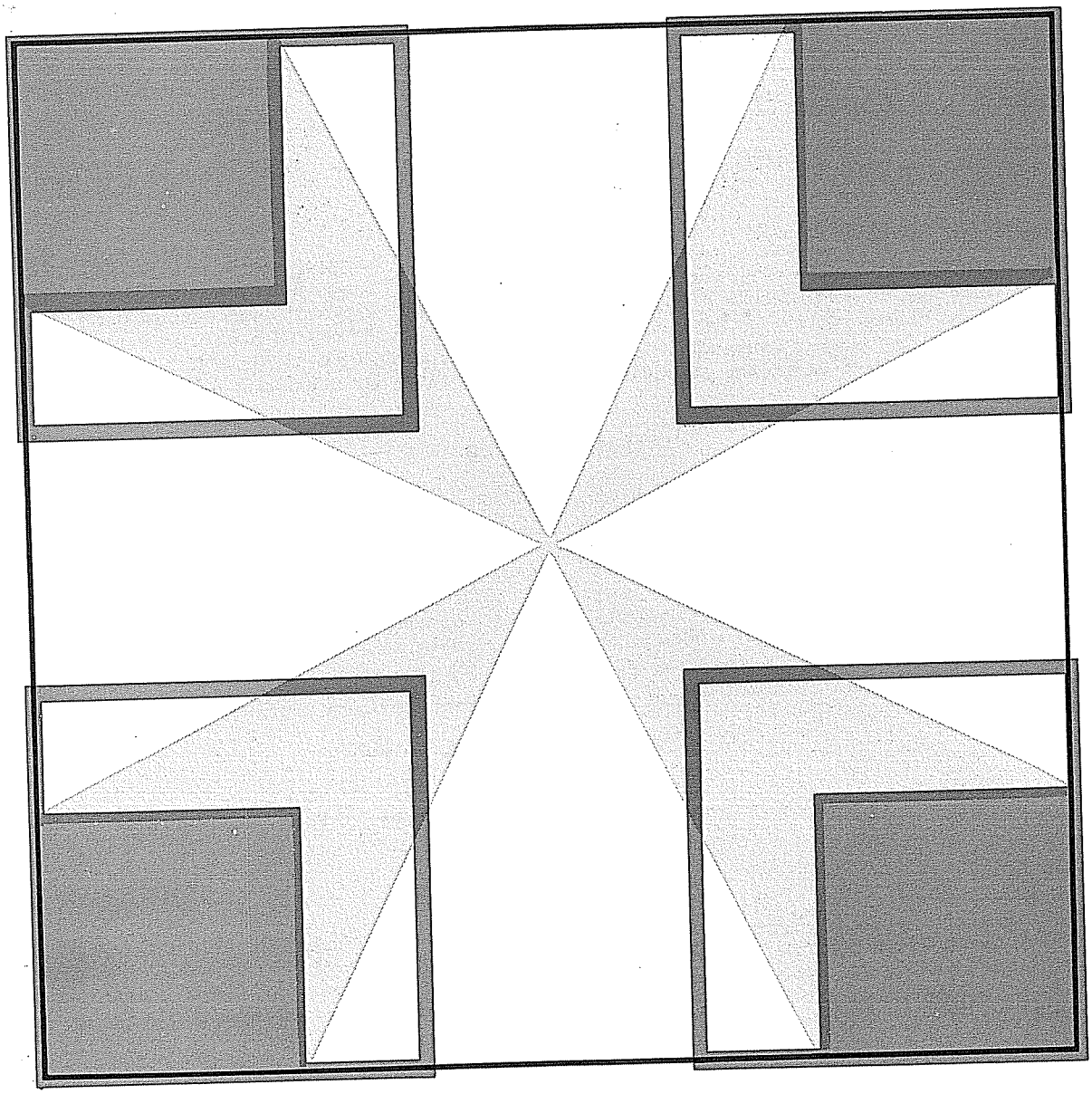
ontario mathematics gazette

VOLUME 29 NUMBER 1 SEPTEMBER 1990



THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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DEADLINES for submission of articles or information to be included in O.M.G. are as follow:
October 15 for the December issue
January 15 for the April issue
June 15 for the September issue

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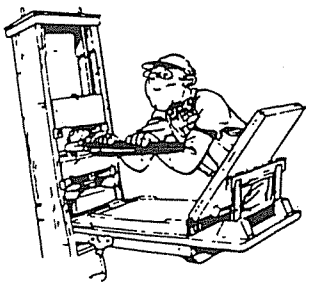
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Anna & Colin Garnham are co-editors of the Abacus for 1990/91.
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by
Barry Onslow

It is with a great deal of sadness that we announce, on page 4 of this issue, the death of Morley MacGregor. Morley, together with his wife Mona, was the driving force and inspiration behind a great many of our association's activities and innovations. He was admired by those who worked with him and will be missed by all.

Although the articles in this month's issue of the *Gazette* are directed towards the secondary teacher, the *Abacus* provides many useful and practical ideas for the elementary teacher. We need articles suitable for the mathematics' teacher in the elementary school and urge our contributing writers to consider the whole field. Many of you have exciting ideas and interesting viewpoints to share with your colleagues and we solicit articles from all of our readers. Don't hide your light under a bushel, put it in print.

Eric Wood's viewpoint expresses his concern with regard to the Radwanski report. We encourage readers to react to our "Viewpoint" column by writing a letter to the editor; we don't mind if you agree or disagree, just write! We are also interested in receiving manuscripts which present opinions on mathematics education issues.

Gord Doctorow and Corinne Fraser describe an alternative approach to teaching mathematics and offer many innovative suggestions as to how teachers might structure and assess their courses.

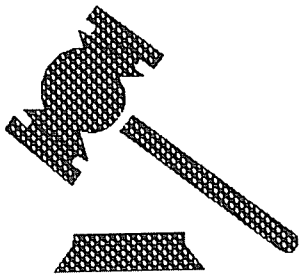
Tim O'Neil and Dipendra Bhattacharya whet the appetite for proof by taking another look at Fermat's Last Theorem, and Edward Cohen examines two practical applications from the "queen of mathematics" - number theory.

Ron Reesor presents an example of the use of manipulatives at the secondary level to provide students with hands-on experience and "visual understanding" while exploring a real-life mathematical problem. L. Keith Roy's article continues in the same vein with an examination of the business world and suggests possibilities for linking the real world to our OAC finite mathematics course. Ron Lancaster provides food for thought with three more creative problems, the third of which, like Reesor's article, links mathematical problem solving with the world on which we live.

Also featured in this month's issue is the revised constitution of Ontario Association for Mathematics Education/Association Ontarienne pour l'Enseignment des Mathematiques. The executive urges members to read the constitution.

Last, but certainly not least, we wish to express our appreciation to Bob Smith who retired as editor of the Gazette in May, 1990. Bob was a colleague at the Faculty of Education at Western, and on the editorial board of the Gazette. We are pleased that he continues to keep in touch and maintains his involvement with mathematics education. At the regional conference in Hamilton, Bob received the prestigious and well deserved honour of life membership in OAME to acknowledge the contributions he has made to mathematics education throughout his career. Our most sincere best wishes go out to Bob and Dorothy during their retirement years.

PRESIDENT'S MESSAGE



As a non mathematician with a deep interest in mathematics, I am most honoured to serve as your president for the next year. Over the past 15 years I have spent many pleasant times working with and getting to know the fine people who make up O.A.M.E. In those years you have been a positive influence and help to me and my sincere hope is to repay some of these kind acts.

Some interesting changes are being tried this year. My first objective is to get all council members more actively involved in the work or our organization. I want to hear their ideas and yours as well. In this latter light, please contact me or any other council members with your input. WE WILL LISTEN.

With this aim toward increased involvement, we have started this year's activities by having a think tank session in June. The basic idea was to involve chairs of standing committees as well as the executive in planning toward the next year. I thank all who attended for making this a worthwhile endeavour and would like to highlight several of the decisions and directions reached.

As noted in the December 1989 *Gazette* the Ministry of Education is looking at "Restructuring the Educational System". With this in mind we have restructured our school division committees to match the Ministry thinking. Thus, our committees in these areas are now Foundation and Formative Years combined, Transition Years and Specialization Years. All of these groups will be working on preparing our position in relation to the Ministry directions. This is one of the key areas in which we are looking for your input.

At the annual meeting in Hamilton, O.A.M.E. voted to support the Ontario Standards Document in the process of being developed by the Ontario Mathematics Coordinators Association. At this time the document has been reviewed by a number of our members and, after revisions, will be approved in readiness for printing in early October and distribution before the end of 1990. This document will provide both O.M.C.A. and O.A.M.E. with a significant tool for advancing the cause of Mathematics in Ontario with both the Ministry and boards of education at all levels.

The year 1991 will be celebrated as the 100th anniversary of an official mathematics organization in the province of Ontario. To commemorate this occasion, O.A.M.E. has planned to have a special issue of *Gazette* in September. This expanded issue will be used to relate the history of our organization for these 100 years. Jack Lesage and Joan Routledge are in charge of this project and have contacted many people for help. Anyone who has anything of interest to contribute to this history can do so by passing on their material to any council member. We will see that it gets to Jack or Joan.

As we move into the 90s and also into our second century, all of council as your representatives are looking forward to making O.A.M.E. a more positive force in the education of Ontario's young people. To accomplish this aim, we hope to involve as many members as we can in carrying out this task. My sincere wish is that each of you take the time to contact any member of council to offer your suggestions to reach our goal.

Thanks,

Mickey Sandblom
President

OBITUARY

Morley MacGregor, 1918-1990

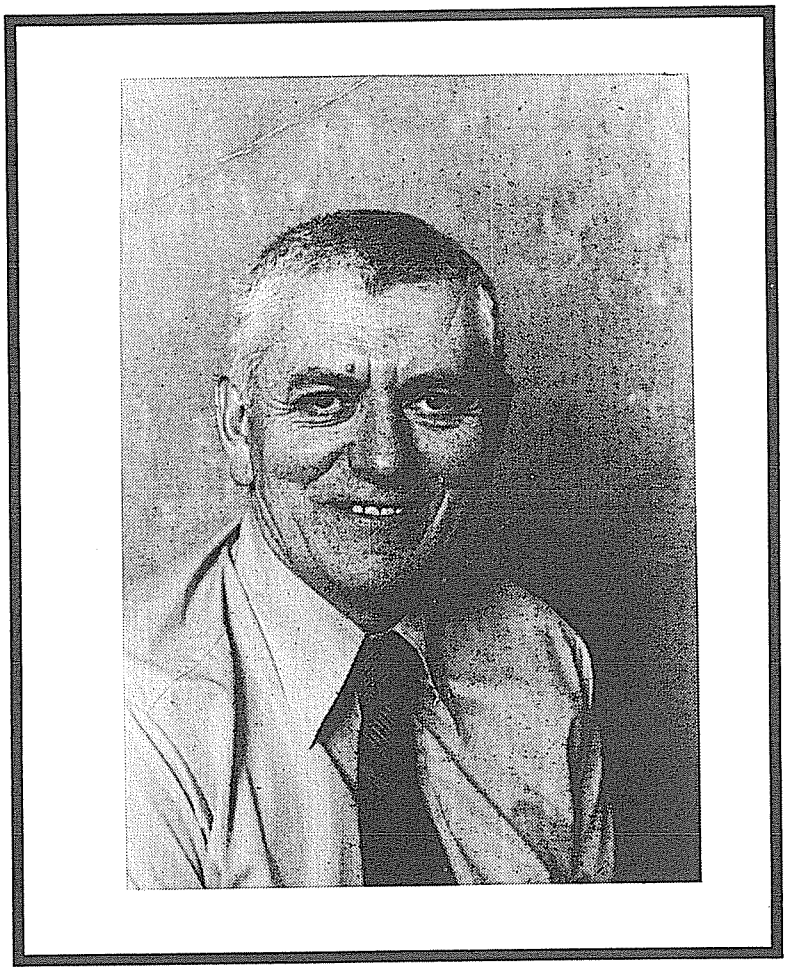
Morley MacGregor died on June 26, 1990, following a heart attack. With Morley's passing OAME/AOEM has lost one of its most distinguished leaders and strongest advocates.

Born in 1918, in Brook township, Ontario, Morley went on to begin his teaching career in that rural area in 1938. He later moved to Sarnia where, in 1950 he became the first principal of Bluewater School, a model for integrating the education system of native Indians into a local Board of Education. He is remembered with much fondness and respect by his former students, parents and colleagues in Lambton County.

Morley did not, however, limit his interests to education. He was also very active politically. He was president of the Sarnia Liberal Association for 25 years and campaign manager for successful members of both the provincial and federal parliaments. From 1973 to 1989 he served the province as a Commissioner of the Blue Water Bridge Authority in Sarnia.

It was within the realm of the OAME though, that we came to know Morley best. In 1964 Morley was elected to the Council of the Ontario Association of Teachers of Mathematics. Soon after his election to the executive in 1968, he became Secretary-Treasurer, a position he held jointly with his wife Mona until 1989. Morley will be remembered fondly by the many colleagues with whom he has worked during his years of service to mathematics education. From the founding of OAME/AOEM on May 13, 1973, Morley became the continuing presence behind the presidents of our organization. His sincerity, frankness and honesty were appreciated and admired. His advice and encouragement supported us during times when it was needed, and he never tired in his efforts to promote and sustain the aims of the association. Those who are fortunate enough to have known Morley are the richer for it. They understand that he leaves a challenge for all of us to become involved in our organization so that it may continue to grow and prosper.

Morley was also a strong and devoted family man. His memory lives on in the hearts of his wife, Mona; their children, Milton, Douglas, Alexander, Eugene, Norma, Larry and Marjorie; and nineteen grandchildren. We extend to his family our deepest sympathy. Morley was a man who was well loved. He will be missed.

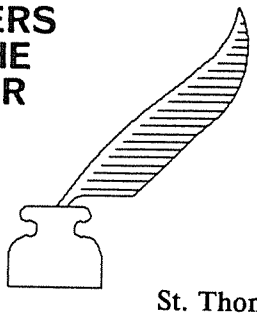


In Memory of Morley, the MORLEY MACGREGOR fund is to be established. OAME/AOEM members and friends are invited to send contributions c/o

D. Attridge Secretary/Treasurer OAME.
1 Southdale Drive,
Markham, Ontario
L3P 1J6

The type and form of this fund will be determined at the OAME/AOEM council meeting in September 1990.

LETTERS TO THE EDITOR



Tues. Feb.27/90
St. Thomas Aquinas S.S.
Brampton, Ontario
L6S 3J7

Dear Editor:

I am writing to clarify an error made in the December '89 issue of the Gazette. In the Problems column regarding Friday the 13th, it says that there could only be 0, 1, or 2 Friday the 13ths in one year. If you count the number of Friday the 13ths in the enclosed Perpetual Calendar (#H) in which January 1st is a Monday and it is a leap year, you'll notice that January, April and July each have a Friday 13th. Other than this, this was a great article.

Faye Forster.

H

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Larchwood Public School
Dowling, Ontario

In memoriam

My first recollection of O.A.M.E. is arriving as a stranger at a Leadership Conference at the Ascot Inn. From that moment on I have always felt at home and amongst friends because of the warm greetings we received from Mona and Morley.

Throughout the years between, Morley has been like a father as well as a friend and associate. He guided me as I worked with the association and finally even convinced me that I should endeavour to become your president. Now he has left our midst. This fine man who advised and pushed us, will be missed. But just as Morley was still handing out advice and suggestions as little as a month before his passing, so his memory will continue to guide us in the future. Morley MacGregor will live forever among us.

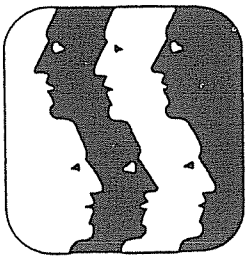
Mickey Sandblom

Ministry of Education
North Bay

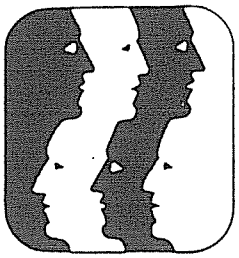
A lot of the success we have had with O.A.M.E. has been due to the informed counsel and warm fellowship which Morley MacGregor brought to the organization. His untiring effort and willingness to help others are qualities which I have always admired. It is impossible to express in a few words the respect of one's colleagues which Morley has enjoyed. I respect morley as a great educator and an even greater Canadian - we will miss a real friend.

Dino Dottori

The Jack McCartney scholarship, which has previously been donated by Science Research Associates (Canada) Ltd., will no longer be offered for competition.



Viewpoint



RADWANSKI AND STREAMING: WHERE IS THE EVIDENCE?

by
Eric Wood
Faculty of Education
University of Western Ontario

The MOE's intention to destream grade 9 is a clear result of the Radwanski Report's (1988) recommendation that the entire secondary school program be destreamed. It is unfortunate, in my view, that a decision such as this one should be made without compelling evidence that is appropriate to the Ontario and the Canadian experience. In the pages that follow I will examine the evidence that Radwanski used to buttress his argument that destreaming is both desirable and possible. I will conclude that his argument falls far short of the kind of reasoned, logical pattern that he claims would be the norm for students educated in his new (read old) school model.

Radwanski uses a number of arguments to support his contention that "the current policy of streaming high school students into academic, general and basic courses of study be abolished, and replaced by provision of a single and undifferentiated high-quality educational stream for all students" (Radwanski, 1987, p.163). The first reason that is used to justify his position is that streaming perpetuates social injustice.

I do not deny the fact that children from lower socioeconomic homes are more likely to be found in the general and basic level programs. Unfortunately, he seems to have made the classic error of confusing causation and correlation because he then jumps to the conclusion that this streaming process somehow causes the perpetuation of the underpriveledged position of these children in society.

The argument goes that because of the trivial nature of the school programs in which the general level students are engaged they are not given access to what could be called high status knowledge. This lack of knowledge then further limits the potential for higher levels of education and as we all know educational attainment is a significant indicator of economic success in future life.

I wonder how this argument explains the recent news story about a metal fabricating plant owner who was lamenting the fact that he had been unable, after much searching, to find two skilled workers for his plant. From what I could see in the television coverage the plant was bright and clean and the work appeared to be varied and not of the back breaking variety. The owner waved his hand towards other workers and claimed that they would earn $50,000 that year. In addition, I cannot remember the number of times I have listened to various government and labour officials expressing their concern over the small number of skilled tradespeople in Canada, in spite of the fact that such technically trained people can certainly earn a good living in this country. These kinds of news items make it hard to argue that lack of access to Euclidean geometry leads to a life of poverty and despair!

When I first read this report I did not dispute many of Radwanski's factual claims; however, his interpretation of the meaning of these facts is disturbing to say the least. Furthermore, his prescriptions for solving the problems to which

he alludes do not seem to show any understanding of the realities of children's learning, school culture and structure, or the nature of teaching. Let me explain why I do not agree with his argument on this point and then illustrate how his suggestions about curriculum are significantly off the mark as far as mathematics is concerned.

In the first place I would dispute his using the research conducted by Jeanne Oakes (1985) in the United States. It is true that Oakes found a big difference in the kind of teaching and learning and the demands that were placed on students in different schools and tracks. However, it is also true that the educational system in the United States bears very little resemblance to our own. For one thing the funding of education by voting on the milage rate each year means that some schools situated in reasonably poor areas have little financial backing. The teachers are poorly paid and consequently probably not of the best quality and the physical resources available to the school are correspondingly poor. The converse is true in rich areas. In fact, the research that Radwanski cites did differentiate between schools in exactly this way, with categories such as "working class school" or "professional elite school." The actual physical descriptions of these different schools made it clear that they were differentially funded and that the differences were exactly what could be expected from such a wide disparity in economic resources. Our funding formulas for education do not allow for such huge disparities. A child from a poor family who attends a large urban composite school does not have any restriction placed on him or her by some overall poverty level of the school -- he or she can elect to take any course at any level -- a clear difference when compared with the U.S. situation.

Even in those U.S. schools that have a wide variety of students and sufficient funds to offer a varied set of programs, it is important to understand how graduation requirements and the push for equity have influenced the U.S. system. Again I do not dispute Radwanski's claim that in non-college bound tracks there is often very little going on (see Sedlak, Wheeler, Pullin and Cusick, 1985 for examples) but this in part is due to the fact that teachers are poorly prepared, poorly trained and poorly paid. Sizer's book "Horace's Compromise" (1984) paints a picture of a veteran teacher who has to work part-time in a liquor store after school in order to make sufficient money to live. This situation is not an atypical one in the U.S. but is very unusual in Ontario. There is little impetus for teachers to demand anything from their students because graduation is based on credit accumulation and this accumulation is based on seat time, not learning. Furthermore, time and time again research has shown that the primary concern of many administrators is for order, not student engagement in academic tasks. The result is a bargain between teacher and students -- the students behave as long as they are not pushed too hard to learn. I do not believe that this has to be the case; however, a significant body of literature suggests that it is closer to the rule than the exception in the United States.

Once again the comparison of the Ontario system of education with the one in the United States on which this research is based is troublesome. For one thing, the credit system in use in Ontario does require certain things from students such as minimum numbers of credits and required credits in certain subject areas. Furthermore, as long as teachers take their professional responsibility to teach seriously, I am convinced (after 15 years of teaching general level classes), that these students can and do learn much content that could only be considered to be high status knowledge.

Even if the description of teachers and teaching that is found in the critical literature from the United States was common in Ontario[1], it is unfair to blame streaming for the fact that some teachers, when faced with difficult students who are reluctant learners, take the easy way out and opt to get along with the students rather than try to engage them in academic tasks. Why would such teachers behave any differently if those same reluctant learners faced them in heterogeneous classes

where the content was even more unpalatable to these students? A careful look at much of the body of critical literature on just this topic does contain some descriptions of teachers who heroically cajole, badger, threaten, and persuade their charges to work at academic tasks -- even in the lower tracks. I would argue that it is not the *content* of these streamed courses that makes the learning experiences that students engage in less than worthwhile, but what teachers in these classes *do* when they are behind closed doors. I would also assert that these same teachers would act no differently, regardless of what system was instituted.

A sharp critic would say that this argument does not support my contention that streaming should be continued because teachers such as the ones that I have described wouldn't be any worse in a destreamed system and, such a system might have positive benefits. It is at this point that I would disagree, because although it is probably true that poor teachers will teach equally badly in a wide variety of contexts, I think that an educational system ought to be organized around what is possible for the majority of dedicated, effective teachers. I would claim that the good teacher who is demanding of all students regardless of their level would be ill served by a destreamed curriculum, particularly in the area of mathematics. Furthermore, the nature of the mathematics curricula is such that attempting to teach the material that is presently in the advanced level programs to all students would result in many students having access to no mathematical knowledge at all.

For one thing, mathematical content has been consistently moved down from higher levels so that some of the material that I studied in grade 12 algebra class is now part of the grade 10 general mathematics curriculum! The advanced level programs contain material that used to be studied in university and so it goes. Unofficially, ministry personnel will admit that the advanced level mathematics curriculum now in place was designed to be suitable for the top 30% of all mathematics students. As it is, 60% of the students are in the advanced level programs and so the teacher is already in a difficult position. It appears that the curriculum designers are attempting to force the demographics of the educational system to be what they think they should be by specifying content that is unsuitable for the majority. How can teachers be expected to teach this content to all, and students expected to learn it, when it was not designed for the average (never mind below average) student?

Trying to teach this kind of material to all students would be impossible. The suggestion made by Radwanski that with the appropriate teaching technology all students can learn a basic body of material may be true for low level numeracy skills. However, there is no evidence whatsoever that this is true for the more advanced mathematical content found at the secondary level. Perhaps we could all learn to play simple tunes on the piano but this doesn't mean that we could all become proficient at a high level of performance just by having a varied program of lessons available.

One of the differences between mathematics and some other subjects has to do with the level of abstraction required. Typically topics from the general level programs are taught from a concrete, applied viewpoint -- a perfectly defensible position from an academic point of view. Topics in the advanced level mathematics programs are more likely to be taught from a more theoretical perspective and by a consideration of the general rather than the specific case. This generality requires a considerable ability on the part of the student to think in the abstract -- a capability that many students at the high school level, now and in the past, simply do not have. The upshot of all this is that mathematics typically becomes an all or none proposition. If a student is unable to master the early stages of a development they are incapable of doing *anything* with the rest.

It is not a case, as Radwanski seems to suggest, of different people getting different things from the same content. I have taught many students who were unsuccessful in the advanced level programs who were excellent

students in the general level programs -- programs that I also taught. I am talking about marks that went from 25% to 90%. No matter what is said about general level programs being easier, they are just not *that* much simpler than the corresponding advanced level program. Furthermore, some of these students decided to return to the advanced level program after such a successful experience in the general program. In almost all cases their performance slipped back to what it had been previously. It cannot be claimed, therefore, that the improved performance was due to maturation or better work habits and so on. Rather, the improvement was due to the fact that the *kind* of mathematics that was being taught was in tune with the way in which these students were able to think mathematically.

Given the choice of having students get little or nothing out of a course that is simply too abstract for them or learning a large body of academically defensible mathematics from a more practical perspective, I find it incomprehensible that Radwanski would choose the former. It is clear to me that he has no idea whatsoever about the content of the various programs or the realities of teaching mathematics at the secondary level. I cannot deny that the general level mathematics programs do contain topics such as income tax, mortgages, municipal taxation, personal finance, vehicle costs and so on. Perhaps we could classify these as low status knowledge; however, if one of the major purposes of schooling is to prepare students to function in society, perhaps these topics ought to be included in the advanced level courses as well. In my view such topics are academically defensible applications of mathematics and worthy of study in their own right. In any event such topics do not comprise even a simple majority of the content of any general level course in secondary school.

It is of more than a little concern that someone charged with reporting to the government can only justify his claim that all students are capable of learning advanced level mathematics by quoting Bernard Shapiro who claims that "... you can learn geometry just as easily with football as you can with building houses." This bizarre suggestion that theoretical geometry (an example of the kind of undifferentiated high quality knowledge advocated by Radwanski for all), which is designed to give students an idea about the nature of proof in mathematics, can be learned *easily* by *any* method whatsoever is only made more ridiculous by the suggestion that considering football plays or planning out houses will do it. Needless to say this comment did little to change my opinion about the seriousness with which we should consider this report.

It is worth noting once again that in the U.S., graduation requirements typically demand only one mathematics course at the high school level (that is, grade 8 and up) and this could in fact be consumer mathematics or a remedial arithmetic course. In Ontario, students must have at least 2 mathematics courses and most take 4 years of mathematics because of apprenticeship requirements or requirements prescribed by other institutions or career paths. The overall result is that students in properly taught general level mathematics classes can both feel successful about their ability to function mathematically and at the same time learn a large body of mathematical knowledge. A body of knowledge which is more than enough for any but the most arcane or technical career, and, most certainly enough to be able to function in even our technological society.[2]

In summary I find the contention that all students can and should learn the same mathematics regardless of their career choice or interests to be acceptable only if the mathematics that they are learning is appropriate to the daily tasks of life that are common to us all. If the curriculum that is presented to the teachers is specifically designed with a strongly academic focus then it is unreasonable to try to teach it to all children regardless of ability or aptitude. It is simply not true that a student intending to pursue a career in business needs to know the same mathematics as one who wants to be an electrical engineer. Furthermore, I deny the

claim that there is no worthwhile mathematical content in the present general level courses -- only someone who has not looked at them could come to this conclusion. It is time that the mathematics education community was more fully involved in these kinds of decisions in a proactive rather than a reactive way. Throughout this province dedicated teachers work hard to ensure that students of all ability levels develop their mathematical understandings as fully as possible. Unfortunately, ill-conceived plans such as the one that the Ministry is now proposing are unlikely to help those students develop to their fullest mathematical potential.

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1. For a variety of reasons, including higher qualifications and better pay, teachers as a group are very different in the two countries.

2. I recently made a presentation to a group of university professors about destreaming and mathematics. To prove my point about general level courses having defensible mathematical content I gave this group an actual test question that I had used on a unit test on ratio and proportion in my grade 10 general class. Only one person out of a dozen or so (many with doctoral degrees) could do the question. If content that cannot be routinely done by people with these educational credentials is not high status knowledge, I don't know what is!

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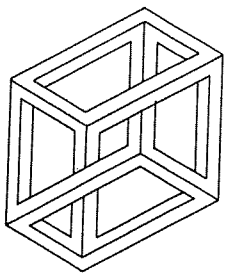
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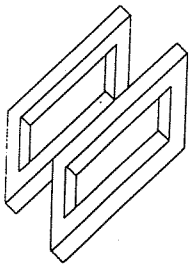
The Ontario Mathematics Gazette is published three times annually (September, December and April). *The Abacus* is a sixteen page insert of hands-on activities, usually with an underlying theme, and distributed as an insert of the *Gazette*.

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TEACHING MATHEMATICS IN AN ALTERNATIVE SCHOOL: ASE1



by

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and
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INTRODUCTION

What is it like to teach mathematics in an independent study environment? Is it possible for students to learn the material on their own? How much teaching do they really need? Is it cheating for them to work with other students? What forms of evaluation should/could be used? Should written tests be timed? Should a rigorous discipline like mathematics be under taken in a flexible, relaxed environment?

ASE1 (Alternative Scarborough Education) is a public Scarborough Board high school spanning grades 11 to OAC (advanced and general levels). As an alternative school, its focus is on flexibility and a large measure of independent study, combined with individual student-teacher consultations. The small size of the school (enrolment is typically between 120 and 130) creates a friendly and informal atmosphere. Students build adult relationships with their teachers based on responsibility and respect.

STRUCTURE

The ASE1 teaching format is based on one workshop a week for each course. The time block varies from one hour to one and a half hours. The rest of a teacher's day is spent consulting with students by appointment. In math, student consultations typically last for one or two 15-minute time blocks. They are usually utilized to help students individually with course difficulties, to help students plan how to manage their course work loads, or to collaboratively mark tests/assignments.

At the first workshop of each course, students receive a detailed course outline indicating all the assignments for which they are responsible. Students then work at their own pace and complete as much of the assignments as they feel necessary before writing a unit test. They write these tests when they are ready, and are given as much time as they require. Tests are not marked until the student books a consultation time to do so.

EVALUATION

Tests are marked with the student present in order to improve the student's level of understanding and to promote effective test writing skills. If the student wishes to improve his/her mark, then the evaluation becomes formative rather than summative and the student will write an equivalent test for his/her summative mark. This procedure, along with the absence of a test-writing time limit, allows students to overcome test anxiety.

Since students are not under a time constraint, they are able to attempt every question on a test. Therefore, evaluation is based on a student's actual performance in relation to the stated objectives of the course. By contrast, the purpose of speeded tests is to rank students against other students in their test-writing pace.

Each course has a final examination, which satisfies the minimum weight stipulated in the OSIS/OAC Ministry requirements. Again, the exams are written when the student is ready, within the constraints of a semester or school year.

Another evaluation instrument is the math journal-log. Each student is required to record attendance at all workshops and consultations along with journal entries. The journal entries should reflect the students' feelings about the sessions and contain a self-analysis of their progress. They should also reflect the student's assessment of the course. Evaluation is in the form of commentary and no mark is attached to this work.

> Students build adult relationships with their teachers based on responsibility and respect.

Alternative schools emphasize educational involvement in the community. In the math program at ASE1, community contact is mandatory and constitutes 10% of the total evaluation. Students are given a set of objectives to attain and assigned the task of finding a project that involves the use of mathematics for a period of 10 hours. They are encouraged to tie it into an area of their personal interest, e.g. art, sports, music.

Among other things, ASE1 students have found themselves assisting in elementary mathematics classrooms, tutoring immigrant students, and investigating the mathematics behind the works of artist M.C. Escher. One student, who is interested in bicycling, developed a formula for a bike shop to calculate spoke lengths for bicycle wheels according to certain performance criteria. Yet another student, taking computer science, used a sound-wave generating program to help a hearing-impaired student to enunciate words correctly.

When students complete their projects they must submit a written summary of their work along with a self-evaluation based on the stated objectives (they may add their own objectives if they wish). The final mark is negotiated between student and teacher. More important than the mark, however, is engaging the student in a service role as a valuable contributor of mathematics. This enhances self-esteem. Furthermore, the student gains insight into the application and need of mathematics in society. If the students' interests are piqued, their curiosity stimulated, and their efforts recognized, then we have achieved some fundamental goals of all education.

ATMOSPHERE

We have concluded that far too much of math teaching is "algorithm crunching". We have observed that students learn algorithmic procedures very well on their own or in tutorial sessions. The most valuable teaching is on the level of problem-solving and analysis. Algorithms are best presented, in our opinion, as a necessary tool toward the solution of a particular problem - so that the algorithm has a context. We prefer to create a sense of discovery in doing mathematics, somewhat in the vein of professional mathematicians' discoveries of algorithms to solve their problems. It is in this context that mathematics flourishes and becomes interesting, fun, and useful.

> Alternative schools emphasize educational involvement in the community.

ASE1 math classes are structured as workshops. At the beginning of the workshops, pairs of students are engaged in trying out new mathematical ideas by working on prepared exercises--both theorems and algorithms. In the meantime, the teacher observes and gives feedback. Formal instruction follows with the aim of consolidating and clarifying what the students have learned. Experience shows that students appear to appreciate being able to try the problems out before having them formally taught.

When workshops are not in progress, the ASE1 math classroom is an open, friendly drop-in environment. Some students choose to work together, others choose to work individually, while still others are engaged in individual consultations with the teacher. They could be working at the large table, or at study carrels. The chief benefit of students working in the math room is that they can ask questions of the teacher when they reach a stumbling block, and therefore maintain their momentum rather than experience the frustration of staying in a rut.

> ...the student gains insight into the application and need of mathematics in society.

People are not the students' only resource, however; math software, videos and additional textbooks are at their disposal. The math room is equipped with a lab of Macintosh and Apple computers that are used to run Calculus and Geometry tutorial software as well as graphing packages. VCR's and monitor are available to run TVOntario segments on trigonometry and vector algebra. In addition, many pamphlets, tutorial books, and supplementary textbooks are available on a variety of math topics for students' use in the room or even for short-term loans.

FLEXIBILITY

ASE1 students have diverse and special needs. Some are self-supporting, some are school phobics, some are re-entering school after a lengthy period of absence, some are engaged in a career (e.g. musicians, actors, and athletes) while completing their schooling, some want to accelerate their course completion schedule, and some want a creative environment. Flexibility is, therefore, a guiding principle of ASE1. How does ASE1 do it?

Math courses may be taken on a semestered or full-year basis. Even when a student chooses initially to go semestered, the student has the option of extending the completion date to the second semester or even of completing the course before the end of the semester. The reason this can work is that students write unit tests when they feel ready, not at a designated time common to everyone. This is also true of the final exam. Even if the student starts the course in the second semester, and is unable to complete it before the end of the school year, the student may contract with the teacher to extend the course completion to the first semester of the next school year.

Occasionally, students find themselves unable to work independently. They may lose their initiative and sense of self-discipline. This is the time when negotiations take place, either on the initiative of the teacher or the student. Negotiations usually begin with a consultation where an informal plan of attack is worked out with unit completions and test dates set out. If a student fails to progress satisfactorily, then a formal contract is worked out, signed by both the teacher and student with appropriate targets and consequences. In the end, the student may drop the course or even go to a regular collegiate. This does not preclude, however, the possibility of the student re-entering at a future date. Indeed some of ASE1's success stories include students who have returned with a more serious attitude after an enforced absence from school.

Our efforts at ASE1 are continual and ongoing. We have no definitive success formula to satisfy all students' needs. Our aim is to continue to experiment and learn with the view of providing an adult, caring and flexible environment for students who are seeking something more individualized to meet their life's goals.

Note: Corinne Fraser, who was a student teacher at the time of the writing of this paper, has since gone on to be part of the faculty at an alternative school in Oakville - Self-reliant Learning Program, General Wolfe High School, 1055 McCraney St.E., Oakville, Ontario. L6H 1H9.

ASE1 MATH

COMMUNITY CONTACT ASSIGNMENT FOR MATH STUDENTS

This is where we develop a sense of involvement and individual responsibility. It is an opportunity for you to go into the community to participate and learn and give of yourself.

OBJECTIVES

What is expected of you? The following set of objectives defines the attitudinal, knowledge and thinking goals this assignment expects you to achieve. On completion of your community contact you should have:

- learned how to make a community contact;
- understood how to negotiate a community contact assignment with an outside institution or individual;
- appreciated the difficulties others have in learning mathematics;
- appreciated your own mathematical capabilities in communicating or learning about mathematical ideas outside ASE1;
- innovated some strategies or approaches in the application or communication of mathematical concepts;
- developed an introspective understanding of your own initiative and abilities in the process of organizing this assignment.

You will be expected to initiate a community contact for which you will contribute no less than 10 hours work. The range of assignments is quite varied. You will be expected to prepare a brief report or log of your activities plus a self-evaluation. *To guide your self-evaluation, explain to what degree you have attained the listed objectives. Feel free to create a set of your own personal objectives and to include a self-evaluation of these as well.* Where appropriate, an evaluation by your community contact will also be expected.

Some examples of activities that you may wish to pursue are:

1. Assisting mathematical teaching in an elementary classroom.
2. Tutoring immigrant students in math to help them catch up.
3. Tutoring disabled students in mathematics (e.g. hearing or visually impaired, wheel-chair users).
4. Investigating the application of mathematics in producing music or art through computers (e.g. Ontario College of Art, University of Toronto, York University).
5. The application of statistics in sports, medicine, social sciences, etc.
6. Participating in a scientific project which employs mathematics; e.g. gathering astronomical data or environmental research.
7. The application of mathematics in puzzles, "magic", games, etc.

Community contact assignments account for 10% of the overall mark. The assignment is to be chosen in consultation with your teacher. You may integrate your assignment with another subject which is appropriate. Don't leave it too late.

MATH JOURNAL-LOG

Math students will be required to maintain and hand in a journal-log as part of their course requirement. A math journal-log will consist of:

- a log of workshops and consultations attended (dates and durations) as well as tests and assignments completed;
- journal entries with your reflections on the course, difficulties you are having, or things that you like. You should be assessing your reactions and your own feelings about your progress.

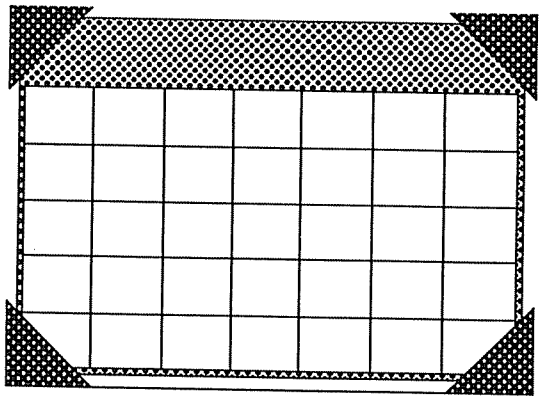
The purposes of this journal-log are:

- to have students develop self-awareness of their progress and performance;
- to make students self-critical in their approach to their work in mathematics with the goal of developing a confident and disciplined attitude;
- to assist the teacher in evaluating student progress and reflecting on the course with a view towards improvement.

There will be no formal assessment of these journals. No mark will be assigned for journal work. However, the journals will be used to provide teacher feedback. Journals will be treated with strict confidentiality. It is expected that they will serve as a vehicle for serious dialogue to benefit each student towards a realistic, confident and successful approach to mathematical studies.

Journal pages should be divided into two columns with log entries on the left and comments, reflections, ideas on the right. A sample page is attached.

Journals will be submitted monthly to your teacher for review.



Gord Doctorow Math Journal

Tues. Sept. 12

10:00 - 11:00 a.m. - Gr. 11 advanced math workshop

1:00 - 1:15 p.m. - consultation

Sept. 12 workshop was a review of polynomial stuff that I already knew. I didn't get a great deal out of it, although I did participate in the discussion. The afternoon consultation was to get me started in planning my test schedule and clarify my community contact choices. I'm still not sure what I want to do, but tutoring booster kids at Porter sounds interesting and so does working with elementary school kids at St. Andrews. I will set up appointments to meet teachers at both places.

Tues. Sept. 19

10:00 - 11:00 a.m. - Gr. 11 advanced math workshop

1:00 - 1:30 p.m. - consultation

1:45 - 2:45 p.m. - Unit 1 test

On Sept. 15, I met with Jane Hill at Porter about the booster program, but she suggested that I wait a few weeks before workings things out since the program has just got going. I think I prefer doing this type of tutoring.

Wed. Sept. 20

9:00 - 9:30 a.m. - consultation to mark Unit 1 test

I went through the review and sample test over the weekend. I feel ready to try out the Unit 1 test.

*

*

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My consultation after the second class plus the class itself straightened out my problem with polynomial division. So I wrote the test which I thought was pretty easy. I like the fact that I don't have to do every problem assigned before I write and that I can write when I want.

E

T

C

The post-test consultation was O.K. I didn't make too many errors. My mark was close to 90% and I learned a little bit about self-checking. So far, I like the course and I'm sticking to my schedule.

EXPLORING FERMAT'S LAST THEOREM

by

Tim O'Neil
Ohio State University
and
Dipendra Bhattacharya
Clarion University of Pennsylvania

The spring of 1988 saw renewed interest on the part of the media in Fermat's Last Theorem, as a Japanese mathematician came forward with his solution to the great problem. He was incorrect, but the event demonstrated that, even after 300 years, the Last Theorem could still draw interest.

Fermat's Last Theorem appears so trivial on the surface that the uninitiated may think that the problem holds no interest. Only after playing with it a while can one see the vast amount of math lurking just below the surface. For this reason, the Last Theorem has a beauty and an intrigue that can motivate a mathematician to new heights.

In this short note, we will examine the Last Theorem, paying particular attention to a couple of interesting patterns and results. It is hoped that these ideas will inspire you or your students to delve into the wonders and mysteries of mathematics with renewed fervour.

FERMAT AND HIS THEOREM

Pierre de Fermat was born in 1601, in the French town of Beaumont-de-Lomagne. He was born into a well-to-do family, and thus received a better education than many of his peers. He practised law in his hometown of Toulouse, and was widely known throughout Europe for the way he performed his duties as Councillor in the town parliament. Fermat was also a member of the nobility, and enjoyed the life of leisure that accompanied this status. He died at Castres in 1665.

Fermat was not a professional mathematician, but the work he did privately won the respect of a select few of his colleagues. However, because he did not publish in his lifetime, Fermat was unknown until after his death.

In 1670, Fermat's son Clement-Samuel published *Observations on Diophantus*, an edition of Diophantus' *Arithmetica* which included Fermat's notes on the discussed topics. In Book II of the *Arithmetica*, problem eight asks "given a number which is a square, write it as the sum of two other squares". In the margin next to this problem was the second of Fermat's 48 observations: "On the other hand, it is impossible for a cube to be written as a sum of two cubes or a fourth power to be written as a sum of two fourth-powers or, in general, for any number which is a power greater than the second to be written as a sum of two like powers. I have a truly marvellous demonstration of this proposition which this margin is too narrow to contain".

His demonstration was never found, and many mathematicians since then have searched in vain for one. For our purpose, Fermat's Last Theorem is stated as follows.

Conjecture: For any integer $n>2$, $x^n + y^n = z^n$ is impossible in positive whole numbers x, y and z.

THE EQUATION $x^n + y^n = z^n$, $n>2$

Let us now examine a few properties of this equation. If $x = y$ then $2x^n = z^n$, and thus $z = x(\sqrt[n]{2})$, which is not a whole number. Therefore it is reasonable to assume that

$x<y<z$. Dividing our original equation by z^n gives

$$(\frac{x}{z})^n + (\frac{y}{z})^n = 1 \qquad (1)$$

Let $a = (x/z)$ and $b = (y/z)$ and substitute these into (1). We now have $a^n + b^n = 1$, where a and b are positive real numbers less than 1.

Now, let us define the function f by $f(n) = a^n + b^n - 1$ where a and b are real numbers as defined above. At $n = 0$, $f(n) = 1$. The first derivative is

$$f'(n) = a^n \log_e a + b^n \log_e b$$

Since $0<a<1$ and $0<b<1$, $f'(n)$ will always be less than zero. Therefore, f is a decreasing function.
In fact, one can show that $f(n) \to -1$ as $n \to +\infty$.

The above results point to an interesting concept. As n increases, $f(n)$ runs between 1 and -1 (see figure 1).

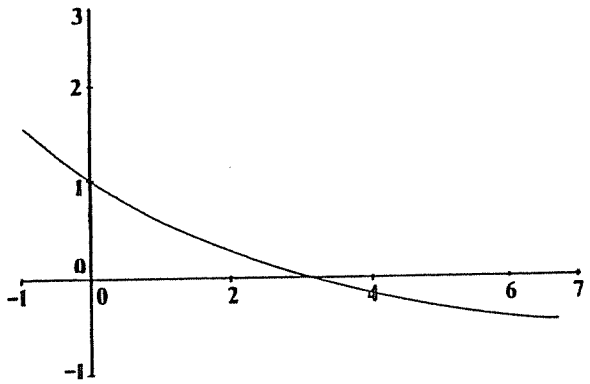


Figure 1

By the Intermediate Value Theorem, there is an N such that $f(N)=0$. For Fermat's Last Theorem to be true, this N cannot be negative or an integer larger than 2.

THE IDEA OF TURNAROUND

Let's now look at a specific problem.[1]

PROBLEM:

We all know that $3^2 + 4^2 = 5^2$.
What about $3^n + 4^n = 5^n$ when $n>2$?

Solution:

$$5^n = (5^{n-2})(5^2) = (5^{n-2})(3^2+4^2)$$

Thus, $5^n = (5^{n-2})(3^2) + (5^{n-2})(4^2)$.

Since $5>3$ and $n>2$, $5^{n-2} > 3^{n-2}$.

This implies that $(5^{n-2})(3^2) > 3^n$.

Similarly, $(5^{n-2})(4^2) > 4^n$.

So $5^{n-2}(3^2 + 4^2) > 3^n + 4^n$.

Therefore $5^n > 3^n + 4^n$.

This result can be applied to any general Pythagorean triple.

We know this:

THEOREM 1:

If (x,y,z) is a Pythagorean triple, then $x^n + y^n < z^n$ for $n>2$.

The obvious question now is what happens when n decreases from 2.

THEOREM 2:

If (x,y,z) is a Pythagorean triple, then $x^n + y^n > z^n$ for $n<2$.

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1. We wish to thank Professor Sahib Singh, who included this result in a talk on Fermat's Last Theorem and thus gave birth to these ideas.

PROOF:

(1) Let n=1. Assume $x + y \le z$.
Squaring both sides, $x^2 + 2xy + y^2 \le z^2$, so clearly $x^2 + y^2 < z^2$, a contradiction.
(2) Let $n = 0$. Clearly true since 2>1.
(3) Let n =-1. Assume $1/x + 1/y \le 1/z$.

$$\frac{1}{x} + \frac{1}{y} - \frac{x+y}{xy} \le \frac{1}{z}$$

Therefore,

$$z \le \frac{xy}{(x+y)}$$

By definition of a Pythagorean triple, x<y<z.
Therefore,

$$x < \frac{xy}{(x+y)}$$

This implies that $x^2 + xy < xy$.
Thus $x^2 < 0$.
But since x is real, this is a contradiction.

(4) Let n be a negative integer <-1.

Assume $x^n + y^n \le z^n$. Let $m = -n$.

Note that $m \ge 2$.

Then,

$$\frac{1}{x^m} + \frac{1}{y^m} \le \frac{1}{z^m}$$

$$\frac{1}{x^m} + \frac{1}{y^m} = \frac{(x^m + y^m)}{x^m y^m} \le \frac{1}{z^m}$$

therefore

$$z^m \le \frac{x^m y^m}{(x^m + y^m)}$$

By Theorem 1, $x^m + y^m \le z^m$ since $m \ge 2$.
Therefore,

$$x^m + y^m \le \frac{x^m y^m}{(x^m + y^m)}$$

Thus,

$$(x^m + y^m)^2 \le x^m y^m$$

and

$$x^{2m} + 2x^m y^m + y^{2m} \le x^m y^m$$

Therefore,

$$x^{2m} + y^{2m} \le -x^m y^m$$

But because (x,y,z) was chosen to be a Pythagorean triple, all three numbers are positive.

This means that

$$x^{2m} + y^{2m} > 0 > -x^m y^m$$

and thus we get a contradiction.

See that we have a pattern here. For n>2, $x^n + y^n < z^n$. At n=2, they are equal; and for $n < 2$, $x^n + y^n > z^n$. The not-equal sign turns around.

From Theorems 1 and 2, we get this result.

THEOREM: If (x,y,z) is a Pythagorean triple, then $x^n + y^n \ne z^n$ for $n \ne 2$.

Exploring this further, we can solve many similar questions (i.e., if $x^2 + y^2 < z^2$ then $x^n + y^n < z^n$ for $n > 2$). We leave these to the reader to pursue (we have truly marvellous demonstrations of these ideas which this note is too short to contain).

CONCLUSION

It is our hope that you, the reader, have had your appetite whetted and will pursue this topic further. If enough of us look at Fermat's Last Theorem, maybe we will one day discover the secret which has been so jealously guarded for 300 years.

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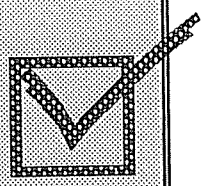
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RESEARCH CLIPS

HOW YOUNG CHILDREN REPRESENT AND SOLVE VERBAL PROBLEMS

Many primary teachers feel that missing addend problems are hard for children. This may be true if the "problem" is simply a symbolic open sentence. But, it is not true if the problem is in a story context. Primary age children can comprehend and solve a wide variety of verbal problems using the means of their own invention. Look at these two problems, both solvable by subtraction and representable by the standard sentence 8 - 5 = ☐.

Join, missing addend: Connie had 5 marbles. Tom gave her some more. Then she had 8 marbles. How many marbles did Tom give her?

Comparison: Jim has 8 marbles. Connie has 5 marbles. How many more does Jim have than Connie?

One of these has implied action; the other does not. The relations between known and unknown portions of the problems are also different. Research done in the U.S. and other countries in the last decade has shown that children can identify these semantic differences. Furthermore, their informal modeling and counting strategies for solving are based upon and reflect these semantic differences. In the first problem, children are most likely to use an adding-on procedure starting with five objects and ending with eight. They then solve by counting how many objects were added on. Or they might count forward, "Five . . . 6, 7, 8." The answer is 3."

Recent research suggests that the mapping of semantic structure onto solution procedures tends to carry over into the symbolic representation of these problems. After brief instruction on the possibility of using non-standard open sentences to represent problems, children seemed to prefer such sentences. Thus, for the first problem above, they write 5 + ☐ = 8. This is much more pronounced for action problems such as a Join or Separate, no matter what the unknown is, than it is for a static problem such as Comparison.

Teachers need to be aware that young children come to school with a rich repertoire of problem solving strategies. Instruction need not be limited to basic "add on" and "take-aways." And it is definitely the case that experience with a wide variety of verbal problems does not have to be delayed until children have mastered basic facts.

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Clip prepared by the Research Advisory Committee of the National Council of Teachers of Mathematics.

RESEARCH CLIPS

LEARNING ABOUT DECIMALS

The topic of decimal numbers has proven to be difficult for students (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Hiebert & Wearne, 1986). Many of the difficulties can be traced to an incomplete or nonexistent understanding of the written decimal symbols. Without connecting the symbols with the quantities they represent, students frequently manipulate symbols by applying memorized rules. Research indicates that students can develop an understanding of decimal numbers if instruction focuses on meaning rather than simply on rules for placing the decimal point within computation problems.

To check the effects of instruction which emphasized the meaning of decimal numbers, fourth-grade classes of varying achievement levels were provided special instruction for two weeks. After spending several days constructing meaning for decimals using base-10 blocks, the students used their understanding to develop procedures for adding and subtracting decimal numbers such as 2.3 + 4.6, and 3 – .2. The procedures mirrored the actions on the blocks (e.g., adding tenths to tenths) rather than following a stated rule such as "line up the decimal points when you add." Eight weeks after the instruction, the students used the meaning of the decimal symbols to solve problems both similar (4.8 – .35) and dissimilar (select the larger of .34 and .8) to those included in the instruction (Wearne & Hiebert, 1989). The investigators conclude that focusing on the meaning of the numerals at the outset enabled the students to use these meanings to solve problems rather than to rely on rules not fully understood.

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Hiebert, J. & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.) Conceptual and procedural knowledge: The case of mathematics. (pp 199-223). Hillsdale, NJ: Erlbaum.

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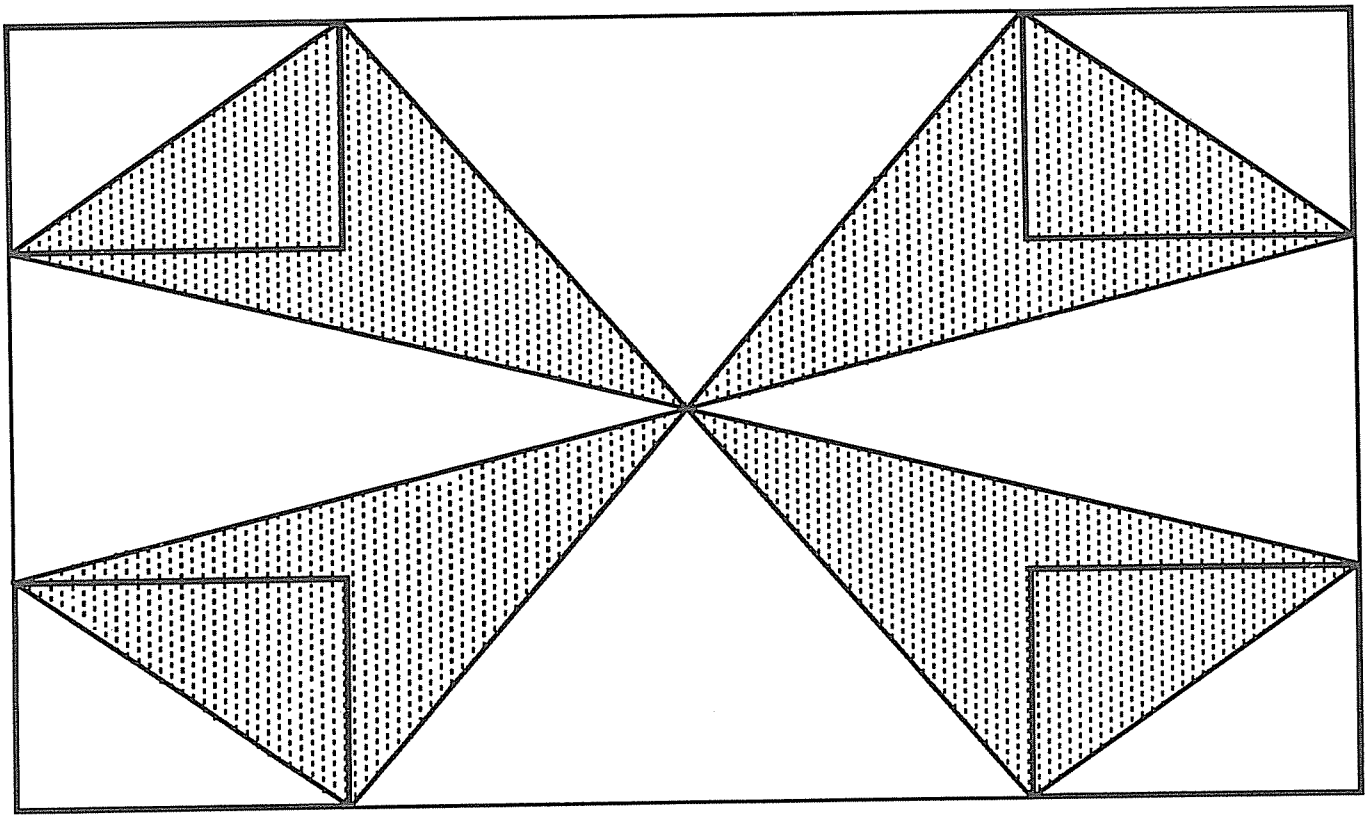
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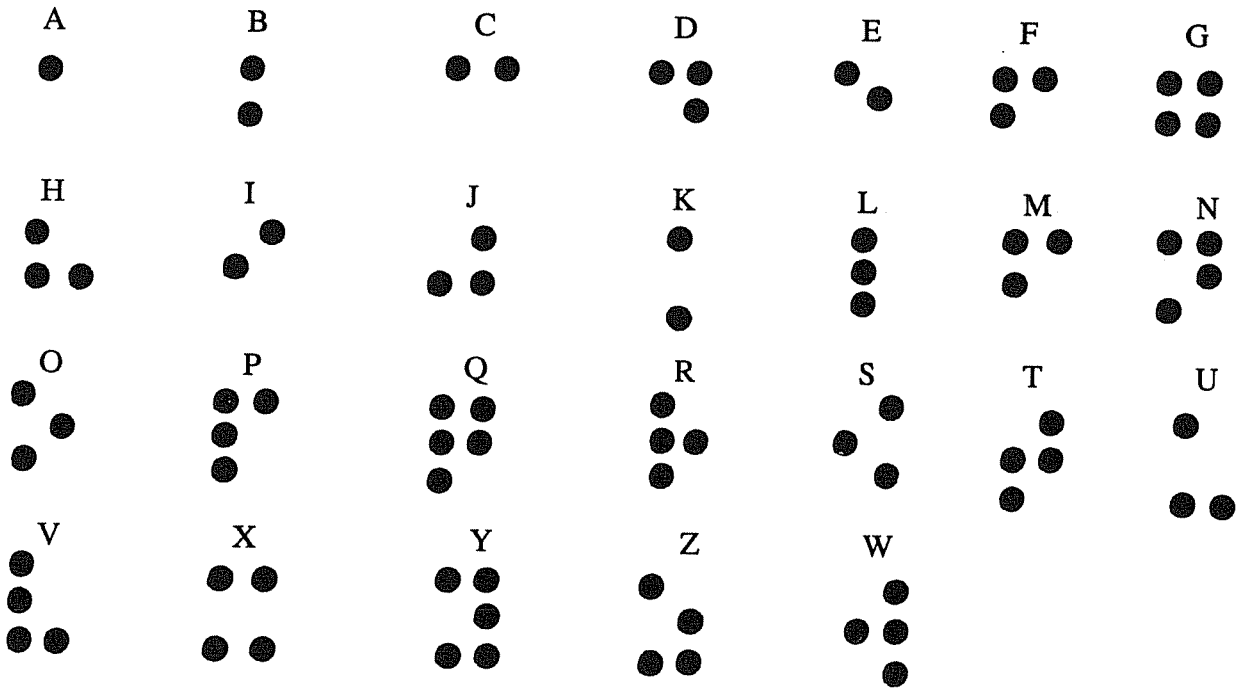
The Braille Alphabet

You have all seen your students struggle at times with algebra. Many students cannot grasp how it is that letters are replaced by numbers and vice versa.

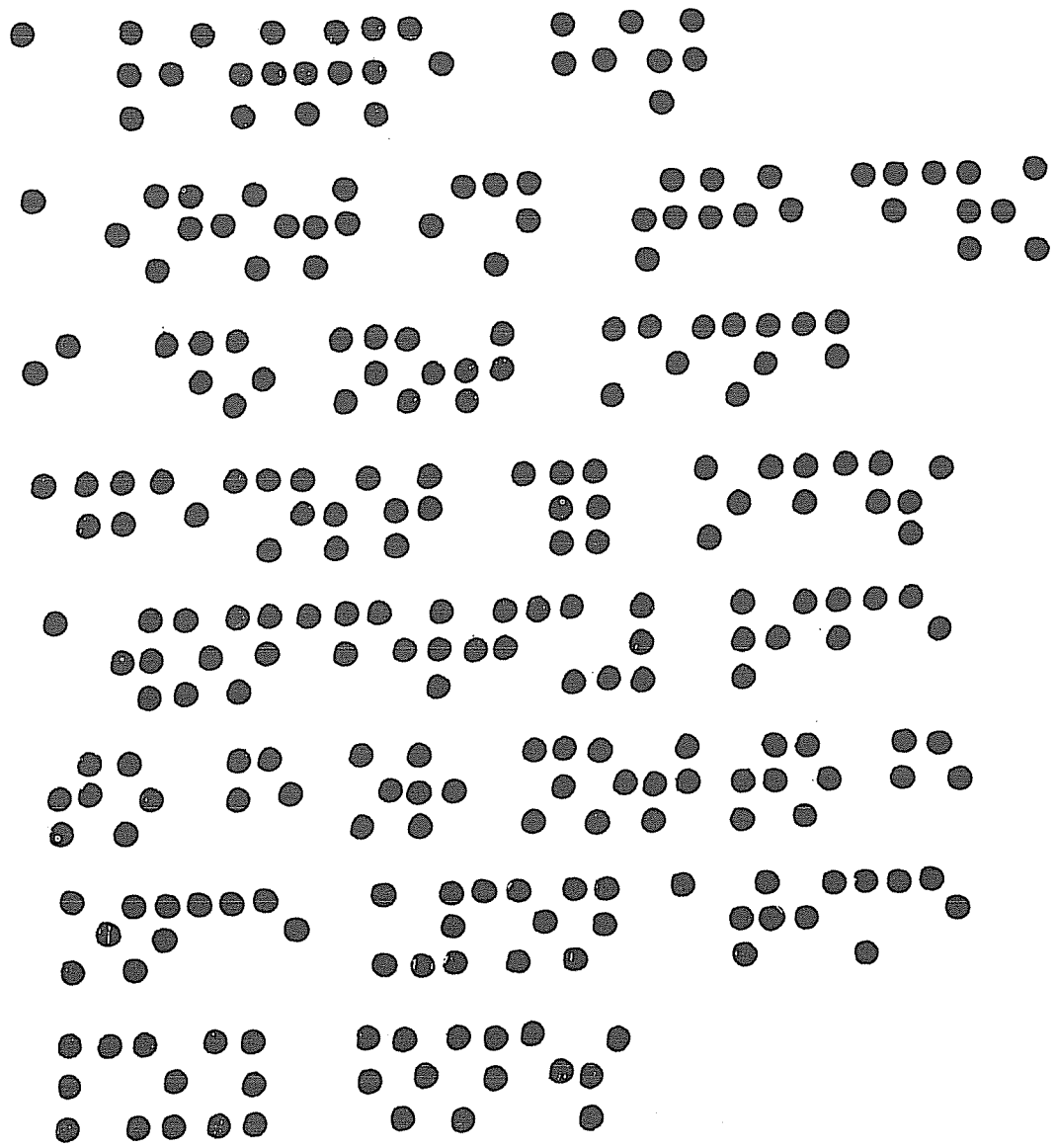
Try to imagine, just for a moment, that you are learning algebra without the benefit of sight. To simply learn to read, a blind person must first be able to intuitively work with algebra. The Braille alphabet, discussed below, is designed using algebraic skills.

The Braille alphabet enables the blind to read and to write as effectively as anyone else. The alphabet itself was developed by Louis Braille in 1829. As a youngster, he became frustrated with the large and bulky raised letters used for him to learn reading and writing skills. Later, while discussing the problem with Charles Barbier, Louis came up with the idea of reading by code. After a number of years of experimenting, he successfully developed the Braille code.

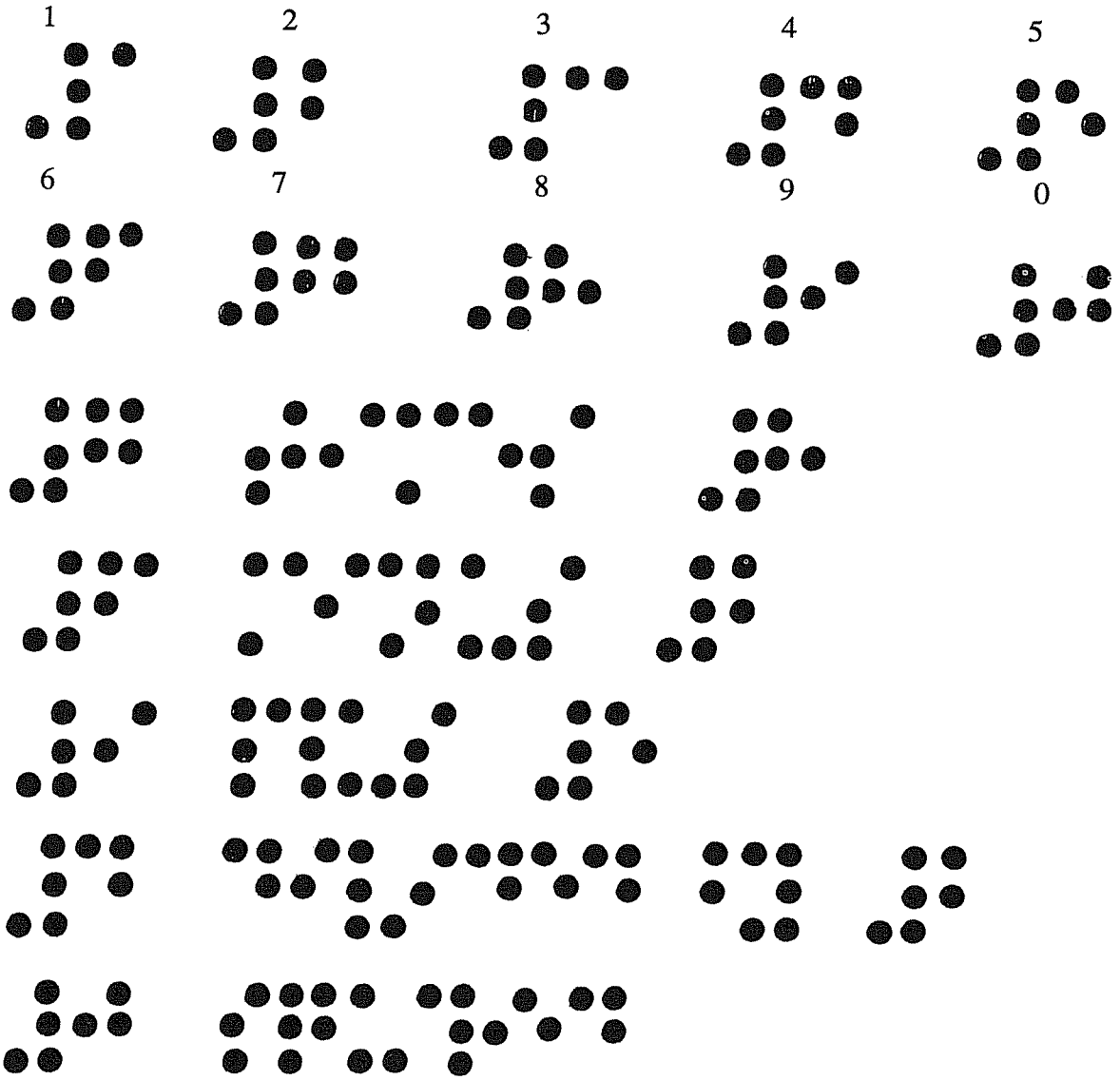
The code itself is based upon the 3 x 2 matrix. The braille "cell" is comprised of six dots placed in a certain order throughout the matrix. Depending upon where the dots are placed, the reader can determine what letter is represented. The alphabet is shown below.



Use the braille alphabet from the previous page to interpret each of the following. What does each sentence represent? A blank line is left between each sentence.



The Braille alphabet is extended to represent numbers. Use the alphabet and the numbers represented below to find the value of each expression.



For your more advanced students, have them look for the pattern that is used to represent letters and numbers using the Braille alphabet.

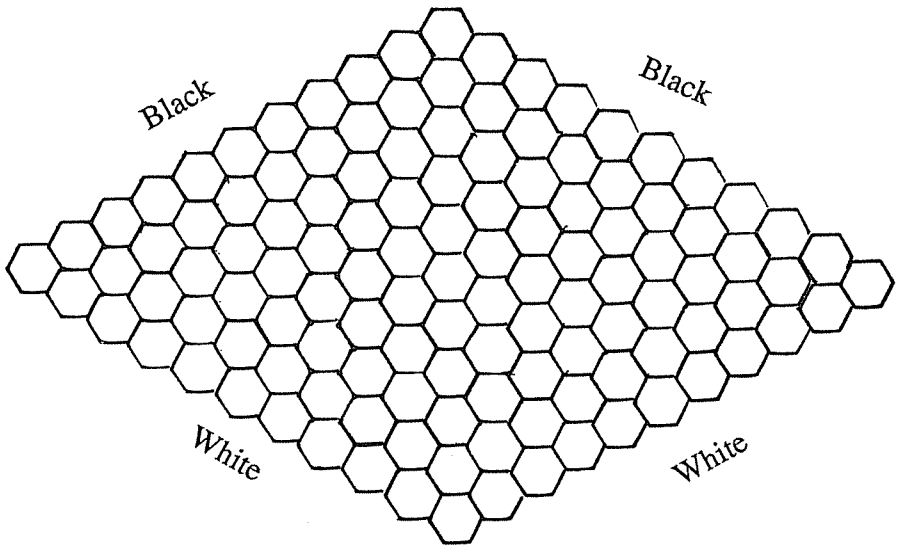
Hex

No, we are not going to put a hex on you. Rather, we are introducing a game that was first presented by Piet Hain, in 1949. Try playing the game with a friend. The rules of the game, and the board, are shown below.

Rules

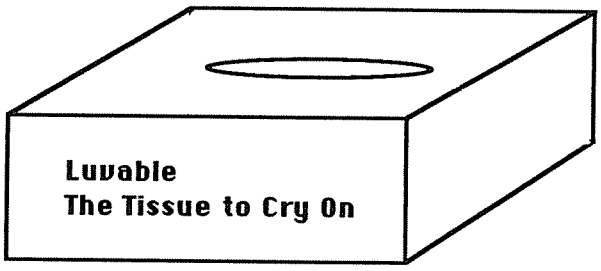
1. Each player alternates colouring a single hexagon.
2. The object of the game is to find a path from the black side of the board to the white, for Player 1, and from the white side of the board to the black for Player 2.
3. The winner finds a path that is not broken. A path is one where you move from one hexagon to another by simply crossing a line.
   - You cannot 'jump' a hexagon.
   - You must cross the 'middle' barrier with 11 hexagons in the column.

Middle Barrier



Math in the Bathroom

You have probably told your students that thay can find mathematics everywhere in their own familiar world. Have you ever thought about how much mathematics can be found in the bathroom?



Look at the tissue box shown to the right. How thick do you think each tissue is? Follow the steps shown below to find out.

Step 1 Measure the height of the tissue box.

Step 2 Find the number of tissues in the box you are measuring.

Step 3 Divide the number in Step 1 by the number in Step 2. How thick is each tissue?

Now try the following problems. Each problem relates to what you could find in your bathroom.

1. (a) Develop a plan to find the surface area covered by the tissues in a box.
   (b) Use your plan. What is the area covered by all the tissues?
   (c) Use your results in (b). Is the area covered by the tissues greater than the area covered by all pages of your local newspaper? Give reasons for your answer.

2. Look at any roll of toilet paper.
   (a) What is the area covered by each sheet?
   (b) How many sheets are on a roll?
   (c) What area is covered by all the sheets?
   (d) Use your results in (c). Is this area greater than the area covered by all pages of your local newspaper? Give reasons for your answer.

3. Use your roll of toilet paper in Question 2.
   (a) What shape is formed by the roll?
   (b) What is the thickness of each sheet?
   (c) What is the surface area of the roll of toilet paper before it is ever used?

4. Use your toilet roll.
   (a) Find the volume of the toilet roll.
   (b) Interpret your result in (a). Do you think that the toilet roll can absorb the volume of water calculated in (a)? Give reasons for your answer.

5. Look at the tube that holds the sheets of paper. The spiral on the outside of the tube is called a "helix".
   (a) Why is it called a helix? (Use your library if necessary.)
   (b) Calculate the surface area of the tube.
   (c) Cut the tube along the helix and flatten the tube. What shape is the result?
   (d) Use your observations in (c). Make a tube for toilet paper that has twice the surface area of the tube in (b).

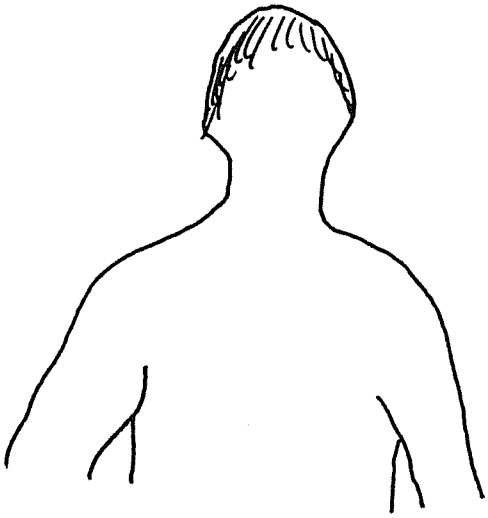
6. Chances are that, at some point, you will find yourself in a bathroom. Did you know that about 70% of your body mass consists of water? This fact, along with the knowledge that 1 L of water has a mass of 1 kg, can be used to find your volume. Use the steps shown below.
   (a) Use a bathroom scale to find your mass.
   (b) The mass of your body is also the number of litres of water in your body. Write your mass in cubic centimtres. (Remember: 1 millilitre = 1 cubic centimetre.)
   (c) Interpret your result in (b). What is your volume in cubic centimetres?

7. Look at the sink in your bathroom.
   (a) Develop a plan to find the volume of water that can fill your sink. (What assumptions do you need to make about a full sink?)
   (b) Use your plan in (a). What is the volume of water that can be held by your sink?

8. Look at the tub in your bathroom.
   (a) Develop a plan to find the volume of water that can fill your tub. (What assumptions do you need to make about a full tub?)
   (b) Use your plan in (a). What is the volume of water that can be held by your tub?

9. (a) Do you prefer to take bath or a shower?
   (b) Predict which you think uses more water - a bath or a shower?
   (c) Check your porediction in (b) using your library. Which uses more?
   (d) Use your answer in (c). Write a short paragraph to explain why is this so?

Math in Your Body



Your body is a system of about 200 bones and 100 joints that make a rather complex system of props, pulleys and levers. This skeleton is needed to ensure that you not only breathe and are able to move, but also to ensure that you do not collapse in heap on the ground.

There are many facts about your body of which you are probably unaware. By answering the questions that follow, find out as many interesting facts as you can. Use your library to investigate many of these, as well as other, facts about your body.

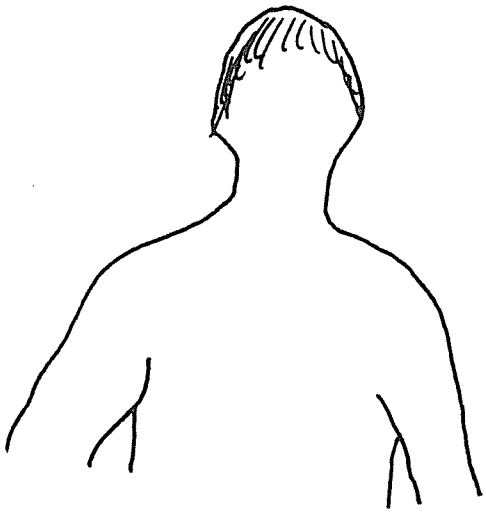
1. Work with a friend. One of you will have to keep track of the time.
   (a) How many times can you blink your eyes in one minute?
   (b) How many times can you blink you eyes in one second?
   (c) Did you known that the fastest human muscle can make your eyelids blink about 5 times each second? How fast are you?

2. (a) Predict how many times you think your heart will beat in a life time.
   (b) Measure your heart beat at rest.
   (c) How many times will your heart beat in a day? in a year? in 70 years?
   (d) How accurate was your prediction in (a)?

3. Use your library.
   (a) What is the distance around the equator of the earth?
   (b) How far would you travel if you walked around the earth 2.5 times? What assumptions do you make in solving this question?
   (c) Interpret your answer in (b). If you laid all the blood vessels in your body end to end, how many kilometres would they stretch?

4. There are about 100 000 hairs on your body. Each hair is strong enough to lift about a 0.2 kg mass.
   (a) Use a scale. Find your mass.
   (b) How many hairs would it take to lift your mass?
   (c) Do you think there are enough hairs on your head to left all the people in your class? Give reasons for your answer.

5. When you perform any physical activity, you will perspire.
   (a) In what activities will you perspire the most?
   (b) Evaluate this expression.
   200 000 x 7 + (314 462 - 2231 x 2 + 1 290 000).
   (c) Each sweat gland averages a length of 0.16 mm. Use your expression in (b). How many kilometres will your sweat glands stretch if they are laid end to end?



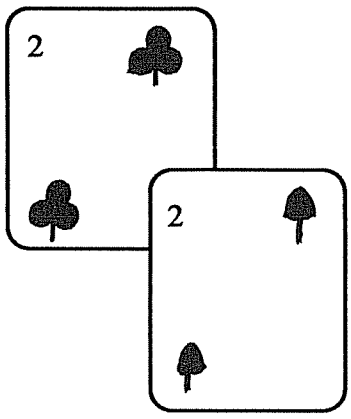
6. Work with a friend.
   (a) Pace off a distance of 100 m.
   (b) Run the 100 m as fast as you can. Your friend will time you.
   (c) Use your results in (b). What is your speed in kilometres per hour?
   (d) The electrical impulses in the nerves of your body travel at a speed of about 400 km/h. In the time it takes you to run 100 m, how far has one electrical impulse travelled?

7. (a) Predict how long you think you spend with your eyes shut each day.
   (b) For how many hours did you sleep last night?
   (c) Stand in front of a mirror. How many times do you blink in one minute.
   (d) On average, each time you blink you will have your eyes shut for 0.03 s. Use your answer in (c). For how many hours will your eyes be shut each day? How accurate was your prediction?

8. Try this experiment.
   (a) Fill a glass about three quarters full of water.
   (b) Sprinkle some black pepper onto the surface of the water.
   (c) Swirl the glass. What do you noitce about the black pepper?
   (d) What conclusion can you draw about the water in the glass?

   The result that you saw in Question 8 can be applied to your ears. There is a fluid in the ear canal that is used to detect and regulate your balance. As you turn your head from one side to the other, your ear canal will move, but the fluid stay still. Use your library. How does this help you maintain your balance?

Card Tricks



Card tricks have provided amusement and fascination for many people over many years. The following shows you two card tricks. They were first presented by Martin Gardner in his book Mathematical Puzzles and Diversions. Martin Gardner, being a magician himself, presented many tricks that worked because of the mathematics involved. After you have tried these two tricks, you may wish to read some of his publications. They will give you many more ideas that you can use with your students.

- Follow the steps given to verify that they work.
- Try them on a friend.
- Then, verify mathematically why they work.

Steps

1. From a well-shuffled deck of cards, have a friend select three cards. Place them face down on a table.

2. Have your friend shuffle the remaining cards and give them back to you.

3. Look at the fourth card from the bottom of the deck. Remove its counter part from the deck. For example, if the fourth card is a 3 of heart, select the 3 of diamonds. If the fourth card is the 7 of spades, select the 7 of clubs.

4. Have your friend turn over the three cards. Each card is given values as follows.
   Ace - 1, Two - 2, Three - 3, ..., Ten - 10, Jack - 11, Queen - 12, King - 13

5. Have your friend start placing cards on each of the three selected cards until 15 is reached. For example, if the card is a Jack, then the Jack represents 11 and four cards are turned over to represent 12, 13, 14 and 15.

6. Repeat Step 5 for all three cards. Then, find the sum of the numbers that have been represented by the three cards. Count out this many cards from the deck.

7. Once you have completed Step 6, announce that the next card in the deck and the card that you removed earlier will be the same number and the same colour.

How is this done?

Here is another card trick you can try. You have probably solved problems that use the same skills used in this card trick.

- Follow the steps to try the card trick.
- Solve the problem shown below.
- Then try to explain why the trick works.

Steps

1. Give a full deck of cards to a friend.

2. Have your friend shuffle the deck thoroughly. Then select 20 cards from the deck.

3. Turn these cards face up and return them to the deck.

4. Your friend then shuffles the deck once again so that the cards that are turned face up are spread randomly throughout the deck.

5. Now have your friend count out the top 20 cards from the deck and hand them to you. You now announce that, with your eyes shut, you will turn over enough cards in the pile just handed to you so that there are exactly the same number of cards in the pile you are holding as the pile your friend is holding.

6. Without looking at the deck, simply turn the pile in your hands over.

7. Now count out the cards that are face up. There is exactly the same number of cards face up in each pile.

To try and determine how this problem works, solve the following problem that you may have seen before.

Colin (whose little wife calls the chinless wonder) has two beakers, each with 100 mL of liquid in them. One beaker contains a red dye solution while the other beaker contains a green die solution. Fifty millilitres of the green die solution are placed in the red die solution. The new solution is then mixed. From this new solution, 50 mL are taken and placed back into the green die solution. Which beaker has the most green die solution in it? Give reasons for your answer.

Spatial Sense

To solve many problems, you will have your students look for patterns. It is through this search for patterns, that students often develop their sense of the space around them. In 1977, Hoffer suggested "seven visual perception abilities" that students use to help develop their sense of the space around them. The seven are as follows.

- Eye-motor coordination
- Figure-ground perception
- Perceptual constancy
- Position-in-space perception
- Perception of spatial relationships
- Visual discrimination
- Visual memory

Much has been written over the years about all seven abilities. In fact, originally, Brennan, Jackson and Reeve (1972) suggested that there were nine "visual perception abilites". At this point, it is not possible to go into all of the perception abilites described. However, one of the abilities is summarized below. On the next 3 pages, you will find activities and problems that can be solved to help develop this one ability. To help your students develop the skill, have them work in small groups to complete each activity.

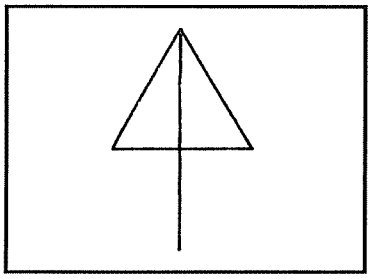
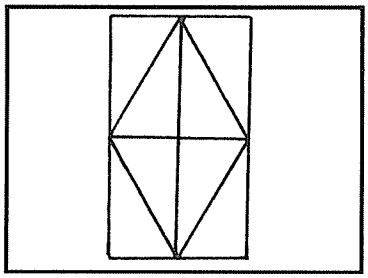
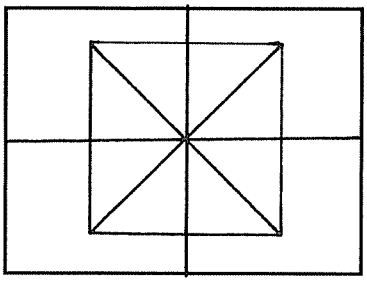
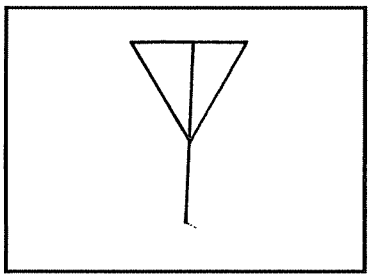
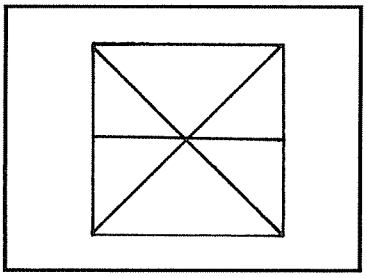
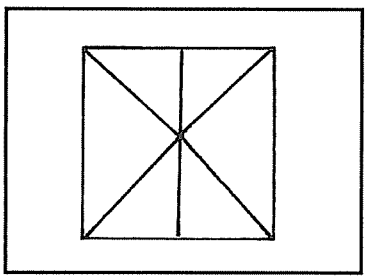
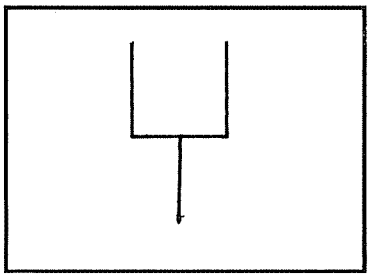
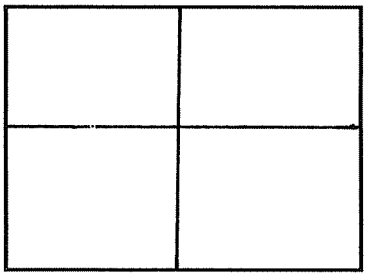
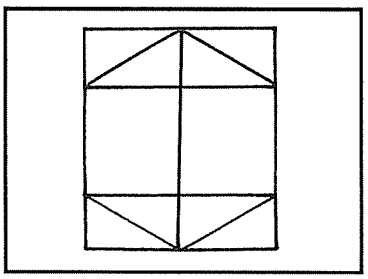
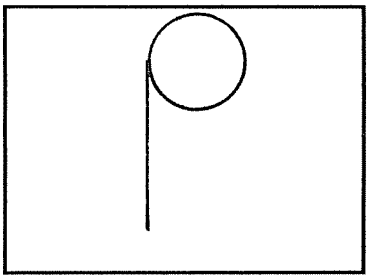
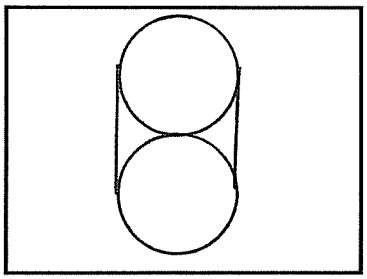
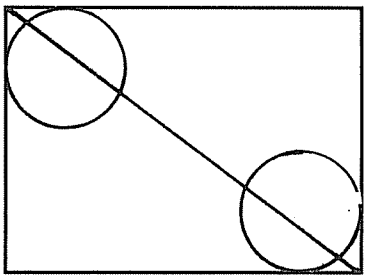
Figure-ground perception

Simply stated, figure-ground perception is the ability to "distinguish between the foreground from the background". This can be done in three of the following ways.

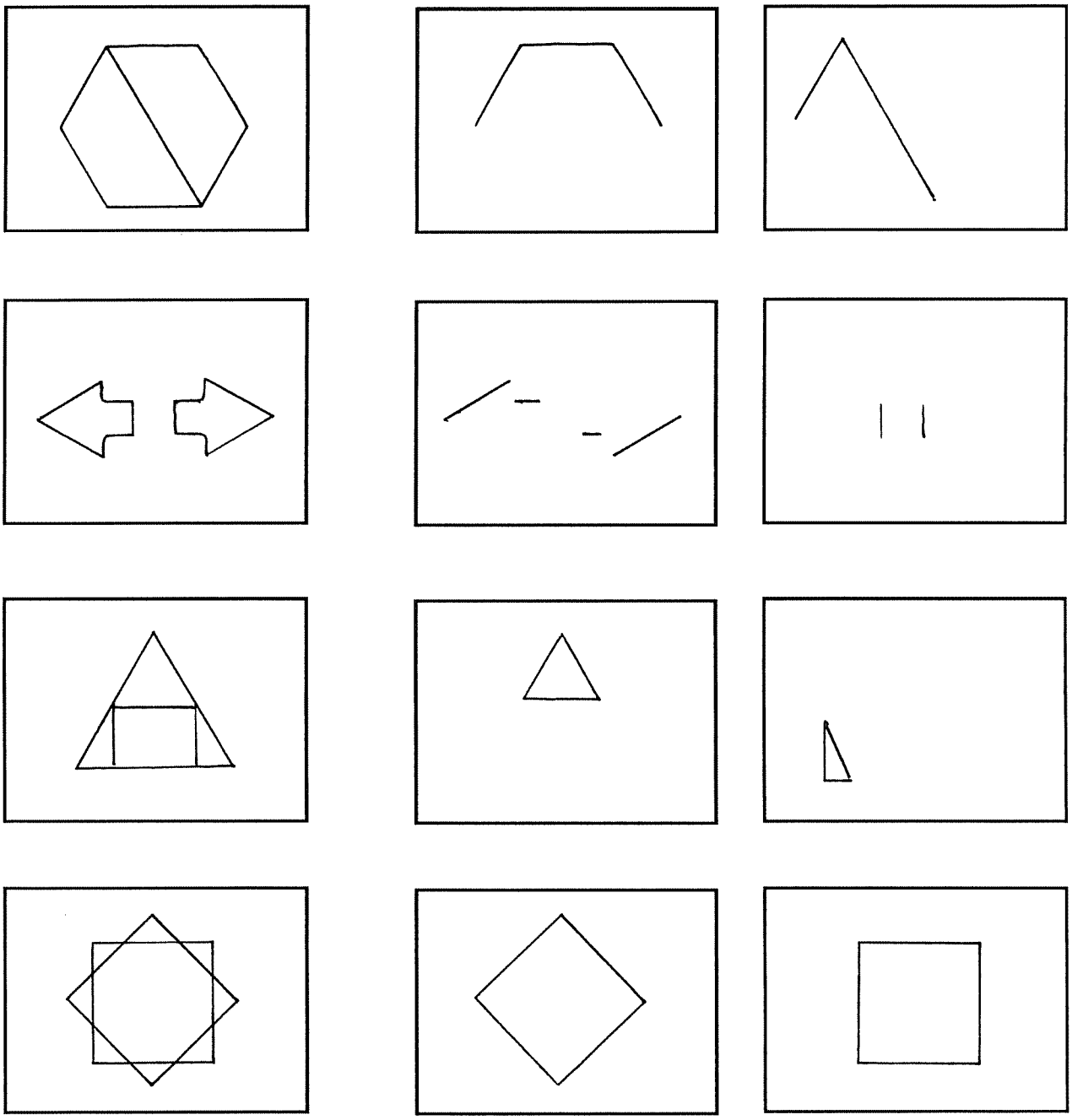
- Identify a figure in a diagram where more than one figure can be seen.
- Complete a given figure from a partial figure.
- Complete a figure from its parts.

Each of the next three pages gives you activities that you can try with your students. Each page is devoted to one of the ways to "distinguish foreground from background". In order to keep your students on track, make copies of each of the next three pages and distribute them. It will be much easier than having your students trace the figures themselves.

On this page, you will identify figures from a given figure. In the box to the left, a figure is shown. Use a red marker to trace the figure in the 2 boxes to the right.

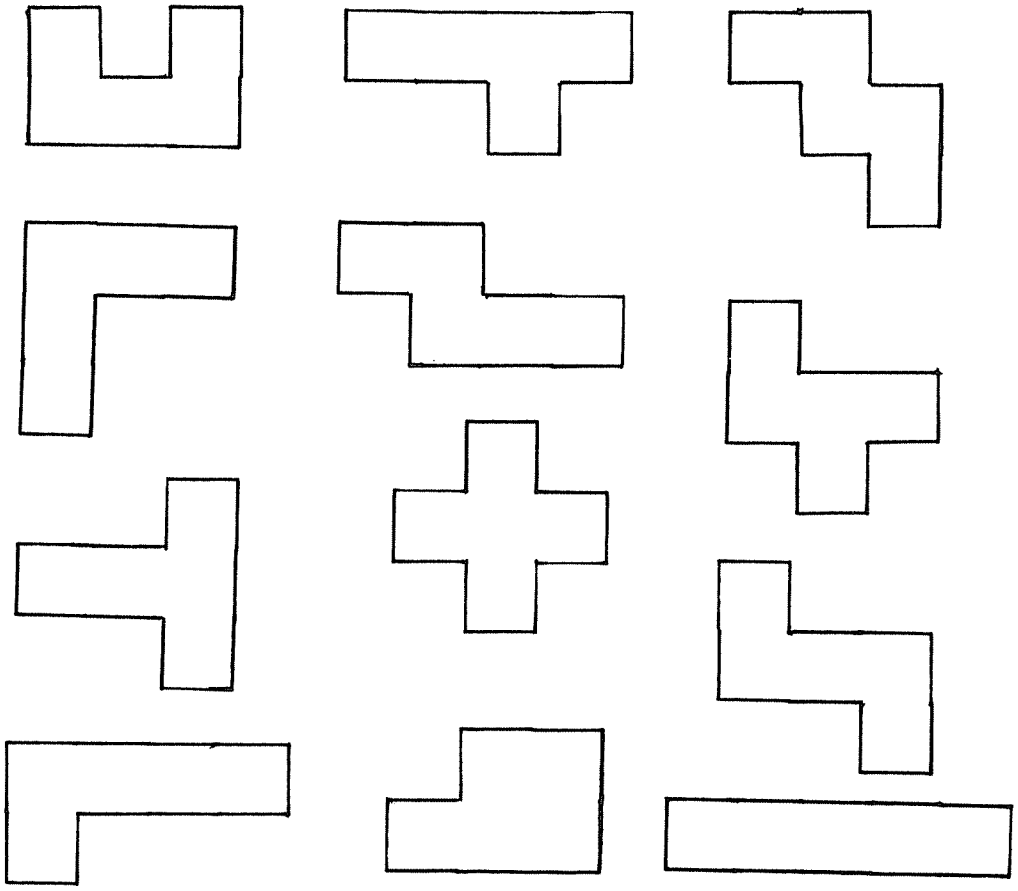













On this page, you will complete the figure given. In the box to the left, a figure is shown. Use a red pen to complete the figure in each box to the right.



The following uses the twelve pentominoes to create figures. The twelve pentominoes are shown below. Cut out the twelve pentominoes and use them to create the following figures.

- 4 different rectangles, each with an area of 60 square units.
- 2 different rectangles, each with an area of 30 square units.
- 1 square with an area of 25 square units.
- A shape of your own. Then, verbally describe the shape to a friend in the class. Have your friend use the pentominoes to construct your shape.



Blame it on the Rain

We are not discussing the popular song being played on many radio stations this summer. rather, we are going to think about other broadcasts made on all radio stations.

How many times have you listened to the weather forecast played on your favorite radio station and heard the following prediction?

"There is a 50% chance of rain today."

Did you think that the weather forecaster was only guessing and that the "odds" were 50-50 that it was going to rain?

In fact, this forecast is interpreted to mean that about 50% of the listening areas is going to be wet at some point throughout the day. For example, if the broadcast signal covers an area of about 250 square kilometres, then about 125 square kilometres are going to get wet.

Pick a radio station in your area. Find out how many square kilometres are covered by the broadcast signal. Use the broadcast area to interpret each of the following weather predictions.

1. "There is only a 10% chance of rain today.

2. "There is a 90% chance of precipitation today.

3. "There is a 70% chance of snow today."

4. "A thunderstorm watch is in effect. There is a 80% chance that damaging hail stones will fall in the surrounding areas."

5. Listen to your favorite radio station for a week.
   - Interpret what is meant by the weather forecast.
   - Record what the weather was like in your area and in the surrounding area.
   - How accurate was the weather forecast in your area? in the surrounding area?

6. Now use your library.
   - Find the Farmer's Almanac.
   - Check the weather forecasts for each day in the Farmer's Almanac.
   - How is the weather forecast in the Alamanac like the one given by your radio station? How is it different?
   - Over the course of one week, which was the more accurate forecast? Give reasons for your answer.
   - Which do you feel is the more accurate forecast? Give reasons for your answer.

PRACTICAL PROBLEMS ARISING OUT OF NUMBER THEORY

by
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This is part of a talk presented at the 16th Annual OAME Conference held May 11 - May 13, 1989 at the University of Ottawa.

Introduction

Number theory and logic are supposed to be the purest of the pure mathematics. Logic, however, lost some of its pureness by its utilization in computer science studies. Few students venture into number theory: this may be due to the fact that the field apparently has no applications. The main theme here is to show that this is false by giving two examples of its usefulness, and presenting references which one can read at a fairly rudimentary level.

Number theory is the study of integers. That sounds simple, but actually may be a very complicated matter. We discuss only *elementary* number theory -- elementary in the sense that it does not use the calculus or beyond. However, this does not mean elementary in the usual sense of the word. It can be rather difficult.

Carl Friedrich Gauss stated that "mathematics is the queen of science and number theory is the queen of mathematics" (Boyer, p.557 and Eves, p.359). I hope that the prophetic words of this genius are still valid. His words had nothing to do with being practical; but, being practical in this age means, to some extent, survival if one is going into number theory. Therefore, I have chosen two practical applications using modular arithmetic to give an impetus to those who wish to study mathematics and who might be questioning whether to enter into the field of number theory.

Application to the Perpetual Calendar

The first application to be discussed is how number theory can be applied to the Gregorian calendar to obtain the **Perpetual Calendar.** Moyer (p.144) states:

> "In 1582 Pope Gregory XIII introduced the calendar named in his honour; it is the system of reckoning days currently in civil use throughout the world. Before the inception of the Gregorian calendar Western civilization depended on a system instituted by Julius Caesar, called the Julian Calendar. For more than 16 centuries this system remained in service in spite of an accumulating discrepancy between the mean length of its year, 365.25 days, and the tropical year, the average interval between successive transits of the apparent sun through the vernal equinox. By 1582 the error in the Julian system had grown to approximately 11 days. This defect was of principle concern to the Pope; if the Julian Calendar had continued in service, Easter would eventually have been celebrated in the summer."

With the help of astronomers, mathematicians and clergymen, Pope Gregory revised the calendar. It started on October 15, 1582 and continues to the present day.

We want a formula which can tell what day of the week a certain date of any year is, using modular arithmetic. We need only two simple definitions and some notations. Then the formula can be stated.

The first definition deals with *modulus* (or *mod*).

Definition 1: a is **congruent** to b (*modulo m*) if and only if m divides $(a-b)$. We write this as $a \equiv b \pmod{m}$ if and only if $m|(a-b)$.

Definition 2: The largest integer less than or equal to the real number x is denoted by $[x]$. Equivalently, $[x]$ is the integer such that $[x] \leq x < [x] + 1$.

We note that in tables giving perpetual calendars, there are 14 types of calendars presented. This is because, in one regular year, January 1 may begin on any one of seven days; the same is true for any leap year. What we propose to do is to give one formula which eliminates all these tables -- and eliminates stating which table a given year falls under.

Let us now establish the procedure for finding the number of the day of the week in the Gregorian calendar. Let d denote the number of the day of the week, where d = 0 for Sunday, d = 1 for Monday, . . ., d = 6 for Saturday. Let D = the number of the day in the month; let M be the number of the month beginning with March. Thus, M(March) = 1, M(April) = 2, . . ., M(December) = 10, M(January) = 11, and M(February) = 12. [We do not have to start with March, but it makes the formula simpler because the extra leap day when added comes at the end of February.]

Let C be the century and Y be the year of the century. Also, let L = 1 for a leap year and L = 0 otherwise. [Leap years are those years divisible by 4 except for the century years not divisible by 400.] Now that we have the notations, the formula can be stated:

$$d \equiv D + [2.6M - 0.2] + [\frac{C}{4}] - 2C + [\frac{Y}{4}] + Y - (L+1)[\frac{M}{11}] \pmod{7}$$

This formula with derivations may be found in Niven, p.312; Rosen, pp.152-156; and Shockley, pp.53-56.

One must be somewhat amused at the function [2.6M - 0.2]. This was found by Reverend Zeller [Rosen, p.156 or Shockley, p.56] and accounts for the sum of the increments (over 28) in the days of the months, which, starting in March, can end in 30 or 31 days.

Let us check two dates: 15 October 1582 (the start of the Gregorian calendar) and 1 July 1867 (the founding of Canada).

1. What day was 15 October 1582?

$$d \equiv 15 + [(2.6)(8) - 0.2] + [\frac{15}{4}] - 30 + [\frac{82}{4}] + 82 - 0 \pmod{7}$$

$$\equiv 110 \pmod{7} \equiv 5 \pmod{7}$$

Hence, this was a Friday, as was noted by Moyer (op.cit.,p.147).

2. What day was 1 July 1867?

$$d \equiv 1 + [(2.6)(5) - 0.2] + [\frac{18}{4}] - 36 + [\frac{67}{4}] + 67 - 0 \pmod{7}$$

$$\equiv 64 \pmod{7} \equiv 1 \pmod{7}$$

Hence, this was a Monday.

Application to Cryptology

The second application I want to talk about is the subject of cryptology, which is the study of secrecy systems. These systems date back to Julius Caesar, but are more necessary than

in military affairs, electronic banking, and diplomacy.

First we introduce a few definitions.
Cryptography deals with the design and the implementation of secrecy systems.

Cryptanalysis deals with the breaking of these secrecy systems.

A message to be altered is called **plaintext**, denoted by $\mathbb{P}$; the **ciphertext**, denoted by $\mathbb{C}$, is the changed or encoded message (to be cryptanalyzed if one does not know the **key**, which gives a particular transformation from the plaintext). The transformation called the **encryption** (or the **enciphering**) map σ: plaintext $\rightarrow$ ciphertext, i.e., $\sigma(\mathbb{P}) = \mathbb{C}$ and the transformation called the **decryption** (or **deciphering**) map τ: ciphertext $\rightarrow$ plaintext, i.e., $\tau(\mathbb{C}) = \mathbb{P}$. Now τ is known by the intended receiver, and unknown by the unintended receiver (who has to use cryptanalysis). All of these systems, of course, translate letters into numbers. We translate *A,B,C,...,Z* into *0,1,2,...,25* respectively. So in the method stated here we always use arithmetic *(mod 26)*.

One of the first known **ciphers** (a transformation for changing a plaintext into a ciphertext) was used by Julius Caesar. It consisted of replacing each letter by the letter 3 places ahead in the alphabet *(mod 26)*. So $C \equiv P + 3 \pmod{26}$, $0 \leq C \leq 25$. It is not difficult to discover the decryption τ even without knowing it.

The Caesar cipher is one of a family of ciphers called **affine transformations** which have the form $C \equiv aP + b \pmod{26}$, $0 \leq C \leq 25$, *with* $\gcd(a,26) = 1$. [We need that the greatest common divisor of a and *26* be equal to *1* so that C runs through a complete cycle of letters when P does. There are then *12* choices for a, *26* choices for b, giving $12 \times 26 = 312$ transformations of this type, one being totally uninteresting (this is the identity transformation with $a = 1$, $b = 0$; hence $C = P$).]

Now let us take stock of what we have. If we have an encryption or ciphertext of a message given by an affine transformation (and I will leave one as an exercise at the end), we would like to decipher it. The letters of the message are of the form $C \equiv aP+b \pmod{26}$. Then the inverse relationship between plaintext and ciphertext is $P \equiv a^* (C-b) \pmod{26}$ where a^* is called the **inverse** and is the unique integer $\pmod{26}$ such that $aa^* \equiv 1 \pmod{26}$. A table of inverses *(mod 26)* follows. It will be used later.

a	1	3	5	7	9	11	15	17	19	21	23	25
a^*	1	9	21	15	3	19	7	23	11	5	17	25

Table 1.

In an attempt to break an encryption, that is to cryptanalyze, we first must know the frequency of letters in ordinary text and compare it with the frequency of letters in the ciphertext. This usually calls for a text with a large number of letters. In various counts of English text, the most frequently occurring letters are (in this order) *E,T,N,R,I,O,A*. Because we are seeking two parameters, a and b, we need only consider the two plaintext letters $E = 4$ and $T = 19$. In our encryption, let us call the most frequently used letter α and the second most frequently used letter β. Then from $C \equiv aP + b \pmod{26}$, we obtain that $\tau(\alpha) = \mathrm{E}$, $\tau(\beta) = \mathrm{T}$, where α and β are known from the count in the given message; thus

$$4a + b \equiv \alpha \pmod{26}$$

$$19a + b \equiv \beta \pmod{26}$$

By a known result, similar to Cramér's rule for the solution of linear equations, (see Rosen, Theorem 3.14, page 137), we solve for the coefficients a and b.
Let $\Delta = 4 \times 1 - 19 \times 1 = -15 \equiv 11 \pmod{26}$. Then, by the table shown previously, Δ^* (the inverse of Δ)=19, because $11 \times 19 \equiv 1 \pmod{26}$.

Then

$$a \equiv \Delta^*(\alpha - \beta)(mod 26), \quad b \equiv \Delta^*(4\beta - 19\alpha)(mod 26)$$

Finally, find a*, again, by the inverse procedure. Then

$$P \equiv a^*(C - b)(mod 26), \quad 0 \le P \le 25$$

We give now an example and an exercise. In both, numbers instead of letters are used. We transfer back to the letters in the solutions by realizing that A,B,C,...,Z are respectively 0,1,2,...,25.

EXAMPLE (adapted from Gottlieb, 1989):

15 19 20 7 20 10 24 1 6 24 15 8 17 20
25 7 8 23 20 3 0 20 15 19 8 3 8 11
4 24 24 10 24 15 10 17 2 15 19 20
4 3 23 4 17 12 20 0 20 17 15 8 11 25 6 7 20
-- 24 12 10 20 17 12 20 20 15 19 4 15
8 11 25 10 12 18 10 17 2 0 20 17 8 11
2 20 17 10 6 24, 21 4 12 18 10 17 2
15 19 20 0 19 20 4 23 10 9 22,
4 17 3 9 20 4 23 10 17 2 15 19 20 0
15 8 3 10 7 20 12 15
15 19 20 0 24 20 9 23 20 24.
1 4 0 20 24 21 7 22 4 17 15 12 8 17 4 17 15

We count each number and find out that the maximums are as follows: there are twenty-two of the 20s and fifteen of the 15s. Therefore,

$$\tau(20) = 4 \text{ and } \tau(15) = 19$$

where 4 represents *E* and 19 represents *T*.

Putting this information into the Cramér-type rule stated previously, we obtain:

$$4a + b \equiv 15 \ (mod\ 26)$$

$$19a + b \equiv 15 (mod\ 26)$$

where $\Delta \equiv 11 (mod\ 26)$, and $\Delta^* \equiv 19\ (mod\ 26)$ *by the table.*

Thus,

$$a \equiv 11^*(20-15)(mod\ 26) \equiv 17(mod\ 26)$$

$$b \equiv 11^*(4\times15 - 19\times20)(mod\ 26) \equiv 4(mod\ 26)$$

Hence, $P \equiv a^*(C - b) \equiv 23(C - 4)\ (mod\ 26)$. Now we can match each letter of the cipher-text with the plaintext letter.

For example, $\tau(20) \to 23(20 - 4) \equiv 4(mod\ 26)$, and $\tau(15) \to 23(15 - 4) \equiv 19(mod\ 26)$. However, we already knew that $20 \to 4 = E$ and $15 \to 19 = T$. We leave the rest to the reader using the formula $P \equiv 23(C - 4)(mod\ 26)$. The plaintext message reads as follows:

THERE IS JUST ONE PROVED METHOD OF ASSISTING THE ADVANCEMENT OF PURE SCIENCE -- THAT OF PICKING MEN OF GENIUS, BACKING THEM HEAVILY, AND LEAVING THEM TO DIRECT THEMSELVES.

JAMES BRYANT CONANT

EXERCISE: Decode the following affine transformation message (given in blocks of five for convenience).

9 13 24 2 3 9 13 24 19 21 7 19 7 24 2
3 25 15 3 3 0 17 20 7 21 19 3 0 21 3
13 0 12 0 15 9 4 3 16 24 2 3 17 16 5
19 7 24 2 3 25 15 3 3 0 17 20 9 13 24
2 3 9 13 24 19 21 7 14 14

Further Applications

Lastly, there are many other applications of number theory, some of which may be found in the references. In Rosen (1988), there are applications to round-robin tournaments, computer file storage, better cryptology

methods, and computer simulation. In Ore (1948) an application to the splicing of telephone cables is discussed. Cryptology is also examined in Vanden Eynden (1987) and Luciano (1987), an elementary article which won a 1988 George Pólya Award, gives a history of cryptology.

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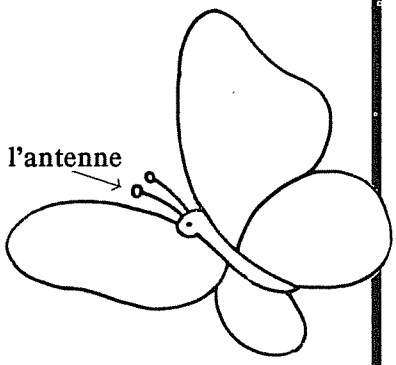
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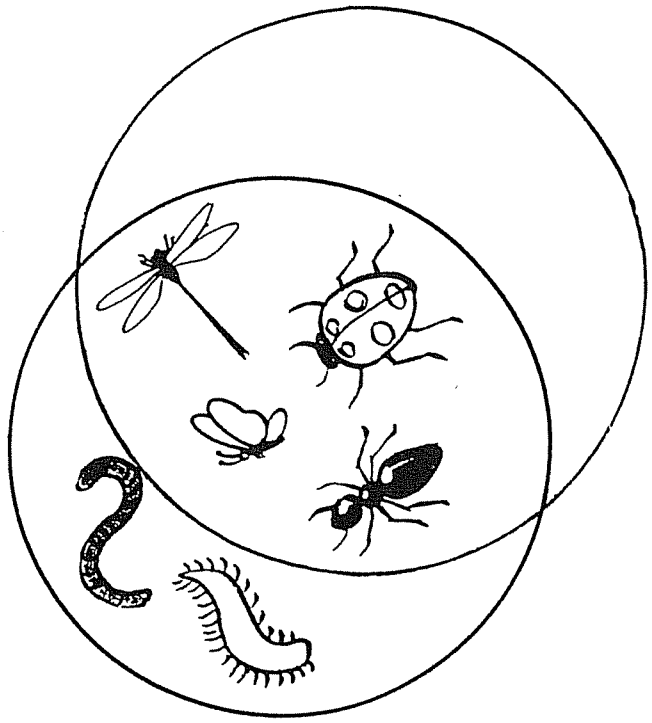
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LA MATHEMATIQUE AU PRIMAIRE: LA CLASSIFICATION

Pour le thème, "Animaux au jardin", on discute les petites bestioles qu'on trouve dehors. Dans le but d'enseigner le concept de la classification, on peut classifier les animaux jardiniers comme petits. Donc, on a une collection ici de petits animaux: des coccinelles, des sauterelles, des tortues, des fourmis et des scarabées.

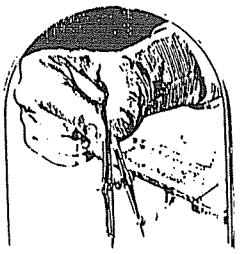
Dans un cercle sur une feuille de papier d'apprentissage, on a dessiné cette collection. Je crois qu'un enfant saisit le concept de la classification plus facilement par rapport aux groupes encerclés qu'avec un graphique. Maintennant, qui peut nommer ceux qui ont des antennes, des cornes, ou des tentacules? Voici le sous-sensemble illustré par un deuxième cercle. Ensuite, on peut imaginer un animal qui a une antenne mais qui n'existe dans la nature. Où donc faut-il le mettre?





Les animaux au jardin font un joli tableau pour une classification mathématique.

Submitted by
A. C. Parker
Agnes Taylor P.S.
Brampton, Ontario.



CIRCUMFERENCE: CAN YOU DO THIS FROM A SATELLITE?



by

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On a recent wilderness fishing trip a forest fire broke out 2km from our camp. Within thirty minutes a reconnaissance plane had surveyed the area and before long a water bomber was on the scene putting out the fire. How had the fire been detected so quickly? Who had sent in the alarm?

After doing a little research I discovered that one possibility was that a weather satellite may have photographed this region. Using a special lens the photos can pinpoint the areas where fires are burning and hence this information can be relayed to the proper authorities. In any event we were saved, and the story has a happy ending.

But let's return to space photography and how it can be used to promote interest and give a hands-on experience in the math classroom. The satellites travel over the north and south poles, 900km above the surface of the earth (recall that the radius of the earth is about 6400km).

To model this:

- use a styrofoam ball (5cm in diameter) to represent the earth
- take a toothpick and break it in half - take each half and stick them in the ball (sharp end first) to indicate the north and south poles - students will need to experiment with this to get the north and south poles directly lined up.



Figure 1. Model of the earth rotating on its axis.

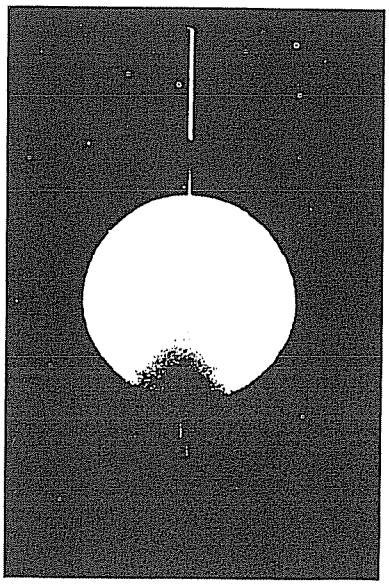


Figure 2. The orbit of the satellite.

It is difficult, visually to line them up through the ball - a good test is to place the model between your thumb and index finger and rotate it. (Figure 1.)

♦ If the earth wobbles, then one needs to adjust the position of one of the poles.

♦ Use a compass to punch several holes near the ends of a garbage tie (the toothpicks should just fit snugly into them) - insert the tie over each toothpick so that it represent the orbit of the satellite between the poles (Figure 2).

♦ Two ties are necessary if the entire orbit is to be shown.The following questions could be asked as the students assemble the model:

1. In which direction does the earth rotate?

2. What is the radius of the ball?

3. What is the ratio of the radius of the ball to the radius of the earth?

4. How far above the ball should the tie be placed in order to represent the orbit of the satellite?

5. What is the circumference of the earth?

6. What is the circumference of the orbit of the satellite?

The model is now completed by marking a point, A, on the ball (in the northern hemisphere) - then slowly rotating the ball and visualizing the position of A from the satellite

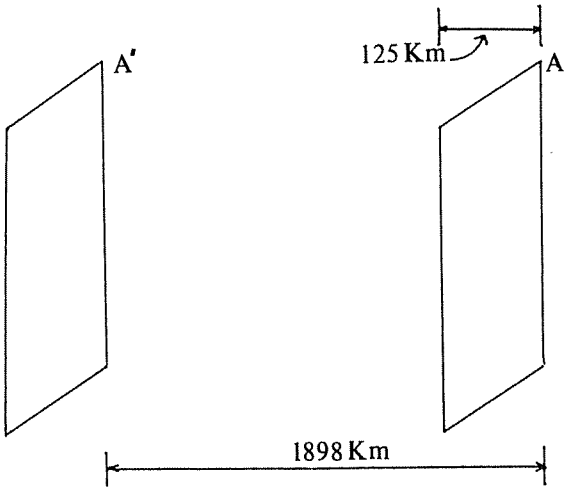
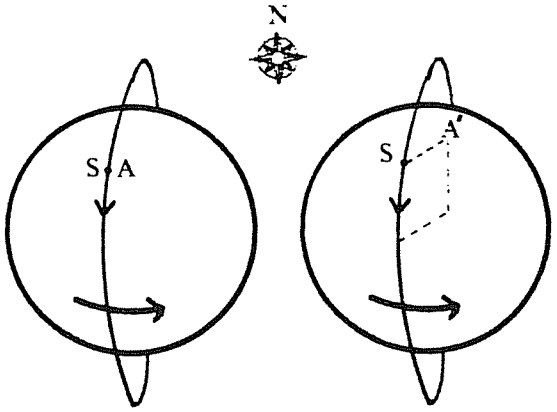


Figure 3. Dimensions of the photograph.

The photograph from the satellite covers an area the shape of a parallelogram of width 125km. By the time the satellite returns to the same position in its orbit the next photo will cover an area that is 1898km away from the first photograph (Figure 3).

The following questions will test the student understanding of the model.



S - position of the satellite

Figure 4. Positions when the picture is taken.

1. Why are the photographs drawn as parallelograms? (Figure 4)

2. What is the surface area of the earth?

3. What is the speed of the earth's rotation in km/h (at the equator) and in degrees/h?

4. How long does it take the satellite to make one orbit?

5. What must the satellite's speed be in order for the next picture to be taken 1898 km away?

6. How many orbits will be required to photograph the entire surface of the earth (at the equator)?

7. How long will #5 take?

8. How many square kilometres are photographed in one orbit of the earth?

9. How many square kilometres of the earth must be photographed in order to cover the entire globe? Calculate how much of this area is 'wasted'.

As a follow up, a number of different extension projects could be assigned;

a) investigate how the ancient Greek (Eratosthenes) determined the circumference of the earth.

b) investigate the uses of satellite photographs - land usage; surveying; extent of blight's, droughts;. . .

c) given a satellite camera which can take pictures that are 150km wide, find a distance between pictures that makes coverage of the entire earth possible and efficient. Explain your solution (answer should be in the range 2000-3000).

The importance of helping students visualize and internalize mathematical concepts is paramount. The use of manipulatives gives the students the opportunity to obtain a 'visual understanding' of the problem, allows for a hands-on experience, and provides more opportunities to improve problem-solving strategies/skills. We, as teachers, must make every effort to present the concepts in such a way that the underlying ideas are discovered from a variety of experiences. This outline gives but one such experience for students studying the topic of circumference.

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RESEARCH CLIPS

THE CONCEPT OF FUNCTION

The NCTM Standards emphasize the importance of functions, yet research indicates that students have difficulty with the concept in a variety of ways.

Secondary and college students view functions as *rules of correspondence* given by formulas which allow direct computation of values. Given a particular value for one variable, often denoted by x, students view functions as the rule that allows them to determine the value of another variable denoted by y or f(x). The focus is on function as a symbolic rule, not on function as a particular kind of mathematical relationship. Other representations such as tables, graphs, and arrow diagrams are not generally viewed by students as functions. Students do not interpret these other forms as alternative ways to represent the same mathematical relationship that a formula describes.

Piecewise functions, constant functions, and functions composed of discrete points present particular problems. Students are strongly "attracted to linearity." When asked to find examples of functions satisfying various constraints, they will provide linear examples almost always, even if it means overriding some of the constraints.

Students do not attend to the subconcepts of function, including domain and range. In one study, students agreed that two functions with identical formulas but different specified domains were the same function. The part of the function definition which requires a unique image for each element of the domain was not at all salient for these students.

Students are comfortable dealing with functions in a "pointwise" manner, but have difficulty when more "global" judgments are required. That is, this research indicates that students can find the value of f(x) for a given value of x, but they have difficulty interpreting the function as a whole scheme. This may be related to the difficulty that calculus students have in looking at secant lines "approaching" the tangent line to a curve at a point, or determining the behavior of a function over an interval.

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RESEARCH CLIPS

THE ZONE OF PROXIMAL DEVELOPMENT

Vygotsky (1978) defines a child's zone of proximal development as the difference between what he is able to do independently and what he can do through social interaction. The zone is a continuum of the degree of help needed for a child to solve a problem. At one end of the continuum, the child is able to solve the task alone, without hints. At the other end of the continuum, the child requires hints so specific that the child becomes an observer of the solution process. If a child's zone of proximal development for a task does not require specific hints, the child should have an opportunity to solve the problem without hints. Instruction should lead development, not follow it.

Rachlin (1988) cites the following example from an interview with Pam. As a first introduction to the concept of factoring, she was asked to find two numbers that are relatively prime. She quickly responded 45 and 2, since their greatest common factor is 1. To see how Pam would generalize the concept of *relatively prime* to monomials, this problem was followed by a request for her to find two monomials that are relatively prime. "How about $3x^3y^4$ and $7c^4d$?", she replied. When asked how she knew that they would be relatively prime she responded that she guessed that in order to be relatively prime the coefficients would have to be relatively prime and the variables would have to be different. It was not necessary to tell Pam how to determine if two monomials were relatively prime, instead a teacher could draw the appropriate meaning from students like Pam.

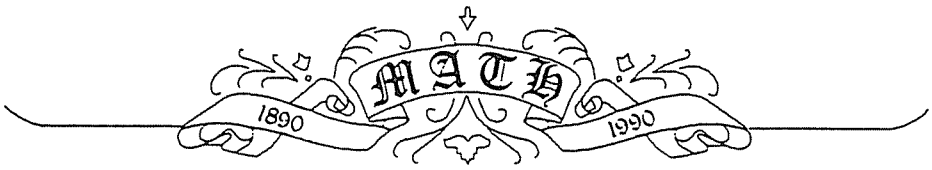
Rachlin found that many high ability students are able to solve problems several days in advance of a teacher. Perhaps this is why we tend to call on them so much in class. It is not that they are quick to learn, but with different questioning they could have intuitively provided their own instruction. In order to develop the higher psychological functions, students need to learn the intellectual skills associated with reflective control and deliberate awareness.

References:

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes.* Cambridge: Harvard University Press.

Rachlin, S. L. (1988). "On the Difficulties of Using the Zone of Proximal Development as a Theoretic Model for the Design of an Algebra Curriculum." Presented at the International Congress on Mathematical Education, Budapest, Hungary.

Clip prepared by the Research Advisory Committee of National Council of Teachers of Mathematics.



ARE YOU AN "OLD" MATHEMATICS TEACHER?
(OR A DESCENDENT OF ONE)

Joan Routledge (416-727-4167) and Jack LeSage (705-728-9958) have been asked by the Ontario Association for Mathematics Education to produce a History of Mathematics Associations in Ontario to celebrate 100 years.

If you have any information, convention programmes, pictures, minutes, vignettes, etc involving any past Mathematics organizations in Ontario (math sections of OEA, OATMP, OATM, OMC, OCI, etc) please write:
Jack LeSage
9 Mountbatten Rd., Barrie Ont. L4M 1T4



COVER DESIGN CONTEST

Readers and/or their students are encouraged to submit their creations.



Designs must be submitted on 8 x 10 paper; the use of colour is restricted to black, white plus one other.



Illustrations drawn with the help of technology e.g. fractals, are welcome.

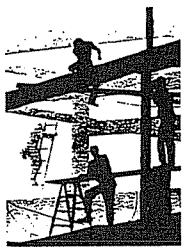
A small prize will be awarded to the submision selected.

PROBABILITY AND THE CONSTRUCTION BUSINESS



by

L. Keith Roy
Department of Mathematical Sciences
Lakehead University
Thunder Bay, Ontario



An illustration such as the following problem could be used as a summary when teaching the section on Probability in the Ontario Academic Course in Finite Mathematics. Such concepts as Venn diagrams, independence, conditional probability, the multiplication rule, complementary events, and tree diagrams are all involved.

The Reliable Construction Company has decided to submit a sealed bid for the construction of a new municipal building. Reliable is quite familiar with all the other local construction companies and feels that, if only the local companies submit competing bids, Reliable will have a 90% chance of its bid being successful. Unfortunately, two out-of-town companies, Ace Builders and Beneficial Construction, have each shown some interest in the project and might independently decide to submit bids of their own. Reliable has been able to purchase some inside information from sources at both Ace and Beneficial and, although the exact details of the intentions of the two companies still remain secret, Reliable has enough information to come to the following conclusions.

(1) Ace Builders seems only moderately interested in this project because of involvement in other more profitable projects. Consequently, the executives at Reliable feel that the probability of Ace submitting a competing bid is only 0.40, and even if Ace does compete (and Beneficial does not), the conditional probability of Reliable's bid being successful would be 0.70 since Ace has tended in the past to include a larger profit margin in its bids than has Reliable.

(2) There seems to be a 60% chance that Beneficial Construction will submit a competing bid. Also there would be some additional cost to transport machinery and workers from Beneficial's home base which is some distance away. This additional cost would have to be taken into account if Beneficial were to bid on the project. Consequently the executives at Reliable estimate that even if Beneficial does submit a bid (and Ace does not), the conditional probability of Reliable's bid being successful would be 0.80.

(3) Even if both Ace and Beneficial independently decide to submit bids, the executives at Reliable estimate that the conditional probability of Reliable's bid being successful would be 0.50.

The executives at Reliable would like to calculate the "overall" probability of Reliable's bid being successful.

The analysis of situations such as the above calls for a tree diagram. Each set of branches in the tree begins with one of the four possible situations involving out-of-town bidders:

(a) neither Ace nor Beneficial choose to bid
(b) Ace submits a bid but Beneficial does not
(c) Beneficial submits a bid but Ace does not
(d) both Ace and Beneficial submit bids

and ends with the conditional event that Reliable's bid is successful, given whichever of (a), (b), (c), or (d) is involved on the initial branch. The conditional probabilities are given above.

The tree diagram with some of the appropriate probabilities appears (in Figure 1.) below.

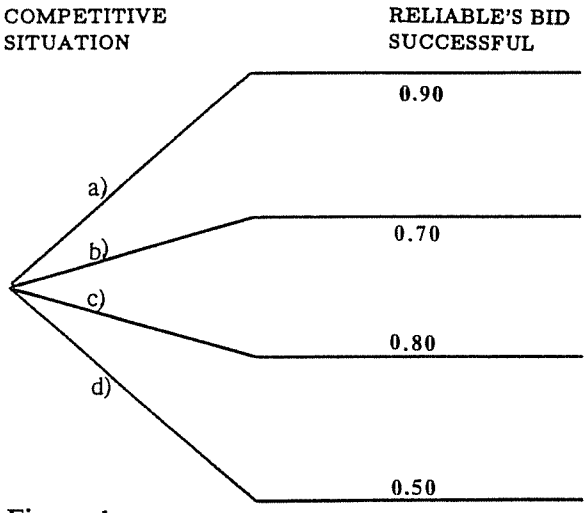


Figure 1.

The probabilities which should be assigned to each of the initial branches are not immediately apparent. Consider the following Venn diagram.

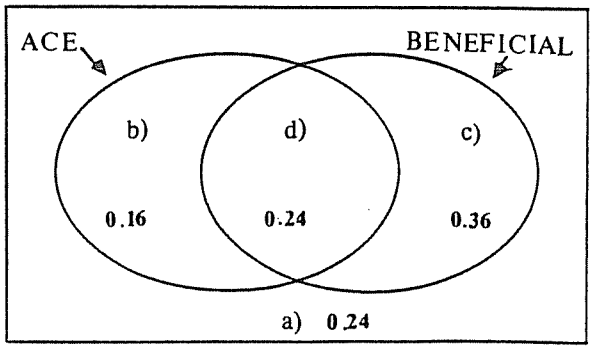


Figure 2.

The ellipse on the left represents the event that Ace Builders will submit a bid, an event which has probability 0.40, while the ellipse on the right represents the event that Beneficial Construction will submit a bid, with probability 0.60. The region where the ellipses intersect represents the event that both Ace and Beneficial will submit bids, that is (d), and since Ace and Beneficial are operating independently, the probability of (d) is (0.40)(0.60) = 0.24 using the multiplication rule for independent events. The probability of (b), that Ace will bid and Beneficial will not, is 0.40 - 0.24 = 0.16, and similarly the probability of (c) is 0.60 - 0.24 = 0.36. The probability of (a), that neither Ace nor Beneficial will bid, is 1-(0.24 + 0.16 + 0.36) = O.24 since the total probability of the four mutually exclusive regions in the Venn diagram is one. The tree diagram can now be completed.

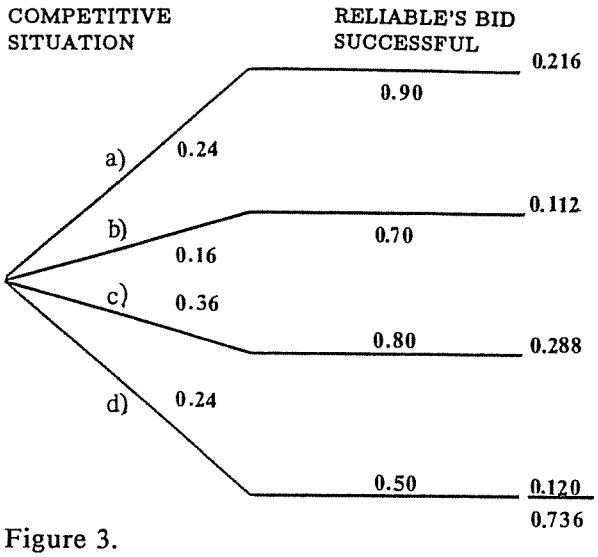


Figure 3.

The number at the right of a set of two branches is the probability that both of the events represented by the two branches occur, which, by the multiplication rule, is the product of the probability of the initial branch with the conditional probability attached to the second branch.

Consequently, the probability of both (a) and Reliable's bid being successful is 0.216, the probability of both (b) and Reliable's bid being successful is 0.112, the probability of both (c) and Reliable's bid being successful is 0.288, and the probability of both (d) and Reliable's bid being successful is 0.120. Since the above four probabilities result from mutually exclusive sequences of events culminating in Reliable's bid being successful, the "overall" probability of Reliable's bid being successful is 0.216 + 0.112 + 0.288 + 0.120 = 0.736 by the third postulate of probability.



PROBLEMS

by
Ron Lancaster
Applewood Heights S.S.
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1 Using a calculator, find the value of

$$\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ$$

2 Using the identity

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

develop a formula for tan 2θ in terms of θ. Do a similar thing for tan 3θ, tan 4θ, tan 5θ, etc. and write up a report on your findings.

3 Because the earth is not a perfect sphere, the value of g, the acceleration due to gravity, is not the same throughout the world. For a given location the value of g depends on the latitude θ and the altitude a (in metres). In 1930 The International Union of Geodesy and Geophysics adopted the following formula for computing g (in cm/s^2):

$$g = 978.049(1 + .0052884\sin^2\theta - .0000059\sin^2 2\theta) - .0003086a$$

Find the value of g for Oakville ($\theta = 43^\circ 27'$ and $a = 74$m)
NOTE:

$$1' = \frac{1}{60}$$

i.e. one minute is equal to one sixtieth of one degree.

SOLUTIONS FOR PROBLEMS

1. I gave this question to my grade 12 mathematics class last year and my students were quite surprised to find that the answer came out to be 0. When I took up the test I gave my students a chance to do some investigative mathematics along with some problem solving by asking them the following questions.

1) Is there anything special about the angles used in the questions?

2) Can you find another similar set of angles for which the answer again comes out to be 0?

3) Can you find a similar result that involves say 7 angles instead of 5?

My students quickly noticed that the angles were consecutive terms of an arithmetic progression ($a = 5°$ and $d = 72°$). This was great because it allowed me to review a topic from the Grade 11 course in a natural way. We tried other values of a, keeping $d = 72°$, and we found the answer always came out to be 0. I then suggested that they try to prove this result. We started out by letting the angles be x, $x + 72°$, $x + 144°$, $x + 216°$, and $x + 288°$. I was very pleased to see that quite a few of my students graphed the curve

$$y = \cos x + \cos(x + 72°) + \cos(x + 144°) + \cos(x + 216°) + \cos(x + 288°)$$

Most students either used a computer or a graphing calculator to obtain the graph. Several students however obtained the graph by adding the five cosine curves! I took a group of students aside and showed them the identity for cos $(x+y)$ and suggested they use this to prove the result. They had some trouble with this and after some discussion we went back to the very beginning and let the five angles be $x - 144°$, $x - 72°$, x, $x + 72°$, $x + 144°$. (Dr. Davison from McMaster would have liked this). I ended up showing them the double angle formula and I eventually had to tell them that

$$\cos 72° = \frac{\sqrt{5}-1}{4}$$

Afterward I showed them how to derive this result.

I urge you to use this problem with your Grade 12 class and your calculus class this year. In fact, you can even use it with your algebra class because the result can be proved using vectors (Walter Howard from Jarvis Collegiate in Toronto would approve of this!). Look closely this year at how you are teaching mathematics and determine if you are in line with the 1985 guidelines and the N.C.T.M. standards.

2. I gave this question to my calculus classes last year and many of my students were surprised and intrigued by the patterns of numbers that emerged in the answers. The results they obtained are given below.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

$$\tan 4\theta = \frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$

$$\tan 5\theta = \frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}$$

$$\tan 6\theta = \frac{6\tan\theta-20\tan^3\theta+6\tan^5\theta}{1-15\tan^2\theta+15\tan^4\theta-\tan^6\theta}$$

If you look at the coefficients you should be able to see Pascal's Triangle! At first I was amazed that Pascal's Triangle came up here, but it all became obvious when I thought about De Moirve's Theorem. I'll leave the details for you. Once again, here is a question that has a great deal of potential things to explore and it allows you to blend together trigonometry with the binomial theorem in a surprising setting. The results can also be verified by graphing!

3. If your students are using trigonometry only to solve the traditional triangle problems then you are clearly "out of phase". This is an excellent question that requires students to read something over and then proceed. You could even give this problem as an assignment. If you do this, let your students determine θ and a for your town or city.

I have been using this question for over 8 years and one thing that I have been asking my students to do lately is to graph
$y = 1 + .0052884 \sin^2\theta - .0000059 \sin^2 2\theta$
for $0° \leq \theta \leq 90°$. Get out your graphing calculator and look closely at the graph. It is quiet interesting! If you don't have a graphing calculator then your phase shift is much too large!

There are a number of other questions that you can ask. For example, give your students specific values for g and a and ask them to find the corresponding value of θ. You could even ask them to find a place on earth that corresponds to these values of θ and a. This will give your students an opportunity to do some research and learn some Geography too!

Teaching Tips

ACTIVE LEARNING

Throughout the school year 1989-90 there has been an on-going discussion about *active learning*. Many educators do not seem to appreciate that consideration of active learning in mathematics was taken seriously with the publication of *Freedom to Learn* by Addison Wesley, Canada in 1969. This remains a classic in Ontario mathematics education.

Here is a brief description of learning mathematics that I prepared for my students. Now I can use it with the new active learning specialists.

LEARNING MATHEMATICS

The learning of mathematics begins in early childhood and should continue into adulthood. Mathematics educators must create vital and enriching mathematics experiences and environments for all students.

Learning mathematics means the internalizing of concepts, skills and attitudes. Mathematics increases the understanding of self, others and the environment. Every student requires the internalization of mathematics achieved by the active processes of experience, interaction, reflection and communication.

Mathematics learners actively construct the tools and knowledge to move along the continuum from concrete to abstract reasoning. However, all mathematics students will benefit from doing mathematics rather than observing mathematics.

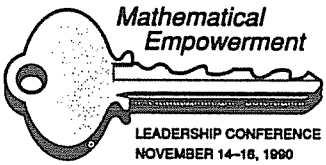
Submitted by:
Dr. Alex Norrie
Co-ordinator of Mathematics
Peel Board of Education

Ontario Association for Mathematics Education

Association Ontarienne pour L'Enseignement des Mathematiques

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- two nights in the Ramada (double room)
- two breakfasts and two lunches
- one banquet and two receptions

WEDNESDAY EVENING

Registration and Reception

THURSDAY

9:00-10:00 Keynote Address:

Geraldine Kenney-Wallace (President McMaster University and Chair of the Science Council of Canada)

10:00-10:15 Coffee

10:15-11:45 Talks and Interaction:

- Archie Chung (Ontario Hydro and Past-President of Operation Researchers of Ontario)
- John Clark (Benchmarks Project Toronto Board of Education)
- Michael Fullan (Dean of the Faculty of Education University of Toronto, Change in Education)
- Nancy Moore (Consultant, Toronto Board of Education, Family Math)

12:00-2:00 Lunch and Publishers Displays

2:00-3:30 Ministry Presentation and Interaction with Laury Roy, Curriculum Policy Development Branch:

"The Restructuring of Education: The Early Years; The Formative Years; The Transition Years"

3:30 • 6:00 Coffee, Publishers Displays, and Free Time

THURSDAY EVENING

6:00-6:30 Cash Bar

6:30-8:30 Banquet and Talk:

Mary Storey "Futures and Women's Issues"

9:00-11:30 Publishers' Reception

FRIDAY

9:00-10:15 Breakout Sessions:

Primary • Doug Herridge (York Region)

Junior • Joan McDuff (Quinte)

Intermediate • Marg Warren (Peel)

Senior • Jack Weiner (University of Guelph)

10:15-10:30 Coffee

10:30-11:45 Breakout Sessions:

Primary • Betty Coombs (Hamilton)

Junior • Helen Baker (Peel)

Intermediate • Max Hicks (Ottawa)

Senior • Rick Sacks (Niagara)

12:00-1:30 Wrap-up Luncheon and Presentation:

Tim Crawford "Classrooms in the Future"

Ontario Association for Mathematics Education

Association Ontarienne pour L'Enseignement des Mathematiques

Secretary–Treasurer
Don Attridge
1 Southdale Drive
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Mathematical Empowerment

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BOARD OF EDUCATION

GRADE LEVELS TAUGHT

CONSTITUTION of
ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
ASSOCIATION ONTARIENNE POUR L'ENSEIGNMENT DES MATHEMATIQUES

MAY 11, 1990

DEFINITIONS:

1. OAME shall mean "Ontario Association for Mathematics Education".
   AOEM shall mean "Association Ontarienne pour l'Enseignment des Mathématiques".
2. A Chapter shall mean a subdivision within the Province of Ontario. A group of OAME/AOEM members, who organize and act in harmony with the aims and objectives of the OAME/AOEM, may be recognized annually as a chapter by a simple majority vote of the members present at the annual general meeting of the organization.

ARTICLE I - NAME

1.1 The name of the organization shall be: "Ontario Association for Mathematics Education", "Association Ontarienne pour l'Enseignment des mathématiques".

ARTICLE II - MISSION STATEMENT

2.1 OAME/AOEM embodies the Ministry of Education's Image of the Learner[1], and endeavours to promote excellence in mathematics education throughout the Province of Ontario. The term "excellence", is this context, refers to the contribution that mathematics educators can, and should make, to the development of those attitudes, skills and concepts related to mathematics.

ARTICLE III - OBJECTIVES

OAME/AOEM is committed to the following objectives:

3.1 To aid the professional growth of mathematics educators by proposing, initiating and co-operating in pre-service and in-service programs designed to improve mathematical understanding and methods of teaching.

3.2 To encourage and to engage in research and development relating to mathematics education and curriculum design, and to make recommendations to appropriate bodies.

3.3 To initiate and to encourage the planned development and evaluation of curriculum teaching materials.

3.4 To provide co-ordination in mathematics education at all levels as well as effective communication among all mathematics educators in Ontario, by promoting the exchange of ideas through publications, conferences and committee involvement.

3.5 To provide liaison with other educational organizations in Ontario and with mathematics education organizations in other provinces and countries.

3.6 To maintain affiliation with the "National Council of Teachers of Mathematics" (NCTM), and to promote projects of mutual concern.

3.7 To maintain an active and representative mathematics educator's voice with
(i) the Ontario Ministry of Education, and
(ii) the various educational bodies in Ontario.

---

[1] The Ministry views the learner as an active participant in education who gains satisfaction from the dynamics of learning. The concept of the learner as a mere processor of information has been replaced by the image of a self-motivated, self-directed problem-solver, aware of both the processes and uses of learning and deriving a sense of self-worth and confidence from a variety of accomplishments.

The Image of the Learner is found on pages 2-4 Issues and Directions - The Response to the Report of the Commission on Declining School Enrolments in Ontario. Ministry of Education June, 1980.

3.8 To encourage the development of local and regional mathematics organizations in Ontario, to grant such organizations chapter affiliate status, and to maintain a mutually supportive relationship with them.

ARTICLE IV - MEMBERSHIP AND MEMBERSHIP RIGHTS AND FEES

4.1 Membership shall be open to any individual or institution interested in mathematics education.

4.2 Membership carries the right to one vote, to serve on any committee, and to receive regular publications.

4.3 Membership fees shall be set annually by council.

ARTICLE V - ORGANIZATION

5.1 **Executive Committee**

The Executive Committee of OAME/AOEM shall consist of:
President
Past-President
President-Elect
Three (3) Directors
Secretary-Treasurer(s) (ex officio)

5.2 **Council**

The Council of OAME/AOEM shall be composed of forty (40) voting members of the organization as follows:

(i) all voting members of the Executive Committee,
(ii) one representative from each chapter of the organization, and
(iii) individual members, elected annually, the number to be such as to bring the total to forty (40).

5.3 **Chapters**

A chapter shall mean a subdivision within the Province of Ontario. A group of OAME/AOEM members, who organize and act in harmony with the aims and objectives of the OAME/AOEM, may be recognized annually as a chapter by a simple majority vote of the members present at the annual general meeting of the organization.

5.4 **Institutional and/or Liaison Representatives**

In addition to the forty voting councillors, there may be a representative invited by the executive committee to attend Council meetings from the following organizations:

(i) Ontario Ministry of Education - Mathematics (MOE)
(ii) Ontario Mathematics Co-ordinators/Consultants Association (OMCA)
(iii) any other body with mathematics responsibility as deemed appropriate by the executive committee.

5.5 **Position of Secretary-Treasurer**

(i) There shall be an annual performance evaluation of the position of Secretary-Treasurer.
(ii) The appointment of Secretary-Treasurer shall be ratified annually by Council at a regular Council meeting on recommendation of the performance evaluation committee.
(iii) The annual honorarium for Secretary-Treasurer shall be reviewed and ratified annually by Council.
(iv) When necessary, on recommendation by executive, a search committee shall be appointed by Council to seek out candidates for the position of Secretary-Treasurer. This committee will bring forth a recommendation to Executive for approval by Council.

5.6 **Committees**

(i) Each year the Executive Committee, with the approval of Council, shall establish such committees as are appropriate to the effective running of the OAME/AOEM organization.

(ii) Committees shall have the right to recommend to the executive the appointment of sub-committees in their area of responsibility. These sub-committees may include persons who are not members of the organization. All such sub-committees are responsible to the Executive Committee and Council.

(iii) A member of the Executive Committee shall act as liaison on each committee.

(iv) At the first committee meeting each committee shall confirm the designated Chair.

ARTICLE VI - MEETINGS

6.1 An Annual General Meeting of the organization shall be held.

6.2 Special general meetings may be called at any time deemed necessary by the Executive Committee.

6.3 The Council shall meet three (3) times per year. Special meetings of the Council may be called by the President or on request of a majority of Council members.

6.4 The Executive Committee shall meet four (4) times per year or at the call of the President.

6.5 Committees of Council and/or Executive shall meet as their function requires and report progress.

ARTICLE VII - NOMINATIONS AND ELECTIONS

7.1 Elections shall be held by mail for the positions of :
President-Elect
Directors
Elected Council Members

7.2 The Nominating Committee shall receive nominations, including a summary of qualifications of each nominee, for all elected positions up to January 15, on which date nominations shall be closed, with the exception that two additional names may be added to the ballot by the voting members at the time of voting.

7.3 A summary of the qualifications of each nominee shall be published and distributed by mail to the membership along with a ballot no later than March 15.

7.4 The Nominating Committee shall endeavour to secure sufficient nominations so as to ensure an election for each elected position.

7.5 The Past-President, and two (2) members selected by the Past-President by February 15, shall act as clerks. The Past-President shall be Chief Clerk.

7.6 Every candidate has the right to a scrutineer when the ballots are counted.

7.7 (i) In the event of a resignation or death of any member of the Executive Committee during a term in office, the Council shall elect one of its members to the vacant office.

(ii) In the event of a resignation or death of any non-Executive Committee member of the Council during a term of office, the Executive Committee shall be responsible for filling the vacancy, subject to ratification by Council.

7.8 (i) Elected non-Executive Committee members of Council are elected for a three-year term, approximately one-third of the number to retire each year.

(ii) Other offices and appointments shall be effective from the time of the election of appointment, until the end of the following Annual General Meeting.

ARTICLE VIII - BUDGET

8.1 The Secretary-Treasurer, at the beginning of each budget year, will establish two major funds:

(i) a Working Fund used for the active daily business of the organization.

(ii) a Special Projects Fund used for projects approved by Council and/or the Executive Committee.

ARTICLE IX - BALANCE OF FUNDS ON DISSOLUTION

9.1 The balance of funds, on dissolution of the organization, is to be donated to one or more educational organizations designated by the Council.

ARTICLE X - AMENDMENTS TO THE CONSTITUTION

10.1 Proposed amendments to the constitution must be received by the membership in writing at least two weeks prior to the Annual General Meeting in which the amendments are to be considered.

10.2 In order to carry a motion to amend the constitution, it must receive a favourable vote of not less than two-thirds of the members present at this meeting.

ARTICLE XI - PUBLICATION OF THE CONSTITUTION

11.1 This constitution, or any amendments to this constitution, when approved at a general meeting of the organization, shall be published in the September issue of the Ontario Mathematics Gazette.

O.A.M.E./A.O.E.M. MEETINGS

DATES	EVENT
September 21	Chapter Reps & Executive
September 22	Council
November 14-16	Leadership Conference
November 16-17	Executive
January 10-11	Leadership Training
January 11	Committee Meetings
January 12	Council
February 1-2	Executive
March 1	Committee Meetings
March 2	Council
April 12-13	Executive
May 9-11	Annual Conference



ONTARIO ASSOCIATION for MATHEMATICS EDUCATION
ASSOCIATION ONTARIENNE pour L'ENSEIGNEMENT DES MATHEMATIQUES

Join the over 1900 mathematics educators, kindergarten through college and university, who enhance mathematics education in Ontario. Your membership:

INCLUDES OAME/AOEM Publication - Gazette and Abacus
SUPPORTS professional Development through the Annual Provincial Conference, the Leadership Seminar, and Regional Mini-Conferences
PROVIDES opportunity for local input with Chapter Affiliation
PROMOTES Mathematics through Excellence in Mathematics Certificates, I Love Math Buttons, and OAME/AOEM Awards,
ENCOURAGES networking and information sharing
STIMULATES Problem-Solving at all grade levels

O.A.M.E./A.O.E.M. MEMBERSHIP APPLICATION FORM

NAME_____
(First name(s)) **(Surname)**

HOME ADDRESS _____
(street)

_____
(town/city) **(province)(postal code)** **(telephone)**

EDUCATIONAL INSTITUTION _____

ADDRESS _____
(board)

_____
(postal code) **(telephone)**

Elementary _____ **Secondary** _____ **CAAT** _____ **University** _____ **Other** _____

Chapter Affiliation (Check one):

CHAMP ○ COMA ○ GOLDEN ○ GVMA ○ NOMA ○ NWOAME ○
PINE RIDGE ○ QSLMA ○ SAME ○ SWOAME ○ WOMA ○ Y^4MA ○

O.A.M.E./A.O.E.M. Membership: New __ Renewal __

Membership Fees (effective May 1,1990 to April 30, 1991)
__ one year $30 ($35 outside Canada)
__ five years $135 ($165 outside Canada)
__ retired and student members $20

Send form and cheque, payable to O.A.M.E., to:
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1 Southdale Drive,
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