media for the transmission of culture's mathematical decree. But they are also human beings and as such experience success and failure, arouse hostility and invoke anger, generate friendship and laughter, cry out for sympathy and understanding, excite our imaginations and challenge our minds. How we respond to them partially determines how we respond to their mathematics. The vivaciousness of Emmy Noether attracted; the bigotry of Cauchy repelled.

Ours is a questioning era. Our religious, social and political convictions, and our mathematical and scientific thought are mingled with insecurity, doubt and uncertainty. Especially in these times the searching question 'What is mathematics?' is a very difficult one to answer. However, we conclude that mathematics is a "living, growing element of culture", enriching and being enriched by

the other cultural elements of the civilization.

A culture existed before mathematics was born, and may exist after mathematics. It has been a "long way from the 'one, two, heap' of the savage to the infinity of George Cantor."[28] In what direction does mathematics proceed now?

FOOTNOTES

1. Raymond L. Wilder, **Introduction to the Foundations of Mathematics,** New York, John Wiley and Sons, Inc., 1952, p. 278.

2. Lucienne Felix, **The Modern Aspect of Mathematics,** New York, Basic Books, Inc., Publishers, 1960, p. 1.

3. André Weil, "The Future of Mathematics", in F. LeLionnais, ed., **Great Currents of Mathematical Thought,** Vol. 1, New York, Dover Publications, Inc., 1971, p. 322.

4. Arthur O. Garder, "The History of Mathematics as Part of the History of Mankind", Mathematics Teacher, Vol. 61 (May 1968), p. 524.

5. Eric T. Bell, "On the Development of Mathematics", in William L. Schaaf, ed., **Our Mathematical Heritage,** New York, Collier Books, 1963, p. 29.

5a. Wilder, p. 270.

6. Francois LeLionnais, "Beauty in Mathematics", In F. LeLionnais, ed., **Great Currents of Mathematical Thought,** Vol. 2, New York, Dover Publications, Inc., 1971, p. 147.

7. William Barrett, "Science and Finitude", in W. L. Schaaf, ed., **Our Mathematical Heritage,** p. p. 206.

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9. William L. Schaef, "Mathematics as a Cultural Heritage", in W. L. Schaaf, ed., **Our Mathematical Heritage,** p. 15.

10. Edna E. Kramer, **The Nature and Growth of Modern Mathematics,** New York, Hawthorn Books, Inc., Publishers, 1952, p. 3.

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12. Edna E. Kramer, **The Nature and Growth of Modern Mathematics,** p. 28.

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16. William L. Schaaf, "Mathematics as a Cultural Heritage", in Schaaf ed., **Our Mathematical Heritage,** p. 18.

17. D. J. Struik, "On the Sociology of Mathematics", in Schaaf, ed., **Our Mathematical Heritage,** p. 65.

18. **Ibid,** p. 64.

19. Arthur O. Garder, "The History of Mathematics as Part of the History of Mankind", p. 525.

20. Wilder, p. 277.

21. Carl B. Boyer, **A History of Mathematics,** New York, John Wiley and Sons, Inc., 1968, p. 598.

22. Edna E. Kramer, **The Main Stream of Mathematics,** New York, Oxford University Press, 1965, p. 209.

23. Wilder, p. 278.

24. **Loc. cit.**

25. Leslie A. White, "The locus of mathematical reality: an anthropoligical footnote", Philosophy of Science, Vol. 14 (1947), p. 295.

26 Wilder, p. 277.

27. Francois LeLionnais, "Beauty in Mathematics", in F. LeLionnais, ed., **Great Currents of Mathematical Thought,** Vol. 2., p. 122.

28. Edna E. Kramer, **The Main Stream of Mathematics,** p. 309.

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EVERYTHING THAT THE GREATEST MINDS OF ALL TIMES HAVE ACCOMPLISHED TOWARD THE COMPREHENSION OF FORMS BY MEANS OF CONCEPTS IS GATHERED INTO ONE GREAT SCIENCE, MATHEMATICS.

Herbart, J.F.

DEPARTMENTS

PROBLEMS? DID YOU SAY

by
Pat Yip

NOTICE: Problems and solutions are solicited from readers. Came across a good one lately? SEND IT IN! (to P. Yip, Department of Mathematical Sciences, McMaster University, Hamilton, Ontario).

Problems:

20.3.1 Circle and Radii

A circle is divided by 2n equally spaced radii ($n > 0$) and one secant. Find the maximum number of non-overlapping areas in the circle.

20.3.2 Fraction and Function

A fraction $f_1(x) = \frac{(2x - 1)}{(x + 1)}$ is given. Define $f_{n+1}(x) = f_1(f_n(x))$ for $n = 1, 2, 3, \ldots$. Find $f_{28}(x)$.

20.3.3 Such is Life

A man passed one-sixth of his life in childhood, one-twelfth in youth, and one-seventh as a bachelor. Five years after his marriage, a son was born who died four years before his father at half his father's final age. How long did the father live?

Solutions: (for problems in Vol. 20.2)

20.2.1 Straight-edge and compass (construction by proposer). (We welcome readers to send in proofs why this construction works.)

Given: two intersecting straight lines OK and OL, and a point P.
To construct: circle or circles passing through P and having OK and OL as tangents.
Assumptions: (1) P does not lie on OK or OL;
(2) P does not lie on the bisector of angle KOL.
Construction: Two circles are possible.

(1) Draw bisector of angle KOL, OX.
(2) Drop perpendicular from P to OX, PQ.
(3) Drop perpendiculars from Q to OP and OL, QA and QB respectively.
(4) Draw B' on OP, such that QB' = QB.
(5) Draw a line through P, parallel to QB', intersecting OX at C. C is the centre of one of the required circles.
(6) Extend QA to QA' so that QA = AA'.
(7) Draw a line through P, parallel to A'B', intersecting OX at C'. C' is the other centre.

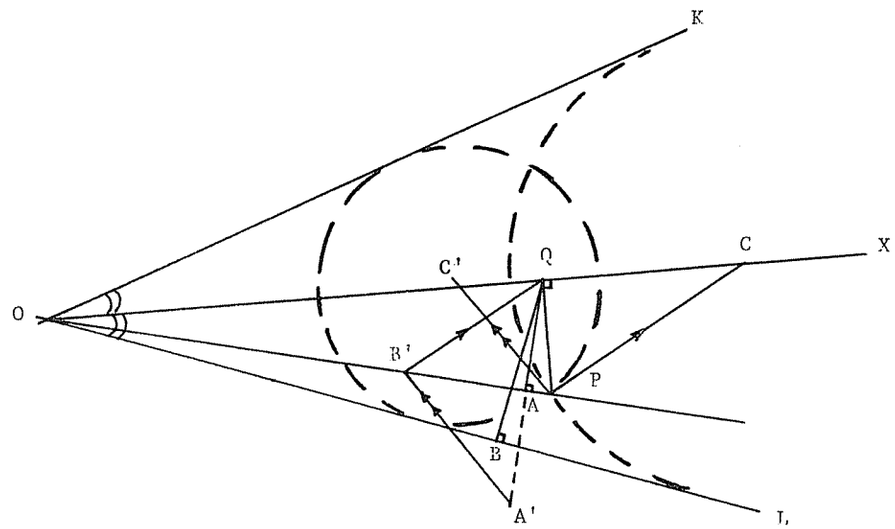


Diagram 20.2.1

20.2.2 Numbers and More:

Given: six digit numbers of the form XYZ, XYZ

To find: a common divisor for all these numbers

Solution: $XYZ, XYZ = XYZ\ (10^3 + 1) = XYZ\ (1001)$

1001 is a common divisor.

20.2.3 Triangles Galore:

Given: in the diagram, $CD = \frac{1}{3}\ CB$, $AE = \frac{1}{3}\ AC$, $BF = \frac{1}{3}\ BA$

To find: area of ΔHLG in terms of ΔABC.

Solution:

Since $CD = \frac{1}{3}\ CB$, $AE = \frac{1}{3}\ AC$, $BF = \frac{1}{3}\ BA$,

$\Delta ABE = \Delta ADC = \Delta CBF = \frac{1}{3}\ \Delta ABC = A_1$

But, $AH = HL = 3LD$, therefore $LD = \frac{1}{7}\ AD$

Similarly, $HE = \frac{1}{7}\ BE$, $GF = \frac{1}{7}\ CF$

Therefore, $\Delta BFG = \frac{1}{7}\ \Delta CBF$, $\Delta AHE = \frac{1}{7}\ \Delta ABE$

$\Delta CDL = \frac{1}{7}\ \Delta ADC$, each $= \frac{1}{21}\ \Delta ABC = A_2$.

Therefore, $\Delta HLG = \Delta ABC - 3A_1 + 3A_2 = 3A_2 = \frac{1}{7}\ \Delta ABC$.

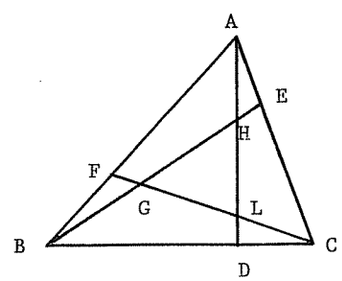


Diagram 20.2.3

INTERVIEW
with
Bob Smith

Marv Westrom graduated from the University of Alberta and worked with IBM in the field of Computer Assisted Instruction (CAI). After some fifteen years of working with the large computers he moved to the Faculty of Education, University of British Columbia. As Marv answered my questions, I could not but note the office of the future. The telephone was answered by a tape recorder so we would not be interrupted, a microcomputer with both a visual screen for output and a matrix dot printer for hard copy was available for instant use, and I wondered when I might feel ready for such an office. I wanted some information, so I asked.

RS: Marv, for my information, please distinguish between CAI and CMI.

MW: Well, Computer Assisted Instruction (CAI) is what most people think of when they think of students using computers. The kids sit down at a computer terminal, ask a question, and the computer prints out information or displays information on a screen. Computer Managed Instruction (CMI) is a much more amorphous kind of thing. The basic idea is that the computer keeps track of individual student differences and assists the teacher with the management of educational resources and curriculum in trying to tailor the instruction for each individual student.

RS: Marv, computer assisted instruction, it seems to me, is the area where the teacher in the classroom is going to be most involved, particularly as the microcomputers are now, or very shortly, will be in classrooms. What advice do you have about CAI?

MW: I think you are quite right about the presence of microcomputers having an effect on CAI getting into the schools. For the last 15 years there have been predictions that within the next 1 or 2 years CAI would make an impact on the schools. All of a sudden the impact was upon us, and it was due solely to the availability of microcomputers. Unfortunately, people who are using microcomputers for CAI don't seem to be paying much attention to all of the background and research work which has been done with large machines. Computer assisted instruction with large machines was traditionally organized into three main areas: Drill and Practice, Tutorial, and Simulation, with a catchall category named Games. Anyway, I think it is useful to understand the main three. Drill and Practice CAI is what the name implies. The purpose is for the student to drill and/or practice something that he already knows. The usual example given is practicing the multiplication table.

RS: That's the sort of thing that our own students can write and some, in fact, do write in their computer science or mathematics courses as a project.

MW: Certainly that's true, and it has the advantage of obtaining the most result for the effort you put into it. The main reason for this is that students spend a lot more time-on-task using a computer drill and practice program than they would normally. I've seen kids spend three and four hours practicing multiplication problems on a microcomputer and I can't think of any other way in which you can get them to invest that kind of effort.

RS: Dr. Peter Lindsay of OISE did a survey of microcomputer use in Ontario Schools in the spring of 1980. Some findings from his preliminary survey agree with your observation.

Most Valuable Aspects of Computerized Instruction (Percentage Time Ranked First)

Instant and Consistent Feedback	27.3
Step by Step Instruction	3.0
Motivating the Student	32.8
Individualizing the Programming	23.9
Other	3.0

MW: The second main area is called Tutorial, and again the name is descriptive. The idea is that the computer acts as an intelligent tutor, holding up one half of a conversation between it and the student. The usual procedure with a tutorial program is that the computer presents some information on the screen, there may even be several pages of this information, after which the computer asks a series of questions. If the student gets the answers correct, the process is continued. If the student gets the answers wrong, then this makes it clear to the computer that the student has

misunderstood something. The computer then presents some more pages of text that provide remediation. For the student who gets all the answers correct, the computer may say "Boy, you really know this stuff" and give him some extra little enrichment or insight into the material or even make explanations at a higher level.

There are some advantages to tutorial CAI that you should be aware of. One is that students can proceed quickly or slowly, of course, because they are proceeding at their own individual pace and the computer is attentive at all times. The student also has a wide choice of content. The tutorial programs, if they are done well, can also deal with various levels of difficulty. A program can be equally as effective with a poor student as with a very good student by choosing questions, and selecting remediation and enrichment material that meets his needs. There is also an advantage of schedule because the students can take the CAI tutorial instruction at a time which is convenient to them. I would say that the main disadvantages of tutorial CAI programs is that they are rather difficult to develop, or difficult to develop well. It is quite easy to develop tutorial programs which merely turn the pages of texts and then ask kids questions that can be answered by "yes" or "no" or a,b,c,d, but turning the pages of texts is not particularly good use of the computer. If you are going to have pages of text to look through, then you are much better to have them in their usual printed form. Putting in extra remediation and enrichment material, material that isn't used for every single student, means considerable extra work must be done. Obviously, putting more remediation and enrichment material in causes the amount of effort to go up geometrically with the content that you are covering. This is one of the main reasons why it takes about 200 hours of design and development time to develop one hour of student tutorial material.

RS: I hear a lot about simulation in the social sciences. Give us an example.

MW: Well, Simulation is indeed the third kind of CAI and it's even more difficult to develop simulation programs than it is to develop tutorial programs. Simulation has an advantage in that nearly any subject can benefit from some kind of CAI simulation program. One of the parameters involved is time compression. There are various things that we want to teach students about that happen over a long period of time and these things are difficult to observe. A good example, is to examine the ecology of a lake. The factors that are affecting the lake are difficult to control in real life and they happen over a long period of time and it is difficult to see any change happen. It's possible to make a simulation program that allows the student to control things like the temperature, the presence of effluents, the growth of algae, oxygen in the air, etc., and to have the ecology of the lake change fast enough so that it's possible to see the effects of changes, even minute changes. In order to make such a program, it's necessary to understand thoroughly and exactly what's happening with that lake, to represent these in equations and heuristics and in computer form. Of course, it's incumbent upon the designer to make sure that his simulation model is as true to life as is feasibly possible.

RS: Could you give us an example of a math simulation experience?

MW: Most real world systems are too expensive and we can't afford to be fooling around with them. I think a good example is flight simulators. Currently, the flight simulators can be made so realistic that student pilots take all of their training on a simulator and the very first real flight that they take is with paying passengers when they are in charge of the plane. I think that shows that the people who run the training programs have quite a bit of confidence in what the pilots learn with their simulators. With a simulation system, it's possible to build in counters and pointers that can let the student know how internal things are changing as the process unfolds and for that reason, the simulation experience can be more educationally valuable than the real experience.

RS: Take a moment to outline your thoughts on the other areas where CAI is used.

MW: I don't want to say too quickly or too strongly that the other areas aren't important. They certainly haven't been that important in the past, but microcomputers have come upon us so recently that it's hard to be certain that the order isn't going to change. One of the areas when we had large machines that was listed as a kind of CAI was labelled "games", and I think that label is probably not useful in talking about educational uses of microcomputers. "Games" has come to mean something quite specific with microcomputers and it means simply games that are played for enjoyment. I think that all CAI's should strive to have the fun and excitement that is shown in games and I think that CAI programmers can learn a lot from looking at games on microcomputers but I think that the term "games" should be reserved and kept to mean things that are not instructional.

RS: Let us now turn to Computer Managed Instruction (CMI)

MW: A teacher looking after the instruction of a class of 25 student has a considerable management problem. These students have different backgrounds, abilities, motivations, individual differences. We all give lip service to the fact that the instruction should be tailored as best it can to all of these individual differences but it's pretty hard for a regular classroom teacher to do that. It's the purpose of CMI systems to assist the teacher in doing that. The basic idea is that the CMI system would have contained in it all of the curriculum information that would be necessary for those students and would be able to assess the students progress. While the computer looked after all of the clerical tasks involved in this, the teacher would then be free to do troubleshooting, to tell the computer how to manage it's students more effectively and make sure that all students were learning as they should. One of the easiest parts of CMI system to develop is the on-line testing. With an on-line testing system, the student sits down at a terminal to try a test on some part of that lesson. The test can be for remediation purposes, for review purposes or for evaluation. The tests run from very easy things, that is, just a little collection of questions put together by a teacher for students to use, to very complicated systems like item banks and automatic retrieval tests based upon teaching specified criterion for items. A more refined type is adaptive testing where the computer selects questions based on the student's answers to previous questions and tries to tailor the test to the student's ability.

RS: Can I use the microcomputer as an aid in teaching as I do the overhead projector?

MW: You certainly can! Teachers are accustomed right now to using overhead projectors in a standard way. It's now possible in a school that has a microcomputer to hook it to a TV screen and use the TV screen as the display in the same way. The teacher would make up the overhead transparencies by simply entering the information needed to be displayed. Simple versions of this kind of program are quite easy to make. Of course, once you've gotten past that, it's possible to do more interesting and complex things. Because the microcomputer can display information so fast, it can give the idea of animation or motion and you can use a changing display to represent changes over time or things that are developing. You can have your own design studio and it's possible for a teacher to make up very interesting and useful presentations to the class. I think that, expecially in mathematics' classes you can spend quite a bit of time looking at, or drawing diagrams to solve particular problems. An example that occurs to me right now is looking at quadratic equations. You can probably only do two or three of those in a regular mathematics period. However, if you have a presentation aid program on a microcomputer, it would be easily possible to enter the a, b and c value of a quadratic and have the table filled out and the graph displayed quite quickly. Once you have that capability, you might examine what happens if we keep everything the same but make the 'a' value negative, or increase it by 1; what happens when we change the 'c' value. We all know what happens in these cases, but what happens when changing the 'b' value? Not many people understand clearly what happens here. With a microcomputer you can have it draw graphs, perhaps several graphs on the same grid, in which the 'b' value changes by 1. You can have the graph animated as it stepped through these values then watch the changes to the quadratic graph as it progresses through these different values.

RS: Marv, that sounds exciting. To create is always much better than merely to present. I can see a direct application too, of course, in motion geometry. Great fun it would be to do that. I'm looking forward to trying it. Could we turn our thoughts in another direction now for a moment or so. I have heard it stated many times that pupils are going to learn most when they write their own programs. Sounds as though teachers won't be necessary.

MW: I have really two attitudes toward that. I think that it's certainly worthwhile for students to write their own programs. In the example on quadratic equations, I know they have been written by students. It certainly is an advantage to the teacher to have their students write their own programs. I think it is an advantage to the students as well but the advantage is more in learning computer science than it is in learning other things. I think that there is a danger of having students get carried away with computer science and it's my feeling that knowing only computer science or knowing computer science just for itself is not very useful. One has to know computer science and something else, whether it's math or science or English or philosophy or whatever, one has to have some subject area. In a way, it's sort of like knowing mathematics by itself. Knowing mathematics without knowing some way to apply the mathematics is not really that useful. I think that the reason people feel so strongly that students should program for themselves is that that's really the only way to learn computer science and to learn how to program. I think it's analogous to learning how to ride a bicycle. You can't ride a bicycle by listening to people talk about it and you can't learn how to ride a bicycle without falling off. The only way to learn how to program a microcomputer, or any computer for that matter, is to do lots of programming, make mistakes, have the computer tell you that you are wrong.

RS: Well, Marv, I'm looking forward to retirement and it seems to me in retirement I should put

some of my skills to use so why don't I become a private entrepreneur and begin writing software.

MW: Well, there certainly is a problem with software and it's easily possible that, as an entrepreneur, you could make a lot of money. I think it's also possible that you could lose your shirt. What's going to happen with software is right now pretty hard to predict. There are three main ways of generating software for use in the schools. One is for the students to write it; there are some advocates and that may be possible. But, of course, there's a problem having students write the software because then, in a sense, they're doing the instruction. I feel, in general, they are not capable of doing it. The second major possible potential source is the teachers. I know that our Ministry of Education in BC likes to think the teachers are going to develop all of these instructional materials but I think that they underestimate the amount of effort and difficulty involved I know teachers are capable of writing textbooks as well but we don't see them doing it and I think it's because writing textbooks is hard. Writing CAI software is even harder and requires all of the skills you need to write a book plus a whole bunch more. Now, the third possibility is to get commercial organizations like publishers or special software people to do this development. I guess that's where you'd fit in as an entrepreneur. It will become a big business. Successful producers will put their stamp of excellence on particular programs and teachers will come to trust the stamp and will buy materials that way. The other way if for the Ministries of Education to get involved and to commission work and to authorize work to be done. But, as an entrepreneur, I'm not sure how you'd interact with either of these scenarios.

RS: Sufficient discouragement. You've mentioned the Ministry of Education and their traditional role in terms of approving materials. I'm sure that this will continue. I know you're invovled in writing curriculum guides here in British Columbia as well as various position papers to help teachers. Take a minute to outline what's happening in BC.

MW: Well, there are lots of things happening in B.C. I have talked to people in the Ministry about curriculum but there are tow main fronts. One is the curriculum for teaching computer science or "Computing Studies" as I prefer to call it, in the schools. There is some work being done and I am working on a committee whose responsibility it is to propose curriculum for Computing Studies in the secondary school. I think there is a perceived need to have curriculum outlines down as low as grade 8 for the teaching of computing studies in the school. The other main area is the integration of computers into the schools at all levels. How are computers being used in English and reading and mathematics and science and social studies and all of the various subjects?

RS: I'd like to end this interview, Marv, by asking a question about what we, as math teachers, might well be doing now to prepare ourselves for what has become known as the computer revolution?

MW: If you go into a room of people from the general public and ask how many people in this room never understood and are afraid of mathematics, you can expect about 50% of the people in the room to raise their hands. A surprising and appalling statistic! If you go into a room full of mathematics teachers you will find that probably none of them will raise their hand to that question but if you ask how many don't know anything about computers, are afraid of computers and really don't want to get involved with them at all, fully half of the people there will raise their hands. To me this is, in a way, like being afraid of a wheelbarrow, or a shovel, or a lawnmower, because computers are just tools. Their microwave ovens and clothes washers have computers in them. The main difference is that you can give a computer instructions instead of having the machine doing something that someone else has told it to do. So, it's merely a matter of learning how to tell the computer what it is to do. I should also say that, when you are learning about computers, the bulk of what you have to learn is not things that can be known through inference or calculation or logic. The bulk of things that you have to know are things that are given by definition. People who make the computers try to make them as easy to use as possible. Once you've learned all those definitions, it's just a matter of playing by the rules, of following the rules, of using the definitions as they are meant to be used and causing the computer to do things you want it to do. There is only one way to learn how to program and that's to get busy and to do it. I think that it's a good idea, particularly for math teachers to do this as soon as they possibly can. The reason is, I think, quite obvious. Math teachers are in the position, right now, of being a new person in town. It's acceptable for you to not know how a microcomputer works because very few people know how it works. But five years from now, if you still don't know, it's not going to be acceptable.

RS: Thank you very much Marv - your challenge I must take up. I'm on my way to the computer department.

Mathematics - Science Department
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London, Ontario

BOOK REVIEWS

CHILDREN'S UNDERSTANDING OF MATHEMATICS: 11-16.

K. M. Hart, John Murray: 50 Albemarle Street, London, W1X 4BD. Paperback. 231 Pages.

Attention: Intermediate Division Teachers

This is an important book as it deals with a hierarchy of concepts from which we, as teachers, can determine proper order of topics and levels appropriate to various children. It is the latest research done with children in their middle years of schooling. It continues the work started in England by Geoffrey Matthews and the Nuffield Mathematics Project in 1974, so allow me to recall pertinent parts of that story.

In the early 1960's it was felt that the traditional concept of introductory mathematics with its familiar emphasis on mathematical calculation was becoming outdated. New and more interesting methods of encouraging mathematical development of younger children had gained ground in the primary classrooms. These ideas had considerable support, mainly because of the hardworking Miss Edith Biggs who was spending all her time and energy in leading courses for the teachers. The basic ideas are to be found in **Mathematics in Primary Schools,** if your memory needs refreshing. (1)

In 1965 Professor Matthews visited Geneva and began a liaison with Piaget which resulted eventually in one of Piaget's researchers working with the Nuffield group to write 'check-ups'. Piaget's work became a key ingredient in the theory behind the project. Professor Matthews believed Science and Mathematics teaching have similar problems so in 1974 he obtained a large grant from the Science Research Council to fund the study "Concepts in Secondary School Maths and Science" (CSMS). Dr. Kathy Hart, who had worked as a teacher educator on the Nuffield Project, came to head the CSMS Mathematics Team.

The Science Team has not yet reported but it is hoped that their work will continue the excellent guides produced by the Schools Council under the titles listed as **Science 5-13,** (2)

There are three main reasons why I believe Kathy Hart's book is worth your time.

1. It is readable, with anecdotes which are recognizable to us as teachers.

2. A researched hierarchy for the teaching of mathematics is needed. One is produced, based on empirical evidence.

3. Research findings do not often help the teacher who must do something to correct problems uncovered. Each chapter contains a section titled Implications for Teaching. This is indeed a practice that all texts might follow.

The Math team was concerned with children's mathematical understanding of ten different topics from number operations and measurement to ratio and reflections. Items were written on these topics as suggested from schemes of work and tried on children during interviews in the schools. The freshness of children's work, of idiosyncratic methods to solve problems and the kinds of anecdotes that we came to enjoy in the Nuffield series, all resulted from these interviews. They also serve the serious purpose of clarifying the nature of the difficulties, be it in the setting of the question or in the actual mathematics.

Pupil answers are discussed in light of what the interviews reveal. An example from the chapter on Place Value and Decimals will give you the flavour.

When asked to 'multiply by ten the number 5.13', (5.13 x 10) none of this group resorted to a long multiplication algorithm. Frances and Kevin appeared quite happy with 5.130 ('...you add a nought on the end - my [illegible]me about that'). Julie and Cecilia each arrive at 50.130 (because ten times five is 50, so you add a nought on the end').

The main aim of the CSMS project was to develop a hierarchy of concepts. Thus, for each of the ten topics, an attempt is made to form a hierarchy based on what the children tested appeared to understand. An approximate matching of each topic hierarchy to every other was obtained by dividing the facility range 0-100 per cent into four stages. This entire interconnection appears in Chapter 13 and should be useful.

Even more useful is the last part of every chapter - Implications for Teaching. In these sections teachers are strongly reminded by the results that, when teaching intermediate pupils, we must not assume that these pupils have mastered basic ideas. The authors point out particular areas of difficulty and imply strategies that might be followed. After five years of work, Dr. Hart's conclusion was: "The overwhelming impression obtained is that mathematics is a **very difficult** subject for most children." This insightful book should help us to discover why, and it does provide clues for better teaching. I hope you read, study and enjoy it as I have. It is most enlightening.

1. **Mathematics in Primary Schools,** 4th Ed., Schools Council, Curriculum Bulletin No. 1, Author - E. E. Biggs, London, H. M. Stationery Office, 1972.

2. **Science 5-13.** Schools Council, London, Macdonald Educational, 1972.

R. S. Smith,
Faculty of Education,
The University of Western Ontario
London, Ontario.

FUN WITH MATHEMATICS

Fun With Mathematics. Shmuel Avital and Mary Stager, editors. Informal four page publication of the Ontario Institute For Studies In Education. Published eight months of the school year and sold in sets of ten copies. Single issue (ten copies) costs $1.50; one year's subscription (ten copies of all eight issues) costs $10.00. To place orders, contact:

Fun With Mathematics, c/o Mary Stager,
Office of Field Services and Research,
Ontario Institute For Studies In Education,
252 Bloor Street West,
Toronto, Ontario. M5S 1V6.

Fun With Mathematics is a most worthwhile publication for the classroom teacher and student in the grades 5 through 8. Teachers are always looking for suitable material for our more gifted students. This is one such item you should consider acquiring either through your own school budget or available through a system-wide distribution.

Fun With Mathematics supplements your regular programme by providing interesting problems, puzzlers, games and other fascinating investigations which span the areas of number, algebra and geometry. This colourful publication can be used to stimulate creative activities for students who have already completed their regular work.

Answers and solutions are provided in the following issue and in fact this is a useful resource for students to work through on their own.

In **Fun With Mathematics,** Dr. Avital and Mary Stager have provided a most useful and timely publication to help the teacher "turn on" the better mathematics students at the 10 to 14 age levels. The price is reasonable, the format manageable and the mathematics is stimulating.

Bob Robinson
Mathematics Consultant
Hamilton Board of Education

THE SAMUEL BEATTY ESSAYS

The Samuel Beatty Essays, 1979-1980. Edited by E. J. Barbeau, published at the University of Toronto with a grant from The Samuel Beatty Fund, 1981 (152 pages, $3.00 paperback).

This book contains nine of the winning essays in the 1979 and 1980 Samuel Beatty Essay Contests along with short notes on each of the seven authors. These authors were primarily students in grades twelve and thirteen at the time of writing although one was a grade seven student. Their essays range from the purely descriptive to the technical. The topics are "Egyptian fractions", "The area of a quadrilateral and Dido's problem", "Surface theory", "Game theory", "Sir Isaac Newton" and "Newton vs. Einstein: comparing the Newton universe and the universe of special relativity".

Detailed information about this annual contest appears in the September, 1981 issue of the Ontario Mathematics Gazette.

E. Mead
Department of Mathematical Sciences,
McMaster University,
Hamilton, Ontario.

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS (NCTM)

by
Bob Robinson

The National Council of Teachers of Mathematics is a professional organization of teachers, individually and collectively dedicated to the improvement of instruction in mathematics at all levels. Through its many publications and services, the Council is a forum for the discussion of new developments, a means of sharing innovative classroom experiences and evaluating trends in the teaching of mathematics.

In conjuction with its many affiliated groups (OAME is one of these affiliated groups) the NCTM maintains a vital, on-going program of in-service training for teachers of mathematics.

Membership is available to all persons interested in mathematics, the teaching of mathematics and/or activities related to mathematics education. There are approximately 40 000 individual members, 25 000 institutional members, and 190 Affiliated Groups receiving NCTM professional services.

BRIEF HISTORY

In 1920, 127 mathematics teachers met in Cleveland, Ohio to establish the National Council of Teachers of Mathematics. The first act of the NCTM was to acquire the **Mathematics Teacher** which for twelve years had been the official organ of the Association of Teachers of Mathematics of the middle states and Maryland.

During the years 1921-30 membership in the NCTM rose to about five thousand, and for some time remained at about that figure. Membership declined for the first time in 1934 when it dipped below four thousand. It climbed back to five thousand, and maintained a membership of about fifty-five thousand during the war. From 1946-57 the interest in mathematics education intensified and was reflected in the steady growth of NCTM memberships from six thousand in 1947 to ten thousand by 1954 to sixteen thousand in 1957. These figures include subscribers to the **Mathematics Teacher** and to the **Arithmetic Teacher** which had been launched in 1954. During the decade of 1958-69, the Council dramatically increased its program of meetings and instituted its program of Name-of-Site meetings (presently called Regional Meetings). Originally there were three annual meetings which attracted some two or three thousand people but with this expanded program of meetings in 1968-69, some thirty-thousand attended. This activity was rewarded by a phenomenal increase in membership and subscriptions and by 1969 the figure stood at eighty-two thousand. During the years 1971-81 membership was at approximately 80 000 and steadily decreased to 65 000 in 1981. The Council is addressing this situation at the present time.

The NCTM has long had many members from countries outside the United States. Recognition of the large Canadian membership was given by the change in structure in 1969 which assures that an elected representative from Canada is always on the Board of Directors. John Egsgard served as the first Canadian President of the NCTM during the years 1976-78.

PRESENT

The NCTM publishes the **Arithmetic Teacher** and the **Mathematics Teacher** both published monthly from September through May. The **NCTM's News Bulletin** is distributed to all members in September, November, January, March, and May. It provides timely news for leaders in Mathematics Education. NCTM Yearbooks deal with topics of both current and lasting interest in mathematics education. The 1982 Yearbook is **Teaching Mathematics In The Middle Grades.** The **Journal For Research in Mathematics Education** is issued five times yearly. There are many, many other general publications available and the NCTM education materials catalogue is available free

on request.

There are 190 affiliated groups throughout Canada and the United States which provide professional services at provincial, state and local levels. There are 10 Canadian affiliated groups representing all the Canadian Provinces except Prince Edward Island. It is hoped that Prince Edward Island will become an affiliate in the not-too-distant future. The NCTM is affiliated with the American Association for the Advancement of Science (AAAS).

The NCTM holds an annual meeting each spring and regional meetings throughout the year. The presentations, workshops, and exhibits provide concentrated in-service experiences in your own back yard.

Information resources dealing with calculators, computers, sex equity, careers in mathematics, free materials, council position statements, and general resource identification are available to members by contacting NCTM.

.Instructional Issues Advisory Committee

*.Professional Development and Status Advisory Committee

.Research Advisory Committee

*.Technology Advisory Committee

*.Regional Services Committee

+.**Mathematics Teacher** Editorial Panel

.**Arithmetic Teacher** Editorial Panel

*.**Journal For Research in Mathematics Education** Editorial Panel

.Educational Materials Committee

*.Conventions and Conferences Committee

*.Mathematics Education Trust Committee

.Finance Committee

*.Elections Committee

* - has Canadian representation
+ - Canadian editor (George Knill) of the monthly applications section

There is also Canadian representation on the American Statistical Association/NCTM Joint Committee. The four advisory committees will recommend projects to be carried out by task forces.

Lynda Kay Neal in an article entitled "What Does NCTM Spell?" (**Arithmetic Teacher,** March, 1970) suggested that the NCTM can represent each of the following:

NEWS AND CONVERSATION TEAMED IN MONTHLIES
NECESSARY COMMENTS ON TECHNIQUES AND MATERIALS
NOTEWORTHY COMMITTEES ON TOPICS FOR MODERNS
NUMEROUS CONVENTIONS FOR TEACHERS OF MATHEMATICS
NEW CONTRIBUTIONS FOR TEACHERS OF MATHEMATICS
NEW AND CURRENT TRENDS IN MATHEMATICS

This is an exciting year for the OAME/AOEM in that we are hosting the first Annual Meeting of

NCTM to be held outside the United States. The dates for this large convention of mathematics educators from throughout the world will take place in Toronto from April 14-17, 1982.

Here are some important addresses for you to write for further information pertaining to the NCTM and the upcoming 60th Annual Meeting.

NCTM Representative:

Dr. Ralph Connelly
Faculty of Education - Mathematics
Brock University
St. Catharines, Ontario
(416)688-5550

Local Arrangements Chairman - 60th Annual Meeting:

Dr. Dave Alexander
Faculty of Education - Mathematics
University of Toronto
371 Bloor Street West
Toronto, Ontario
M5S 2R7
(416)978-4068

NCTM Headquarters:

1906 Association Drive
Reston, Virginia 22091
(703)620-9840

Hamilton Board of Education,
Hamilton, Ontario.

DELEGATE'S REPORT ON THE NCTM 59TH ANNUAL MEETING

St. Louis, Missouri, 1981-04-22-25

by

Bill Dale

The NCTM is divided into regions, each of which has varying numbers of affiliates to the NCTM. Canada forms one such region, having 11 affiliates. The BCAMT is one.

Each affiliate may send one delegate or an alternate to the delegate assembly. One voting card is permitted per affiliate. Later in this report you will read how the 32nd Delegate Assembly dealt with the resolutions presented to it.

The Canadian Caucus informs new delegates of procedures to expect at the delegate assembly and discusses issues of regional concern.

Two announcements made at caucus are worth noting. Connie Shaver, past president MAMT, was appointed Canadian representative to the Regional Affairs Committee of the NCTM, and Ron Wittner, NCTM representative for SMTA, was appointed the new Canadian representative to the New Regional Services Committee, formerly the Committee on Affiliated Groups. Thus Ron replaces Bob Robinson, of Hamilton, Ontario.

A progress report was given on the 60th Annual Meeting to be held in Toronto 1982-04-14-17, the first time the AGM will be held outside the U.S.A. Information was shared on membership, airfares to Toronto and speakers lists provided by the Canadian affiliates. There should be a high Canadian profile at the 60th.

Delegate Assembly

Resolutions and their disposition follow:

Resolution A. The NCTM establish a periodic journal focussing on instructional uses of the com-

puter in the mathematics curriculum (K-college).

Tabled until action on Resolution P.

Resolution P. The Mathematics Teacher and Arithmetic Teacher include in each issue a regular feature addressing the use of computers in mathematics education.

Carried

Resolution B. The NCTM Board seriously consider publishing three separate journals for school teachers on three levels: Grades 1-6, Grades 7-9, Grades 10-12.

Defeated for fear of insufficient funding.

Resolution C. The NCTM call for a conference of key personnel from the business community and from national organizations and government agencies concerned with the teaching of mathematics to address the issue of the shortage of qualified teachers of mathematics in our schools and to propose solutions.

Carried

Resolution D. The NCTM undertake a program to increase the public's awareness of the shortage of mathematics teachers. This information should be publicized through newspaper articles, radio, and television.

Carried

Resolution E. NCTM collect from each state and publish or distribute the following information: (1) certification requirements for computer science teachers, if such a certificate is available; (2) number and type of computer science courses required for elementary certification; (3) number and type of computer science courses required for secondary certification in mathematics; (4) the extent to which computer science courses can be used to satisfy mathematics requirements for certification at the secondary or elementary levels.

Defeated - confusion re computer vs mathematics-content courses

Resolution F. The NCTM set up a task force to review the different state requirements for mathematics teaching certification at the different grade levels...make recommendations for college preparation in mathematics for teachers.

Defeated-work is being done on this at the present time

Resolution G. NCTM recommend that state certification agencies make provision for a mathematics specialist endorsement on teaching credentials for elementary teachers.

Carried

Resolution H. The NCTM develop guidelines for the role of a specialist in teaching elementary school mathematics.

Carried

Resolution I. The NCTM Board of Directors urge the appropriate federal agencies to examine the operation of the Metric Education Program through a widely representative task force including leading metric educators, the U.S. Metric Board Education representative and others concerned about the slow progress of metrics in the U.S. for the purpose of establishing program regulations that will increase the effectiveness of the implementation of the Metric Education Act of 1978.

Carried

Resolution J. NCTM encourage Congress to provide special financial support to school districts for unique educational needs of limited English proficiency (LEP) students (e.g., Southeast Asian students, Spanish-speaking students.) **Note:** Orlando schools have 62 languages to address, and Los Angeles has identified the need to teach in 84 languages.

Defeated - (49-50) identified as a local problem in some larger U. S. centres

Resolution K. NCTM develop guidelines and recommendations regarding vertical acceleration in mathematics instruction at Grades K through 12.

Carried

Resolution L. In place of an increase in dues, the NCTM look into the possibility of restructuring priorities and cutting costs in other areas so as to maintain the current rate of membership.

Carried

Resolution M. The NCTM offer to the membership a dues structure which includes a category of

membership only, without journals or research bulletin, that is less expensive than the current dues structure.

Defeated - no journal, no membership

Resolution N. NCTM make available a list of more than one place to stay at all regional and annual meetings. College dormitory facilities and costs should be included if available.

Carried

Resolution O. The NCTM continue, and consider expanding, the concept of allowing other interest-group members to register for regional meetings at the same fee as NCTM members.

Carried

Resolution P. The NCTM include in its journal The Mathematics Teacher articles that specifically address the needs of the two-year college mathematics teachers.

Carried

Resolution Q. The Board of Directors of NCTM be urged to rescind the action requiring speakers to pay registeration fees at the NCTM Annual Meeting.

Carried.

At 16:00 the regional caucus meetings were resumed to conclude discussion on the day's proceedings and for announcements of coming events during the conference.

BCAMT announced a wine-and-cheese gathering for delegates from Canada, to bring together, in a relaxed setting, the presidents and editors of Canadian mathematics publications to discuss the establishment of an all-Canadian mathematics education journal.

The gathering was held on Friday, April 24 from 17:00 to 19:00.

Thank you. I encourage you to support future delegates to the NCTM annual meetings as you have supported me in the past.

Reprinted from *Vector* Vol. 23#1, Fall 1981

CALENDAR OF EVENTS

April 14-17, 1982 **National Council of Teachers of Mathematics (NCTM) 60th Annual Meeting** at the Sheraton Centre in Toronto. Information: Local Arrangement Chairman, Dr. Dave Alexander, Faculty of Education-Mathematics; University of Toronto, 371 Bloor St. West, Toronto M5S 2R7. Tel. (416) 978-4068.

April 17, 1982 **Ontario Association for Mathematics Education** (OAME) will hold its Annual Meeting at 12:45 p.m. on Saturday, April 17 at the Sheraton Centre in Toronto. (For exact room location, inquire at the hotel desk.) The program will include (i) A review of the OAME financial status. (ii) Official appointment of 1982-83 officers; (iii) Declaration of 1982-83 chapters; (iv) Constitutional Changes. For further information contact George Knill, OAME President, 247 Bright St., Sarnia, Ontario N7T 4E9, or M. MacGregor at the same address.

April 18, 1982 **Research Council for Diagnostic Prescriptive Mathematics (RCDPM) Meeting:** State University of New York at Buffalo. The Program Committee solicits papers that address all aspects of mathematics instruction for exceptional learners, diagnostic assessment procedures and remedial instruction activities. Proposals of 1 to 2 page length are to include statement of purpose, educational significance, description of subjects research methodology, results and conclusions -should be sent by December 1981 to: RCDPM Program Committee, c/o L. G. Callahan, 505 Baldy Hall, State University of New York, at Buffalo, Amherst, New York, 14260.

May 12, 1982 **QSLMA Meeting:** Gananoque 4:00 - 9:00. Information: Peter Battey-Pratt (613) 386-3054 or David Handley (613) 962-9581.

May 31 - June 1, 1982. **Canadian Mathematics Society Education Days,** Carleton University, Ottawa. Secondary School Teachers are invited to join the Canadian Mathematical Society for its regular session on education. The programme on Monday will include a report on the Science and Education Study by Dr. Graham Orpwood of the Science Council on Tuesday morning, we will look at the Second International Mathematics Survey in British Columbia and Ontario. And there will be more. For details and a copy of the complete programme, write E. J. Barbeau, Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 1A1. (416) 978-8601.

May 30 - June 1, 1982 **Canadian Mathematical Society Annual Summer Meeting;** Carleton University, Ottawa. For information: Dr. D. L. Nel, Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario.

June 3 - 6, 1982 **Canadian Mathematics Education Study Group - Groupe d'etude can. en didactique des mathematiques;** Queen's University, Kingston, Ontario.

June 28 - 30, 1982 **National Education Computing Conference;** Kansas City, Missouri.

August 8 - 12, 1982 **Twentieth Annual AMTNYS Summer Workshop;** Potsdam, New York. Information: Dr. James M. Parks, Director, 1982 AMTNYS Summer Workshop, Department of Mathematics, State University, College of Arts and Sciences, Potsdam, N. Y. 13676.

MISCELLANEA

ERIBC Grant Money Available

Funds from the ERIBC grants program are available to all people in education interested in conducting research. Obtain discretionary grant and major grant guidelines and application forms by writing the Educational Research Institute of B. C. at Suite 400, 515 West 10th Avenue, Vancouver, B. C. V5Z 4A8, phone 873-3801.

For the Learning of Mathematics - An International Journal of Mathematics Education. An international Journal of Education has been recently started by the FLM Publishing Association under the auspices of The Canadian Mathematics Education Study Group. This publication aims to promote inquiry and research at all levels of mathematics and mathematics education. For further information: Dr. David Wheeler, Editor, Department of Mathematics, Concordia University, 7141 Sherbrooke St. West, Montreal, Quebec. H4B 1R6.

Letter to the Editor

Dear Professor Heinig:

It seems that gremlins have struck the print shop which turned out the December 1981 issue of the Gazette. Page 4 of my article should have lines through columns 1, 5, 6 and row 3. The second sentence after table 4 should read, "The effect of these two actions is to subtract our smallest uncovered entry from all uncovered entries and to add our smallest uncovered entry to every entry where lines intersect." Printing errors such as "whould" instead of "should", "strike" instead of "stroke" are less important.

Ritual incantations during the printing process by bearded men in white robes and rubber boots may keep the gremlins away in the future, but I wouldn't count on it.

Sincerely,

L. Keith Roy
Department of Mathematical Sciences
Lakehead University

MEMBERSHIP RENEWAL

As you renew old friendships and meet new colleagues at the NCTM and OAME annual meetings in April at the Sheraton Centre in Toronto, we hope that you will again rejoin the OAME. Look for the convenient area where you can renew your membership in this important and vital professional organization.

If you are unable to attend the Annual Meeting, we would appreciate if you would forward your renewal fee prior to May 31 to save the Association the cost of sending a reminder. Just fill in the information below, enclose cheque or money order for $15.00, payable to OAME and mail to OAME, 247 Bright Street, Sarnia, Ontario. N7T 4E9.

Name _____

Same address as before _____or new address

_____

Chapter Affiliation (if applicable) _____

This year for reasons of economy, your receipt, certificate and your seal will be mailed with your copy of the September issue of the **Gazette.** Until then, we trust your cancelled cheque will serve as receipt that your renewal has been received.

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
ASSOCIATION ONTARIENNE POUR L'ENSEIGE DES MATHEMATIQUES

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Dr. ☐ Miss ☐
Mr. ☐ Mrs. ☐

_____

Address _____

_____ Postal Code _____

Educational Affiliation _____

Chapter Affiliation (if known) _____

Please detach and mail, with a cheque or money order, to: OAME, 247 Bright Street, Sarnia, Ontario N7T 4E9.

NATIONAL COUNCIL OF
Teachers of Mathematics
REGIONAL SERVICES COMMITTEE

60TH ANNUAL MEETING - TORONTO, APRIL 14-17, 1982

DELEGATE ASSEMBLY

The Delegate Assembly will take place on Wednesday, 1982 04 16 at the Sheraton Centre. The schedule is as follows

08:30 - 10:30	Canadian Caucus	Dominion Room (Sheraton)

Note: Be sure to invite the alternate representing your affiliate group to attend both the Caucus and Delegate Assembly.

10:30 - 14:30	33rd Delegate Assembly	Dominion Room
12:00 - 13:30	Lunch Break	

There is no Delegate luncheon planned.

14:30 - 15:30	Canadian Caucus	Dominion Room

THURSDAY, 1982 04 17

8:30 - 10:30	Annual Business Meeting open forum and Presidential address	Canadian Room Royal York

Please send the name of your delegate and alternate to me. (The original should have gone to Reston)

SESSIONS ARRANGED BY NCTM COMMITTEES

See P. 10 and P. 11 of Program Booklet