Vol. 34 #3, April 1996

Ontario Mathematics Gazette

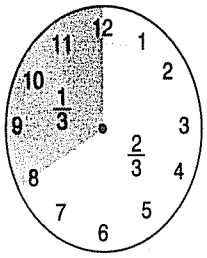
- Supplementing welfare income Ontario style: a mathematical investigation

And...

- A novel approach to limits at 0 and ∞

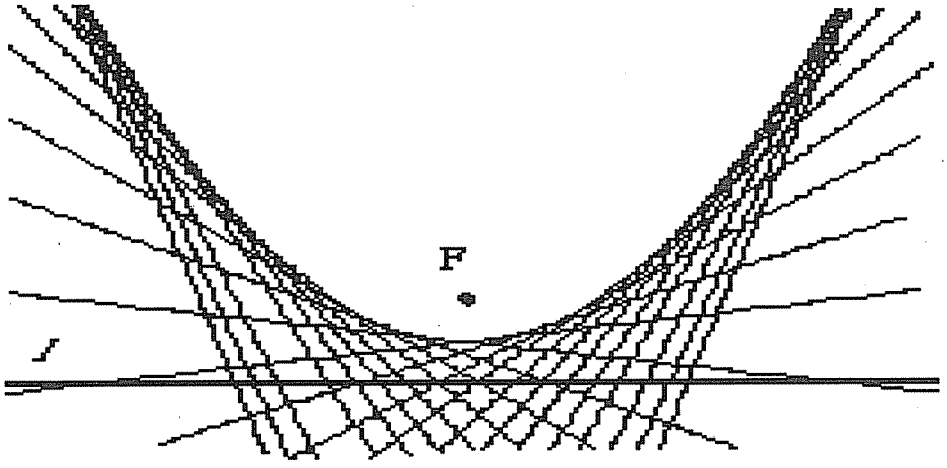
And.....

- A timely approach to fractions



And.....

- *The Geometer's Sketchpad.*



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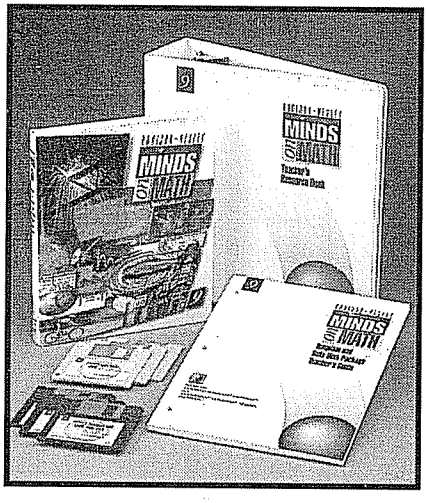
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▲ Editorial



I just had an interesting conversation with a teacher from the Oshawa area. She was concerned about the core content in her OAC Algebra and Geometry Course. The problem? A few (by no means all) of her former students reported that their university professors in first year algebra courses routinely assumed familiarity with topics not in the Algebra and Geometry curriculum. For example, one professor expected students to have an elementary knowledge of eigenvectors and eigenvalues. When the students, characteristically (sorry, couldn't resist the pun), protested that these topics were unfamiliar, the response was, in effect, "Too bad. Here is a text reference. Go learn the material on your own."

The teacher's question? Should she teach non-curriculum topics in place of, for example, induction, which these same students said was taught as if totally new?

My answer: an emphatic no! There are over thirty universities and colleges around the province. Each designs its own courses. The developers of these courses all have access to the curriculum guidelines published by the Ministry of Education. It is their job to construct courses whose content builds upon the foundation of the guidelines. You serve your students best by covering the Ministry curriculum.

Unfortunately, the reality is that there are post-secondary instructors who will, ignorantly and/or arrogantly, pay no attention attention to the content of the OAC prerequisites for their courses. Some blithely assume that their students in first year know what the instructors knew (or think they knew—memory is fallible). When some students report as did those of the teacher who inspired this editorial, you will again serve them well by first checking whether the claim is true. If it is, write a stern letter to that instructor, cc'd to the chair of the department and the president of the university or college, about the lack of teaching professionalism inherent in ignoring the curriculum of the prerequisites.

Redesign of first year courses to reflect the content of the high school courses should be ongoing, just as the high school and elementary school curricula must be regularly updated with the times. With the death of the OAC's probable (imminent?), the lines of communication particularly between university, college and secondary school teachers need massive upgrading. All groups should be consulting one another when constructing courses that depend on each other. And all teachers at any level should be cognizant of the content of the prerequisites.

Jack

PRESIDENT'S MESSAGE



SUSAN STUART

The phones, faxes and e-mail have been put into high gear these past few months with all the happenings, rumours and announcements in the province. The everyday pressures of the classroom and the worries about preparation time reductions, staff cuts and salary roll-backs made a March Break a welcome time. I hope that yours was restful and enjoyable.

The "education toolbox" turned out to be empty, at least in the opinion of most of the school board officials. That leaves us, the teachers, still up in the air about the future. Will we have time to plan for changes in the math curriculum, to learn about authentic assessment in math, or get our students prepared for province-wide assessments? Only time will tell.

The province-wide assessment plans are well under way and many of our members were actively involved in the bidding for the contract to develop an assessment unit for grade 3 (reading, writing, math). The Education Quality and Accountability Office has awarded the contract to a consortium from the Durham, Halton, Lakehead, the City of North York and Scarborough Boards of Education. OAME is committed to offering assistance to these writers and to the EQAO Office when review of the assessment material is needed. As well, many OAME members attended grades 3 & 6 Assessment Consultation meetings held across the province. Thank you to those who represented this organization. The feedback indicates that similar discussions took place at all the meetings. Some of the concerns and recommendations focused on adequate in-service for administering and marking of the assessment, appropriate communications with the public about the aims of the assessment, appropriate assessment of French Immersion students, ensuring that the grade 3 teacher does not feel the full responsibility for the results of the assessment, and using a common method of reporting the results to the parents.

Provincial math assessment will also take place in randomly chosen grade 6 classrooms in the spring of 1997. The assessment tool for this will be developed through the E.Q.A.O. If there are any members who would be interested in working on this, contact me or a chapter representative. If OAME is asked for names of teachers to develop the grade 6 material, we should be ready.

The Ministry continues to focus on changes to Specialization Years and has established The Secondary Reform Advisory Group, chaired by Bob Kennedy from North Bay. OAME sent the Ministry a list of secondary school exit outcomes for a sample of mathematics strands, along with a brief outline of the way course-specific outcomes should be decided upon. A copy of this will be printed in the June edition of the Gazette. Your comments would be greatly appreciated. The response from those at the Ministry who have been given the job of dealing with the reforms has been positive. They praised OAME for taking the initiative, leading the way rather than waiting for the Ministry to tell educators how things will be done. Thanks in particular to Peter Saarimaki and Marg Warren for wrestling with the vision formulated at the O.M.C.A. retreat.

I would like to take this opportunity to extend the warmest thank you and well-wishes to Jack Weiner, editor of the *Gazette*, and Jennifer Weiner, editor of the *Abacus*, for their dedicated work. As you may know, they are both leaving these positions, although they have agreed to "break in" the new editors in the spring. The *Gazette/Abacus* is known to be a mathematics journal of the highest quality. It is seen as an exemplar for other subject journals across North America. It is truly because of the editors. Thanks, Jack and Jennifer, for the years of outstanding work on behalf of math education. And our new editors are .... (See the June issue.)

The May conference is close at hand. If you have not attended before, you should - it is the best subject conference in the province! If you have attended, I know you are looking forward to another great time. W.O.M.A. certainly has interesting events planned for us.

During this past year I have worked with an outstanding executive and a hard-working, dedicated Council. It has been a privilege to represent an association which is a positive influence and an educational force in this province.

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▲ Have Your Say Chess - The World's Greatest Problem Solving Activity*



RON WALLACE
HILL STREET SENIOR PUBLIC SCHOOL
CORUNNA, ONTARIO

Despite its illustrious history, chess, perhaps like soccer, is still seen as a poorly understood import to Canada, but it is gaining recognition in the classrooms of Ontario, especially in the classrooms of Hill Street School.

One does not have to be a chess master to teach this game. Like any good teacher or coach the most important skill needed in getting young people involved is enthusiasm on the part of the instructor. Children love to play games and in game playing the learning is often so subtle that we as parents and teachers often do not see, initially, what has transpired.

I have been involved in teaching and coaching this game for 32 years, having taught over 600 students, and have worked with over 1300 chess enthusiasts. During this period of time I have made some interesting observations.

What can chess teach our students?

- concentration
- long range planning
- predicting outcomes
- drawing conclusions
- memory skills
- common sense
- quiet activities can be fun
- importance of controlling numerous variables
- analyzing situations
- spirit of true sportsmanship
- appreciation of the talents of others
- computer skills (Chessmaster 2100)
- lasting and long range friendships (by mail, by computer)
- significance of body language in communication
- point of view
- value of changing one's point of view to find solutions

Reprinted from the WOMA Sine Post

These are indeed skills that permeate all areas of the curriculum.

What conclusions do I draw from my observations?

- chess can bring families together
- chess should be an integral part of the school curriculum
- chess can and should be taught to every student in our schools beginning at grade three
- chess can change a physically aggressive student's behaviour
- chess can elevate self-esteem
- chess playing puts people of different cultural backgrounds on equal footing

Research has shown that chess playing can improve reading scores as well as math scores.

No computer chess game can match the power of the human mind. World chess champ Gary Kasparov said, "Deep Thought (the name of the computer game) lacks tactical vision and above all, common sense. You need fantasy and intuition."

Why not make an effort to bring this game into your classroom? Who possesses a greater realm of fantasy than an eight year old child? Let's capitalize on it. Check and Checkmate!

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▲ Envelopes of the Conic Sections and their Reflector Properties using *The Geometer's Sketchpad*

JOHN DEL GRANDE AND DAVID RASMUSSEN



John Del Grande has been a major contributor to mathematics education in Ontario and beyond for over fifty years. He is the author of several textbooks and has published articles in many journals, including the Ontario Mathematics Gazette *and* The Mathematics Teacher. *In 1992 he was elected a Life Member of OAME.*



David Rasmussen has taught mathematics in northern Cape Breton Island for twenty three years. He is a graduate of St. John's College, Annapolis, Md. and the Harvard Graduate School of Education.

A well known paper folding exercise yields fold lines that envelop a parabola, ellipse, or a hyperbola (Sobel, 1975, Del Grande, 1992). The object of this paper is to draw the conics by simulating the paper folding on a computer and then investigate some of the properties of the conics. The steps in the paper folding that yield conic envelopes follow.

For a **parabola:** a point and a line are drawn on paper. The paper is repeatedly folded so that the point maps onto points on the line. The crease lines envelop a parabola, Figure 1.

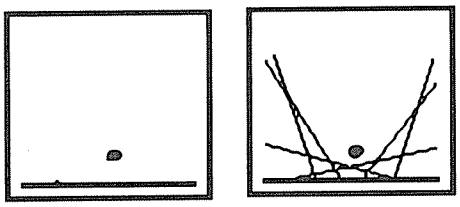


Figure 1

For an **ellipse:** a circle and a point inside it are drawn. The paper is repeatedly folded so that the point maps onto points on the circle. The crease lines envelop an ellipse, Figure 2.

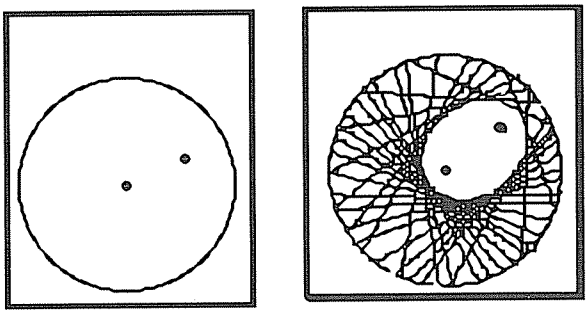


Figure 2

For a **hyperbola:** a circle and a point outside it are drawn. The paper is repeatedly folded so that the point maps onto points on the circle. The crease lines envelop a hyperbola, Figure 3.

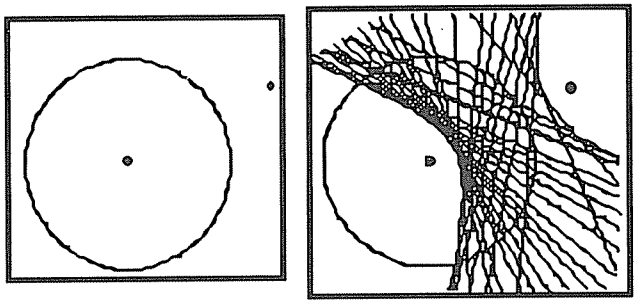


Figure 3

These activities can be executed on waxed paper where the crease lines show up clearly. The waxed paper can be used on an overhead projector to show the resulting figure.

When a point is mapped onto another point using paper folding, the fold line is a line of symmetry or mirror line in which one point is the image of the other. Thus on plain paper the Mira can be used to map one point onto another and the crease lines can be replaced by a set of mirror lines or lines of reflection.

A proof, at the grade 11 level, that the crease lines are tangents and envelop the conics can be found in *"Reflector Properties of the Conics"* (Del Grande, 1992).

In writing a paper on these activities it was necessary to draw the envelope diagrams for a computer print-out. The difficulty in accomplishing this task led to an investigation using *The Geometer's Sketchpad* software (Jackiw, 1991). Surprisingly there were some unexpected spin-offs and extensions to the activities that were facilitated with the software. *The Geometer's Sketchpad* is a geometry teaching and learning program for Macintosh computers. It employs traditional Euclidean tools, dynamic transformation procedures, and metric tools to help the user explore geometric constructions and relationships. To illustrate the dynamic features of Sketchpad, the properties of medians of a triangle

are investigated.

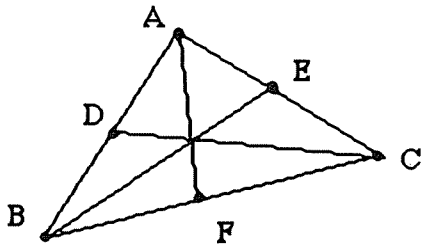


Figure 4

- Draw a triangle ABC and its medians $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$.

  The segment BC can be held fixed and everything else in the diagram varied.

- 'Drag' the point A and move it around.
  The whole triangle ABC and its medians, except for $\overline{BC}$, move accordingly.

  With $\overline{BC}$ fixed the sides of the triangle, Figure 4, and the medians stretch or contract as if they were rubber bands. No matter where the point A is moved the medians always intersect in a point!

  With a little additional work the ratio into which the medians are divided and the areas of the six interior triangles can be investigated.

An envelope of lines outlining a parabola, shown in Figure 1, can be obtained using *The Geometer's Sketchpad.* The procedure to draw this envelope can be described in general terms.

- A point F and a line l are drawn, Fig. 5a.
- Any point P is selected on the line l , Fig. 5b.
- The perpendicular bisector of $\overline{PF}$ is drawn, Fig. 5c.
- Hide $\overline{PF}$ to keep the diagram from becoming cluttered, Fig. 5d.

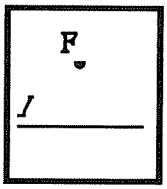


Fig 5a

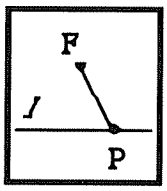


Fig 5b

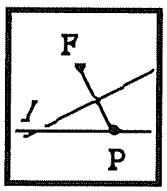


Fig 5c

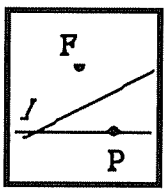


Fig 5d

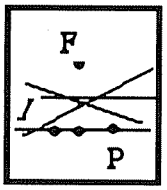


Fig 5e

- Repeat the previous steps for several points on l . This gives a set of perpendicular bisectors corresponding to the crease lines in the paper folding exercise, Fig. 5e.

A Sketchpad program can be written to produce crease lines that envelop a **parabola.** To use the following instructions only minimal familiarity with *The Geometer's Sketchpad* is required. The menu used is shown in bold.

- Draw a point F and a line l .
- Mark a second point P on l .
- Select the points F and P and **Construct** a segment joining F and P.
- Select $\overline{FP}$ and Construct the midpoint.
- **Construct** the midpoint and $\overline{FP}$ and **Construct** the perpendicular bisector of $\overline{FP}$.
- Select the perpendicular bisector and choose **'Trace Locus'** under the **Display** menu.
- Slowly drag P along the line l and watch the envelope grow.

  A set or family of lines is drawn each of which is the perpendicular bisector of $\overline{FP}$ where F is a fixed point and P is any point on the line l. Each line seems to be tangent to a curve. The curve is a parabola.

  The envelope seems to be a parabola.

This is a true example of dynamic geometry.

Additional steps must be taken to produce a drawing that can be printed. A printable parabolic envelope can be obtained using the following steps. (In the recently released version 3 of *The Geometer's Sketchpad* these steps can be implemented using the **LOCUS** command in the **CONSTRUCT** menu.)

- Draw a point F and line l.
- Mark 20 or 30 points on the line l . (approximately evenly spaced by 'eyeballing')
- Under the **File** menu select **New Script** and press **REC**ord on the script control bar.

  A program will be written for the constructions that follow.

- **Construct** a segment from F to one of the points on the line l .
- **Construct** the midpoint of the segment and the perpendicular bisector of the segment.
- **Hide** the segment and the midpoint. (use the **Display** menu)
- Press **STOP** on the **script** control bar.

  The scripted program can now be executed for the other points on line l .

- Select the point F and another point on the line l .

- Hold down the **option** key and click **'FAST'** on the script control bar.

  A second perpendicular bisector appears.

- Select F and each point in turn on l and repeat the last step. **(option - FAST)**

The resulting diagram looks like the envelope of a parabola. (Figure 6).

Although the parabola is not actually drawn the eye "sees" the parabola as a curve enveloped by a set of lines. Each line appears to be tangent to the curve.

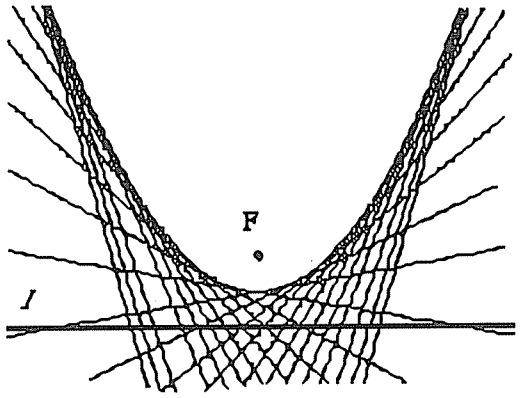


Figure 6

The size and shape of the parabola (envelope) can be changed by dragging the point F or dragging the line l .

It can be shown that each perpendicular bisector is a tangent to the outlined parabola.(Del Grande, 1992). However, *Sketchpad* can be used to convince one that the envelope is indeed a parabola.

INVESTIGATION

Use Figure 6. (See Figure 7.)

- Select and hide the points on line l .
- Mark, P, any point on the envelope.
- Drop a perpendicular $\overline{PC}$ to l.
- **Measure** $\overline{PC}$ and $\overline{PF}$.

  $PC = PF$

  $\therefore$ P is on the perpendicular bisector of $\overline{FC}$
- Drag F to get envelopes of different shapes and sizes. (Figure 7)

  How do $\overline{PC}$ and $\overline{PF}$ compare in length as the envelope changes size and shape?

  The lengths of $\overline{PC}$ and $\overline{PF}$ *change and are automatically recorded on the screen.*
  $PC \approx PF$ *for each parabola.*

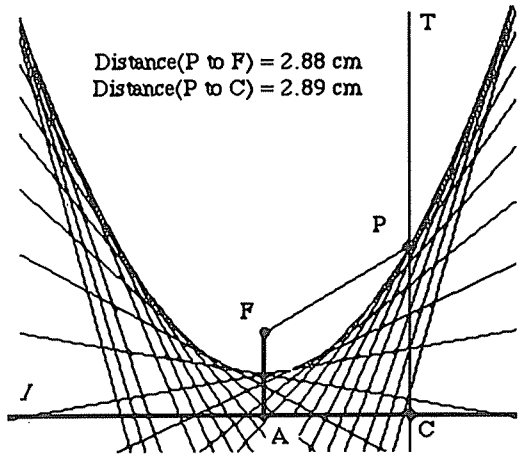


Figure 7

These investigations suggest that PC = FP. That is the distances from any point P on the curve to a fixed point F and to a given line l are equal. The curve with this property is the parabola. The line l is the directrix.

To show the reflector property of the parabola we first recall, from physics, that when a surface is curved, the curved surface can be considered to be flat at the point at which a light ray strikes it. The flat spot has the same direction as the tangent at the point. The surface reflects 'locally' as if it were the tangent to the curve. (Figure 8)

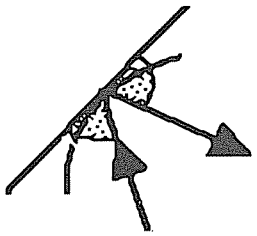


Figure 8

Look at a single tangent line, $\overline{KD}$, on which the point P lies. (Figure 9).

Draw $\overline{FA} \perp$ l. The axis of the parabola is along $\overline{FA}$.

If a ray of light leaves F along $\overline{FP}$, is the line $\overline{PT}$ the direction of the reflected ray?
Measure $\angle$ FPK and $\angle$ TPD.
Drag the point C along the line l and note how the angles FPK and TPD compare.

Angle (FPK) = 25 degrees
Angle (TPD) = 25 degrees

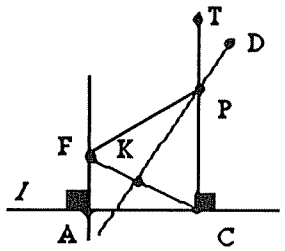


Figure 9

The angles are equal for all positions of C along l.

Thus , a ray along $\overline{FP}$ *is reflected in the tangent line at P in the direction of the line PT.*

But $\overline{PT} \perp l$.

$\overline{PT} \parallel \overline{FA}$

Using Euclidean geometry these two angles can be shown to be equal.

Δ FPC *is isosceles*

$\angle$ FPK = $\angle$ CPK

$\angle$ CPK = $\angle$ TPD

$\therefore$ $\angle$ FPK = $\angle$ TPD

If C is dragged along the line l, the locus of P should be a parabola. To show this, select P, choose ***'Trace Locus'*** under the ***Display*** menu, and drag C along l. (The locus of P shows on the screen as a series of points that trace a parabola.) This illustrates that P is on the parabola.

Thus a light beam emanating from the focus of a parabola, upon striking the parabola, is reflected so that the reflected ray is parallel to the axis of the parabola.

To draw an envelope of lines that outline an ellipse, the same procedure is used as for the parabola except that a circle is used with a point F inside. Proceed as follows:

- Draw a circle and a point F inside the circle.
- **Construct** a segment from F to any point on the circle.
- **Construct** the midpoint of the segment and **Construct** the perpendicular bisector of the segment.
- Select the perpendicular bisector and choose **'Trace Locus'** from the Display menu.
- Drag the point on the circle around the circumference.

The perpendicular bisectors seem to envelop an ellipse.

In order to produce a printable ellipse, the steps are similar to those for the parabola.

- Start with a circle (centre C) and an interior point F.
- Space 30 or so points evenly around the circle.
- Use a script routine similar to the parabola script routine to construct the perpendicular bisector of each segment between F and a circumference point.

The resulting diagram is an elliptic envelope. (Figure 10)

The size and shape of the envelope can be changed by dragging the point F or the centre, C, of the circle. It appears that F and C are the foci of the ellipse. As long as F remains inside the circle we get an elliptic envelope.

What is the envelope if F is moved to coincide with the centre? What is the envelope if F is moved onto the circle?

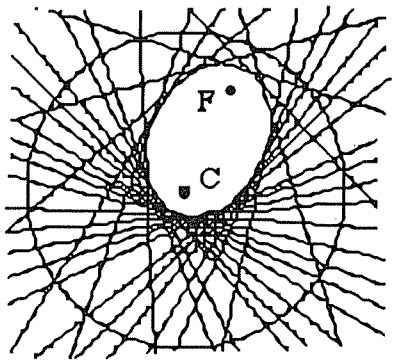


Figure 10

To verify that the envelope is indeed an ellipse with foci at F and at the centre, C, of the circle, a simple test can be applied. Recall that an ellipse can be defined as the locus of a set of points such that the sum of the distances from each point to the foci is a constant. It can be shown that this sum is actually equal to the radius of the circle. (Del Grande 1992)

To test the sum of the distances definition consider the following procedure:

- Mark an arbitrary point P on the envelope and any point A on the circle. (Figure 11)
- Find the distances FP, CP, and the radius, CA, of the circle.
- Compare the sum FP + CP with the radius of the circle, CA.

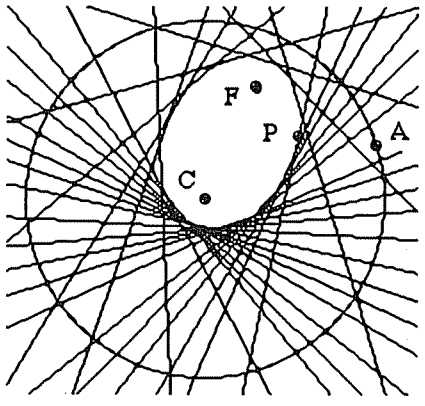


Figure 11

Distance (P to C) = 1.75 cm
Distance (P to F) = 1.03 cm
Distance (C to A) = 2.78 cm

This test shows that for any point P on the envelope FP + PC $\approx$ CA (the radius).

This strongly suggests that the envelope is indeed an ellipse with foci at F and C. It also suggests that point P is the point of tangency (Del Grande, 1992).

The reflector property of the ellipse can be determined using the diagram in Figure 12.

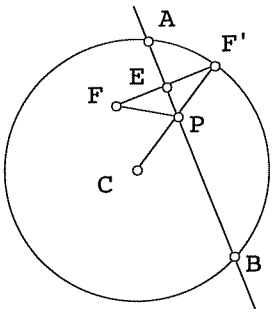
- Draw a circle with centre C, point F inside the circle and F' on the circle.
- Draw the perpendicular bisector of FF' cutting the circle at A and B.
- Join C, F' cutting $\overline{AB}$ at P.

Since P is on the perpendicular bisector of $\overline{FF'}$ *then* $FP = PF'$

$\therefore\ PF + PC = PF' + PC = CF'$ *(radius of the circle)*

P is a point on the ellipse with foci F and C.

To illustrate this fact, select P, choose **'Trace Locus'** and drag F' around the circle. The locus of P is shown as a set of dots on the screen and the set makes up an ellipse.



Notice
also that $\angle FPE = \angle F'PE$
and $\angle F'PE = \angle CPB$
$\therefore\ \angle FPE = \angle CPB$

Figure 12

A light beam emanating from focus F and striking the ellipse at P reflects from the surface (the tangent) so that the angle of incidence, $\angle$ FPE equals the angle of reflection, $\angle$ CPB. Thus, the reflected beam passes through the second focus, C.

A light beam emanating from one focus of an ellipse, upon reflection from the surface of the ellipse, passes through the second focus.

A simple way of drawing a hyperbolic envelope is to start with the elliptic envelope (Figure 10) and drag the point F outside the circle. (Figure 13)

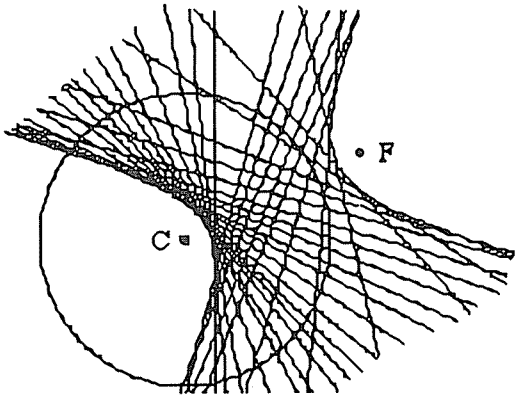


Figure 13

- How is the shape of the hyperbola affected as F is dragged away from the circle?
- How is the shape affected as F approaches the circle?

A hyperbolic envelope can also be obtained by starting with a circle and a point, F, outside the circle and using the same steps as were used for the elliptic envelope. That is, draw a set of mirror lines in which the image of F lies on the circle.

INVESTIGATIONS

1. Select several points on the hyperbola. Show by measurement that the difference of the distances of points on the hyperbola from F and C is a constant. Compare this constant with the radius of the circle.
2. Make a diagram showing the circle with centre C and a point F outside the circle. Draw the perpendicular bisector of FF' cutting the circle at A and B. The line AB is tangent to the hyperbola. (Figure 14). That is, P is the point of tangency of the tangent line AB and the hyperbola. Select P, choose **'Trace Locus'**, and drag F' around the circle. What is the locus of P?

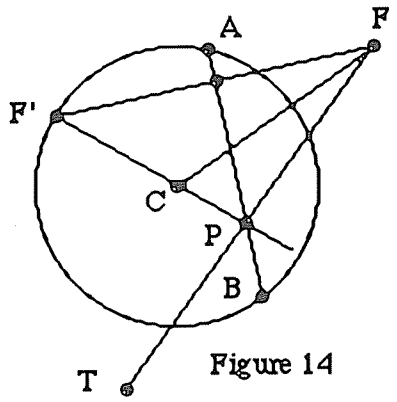


Figure 14

3. Use Euclidean geometry to show that PF - PC is a constant (the radius of the circle).
4. Extend $\overline{FP}$ to T. Measure $\angle$ F'PA and $\angle$ BPT.
   Select P and choose **'Trace Locus'** and drag F' around the circle.
   How do the measures of $\angle$ F'PA and $\angle$ BPT compare?
5. If a ray of light emanates from C and strikes $\overline{AB}$ at P locate the reflected ray.
   Verify the reflector property of a hyperbola.

   A light ray emanating from the focus of a silvered hyperbolic surface will, upon reflection from the hyperbolic surface, move on a line which, when extended, will pass through the second focus.

This paper illustrates how The Geometer's Sketchpad can be used in geometric investigations. There are many properties of conics and other geometric figures that can be discovered and revealed through the use of this software. The facility to change figures in a dynamic way opens up many possibilities for conjecturing and investigating problems in geometry. .

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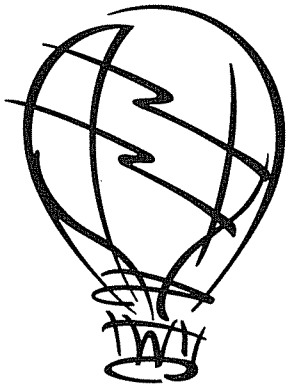
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▲ Mathematics and the Common Sense Revolution: *Relevance and Numeracy*

JOSEPH CUNSOLO
DEPARTMENT OF
MATHEMATICS AND STATISTICS
UNIVERSITY OF GUELPH



Joe Cunsolo is an associate professor of Mathematics at the University of Guelph, where he specializes in mathematics education. He has won several teaching awards, including the prestigious 3M award for excellence in University Teaching in Canada.

The challenge for us, as mathematics educators, is to keep students interested and involved in mathematics. One way to achieve this is to present the mathematics in a context that is meaningful and relevant. This means that we must always be on the lookout for understandable, and mathematically accessible real-world problems, in which we can introduce the mathematics and the students can apply it. These problems should, if possible, also demonstrate to the students the importance of their being "numerate" in today's society. In other words, they must be convinced of the dangers of being "innumerate" which can be thought of as the mathematical counterpart of being illiterate.

The University of Guelph's Numeracy Learning Objective states in part, "...numeracy may be defined as the ability to use mathematics at a level and a manner appropriate to good citizenship and vocational fitness. .... Numeracy...is an essential attribute of the informed and responsible citizen. A correct understanding of the proper use of numbers is necessary in a culture in which information routinely comes in numeric form and significant decisions of social policy often have quantification at their base. Without the ability to comprehend the use of quantitative data and to detect instances of misuse, we may have to forego opportunities for independent judgment... "

A topic that has both these components is the Ontario Progressive Conservative's reduction of the welfare benefits by 21.6% beginning October 1, 1995. This controversial decision generated heated attacks by the Ontario Liberal and New Democratic Parties, demonstrations at Queen's Park, heated debates on TV talk shows, a court challenge under the Charter of Rights, editorials, and letters to editors, (to name just a few). Many charges and counter charges were made about the impact of the cuts and about the equivalency of the original and the new benefit levels. Some of these charges, especially the equivalency arguments, have mathematical underpinnings.

The following is presented in a "Case Study Format" with the topic unfolding in chronological order and with actual government news releases simulated and slightly condensed to create realism and mimic the way the citizenry would receive the information. The material was deliberately organized in this manner to facilitate distribution to students. As a bonus, students will be familiarized with a part of the Ontario political scene. The presentation is kept politically neutral and involves the students "modelling" parts of the welfare system and mathematically verifying and analyzing statements made by the Progressive Conservative Government, the Liberal and New Democratic Parties, and others concerning the cuts to the General Welfare Payments on October 1, 1995, which was part of the Common Sense Revolution.

The mathematics required is accessible by students from Grade 10 upwards. The mathematics involved includes setting up linear functions that model the system, evaluating them under specific conditions, solving them, and using spreadsheets and their graphing capabilities or manually graphing these functions. Using the graphs, the students are asked to make comparisons of benefits before and after the changes, and then they are asked to be the "Minister of Community and Social Services" and make changes to the model to achieve specific stated government policies. The questions asked can all be used or judicially chosen, and with small modifications many more questions can be generated. The material is appropriate for Grade 10, but this topic could have relevance to all groups as supplementary material in its entirety or by emphasizing certain aspects such as modelling or decision making. The component that is important to all our students is their exposure to the "politics of numbers". They have been conditioned to believe that, in mathematics, if they do everything correctly they will get the "right answer". They will see that, in politics, there may be agreement on the mathematical answer, but there can be vehement disagreement on the interpretations of that answer.

Changes to the Ontario General Welfare Benefits
A Case Study

Background:

Prior to their election in June, 1995, the Ontario Progressive Conservative Party and, (now Premier) Mike Harris distributed a booklet entitled the Common Sense Revolution in which their party's platform was outlined. Under Welfare Benefits (page 11), as part of their welfare reform, the following statements were highlighted:

- "Ontario pays the highest welfare benefits not only in Canada, but anywhere in North America."
- "Our plan will set welfare benefits at 10% above the national average of all provinces."
- "However, we will allow anyone on welfare to earn back the difference between the current rate and the new, lower rate without penalty and without losing their eligibility."
- "Any welfare recipient who is willing to work will be able to maintain their income at the current level."

On October 1, 1995, through the office of The Honourable David H. Tsubouchi, Minister of Community and Social Services, the General Welfare rates were cut by 21.6% in most cases. The following are slightly condensed and modified versions of two circulated government information sheets.

You need to know August 1995

These changes will mean that Ontario's social assistance rates are still about 10 per cent higher than the average of the other nine provinces in Canada.

Here are some examples of how monthly maximum rates will change.

Category Maximum Rate	Before Oct. 1/95 Monthly Rate	After Oct. 1/95 Monthly Rate
Single, Employable	$663	$520
Sole-Support Parent (one child)	$1,221	$957
Employable Couple (no children)	$1,150	$901
Employable Couple (2 children)	$1,549	$1,214

You can earn back the difference

You might be getting a smaller cheque, but you do not have to live with less. If you get a job and make some money, you will be able to earn back the difference between your new social assistance amount and the old one.

You need to know about changes to STEP August 1995

What is STEP?

The Supports to Employment Program (STEP) helps people on General Welfare Assistance (GWA) to move into the work force.

How does STEP work?**

When you are working or attending paid training, STEP allows you to deduct certain amounts from your net earnings: (i) a basic exemption, (ii) an extra 25% exemption, (iii) child care costs for children under twelve, (iv) work expenses related to disabilities.

What are the Changes to Step?

The basic exemption has been increased effective October 1, 1995.

Category Basic Exemption	Before Oct. 1/95 Monthly Rate	After Oct. 1/95 Monthly Rate
Single, GWA	$50	$120
Sole Support Parent (1 child)	$120	$230
Two Adult Family (no children)	$100	$200
Two Adult Family (2 children)	$100	$200

**(i) The basic exemption is the monthly amount of money an individual can earn without having the welfare payment reduced.

(ii) Also an individual is allowed to keep 25% of the monthly money earned above the basic exemption without penalty. (This 25% amount was also allowed before October 1, 1995.) In other words, the monthly welfare payment is reduced by 75% of the money earned above the basic exemption.

Example

(a) In November, 1995, a single welfare recipient receives $520 a month and qualifies for a basic exemption of $120 a month (from above chart). If, in that month, he earned $120, he would also receive the full $520.

The total monthly income would be: $120 + $520 = $640
(earned + welfare benefit)

(b) If he had earned $160 in that month, the government would reduce its payment by using the following calculations.

Monthly Earnings:	$160
Less Basic exemptions:	$120
Less 25% of amount over basic exemption: 0.25($160 – $120) = 0.25($40)	$10
Amount Subtracted from that month's welfare cheque:	$30
The value of that month's welfare cheque would be:	$520 – $30 = $490
The total monthly income would be:	$160 + $490 = $650 (earned + welfare benefit)

The Progressive Conservative government's welfare reforms generated heated attacks by the Ontario Liberal and New Democratic Parties, demonstrations at Queens Park, heated debates on TV talk shows, a court challenge under the Charter of Rights, editorials, and letters to editors, to name a few. Many charges and counter charges were made about the impact of the cuts and the equivalency of the original and new benefit levels. On February 8, 1996, an Ontario Court ruled against a group who argued that the welfare cuts violated the charter of rights.

In the following problems you will mathematically examine statements made about this issue.

1. Earning Back the 21.6% Reduction

Both the Community and Services Minister David Tsubouchi and Premier Mike Harris, in response to criticisms that the welfare cuts were too drastic, responded that welfare recipients could earn back the amount that their welfare benefits were reduced on October 1, 1995, ***by working a few hours a week at minimum wage.***

For an individual whose maximum monthly welfare payment was reduced from $663 to $520 and assuming a four week month, how many hours a week at a minimum wage of $6.85 per hour would he have to work to end up with the old amount of $663?

2. Comparison of Total Incomes before and after October 1, 1995

In the Common Sense Revolution, it was stated that, ***"Any welfare recipient who is willing to work will be able to maintain their income at the current level"*** *(i.e., prior to the October 1, 1995, welfare reduction).*

James Laxer, Professor of Political Science at Atkinson College, York University, wrote a column against the Government's policy on welfare in the Toronto Star on November 19, 1995, entitled "Myths Shroud Reality of Tory Welfare and Workfare". Within this article, he states "...Myth Number 3: Welfare recipients who find work can earn back the amount their benefits were cut by the Tories on Oct. 1. On more than one occasion, Social Services Minister David Tsubouchi has said welfare recipients can work 'to earn back what was cut.' The minister's claim is false. Here is why. Both before and since benefits were cut by 21.6%, welfare recipients have been allowed to "top up" their benefits by working... " (The example he used to support his statement is expressed as questions (a) and (b).)

(a) Before October 1, 1995, a single individual received a maximum monthly welfare payment of $663. If he or she had earned a net amount of $700 in a month prior to Oct. 1, 1995, by how much would the welfare cheque be reduced and what would be the total income for that month?

(b) After October 1, 1995, this individual's maximum monthly welfare payment was reduced to $520 and say, in a month, he or she earned $800 net. How much was the welfare cheque reduced and what was the individual's total income that month?

After his example, Professor Lax concluded Myth Number 3 with the following, "As can be seen, the welfare benefit will drop to zero before the recipient's income reaches the level existing before the cut. (The foregoing is true for any set of numbers, whether the recipient is single or supports a family)." The following questions will help examine this statement.

(c) At what monthly net earning would an individual's welfare cheque be reduced to zero
(i) for a $663 monthly benefit before Oct. 1/95 and (ii) for a $520 monthly benefit after Oct. 1/95?

(d) For a single individual who is eligible to receive a maximum welfare benefit of $663 before October 1, 1995, and $520 a month after October 1, 1995, write the equation (model) that gives the total monthly income (earned + welfare benefits) as a function of monthly earned income in the following time periods:
(i) before October 1, 1995 (ii) after October 1, 1995.

(e) For each of the following time periods and for the single individual in (d), set up a spreadsheet which calculates the monthly amount of the welfare cheque and the total income

(earned + welfare benefit) for monthly earnings starting at $0 and increasing by increments of $50 to $1,000: (i) before October 1, 1995 (ii) after October 1, 1995.

(f) Use the data in (e) to graph the total income as a function of monthly earnings for each of the time periods. For comparison purposes put the two curves on the same graph.

(g) If an individual needed a total monthly income of $788 to meet expenses, how much money did he or she have to earn each month before October 1, 1995, to have a total income of $788? After October 1, 1995, how much would he or she have to earn each month to have a total monthly income of $788?

(h) Form two groups and using the calculations and graphs, have one group defend the statement in the Common Sense revolution and the other group defend Professor James Laxer's statement.

3. STEP Program Exemptions Increased

The criticisms continued both in the Ontario Legislature (Ontario Provincial Parliament) by the Ontario Liberal and New Democratic opposition parties and in the media. On December 6, 1995, in the Ontario Legislature, The Hon. David Tsubouchi announced that the basic exemptions in the STEP program would be increased on January 1, 1996. Welfare recipients who had been working would be given rebates retroactive to October 1, 1995, as the Government was keeping its commitment to allow people to earn back the difference between the old rate and the new rate without clawback. When asked why he was making changes to the earn-back provisions, Tsubouchi responded that since the rate changes took effect (in October, 1995), his Ministry had been observing the impact on social assistance recipients. As a result, the government believed that there was more that it could do to improve incentives for people on social assistance to get into and remain in the work force. The new exemption rate under the STEP Program would be the difference between the old GWA rate and the new GWA rate.

The following is a slightly modified and condensed flyer that was distributed to the local Social Services Offices .

MESSAGE BROADCAST
December 11, 1995

Changes which take effect on January 1, 1996, have been made to the STEP Flat rate exemptions to allow all recipients to earn back the difference between the old social assistance rates and the new ones. Provisions have been taken to incorporate the new flat rates retroactively to October 1, 1995, for recipients who had earnings in October, November, and/or December.

Basic Exemption Rates			**New Rate****
Category	Before Oct. 1/95	Oct. 1 to Dec 31/95	Jan 1, 96
Basic Exemption	Monthly Rate	Monthly Rate	Monthly Rate
Single, GWA	$50	$120	$143
Sole Support Parent (1 child)	$120	$230	$275
Two Adult Family (no children)	$100	$200	$249
Two Adult Family (2 children)	$100	$200	$346

** For example, for a single GWA , the old maximum welfare benefit was $663 and as of October 1, 1995, the new maximum welfare benefit was $520. The difference of $663 – $520 = $143 is the new Basic exemption rate. Note that if you calculate the differences for the two categories with children, you will find they are slightly lower than the rates listed above. This increase is due to the changes in allowances for children.

(a) Because the new exemption rates were retroactive to October 1, 1995, the Social Services had to calculate refunds for anyone who had earned money in October, November and December of 1995. If a single individual in the $520 GWA category had net earnings of $300 in November and $550 in December of 1995, how much will he or she receive as a refund from the Welfare Office?

(b) Under this new basic exemption rate, at what monthly net earning will this individual's welfare cheque be reduced to zero?

(c) With the new exemption rate and for a single individual who is eligible to receive a maximum welfare benefit of $520 a month, write the equation (model) that gives the total monthly income (earned + welfare benefits) as a function of monthly earned income. (See question 2 (d).)

(d) For the time period after January 1, 1996, and for a single individual, set up a spreadsheet (as in question 2 (e)) that calculates the monthly amount of the welfare cheque and the total income (earned + welfare benefit) for monthly earnings starting at $0 and increasing by increments of $50 to $1,000.

(e) Use the data in (d) to graph the total income as a function of monthly earnings. For comparison purposes, put this curve on the same graph with the curves generated in question 2(f).

4. What if you were Minister of Community and Social Services?

In a December 6, 1995, statement to the Legislature, the Minister, David Tsubouchi, announced the increase in the basic exemptions that were to take effect January 1, 1996. He stated, "Our government is committed to helping people break their dependency on welfare.... We are ... embarking upon a fundamental reform of the welfare system so that people will have the opportunity to become self-sufficient. Today, I would like to inform members of adjustments that will give people on welfare more incentive to return to the work force." That is, he was referring to the increase in the basic exemption as the reason for the improved incentive to return to the work force.

Assume as Minister that you are allowed to change the basic exemption figure and/or the 25% figure which was the percentage of the net earnings above the basic exemptions that could also be deducted. What basic exemption and what percentage extra deduction would you have chosen "to increase the incentive for recipients to return to the work force"? Sketch the graph of the total income using your new figures and compare it to the graph and incomes for the time prior to October 1, 1995. Compare your decisions with others in the class justifying why your decisions will be more effective in achieving your goal. Remember, there is no one right answer.

Conclusion

In a complex subject such as this, you will learn that there are no definite conclusions on which everyone will agree. No one will argue with the accuracy of the calculations (assuming they were done correctly), but it is the interpretations of your results that leads to the differences of opinions. Therefore we are left not with conclusions, but with more questions. When the results of the mathematical model you produced impacts on people, it is not sufficient to say the numbers are correct. You must also justify that they are "fair". Who decides if they are fair? As you can see, we have just scratched the surface of welfare reform. You have equations, graphs, and calculations in front of you. Based on your mathematics, have you made a decision about where you stand on this highly contentious issue of welfare reform? Is there consensus in the class? Does the mathematics tell the whole story?

Is the Welfare Reform Policy Working?

Here is an example in the importance of interpretation. How could one judge if the Government's welfare reform policy is working? Remember their intent was to get people off welfare and back into the work force. In a news release dated January 3, 1996 and titled *Ontario's Social Assistance Caseload Drops Again in December,* Mr. Tsubouchi stated, "The caseload has decreased by 10 percent since the end of June, 1995. Over 113,000 people have left Ontario's welfare system and social assistance costs have declined by almost $100 million over the same period. ... It is encouraging that people are continuing to break their dependency on Ontario's Welfare system. Some people have found work and our welfare reforms are having some impact."

Based on those decreasing caseload figures, how would you answer the question, "Is the Welfare Reform Policy Working?" Before you answer, let us look at someone else's interpretations of these same figures.

Kelly Toughhill works for the Toronto Star Newspaper in the Queen's Park Bureau. In an January 6, 1996, article in the Toronto Star entitled *Shrinking Welfare Rolls an Ontario Mystery,* Kelly Toughill states, "The Good News is that Ontario Welfare rolls are in a free fall. It is the sharpest, fastest drop in social assistance in the history of the province, a massive exodus that means taxpayers are supporting 115,000 fewer people today than just seven months ago. The bad news is that few of these leaving welfare have managed to get a job - and no one is keeping track of exactly how or where they are surviving without welfare or work. ... Statistics kept by the Ministry of Community and Social Services clearly show people are leaving social assistance. What they don't show is what happens to them after they leave and how, if they are not working, they manage to survive. ...".

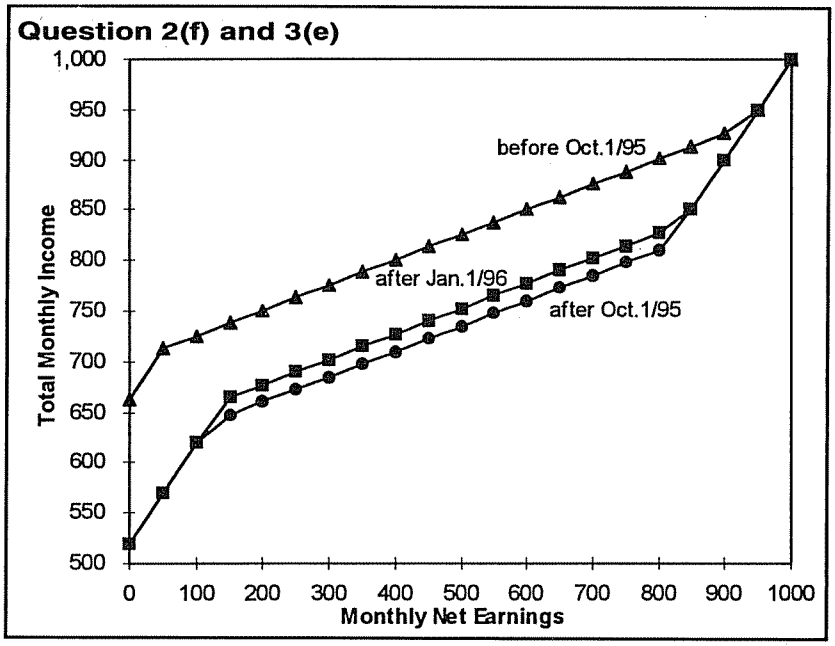
Well, is welfare reform working? You, as a citizen of Ontario, will be bombarded with facts, figures, and opinions and you will have to make the ultimate decision. The issues are complex and the conclusions will differ, but there is common agreement on the importance of our citizens having the mathematical ability to help them interpret and decipher the information they receive and make informed decisions!

Answers

1. To end up with $663 you need a monthly net earning of, E, where $E + 520 - 0.75(E - 120) = 663$. $E = \$212$
The net weekly earning is 1/4($212) = $53.
At $6.85 an hour it would take 53/6.85 = 7.74 hours a week to earn back the reduction.

2. (a) Before Oct. 1/95:
Note that since the recipient is allowed to keep 25% of (Amount Earned – $50), the welfare benefit will be reduced by 75% of (Amount Earned – $50).
The reduction of the welfare benefit is
$0.75(\$700 - \$50) = \$487.50$
Total income for the month is
$\$700 + (663 - \$487.50) = \$875.50$

(b) After Oct.1/95
The reduction of the welfare benefit is
$0.75(\$800 - \$120) = \$510$
Total income for the month is
$\$800 + (\$520 - \$510) = \$810$



	Question 2. (e) (i)		Question 2. (e) (ii)		Question 3. (d)	
	Before Oct.1/95		After Oct.1/95		After Jan.1/96	
Monthly Earnings	*Welfare Benefit*	*Total Income*	*Welfare Benefit*	*Total Income*	*Welfare Benefit*	*Total Income*
0	663.00	663.00	520.00	520.00	520.00	520.00
50	663.00	713.00	520.00	570.00	520.00	570.00
100	625.50	725.50	520.00	620.00	520.00	620.00
150	588.00	738.00	497.50	647.50	514.75	664.75
200	550.50	750.50	460.00	660.00	477.25	677.25
250	513.00	763.00	422.50	672.50	439.75	689.75
300	475.50	775.50	385.00	685.00	402.25	702.25
350	438.00	788.00	347.50	697.50	364.75	714.75
400	400.50	800.50	310.00	710.00	327.25	727.25
450	363.00	813.00	272.50	722.50	289.75	739.75
500	325.50	825.50	235.00	735.00	252.25	752.25
550	288.00	838.00	197.50	747.50	214.75	764.75
600	250.50	850.50	160.00	760.00	177.25	777.25
650	213.00	863.00	122.50	772.50	139.75	789.75
700	175.50	875.50	85.00	785.00	102.25	802.25
750	138.00	888.00	47.50	797.50	64.75	814.75
800	100.50	900.50	10.00	810.00	27.25	827.25
850	63.00	913.00	0	850.00	0	850.00
900	25.50	925.50	0	900.00	0	900.00
950	0	950.00	0	950.00	0	950.00
1000	0	1,000.00	0	1,000.00	0	1,000.00

2. (c) (i) Let E denote the monthly net earning. The required E satisfies

$$663 - 0.75(E - 50) = 0.$$

At a monthly net earning of $934, the welfare benefit will be reduced to zero.

(ii) The required E satisfies

$$520 - 0.75(E - 120) = 0.$$

At a monthly net earning of $813.33, the welfare benefit will be reduced to zero.

(d) Let E denote the monthly net earnings.

(i) Total Monthly Income =

$$\begin{cases} E + 663 & \text{for } 0 \le E \le 50 \\ E + 663 - 0.75(E - 50) & \text{for } 50 < E \le 934 \\ E & \text{for } E > 934 \end{cases}$$

(ii) Total Monthly Income =

$$\begin{cases} E + 520 & \text{for } 0 \le E \le 120 \\ E + 520 - 0.75(E - 120) & \text{for } 120 < E \le 813.33 \\ E & \text{for } E > 813.33 \end{cases}$$

2. (g) Before Oct. 1/95: from the spreadsheet values in **2. (e) (i),** you would have to earn $350 per month net.

(ii) After Oct. 1/95: solve $E + 520 - 0.75(E - 120) = 788$. You would need to earn $712 per month net.

3. (a) Since the welfare benefit was reduced by $0.75(E - 120)$ those two months and it should have been reduced by $0.75(E - 143)$, the difference each month is

$$0.75(E - 120) - 0.75(E - 143) = 0.75(23) = 17.25$$

The total refund is 2($17.25) = $34.50

(b) The required E satisfies

$$520 - 0.75(E - 143) = 0.$$

At a monthly net earning of $836.33, the welfare benefit will be reduced to zero.

(c) Total Monthly Income =

$$\begin{cases} E + 520 & \text{for } 0 \le E \le 143 \\ E + 520 - 0.75(E - 143) & \text{for } 143 < E \le 836.33 \\ E & \text{for } E > 836.33 \end{cases}$$

(d) See spreadsheet.

(e) See above graph.

▲ About Problem Solving: A Curious Sequence



JOHN GRANT MCLOUGHLIN

Have you encountered the Farey Sequence in your mathematical travels? This sequence caught my attention recently. Apparently it was discovered by Farey in 1816 while working with tables of decimal quotients. Posamentier and Stepelman (1986) offer the following definition:

> *The Farey Sequence F_n of order n, is defined as the ordered set consisting of 0/1, the irreducible proper fractions with denominators from 2 to n, arranged in order of increasing magnitude, and 1/1 (p.368).*

Consider the Farey Sequence of order 6, for example. The sequence would include 0/1, 1/1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6 and 5/6. It is essential to list these terms in increasing order (i.e., 0/1, 1/6, 1/5,...,1/1) to create the Farey Sequence. An intriguing relationship emerges when the terms are placed in this order. Select any term between 0/1 and 1/1 in the sequence. Create a fraction by taking the sum of the numerators of the adjacent terms over the sum of the adjacent denominators. The reduced form of this fraction will be the term you had selected. For example, (0+1)/(1+5) = 1/6. This property holds for Farey Sequences of all orders greater than 1. If you find a break in this pattern, it is wise to check the ordering of the terms and possibly the arithmetic.

Creating the sequences of various orders may provide a challenge to those who are not so familiar with fractions. Arranging the terms in order is an excellent way to develop number sense and skills with comparing fractions. The task is not really as cumbersome as it may initially appear. Suppose that the Farey Sequence or order 6 is in place. Then it is only necessary to insert 1/7, 2/7, 3/7, 4/7, 5/7, and 6/7 appropriately to obtain the next higher order sequence. The positions could be determined by comparing quantities. It may also be helpful to consider the aforementioned property of the sequence. It would follow that 5/7 would be placed between 2/3 and 3/4 because 2 + 3 = 5 and 3 + 4 = 7. While prime orders contribute many new terms, others such as 8 or 12 will not modify the sequence considerably.

Farey's sequence seemingly consists of a relatively random number of terms. In fact the number of terms approaches $\frac{3n^2}{\pi^2}$ as n gets larger. When n = 6, the number of terms is 13. The approximation is decent. However, its accuracy is quite remarkable for larger values such as n = 100. The actual number of terms, 3043, is less than 4 off the approximation. When n = 101, we get an additional 100 terms, thus, further improving the accuracy of the approximation. The reader may wish to consider what happens for n = 102, 103, 104,...

The Farey Sequence lends itself to consideration at various levels of mathematical learning. Identifying the terms for a particular order would be suitable for middle grade students. The sequencing of the terms offers a more advanced challenge. Students could move towards making conjectures about the number of terms, observations about the contributions of various orders based on factoring or number theory, or move in other directions of investigation. Others may be challenged to prove some of the properties or make their own observations to substantiate.

As promised, we shall revisit the coins' problem initially posed in April 1995.

The total value of the nickels, dimes, and quarters in a cash register is $19.95. Interestingly enough the total value of the quarters is an exact number of dollars with no change left over. similarly the total value of the dimes is an exact number of dollars. There is a total of 167 coins made up of nickels, dimes and quarters in the cash register. Determine the respective number of nickels, dimes and quarter in the cash register.

What is the rationale for the inclusion of such a problem? The problem does not give the solver any direct relationships between the values of two types of coins (e.g., the total value of the dimes is one dollar more than that of the dimes.) Nor does it offer any direct comparison of quantities such as the number of nickels is 5 more than the number of quarters. Such qualities were intentionally omitted in my own effort to pose the problem. It is their absence that gives the problem some distinctiveness from those more likely to be found in textbooks.

The statement of the problem was intended to lend itself to reasoning based on number theory rather than algebra. One solution is given by 40 dimes, 48 quarters, and 79 nickels. The problem was built around this solution. The surprise came when Eileen Shannon, of Westmount Secondary School in Hamilton, informed me that she and her husband, Harry, had both solved the problem independently. The catch was that they had different answers. Multiple solutions were not anticipated by this problem poser. Anyhow, it seemed reasonable to consider a second problem, namely that of verifying these solutions were the only ones. Indeed that is the case. Readers can convince themselves that the 95 cents must be made up of 19 nickels; any additional nickels must account for an exact number of dollars. Therefore, the possible number of nickels must come from the numbers 19, 39, 59,... A table can be made to consider possibilities. The second solution of 19 nickels, 120 dimes, and 28 quarters will emerge through this process.

The algebraically astute can also get to this result. Jim Sherrill, of the Faculty of Education at the University of British Columbia, sent along a detailed algebraic solution, the framework of which is provided here. Suppose that there are x dimes, y nickels, and z quarters. Two equations follow:

(1) x + y + z = 167

(2) 10x + 25y + 5z = 1995

He proceeds to note that the number of dimes is a multiple of 10 and the number of quarters is a multiple of 4 because of the exact dollar amounts. Substituting x = 10a and y = 4b (a,b are natural numbers) and simplifying produces another equation:

(3) 20a + 20b + z = 399

Subtracting (1) from (3) and substituting for x and y in (1) yields the equation 5a + 8b = 116 or 5a = 116 - 8b.

Note that a and b are positive integers. The only possible cases to consider become b = 2, a = 20; b = 7, a = 12; b = 12, a = 4. Recall that there are a total of 10a dimes. This eliminates a = 20 because the value of 200 dimes exceeds $19.95. The latter cases produce the two feasible solutions.

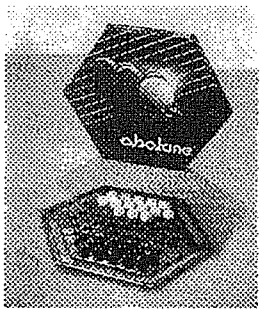
Problems are not intended to be closed. Feel free to suggest variations or offer alternative solutions. Do you have any curious sequences or mathematical investigations to share? Your input is welcomed - oscillating pairs, sequences, investigations,...

Reference

Posamentier, Alfred S. & Jay Stepelman, *Teaching Secondary School Mathematics: Techniques and Enrichment Units (2nd Ed.)*, Columbus, OH, Charles E. Merrill Publishing Company, 1986.

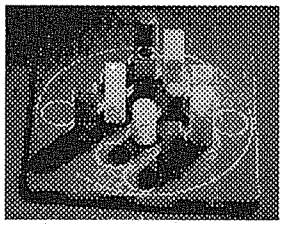
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▲ ASYMPTOTIC LIMITS: ZERO AND INFINITY

MARC HURWITZ

PRESIDENT, MATH PLUS LEARNING CENTRES



Marc attended the Universities of Guelph, McMaster, and Maryland (College Park) specializing in mathematics and theoretical physics. A resident of St. Mary's, he currently divides his time between tutoring math, writing poetry, and family.

Two limits in high school Calculus invite special attention—infinity and zero. This article will present techniques for dealing with both limits that are more robust (and easier to use) than standard textbook pedagogy. These alternatives for solving limit questions are "asymptotic solutions"— i.e., the focus is ONLY on the behaviour of the functions as they approach the limiting values.

Asymptotic analysis is routinely used in quantum physics, boundary layer problems, mathematical approximations, differential equations, population dynamics, etc. Fortunately, a simplified subset of the basic ideas can produce pretty powerful results at the introductory level too.

Basically, the asymptotic approach suggests in the limit, we can restrict our focus to higher order terms for limits to infinity or lower order terms for limits to zero. Remarkably, this produces a lot of information about the function but with much less work and fuss. Here are the basic 'asymptotic' rules:

(I) Limits to infinity

When adding polynomial terms, ignore them all except for the highest power.

(II) Limits to zero

When adding polynomial terms, ignore them all except for the lowest power. If the final expression is still indeterminate, then you have to consider other subtleties.

Two other special approximations appear in limits to zero, $\sin(x) = x$ and $e^x = 1+x$ (NOTE: in this article the equal sign should be interpreted to mean "asymptotically equal" whenever appropriate.)

Abacus

AIDS Awareness

Acknowledgement: This activity originally appeared in The California Mathematics Council Communicator, *September 1995, p. 45. It was submitted by Beverly Gleason and Helen Olmsted from Santa Monica High School.*

Strand: Statistics and Probability

Skills: Conducting an experiment, analyzing patterns

Grades: 11-12 [but could be adapted for use in grades 7-10 as well]

Materials: vials, eyedroppers, sodium hydroxide, phenolphthalein

Description:

When Marla Weiss of Beverly Hills High School shared this activity with teachers at CSU Northridge, it seemed the ideal activity for mathematics and science students during AIDS Awareness Week in late April. Despite the harmless appearance of each clear vial and the exchange of liquids with friends they trusted, students were impressed with how quickly the "virus" spread.

Each student received a small vial of clear liquid. All vials were filled with water except for one which was filled with 6 ml of sodium hydroxide (NaOH). For larger classes, several vials should hold sodium hydroxide.

During round #1, students used eyedroppers to exchange two droppers full of liquid with another student. During round #2, students do another exchange with a different student. After the third exchange, the student "physician" places one drop of the indicator phenolphthalein in each vial. Those which contain the "virus" turn bright magenta to indicate contamination.

Students calculate to determine how they became infected and they analyze patterns to determine who carried the virus. This experiment was an effective way for students to see how quickly the virus travels through the exchange of fluids.



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Please, Chew Your Data!

Acknowledgement: This activity created by E. Suzanne Henderson from East Prairie Elementary School in Tuscola, Illinois originally appeared in the November 1995 issue of The Illinois Mathematics Teacher *on pp. 3-7.*

Active involvement of the students in a lesson which promotes prediction, data gathering and analysis, and measurement skills is the focus of the following lesson. The *Standards* stress that these skills be taught in a real world situation. Connecting the child's real world to these areas makes the lesson exciting and valuable to the child.

"Does bubble gum weigh more before it is chewed or after it has been chewed?" This question will get the attention of any intermediate age student. They will have an opinion and they will be great at convincing one that they are right.

"Of course it weighs more after it is chewed because of the saliva that is in it." "But no, that can't be because it always seems smaller after you chew it."

This type of dialogue about a "sticky" problem will really motivate the students. The following "Bubble gum lesson" has proven to be a perennial favourite in my fifth grade classroom. Since gum comes in packs of five, the class is divided into groups of five for this activity. Due to class sizes it may be necessary to have some groups of four. This is not a problem as there is always someone in a group willing to chew two pieces of gum. Each group will be given a different kind of bubble gum. There should also be five balls of gum from a machine and one pack of sugar free gum. Before chewing the gum the students will predict what will happen to the weight of the gum when it has been chewed. Fifth graders generally come up with the consensus that saliva will go into the gum and make it weigh more. After some predictions have been made, the activity proceeds following this procedure:

1. Each group of five will be given one pack of gum or the five gum balls from a gum machine and five small paper cups. Condiment cups work well.
2. Each child gets one piece of gum and one small cup. The students are to first make an estimate of the weight of their own piece and then multiply by five to get a pack estimate. This should be recorded on the group data sheet. (Figure 1)
3. Each group weighs all five empty cups.
4. Each group member places his/her unwrapped gum in the paper cup and the group weighs all five pieces at once to get the unchewed pack weight. The weight of the empty cups must be subtracted to get the gum weight.
5. After all groups have completed the "weigh-in" the students will all begin to chew their gum. Again to retain reliability the chewing will begin on signal and continue for eight minutes. Use a timer; it seems more official to the students.

During the chewing time the class will be engaged in discussion on ways to organize and display the data. The students may be very creative on this point. One good way to display the data would be with a double bar graph (Figure 2). One bar would be the total unchewed weight and the other bar could be the total chewed weight of each pack. This could also be achieved with a pictograph set up in the same fashion.

While chewing, the students might also take a minute to reflect back to the original prediction about what will happen to the weight of the gum. Some may wish to change their prediction.

When the eight minutes has passed the activity continues:

6. Each child places his gum back into his paper cup and the group reweighs the pack of gum. The students must again subtract the weight of the empty cups. The results are recorded.

7. All cups with gum still in them are thrown away!

After all groups have completed the weighing the class compiles all the data so that it can be organized, displayed, and interpreted. Again use the group data sheet, Figure 1, to compile the data. A class graph can be created with each group displaying their own data on a group graph. Using this class graph, the class now enters into a discussion on the uses of such information. Who could possibly benefit from knowing what the students have discovered? The students will come up with a variety of answers.

Each group will be given the task to work on one mathematical situation and one connected activity and then to make a presentation to the class based on their work. The math related task would be to use the data sheet and to create one problem solving activity that all the students could solve using their own data sheets. Examples of these situations would be to find which gum had the highest percentage of loss during chewing; to organize a survey in which to find the favourite gum in the fifth grade from the ones tested; to make an attribute chart which has such categories as colour, shape, size and flavour included or to make a chart showing price comparisons. Calculators need to be available for such things as the percentage problems.

Each group would have its own individual second task. The second related task could come from the following ideas:

1. Research the origin of gum. Find where the raw materials come from and display that information on a map.
2. Research the various sugars that might be present in gum.
3. Devise a rating card that would indicate the favourite gum. This could be done with a number system ranging from 1-4 and rating such things as flavour, texture, ease of bubble blowing, bubble size, and length of time the flavour lasts.
4. Research what pioneers used for chewing gum, and find out how it was made.
5. Write a television commercial for their gum flavour. A video tape presentation would be good for this.
6. Write an instruction manual on "How to Blow Bubbles".

Other fun things to go along with this gum lesson would be to estimate the number of gum balls in one pound [or kilogram], write poetry about gum or bubbles, or to write the principal or teacher a letter trying to convince them that gum chewing in school is a good idea. It is also fun to listen to a recording of "Does Your Chewing Gum Lose Its Flavor on the Bedpost Overnight?" written by Rose and Mary Bloom. The lyrics can be found in *Kid's America.*

This rather "sticky" lesson will have your intermediate age students chewing up math. This is real life math that is quite relevant to the middle grade student. It gives them opportunities to collect data, interpret it, and display it. It also allows for connections with other academic areas, and it certainly reinforces skills which are stressed in the *Standards* such as measurement and data collection and analysis.

References:

Caney, Steven. *Kid's America.* Workman Publishing Company, 1978, p. 176.

Project AIMS. *Jaw Breakers and Heart Thumpers.* Fresno, CA: AIMS Education Foundation, 1987, pp. 34-36.

Figure 1 Which Will You "Chews"?

Consumers:_____

Estimated weight of your unwrapped piece _____.

Estimated weight of your group's package _____ (You only need ***your*** piece to do this.)

Instructions:

1. Each member is to get one piece of gum, unwrap it, and estimate the weight to the nearest gram. Then determine what the estimated pack weight is. DO NOT CHEW YOUR GUM YET!!
2. Each group must weigh the five empty cups together. Record this number for use in the next instruction.
3. Put your piece of gum in your paper cup. Everyone will weigh their gum at one time in the balance pan. Subtract the weight of the empty cups so that you only have the weight of the gum. Record your total pack weight.
4. Wait for the signal to chew your gum. Keep your paper cup.
5. When chewing time is over you will reweigh your gum as you did the first time and record your results. ***Remember*** to subtract the weight of the empty cups.
6. We will compile the rest of the chart for the other flavours in class together.

Whole Class Data Chart

	Brand	Flavour	Unchewed Pack Weight	Chewed Pack Weight
1				
2				
3				
4				
5				
6				

Figure 2

Which Will You "Chews"?

		0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51
Flavour																			
	Total																		
	Chewed																		
	Total																		
	Chewed																		
	Total																		
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Key:

Each block equals 3 grams. **Total Weight** ☐ **Chewed Weight** ☐

Early Years Teaching Ideas

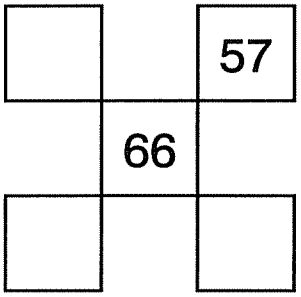
Acknowledgement: This article is reprinted from the January 1996 issue of The Math Journal of the Manitoba Association of Mathematics Teachers *(pp. 4-5). It was compiled by Dianne Soltess.*

Our Early Years Representative has submitted some sample test questions , based on the St. James, Gr. 3 Assessment, 1995. These questions are used at a level 3, to assess Patterns and Relations, and Problem Solving.

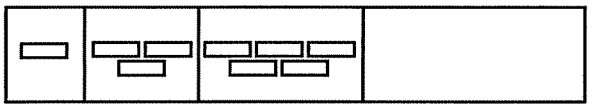
Mathematics - Grade 3

Patterns and Relations

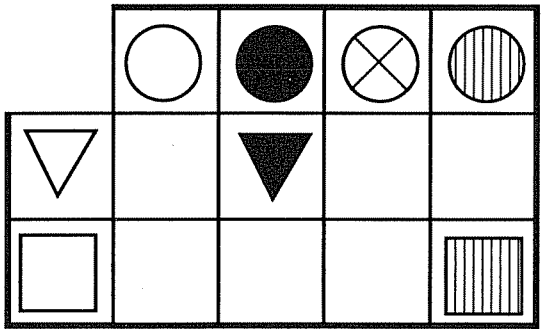
1. This is a piece cut from a hundred chart. Fill in the missing numbers.



2. Draw the next figure in the pattern.

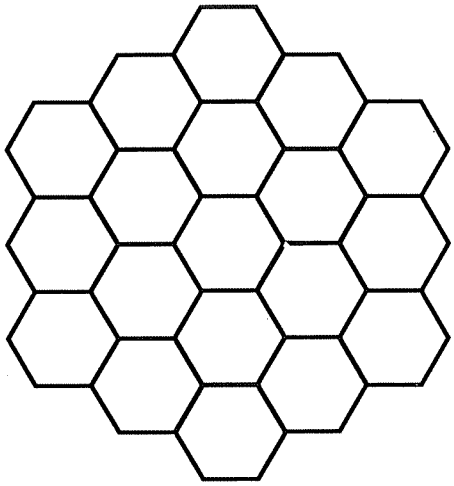


3. Complete the chart.



4. Bees were making honeycomb. They started on day 1 with the middle cell. Each day they added another row of cells all around the honeycomb.

   How many cells will there be after the 6th day?
   There will be _____ cells.



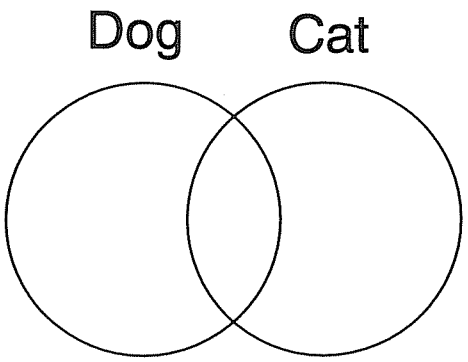
5. On Friday, 15 people are in the Sea Sub on the first trip down, 13 people are on the second trip, 11 people are on the third trip, and 9 people are on the fourth trip. If the pattern continues, how many people will be on the sixth trip?

Trip	People
1	15
2	13
3	11
4	
5	
6	

   There will be _____ on the sixth trip in the Sea Sub.

6. Paul surveyed 16 students in his class. He asked whether they had a cat or a dog for a pet. He got the following results: Dog - 6, Cat - 4, Both a dog and a cat - 2, and No dog or cat - 4.

Record the results on the Venn diagram below.



7. Look for a pattern and then write the next line of the pattern.

   1 + 3 = _____

   3 + 5 = _____

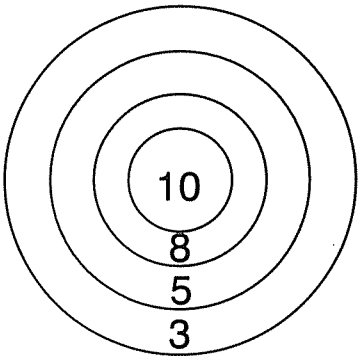
   5 + 7 = _____

   _ + _ = _____

8. Dart Throw

   Bob threw three darts. If his score was 20, where could his shots have landed?

   The darts could have landed on _____, _____, and _____.

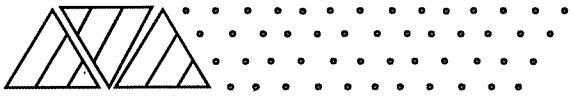


9. Look at the table. Look for a pattern to help you find the secret rule. Complete the table and write the rule.

IN	3	4	5	6	9
OUT	9	12	15		

Rule: _____

10. Draw the next 2 figures to continue the pattern.



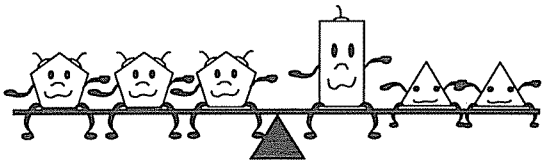
11. Rules:

    Balance the seesaw.

    Robots that are the same have the same weight.

    Robots that are different have different weights.

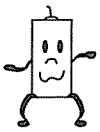
    All robots weigh more than zero kilograms.





If on the seesaw weighs 5 kilograms, what could the other robots weigh?

Give one possibility.



_____kg.

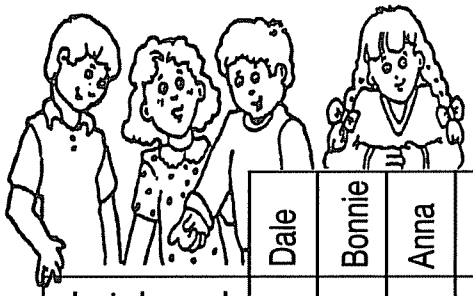


_____kg.

12. Solve the following logic problem.

    Four children rode to the park. Read all the clues. Then make x's on the chart to match the children with the things they rode to the park.

1. Dale loves to ride his bike.
2. Dale lives across the street from the person with the skateboard.
3. Bert lives next door to Dale.
4. Bonnie just got her new scooter.



	Dale	Bonnie	Anna	Bert
skateboard				
bike				
roller skates				
scooter				

THE GAME

Acknowledgement: This activity was first published in the California Mathematics Council Communicator, *March 1996 edition. It was written by James Baird of Hidden Creek, Conejo Valley Unified.*

Strand: Number, logic

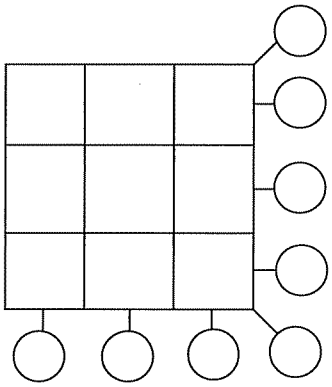
Skills: Reinforce basic facts and computational skills

Grades: 3 - 8

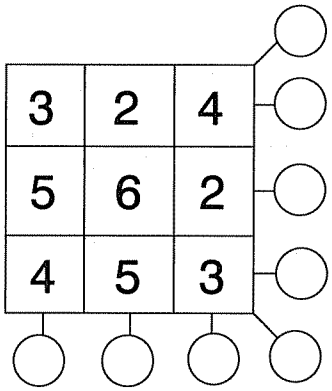
Materials: Dice, paper, pencil

Description:

On an overhead or on the chalkboard, make a grid like the one below. The students make a copy on their paper. Be sure they make only eight balloons.

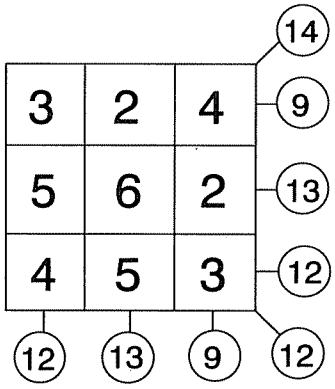


The teacher then rolls a die nine times. As the die is rolled, each number that is rolled is placed into one of the nine squares on the grid. This continues until the grid is complete. For example, if you rolled 5, 3, 4, 5, 2, 2, 6, 4 and 3, the grid would look like this:



The students make their own decisions as to where they will place the numbers as they are rolled. Once a number is placed on the grid it can not be moved.

The numbers in each row, column, and diagonal are added separately, and the respective totals are placed in the appropriate balloon as shown below.



Any totals that are duplicated at least twice are used to determine the game score for the round. For example, from the grid above there were two 9's, two 13's and three 12's. These totals are added together to get the game score for the round:

9 + 9 + 13 + 13 + 12 + 12 + 12 = 80.

The high score wins that round.

The game may be played several times. Each player saves his/her total score from each round. These can be added to determine a high score winner. Other possibilities for declaring a winner are to use the mean, median, mode and/or range of each individual's scores. Another alternative is to ask each group to compute their total scores and averages.

The luck of the game is in the roll of the die. The strategy is in placing the numbers in positions that allow repeating sums.

The teacher can play along with the students because they love to beat the teacher's score. After a period of time the student with the highest score is allowed to roll the die for the next round of the game.

Extension:

A die with larger numbers, other than 1 - 6, or a die made with positive and negative numbers, can be used.

(I) Limits to Infinity

Find the value for each of the following limits.

(1) $\lim_{x \to \infty} \dfrac{x^2 + 2x - 3}{3x^2 + x - 5}$

(2) $\lim_{x \to \infty} \dfrac{3x^3 + 2x + 1}{x^2 + 5x - 7}$

(3) $\lim_{x \to \infty} \dfrac{(2x - 5)(3x + 2)}{(2 - 3x)(x + 1)}$

(4) $\lim_{x \to \infty} \dfrac{e^x - e^{-x}}{e^x - e^{-x}}$

(5) $\lim_{x \to -\infty} \dfrac{e^x - e^{-x}}{e^x - e^{-x}}$

(6) $\lim_{x \to -\infty} \dfrac{\sqrt{x^2 + 4}}{x + 4}$

Example (1) is the simplest case and either the standard procedure, i.e., dividing the top and bottom by the highest power of x, or the asymptotic method are pretty much equivalent.

Example (1) Standard Method:

$$\lim_{x \to \infty} \frac{x^2 + 2x - 3}{3x^2 + x - 5} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2} + \frac{2x}{x^2} - \frac{3}{x^2}}{\frac{3x^2}{x^2} + \frac{x}{x^2} - \frac{5}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{2}{x} - \frac{3}{x^2}}{3 + \frac{1}{x} - \frac{5}{x^2}}$$

$$= \frac{1 + 0 + 0}{3 + 0 + 0}$$

$$= \frac{1}{3}$$

Example (2) is more interesting because we can begin to see that asymptotic techniques are more efficient (easier too!)

Example (1) Asymptotically:

$$\lim_{x \to \infty} \frac{x^2 + 2x - 3}{3x^2 + x - 5} = \lim_{x \to \infty} \frac{x^2 + 2x - 3}{3x^2 + x - 5}$$

$$= \lim_{x \to \infty} \frac{x^2}{3x^2}$$

$$= \lim_{x \to \infty} \frac{1}{3}$$

$$= \frac{1}{3}$$

Example (2) Standard Method:

$$\lim_{x \to \infty} \frac{3x^3 - 2x + 1}{x^2 - 5x + 7}$$

$$= \lim_{x \to \infty} \frac{\frac{3x^3}{x^3} - \frac{2x}{x^3} + \frac{1}{x^3}}{\frac{x^2}{x^3} - \frac{5x}{x^3} + \frac{7}{x^3}}$$

$$= \lim_{x \to \infty} \frac{3 - \frac{2}{x^2} - \frac{1}{x^3}}{\frac{1}{x} - \frac{5}{x^2} + \frac{7}{x^3}}$$

$$= \frac{3 - 0 + 0}{0 - 0 + 0}$$ Oops! Will this be +ve or –ve ∞?

$$= \lim_{x \to \infty} \frac{3 - 0 + 0}{\frac{1}{x} - 0 + 0}$$

$$= \lim_{x \to \infty} (3x)$$

$$= +\infty$$

The step where the sign of the infinity is determined is critical for curve sketching. In fact, a student could only complete this problem satisfactorily by realizing that the dominant term in the denominator is the one with the highest power!

Some textbooks realize this and replace the standard method with a modified prescription: "Divide top and bottom by the highest power of x in the denominator only." This would have worked in Example (2) and it is an improvement but it is not (as we will see below) as efficient as using asymptotics.

Example (2) Asymptotically:

$$\lim_{x\to\infty}\frac{3x^3-2x+1}{x^2-5x+7}=\lim_{x\to\infty}\frac{3x^3-2x+1}{x^2-5x+7}$$

$$=\lim_{x\to\infty}\frac{3x^3}{x^2}$$

$$=\lim_{x\to\infty}3x$$

$$=+\infty$$

I try to avoid having to work for extra information—in my opinion, the best solution method is the laziest one! Here, not only was the answer easy to determine, but without any extra effort the second last line indicates that the function has an oblique asymptote with a leading term 3x. If the second last line had contained a term such as $3x^2$, then I would have known that the function has a parabolic asymptote. The general functional form of the asymptote is a great bonus in curve sketching!

In the third example the asymptotic method is by far the easiest to use and less likely to produce errors. It also follows a dictum I often tell my students, "Never expand a denominator."

Example (3):

$$\lim_{x\to\infty}\frac{(2x-5)(3x+2)}{(2-3x)(x+1)}=\lim_{x\to\infty}\frac{(2x-5)(3x+2)}{(2-3x)(x+1)}$$

$$=\lim_{x\to\infty}\frac{(2x)(3x)}{(-3x)(x)}$$

$$=\lim_{x\to\infty}\frac{6}{-3}$$

$$=-2$$

I particularly like the next two examples because they were shown to me by a student who had been having trouble solving limit problems in class. She managed to solve both questions by herself, a real boost to her self-confidence early in the semester. The key point she noticed was that positive exponents are bigger than negative exponents, and then she applied asymptotics.

Example (4):

$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

$$=\lim_{x\to\infty}\frac{e^x}{e^x}$$

$$=\lim_{x\to\infty}1$$

$$=1$$

The situation reverses itself when the limit goes the other way.

Example (5):

$$\lim_{x\to-\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to-\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

$$=\lim_{x\to-\infty}\frac{-e^{-x}}{e^{-x}}$$

$$=\lim_{x\to-\infty}-1$$

$$=-1$$

The final example is worth trying both ways. Here, for the first time, is a problem where dividing by the highest power actually seems to produce the wrong answer! In fact it does not, but I gave it to a number of my friends and was surprised how many had serious difficulty solving it rigorously without asymptotics.

Example (6) Standard Method:

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+4}}{x+4}=\lim_{x\to\infty}\frac{\sqrt{\frac{x^2}{x^2}+\frac{4}{x^2}}}{\frac{x}{x}+\frac{4}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{1+\frac{4}{x^2}}}{1+\frac{4}{x}}$$

$$=\frac{\sqrt{1}}{1}$$

$$=1\ ??!!$$

The actual answer is -1 but the 'mistake' is not in the second last line as you might be tempted to think – the square root function is positive by definition. In fact, the sign of the answer was lost in the very first step! If you ever give a question like this to your class you probably also found that many students had a difficult time identifying the highest power of x.

Here is the same question solved asymptotically.

Example (6) Asymptotically:

$$\lim_{x \to -\infty} \frac{\sqrt{x^2 + 4}}{x + 4}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2 + \cancel{4}}}{x + \cancel{4}}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x \to -\infty} \frac{|x|}{x}$$

$$= \lim_{x \to -\infty} \frac{-(x)}{x}, \text{ since } x \text{ is negative.}$$

$$= \lim_{x \to -\infty} -1$$

$$= -1$$

(II) Limits to Zero

When limits tend to zero, the solution technique is not as general. The principle used in the case of infinite limits is often too crude to pick up subtle second or third order behaviour which may actually dominate the limit. Most questions requiring students to find a derivative from first principles, for example, fall into this category. The one saving grace is that failed attempts to apply the basic asymptotic idea leads back to an indeterminate form, i.e., the student should realize that more is required.

Two of the most common problems involve the limits of trigonometric and exponential functions. Here are the three fundamental asymptotic identities to keep in mind:

(1) lower order terms dominate

(2) $\lim_{x \to 0} \sin x = \lim_{x \to 0} x$

(3) $\lim_{x \to 0} e^x = \lim_{x \to 0} (1 + x)$

The trigonometric identity is proved in most OAC Calculus courses using the Sandwich Principle. The exponential identity can be deduced by examining the limit to zero of $(e^x - 1)/ x$.

These special limits have "real-world" importance: the function sin(x)/x, for example, is the square root of the intensity of a cross-section of light passing through a simple diffraction grating. Because sin(x) tends to x in the zero limit, this composite function is well-defined for all real values of x and has a finite integral over the real line. The approximation of sin(x) is also used in the solution of pendulum problem and many other physically important situations.

Solve for the following limits.

(7) $\lim_{x \to 0} \frac{x^2 + 2x - 3}{3x^2 + x - 5}$

(8) $\lim_{x \to 0} \frac{\sin(3x)}{x}$

(9) $\lim_{x \to 0} \frac{x \sin(2x)}{\sin^2(x)}$

(10) $\lim_{x \to 0} \frac{\sin^3(2x^2)}{x^4 \sin(x) \tan(3x)}$

(11) $\lim_{x \to 0} \frac{\cos(x) - 1}{x^2}$

(12) $\lim_{x \to 0} \frac{\sin(x)e^x - x \cos(x)}{x^2}$

The first of these is for demonstration purposes only because questions are rarely this straightforward.

Example (7):

$$\lim_{x \to 0} \frac{x^2 + 2x - 3}{3x^2 + x - 5} = \lim_{x \to 0} \frac{\cancel{x^2} + \cancel{2x} - 3}{\cancel{3x^2} + \cancel{x} - 5}$$

$$= \lim_{x \to 0} \frac{-3}{-5}$$

$$= \frac{3}{5}$$

Example (8):

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{x \to 0} \frac{3x}{x}$$

$$= \lim_{x \to 0} \frac{3}{1}$$

$$= 3$$

Every time I see this type of question the simplicity of the method always strikes me, but the next two examples really make the point forcefully.

Example (9):

$$\lim_{x \to 0} \frac{x \sin(2x)}{\sin^2(x)} = \lim_{x \to 0} \frac{x \cdot 2x}{x^2}$$

$$= \lim_{x \to 0} \frac{2}{1}$$

$$= 2$$

Example (10):

$$\lim_{x \to 0} \frac{\sin^3(2x^2)}{x^4 \sin(x) \tan(3x)} = \lim_{x \to 0} \frac{(2x^2)^3}{x^4 \cdot x \cdot 3x}$$

$$= \lim_{x \to 0} \frac{8x^6}{3x^6}$$

$$= \frac{8}{3}$$

If you note that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ in the second line then it follows that the limit as $x \to 0$ of $\tan(x) = x$.

Example (11):

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2} = \lim_{x \to 0} \frac{\cos(x) - 1}{x^2} \cdot \frac{\cos(x) + 1}{\cos(x) + 1}$$

$$= \lim_{x \to 0} \frac{\cos^2 - 1}{x^2(\cos(x) + 1)}$$

$$= \lim_{x \to 0} \frac{-\sin^2(x)}{x^2(\cos(x) + 1)}$$

$$= \lim_{x \to 0} \frac{-x^2}{x^2(\cos(x) + 1)}$$

$$= \lim_{x \to 0} \frac{-1}{\cos(x) + 1}$$

$$= -\frac{1}{2}$$

If we isolate the first and last lines of the solution to this question we can derive a second order asymptotic relation for the cosine function. As $x \to 0$, $\cos(x) \to 1 - \frac{x^2}{2}$. Finally,

Example (12):

$$\lim_{x \to 0} \frac{\sin(x)\, e^x - x \cos(x)}{x^2}$$

$$= \lim_{x \to 0} \frac{x(1 + x) - x(1 - x^2 / 2)}{x^2}$$

$$= \lim_{x \to 0} \frac{x + x^2 - x + \cancel{x^3 / 2}}{x^2}$$

$$= \lim_{x \to 0} \frac{x^2 + x^3 / 2}{x^2}$$

$$= \lim_{x \to 0} \frac{x^2}{x^2}$$

$$= \lim_{x \to 0} 1$$

$$= 1$$

I hope you enjoyed this (brief) tour of asymptotics.

A Timely Model for Division of Fractions

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Division of fractions is a challenging topic in arithmetic for many intermediate students. It's no wonder. Both the operation of division and the concept of a fraction have their own individual difficulties. Putting them together requires the synthesis of a number of critical understandings.

Recommendations on teaching this topic consistently emphasize the importance of conceptualization over procedures (NCTM, 1989; Coburn, 1989; Graeber and Tanenhaus, 1993; Bezuk and Bieck, 1993). Bezuk and Bieck, who reviewed research in this area, included other specific recommendations: using real world applications, assisting students with the meaning of the operation of division, and adopting a pattern searching approach.

Other general recommendations come from the NCTM *Professional Standards for Teaching Mathematics* (1991) which promotes a vision of mathematics classrooms where teachers pose challenging tasks, where students use models, metaphors, diagrams and one another as resources for problem solving, and where the truths of mathematics emerge as a result of discourse among the participants. A major element in this environment is reflection, looking back at what has been done and making sense of it in ways that can be retained, applied, and transferred.

The following sequence of tasks and questions attempts to implement these suggestions. No rules are provided. The intent is to provide a path for students which allows them to construct their own meanings. Reasoning, pattern searching, estimation, and communication are basic elements.

The storyline is in-school television broadcasting and its related timetabling challenges. This context helps make meanings accessible to students and allows curriculum integration with language arts, technology, and other activities. As time is the measure of interest, dial clocks (still common on classroom walls if not on wrists!) make a natural visual aid. These have the advantage of extending earlier experiences with circle models for fractions. Also, students have the option of translating time periods into minutes which provides alternative ways of solving the problems. (For other examples see Steiner, 1987, who used a money model, Bezuk and Armstrong, 1993, who used road paving, and Hyde and Nelson, 1967, who used an egg carton model.)

The meaning of division embedded in these situations is successive subtraction, often called the measurement model. (In contrast, the partitive model suits situations involving "sharing.") The measurement model is much easier for students in initial work with division of fractions (Trafton and Zawojewski, 1984). (For applications using the partitive meaning see Ott et al., 1991.)

These materials include "Think it over" questions designed to incorporate the discourse and communication recommendations of the Standards. These questions promote students' clarifying, deepening, and generalizing their understanding. Although they appear here interspersed with the other tasks, they may, instead, be asked orally by the teacher or be used as journal writing prompts as students reflect on and summarize their thinking.

The five activities are as follows:

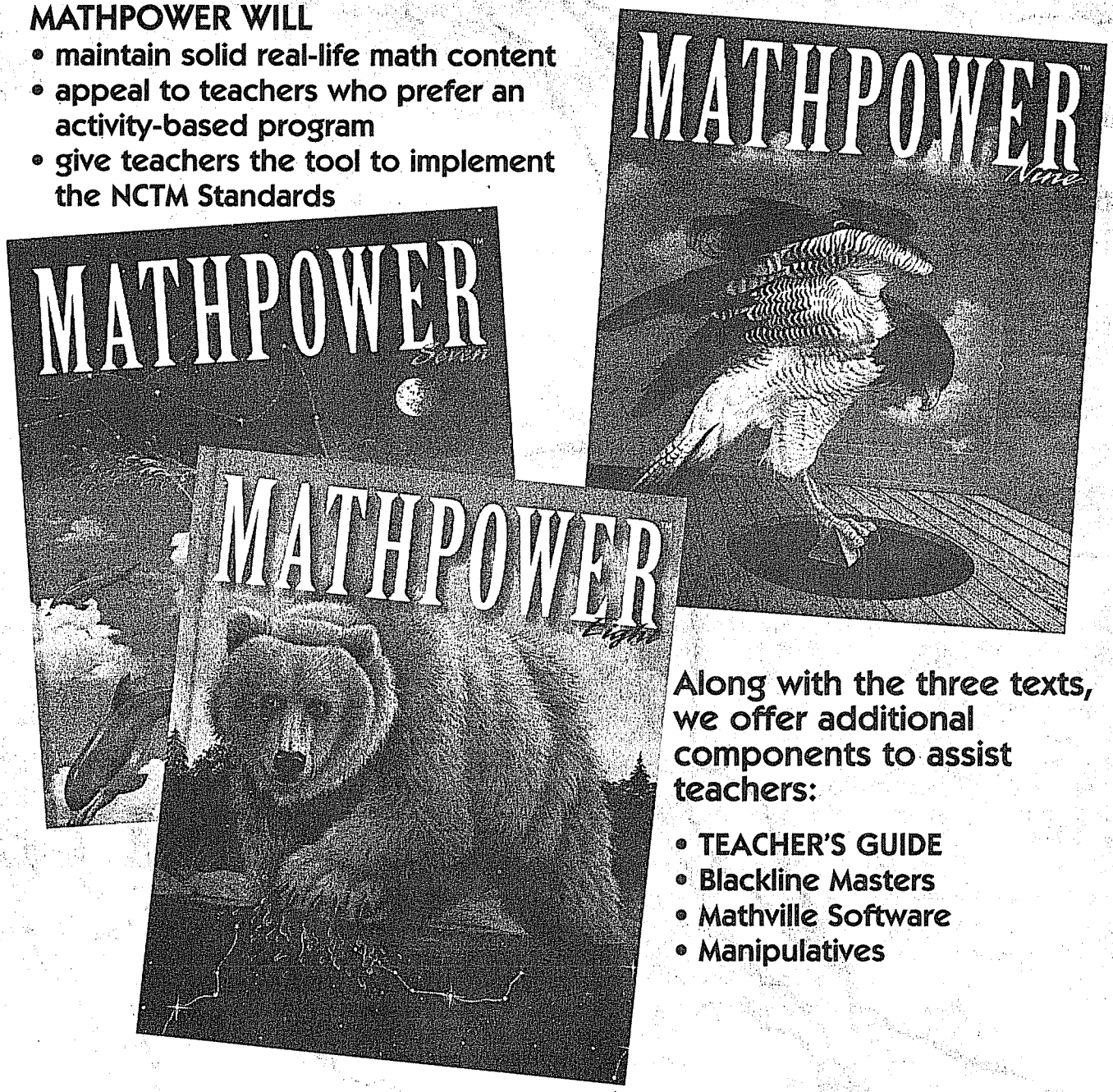
	Title	Objectives
A.	**Minutes and Hours**	**Review equivalent minutes and fractions of an hour.**
B.	**Long Programs**	**Associate the meaning of division with time measurement situations.**
C.	**Short Progams**	**Find patterns in the number of periods of length 1/n in 1, 2 or 3 hours.**
D.	**Mixed lengths (1)**	**Find patterns in the number of periods of length 1/b and a/b in a given number of hours.**
E.	**Mixed Lengths (2)**	**Find patterns in problems that do not have whole number solutions.**

Figure 1.

Figure 5 provides a worksheet of dial clocks that students may use as a problem solving tool.

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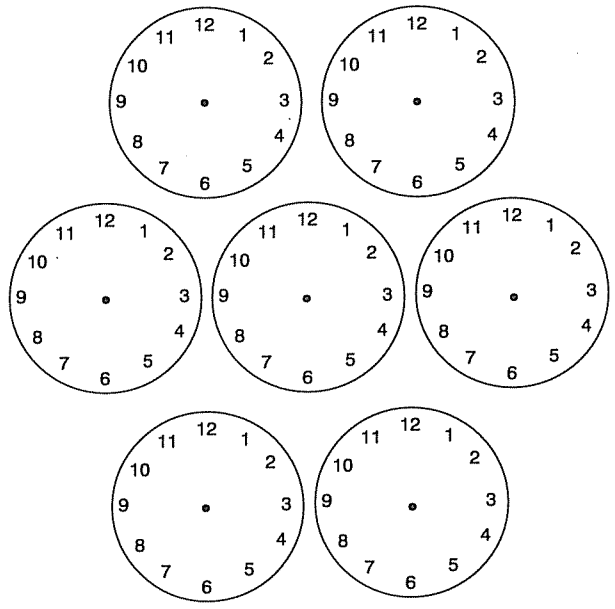
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Timing to a T

Your school community produces its own television programs. Your class has been selected as scheduling consultants. Your job is to schedule programs of different lengths into various time periods.

A. Minutes and Hours.

Classes, individual students, parents, teachers, and administrators will be broadcasting. They need different length time slots, so your team has to be able to use both minutes and fractions of an hour. Use the dial clock faces below to illustrate different time periods. Express each time period as a fraction of an hour and in minutes.



	Time in Minutes	Fraction of an hour
1.	30 minutes	_____
2.	20 minutes	_____
3.	_____	1/4 of an hour
4.	_____	1/5 of an hour
5.	10 minutes	_____
6.	45 minutes	_____
7.	_____	2/3 of an hour

Figure 2.

B. Long Programs

Community groups want to plan some long programs. Some of their scheduling problems are given below. Solve each problem by whatever method you like. Then, write an arithmetic problem to represent each question.

1. How many 2 hour programs fit in 8 hours?
2. How many 4 hour programs fit in 12 hours?
3. How many 3 hour programs fit in 24 hours?

Think it over.

4. What operation are you using? Why?
5. Write a scheduling problem that could be represented by 12 divided by 2.
6. The Parent Club is given 8 hours on special Saturdays. They are debating whether to have programs of 4 hours, 2 hours, or 1 hour. How many of each could they have? How does the number of programs change as the length of the programs gets shorter?

C. Short Programs

Individual students want to present short television programs. Some of their scheduling problems are given below. Solve each problem by whatever method you like. Then, write an arithmetic problem to represent each question.

1. How many 1/2 hour segments fit in one hour?
2. How many 1/3 hour segments fit in one hour?
3. How many 1/4 hour segments fit in one hour?

Think it over.

4. What operation are you using? Why?
5. What pattern do you notice? Explain. Use your pattern to solve the following.
6. How many 1/10 hour segments fit in one hour?
7. How many 1/60 hour segments fit in one hour?
8. With a one hour slot, how does the number of programs change as the lengths of the programs get smaller and smaller?
9. How many programs of length 1/b fit in 1 hour?
10. How many 1/2 hour segments fit in 2 hours?
11. How many 1/3 hour segments fit in 2 hours?
12. How many 1/4 hour segments fit in 2 hours?
13. Reread #10-12 and #1-3. How do the questions compare? How do the answers compare? Explain.
14. Suppose all the questions in #1-3 allowed 3 hours instead of 1 hour. How would the answers change?
15. Describe any patterns you have found.
16. How many programs of length 1/b fit in 2 hours? in h hours?

D. Mixed Lengths (1)

Classes are getting longer time slots and some of them want longer periods for their shows. Solve the following:

1. How many 1/3 hour segments fit in 4 hours?
2. How many 2/3 hour segments fit in 4 hours?

3. How many 1/4 hour segments fit in 6 hours?
4. How many 3/4 hour segments fit in 6 hours?
5. How many 1/5 hour segments fit in 4 hours?
6. How many 4/5 hour segments fit in 4 hours?
7. Summarize 1-6 in a chart. Write each arithmetic problem in the space indicated.

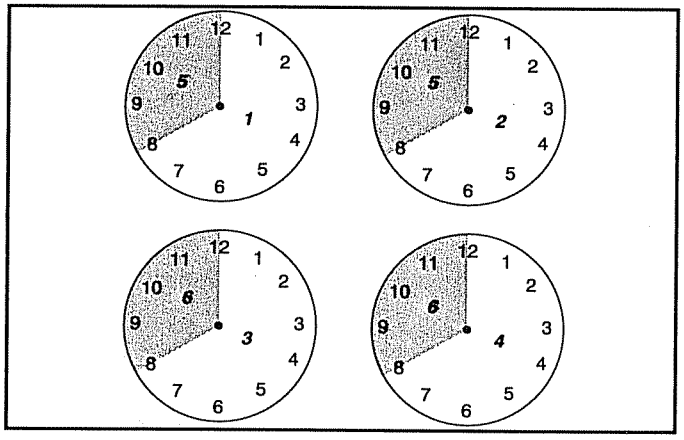
1.	2.	3.
4.	5.	6.

Figure 3.

Think it over.

8. Explain any patterns you see in the top row.
9. Explain how to use a result in the top row to find the result below it.
10. Write and solve a problem that could be represented by 5 divided by 5/6.
11. For #2, Marcus used the following drawing.

Figure 4.



Kim used a different method. She reasoned: There are 3/3 in 1 hour, so there are 12/3 in 4 hours. "How many 2/3's in 12/3?" is the same as "How many 2's in 12?" So the answer is 6. How was Marcus thinking about the problem? How was Kim thinking?

12. How many programs of length 1/b hour would fit in h hours? How many programs of length a/b hour would fit in h hours?

E. Mixed Lengths (2)

School administrators are also going to get television programming time.

They want to use more unusual lengths.

Continue to use circles or other models and compare with your findings from part D. Try to solve each problem in more than one way.

1. How many 3/4 hour programs fit in 2 1/4 hours?
2. How many 3/4 hour programs fit in 1 1/2 hours?
3. How many 3/4 hour programs fit in 1 hour?
4. How many 2/3 hour programs fit in 3 hours?
5. How many 2/3 hour programs fit in 1 hour?
6. How many 2/3 hour programs fit in 1/2 hour?
7. How many 1 1/2 hour programs fit in 3 hours?
8. How many 1 1/2 hour programs fit in 2 hours?
9. How many 1 1/2 hour programs fit in 1 hour?

Think it over.

10. Write and solve a problem that could be represented by 2 divided by 3/10.
11. Three students were debating about a scheduling problem: How many 1/2 hour programs fit in a 15 minute time slot? Carla said none because 15 minutes is too short. Mai said you could put half of a 1/2 hour program into 15 minutes because 15 minutes is half of 30 minutes (or half an hour). Kyo said you could put a quarter of a program on because 15 minutes is a quarter of an hour. Were any of them right? If so, who was right and why? If not, what do you think?
12. Think about the number of 5/6 hour periods in 13 hours. Without working on the problem decide if the answer is over or under 13. How did you decide? Estimate the actual answer.
13. Write a programming problem that could be represented by 3 divided by 1 3/4.
14. Michelle noticed that the answers to #1-3 could be written 9/3, 6/3, and 4/3. She created and solved a similar set of problems: How many 1 1/4 hour programs fit in 3 hours, 2 hours, and 1 hour? She got 12/5, 8/5, and 4/5. Was she right? Are there any patterns? Explain.
15. Estimate the number of 1 2/3 hour periods in 8 hours.
16. Suppose a new classmate joined your team tomorrow. How would you explain what you have been learning?

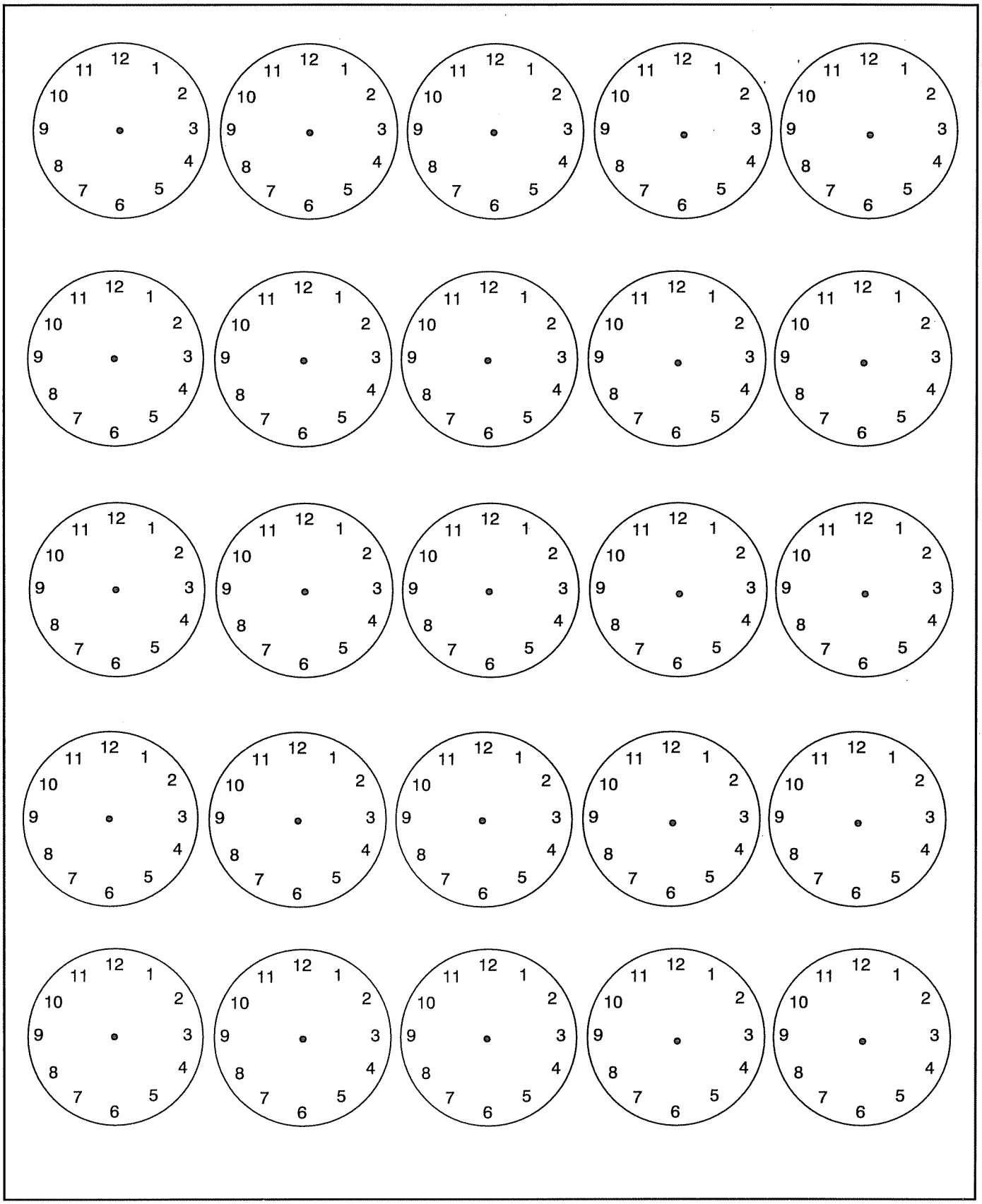


Figure 5. Clocks for student work.

Answer Key

A. **1.** 1/2 h; **2.** 1/3 h; **3.** 15 min; **4.** 12 min; **5.** 1/6 h; **6.** 3/4 h; **7.** 40 min.

B. **1.** $8 \div 2 = 4$; **2.** $12 \div 4 = 3$; **3.** $24 \div 3 = 8$; **4.** Division because we are finding the number of equal parts to make a whole; **5.** Answers may vary. Here is a sample: How many two hour programs fit in 12 hours?; **6.** 2, 4, or 8 programs respectively; as the length of the program gets shorter, the number of them increases.

C. **1.** 2; **2.** 3; **3.** 4; **4.** Division because we are finding the number of equal parts to make a whole; **5.** Answers may vary. Here is a sample: As the lengths of times get shorter, the number of them increases; **6.** 10; **7.** 60; **8.** The number of periods of length 1/n in one hour is n; **9.** b; **10.** 4; **11.** 6; **12.** 8; **13.** One hour has been replaced with two hours; the answers are twice as big. **14.** The answers would be three times as big; **15.** Answers may vary. Here is a sample: There are b periods of length 1/b in an hour so in more hours multiply b by the number of hours. **16.** 2b; bh.

D. **1.** 12; **2.** 6; **3.** 24; **4.** 8; **5.** 20; **6.** 5; **7: 1.** $4 \div 1/3 = 12$; **2.** $4 \div 2/3 = 6$; **3.** $6 \div 1/4 = 24$; **4.** $6 \div 3/4 = 8$; **5.** $4 \div 1/5 = 20$; **6.** 4 4/5 = 5. **8.** Answers may vary. Here is a sample: Dividing by a number 1/n is the same as multiplying by n, its reciprocal. **9.** To find a result in the bottom row, divide the answer in the top row by the new numerator; **10.** Answers may vary. Here is a sample: How many 50 minute periods are there in 5 hours? **11.** Answers may vary. Here is a sample: Marcus thought about the problem with pictures; Kim was thinking about the problem using common denominators. **12.** bh; **13.** bh divided by a.

E. **1.** 3; **2.** 2; **3.** 1 and 1/3; **4.** 4 and 1/2; **5.** 1 and 1/2; **6.** 3/4; **7.** 2; **8.** 1 and 1/3; **9.** 2/3. **10.** Answers may vary. Here is a sample: How many 18 minute programs fit in 2 hours? **11.** Answers may vary. Here is a sample: Carla is right if a program can't be broken apart; Mai is right if you can take a fraction of a program; Kyo is using the remainder and not seeing it as part of the whole half hour; **12.** The answer is over 13 because 5/6 is less than an hour so there are at least 13 periods in 13 hours. After taking out the 13 5/6 here are 13/6 left. That is at least two more sets of 5/6. An estimate might be 15. **13.** Answers may vary. Here is a sample: How many 1 and 3/4 hour programs are there in 3 hours? **14.** Yes. Answers may vary. Here is a sample: As the time slot decreases the number of programs decreases. **15.** Answers may vary. Here is a sample: There are over 4 because there are four two hour periods and 1 2/3 is close to be smaller than 2; **16.** Answers may vary.

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▲ Problem Solving and Portfolios



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The following three articles are from the *Transition Years Committee of OAME*. The first, written by Joe Hall, is a covering article about problem solving in general. The second article is on *Assessment of Portfolios* by Bill Morrison and Lisa Ritchie. The third article consists of a problem solving work sheet and a rubric for assessing problem solving. These were prepared by David Zimmer.

Problem Solving

One of the six strands in the Provincial Standards in Mathematics is problem solving. Problem solving is different from the other strands in that it is taught through each of the other strands. Students are to use their mathematics knowledge and skills to help them find solutions to problems in their daily lives and to make decisions based upon those solutions.

Most mathematics teachers do not feel uncomfortable incorporating problem solving in their curriculum. They are familiar with George Polya's research in problem solving.

Polya's multi step process is...

1) Understand the problem.

2) Make a plan.

3) Carry out the plan.

4) Look back over the work done.

5) Communicate the solution.

The problem of assessing problem solving is difficult for most teachers. Since many of the problems are open ended there is no final product to mark. What you are attempting to do is capture the thought processes that go on in the student's head. I think the last step in Polya's process is a very important one for teachers. Whether the student arrives at a solution to the problem or not, each one is able to communicate the process they used. Communication may be writing about their procedure, or presenting it verbally, either to the whole class or to the teacher in an interview. You may ask how they solved the problem and if they could think of another strategy to solve the problem.

OAME's publication, *Linking Assessment and Instruction in the Transition Years,* is an excellent source for different problems and different methods of assessment. Another good source of materials comes from Exemplars in Underhill, Vermont. Both of these sources give examples of different levels of students work and suggests a rubric for evaluation.

One thing the current emphasis on problem solving does for the teacher is to make the teacher aware that there may be more than one correct answer and if there is only one answer then there may be many different ways of arriving at that answer.

Linking Assessment and Instruction in the Transition Years is available from OAME, c/o Dave and Bonnie Alexander, P.O. Box 96, Rosseau, Ontario, P0C 1J0.

Information about Examplars may be obtained by writing Examplars, R.R. #1, Box 7390, Underhill, VT 05489.

Assessment - Portfolios

The use of portfolios can be a valuable assessment strategy in any mathematics course but it is especially helpful in a grade 9 destreamed class. A portfolio can help to address the problem of evaluating students who may vary greatly in their knowledge of mathematics.

The portfolio becomes a showcase for student work. There usually will be no minimum or maximum number of items but students need to be given some direction about what to include in their portfolio. Students who are lacking background knowledge in one or more areas of mathematics, can do extra work in those areas, while students who have an excellent background can concentrate on problem solving and enrichment materials. An extension of such an application is to provide students, who perform poorly on written tests, the opportunity to do an extra assignment on the same topic and place it in their portfolio. This allows the teacher to evaluate whether the student now understands the topic and emphasizes the importance of continuous improvement in mathematics. Those who experienced success on the test can apply their skills to related problems. It is also important that they keep a log showing what is in the portfolio and the date when it was submitted. A common method for such tracking is to have the students tape a piece of paper to the inside of the front cover of their portfolio (file folder) to document all entries.

The portfolio is not limited entirely to mathematical questions. The portfolio may also include journal type entries in

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which students share their views, ideas and thoughts as they learn mathematics, solve problems, and become aware of the applications of math in life (career projects, daily applications, etc.). This is an excellent opportunity for the students to provide their own grade 10 level recommendation and support such a recommendation with reasons for their level placement. The report can be graded on completeness and reasoning so all students, regardless of mathematical ability, can experience success.

Developing criteria for evaluating portfolios can be a challenge. For some of the work, teacher comments may be used, but, to make the portfolio part of a grade, it is necessary to develop some kind of rubric for evaluation. A holistic rubric works fairly well but it takes time to develop and apply. Using letter grades or levels makes the task manageable. An example of the use of levels is as follows:

Suggested Portfolio Evaluation Rubric

Level 0:
- empty portfolio

Level 1:
- a few extra worksheets have been completed
- no problem solving
- no creative work

Level 2:
- extra worksheets have been included
- some problem solving has been completed

Level 3:
- the portfolio includes extra work on many course topics
- copies of tests on which the student had a poor performance are accompanied by complete extra assignments sheets
- some problem solving has been attempted
- special projects (e.g., the use of statistics in the media) are included

Level 4:
- the portfolio is well organized and exciting to look through
- it includes solutions to problems (type 2 and contest questions)
- it includes evidence of enrichment (from text and other sources)
- there is evidence of student use of resources such as computers, calculators, etc.
- special projects are included.

Since one of the main focuses of a portfolio is to see the best that a student can offer, you must keep in mind the ability of the individual when you are evaluating. If a student with a weak background produces work above what they have in the past, then they should not be penalized for lack of type 2 problem solutions or contest questions. The focus must be more on the fact that, for their ability level, they have greatly advanced.

This article outlines how portfolios are being used in grade 9 destreamed mathematics classes but the ideas presented are applicable to most courses. Portfolios will be especially helpful in courses where the assessment includes the tracking of each student's achievement of specific outcomes.

Mathematics Problem-Solving Work Sheet

Problem Statement

My Interpretation of What the Problem is Asking me to Do

Diagram/Sketch/Visualization to Help Me Understand the Problem

My Plan for Solving the Problem

My Detailed Solution with Explanations

My Conclusion or Answer

Evaluation of My Solution and My Process

Independence	_____	Correctness/Completion	_____
Communication	_____	Organization	_____
Understanding the Problem	_____	Extensions/Variations	_____

Problem-Solving Assignment Evaluation					
Category	**0**	**1 to 4**	**5 to 6**	**7 to 8**	**9 to 10**
Number of Problems Attempted and Level of Completion	No problems were attempted	Few problems were attempted or completed.	Some problems were tried or completed, but a better effort was anticipated.	A reasonable number of problems were tried and completed.	Many problems were tried and completed.
Appearance and Organization of the Final Product.	The assignment was messy and disorganized to the extent that it was very difficult to evaluate.	The assignment was somewhat messy and disorganized. It was difficult to evaluate as a result. Questions may have been out of order.	While the appearance and organization could have been improved, the assignment was generally satisfactory.	The projects was well organized. Solutions appear in the proper order. The student had clearly taken care to rewrite final versions of solutions.	The assignment indicates that the student has taken exceptional care in the preparation of final versions of solutions. Diagrams are drawn neatly. Colour has been used appropriately.
Quality of Solutions: **• Format** **• Clarity of explanation** **• Use of language**	Only a minimal amount of work has been shown.	Insufficient explanation or detail has been provided. Reasoning is unclear or faulty. There are examples of poor mathematical form or language use.	The student has generally used proper form. While improvements are possible, solutions contain sufficient detail to be understood.	Solutions generally display use of proper form. Language used is clear and easy to understand. Reasoning is logical and conclusions are well justified.	Solutions and explanations are complete and detailed. Proper mathematical form is maintained throughout the assignment. Language used is clear and easy to understand.
Demonstrated Level of Insight, Understanding and Creativity	Little or none.	Solutions tend to show marginal understanding of concepts involved.	Solutions indicate a fairly straight forward approach to the problems. Student appears to understand some of the key concepts.	Some solutions reflect a novel approach to the problem. Student appears to understand most concepts.	Student appears to understand all the key concepts. Solutions indicate insight into the nature of the problems. There are several examples of creative approaches.

Council Comments



PETER SAARIMAKI
COORDINATOR OF MATHEMATICS
TORONTO BOARD OF EDUCATION

These notes were taken at the OAME Council meeting held Saturday January 27, 1996. Sincere thanks must be extended to my morning scribe, Ron Sauer, for his recording of events, but all interpretations are mine. Comments about Council and its activities should be directed to the President. Comments about these comments can be sent to the editor.

General

NCTM is reviewing the Curriculum and Evaluation Standards and invites anyone to submit comments and suggestions. Contact the executive for specific questions if you need them. Council was also reminded that orders for NCTM materials should be forwarded through our NCTM rep, Mickey Sandblom, so that OAME receives the commission.

Don Attridge sent his sincere thanks to Council for the naming of an award in his honour. After all that Don has contributed to mathematics, education and OAME over the years, it is certainly well merited.

Ministry Directions

There was general discussion on provincial trends, tests, and directions, and OAME's role in all of this. Susan Stuart and members of the executive have been in touch with, talking to and writing people in the Ministry and the Assessment Office in an attempt to continue to influence policy decisions.

Local members of OAME have also been instrumental in expressing our concerns and suggestions in meetings with Joan Green and others regarding the grade 3 literacy/numeracy test for March of 1997. See the President's Message for more details, or contact your councillor.

The OAME has several methods to find out about, react to and influence Ministry agendas, including: executive correspondence on behalf of Council, the Specialization Years committee, our OAME representative to the OTF Curriculum Forum (me), and our members. Each of us should take the opportunity to get up-to-date information (from councillors, school board members or your MPP). As informed educators we CAN have an influence on what provincial expectations are, and therefore what goes on in the classroom.

OAME 1996 conference - LONDON

It was reported that the conference plans are proceeding at a sure pace with conference booklets distributed and registrations coming in. Will you win the balloon ride?

OAME 1997 conference - TORONTO

Welcome to "A World of Wonders", the theme for 1997, Watch for further announcements at the London conference and in the *Gazette*.

Leadership Conference 1996

The theme will be Assessment with workshops based on assessment documents at several levels. To ease the supply teacher crunch, it will run from Thursday evening November 7 to Saturday November 9. This is an opportunity for teachers to become resources for their home school or board, and therefore look to funding from their boards as a result.

Special Projects

The Transition Years document has gone into a second printing and the Formative Years project is field testing assessment activities.

Committees

Early/Formative Years

The *Abacus* is looking for articles and this committee suggested inviting Faculty of Education students to submit ideas*. They are also approaching OAJE to see if a joint article is possible. Where are the write-ups of all the wonderful things going on in YOUR classroom or that of a colleague?

(*Editor's note: To give credit where credit is due, Jennifer has proposed this every year of her three year tenure as editor of the Abacus. Moreover, she has sent personal letters to the appropriate professors at most of Ontario's Faculties of Education, urging them to make preparation of suitable materials for the *Abacus* a class assignment. To her and my dismay, the response has been non-existent.)

Transition Years

An article for the *Gazette, Problem Solving and Portfolios*, has been prepared by the Transition Years Committee, with more articles to come. Good work! (Editor's note: See pages 29–31 of this issue of the *Gazette*.)

Specialization Years

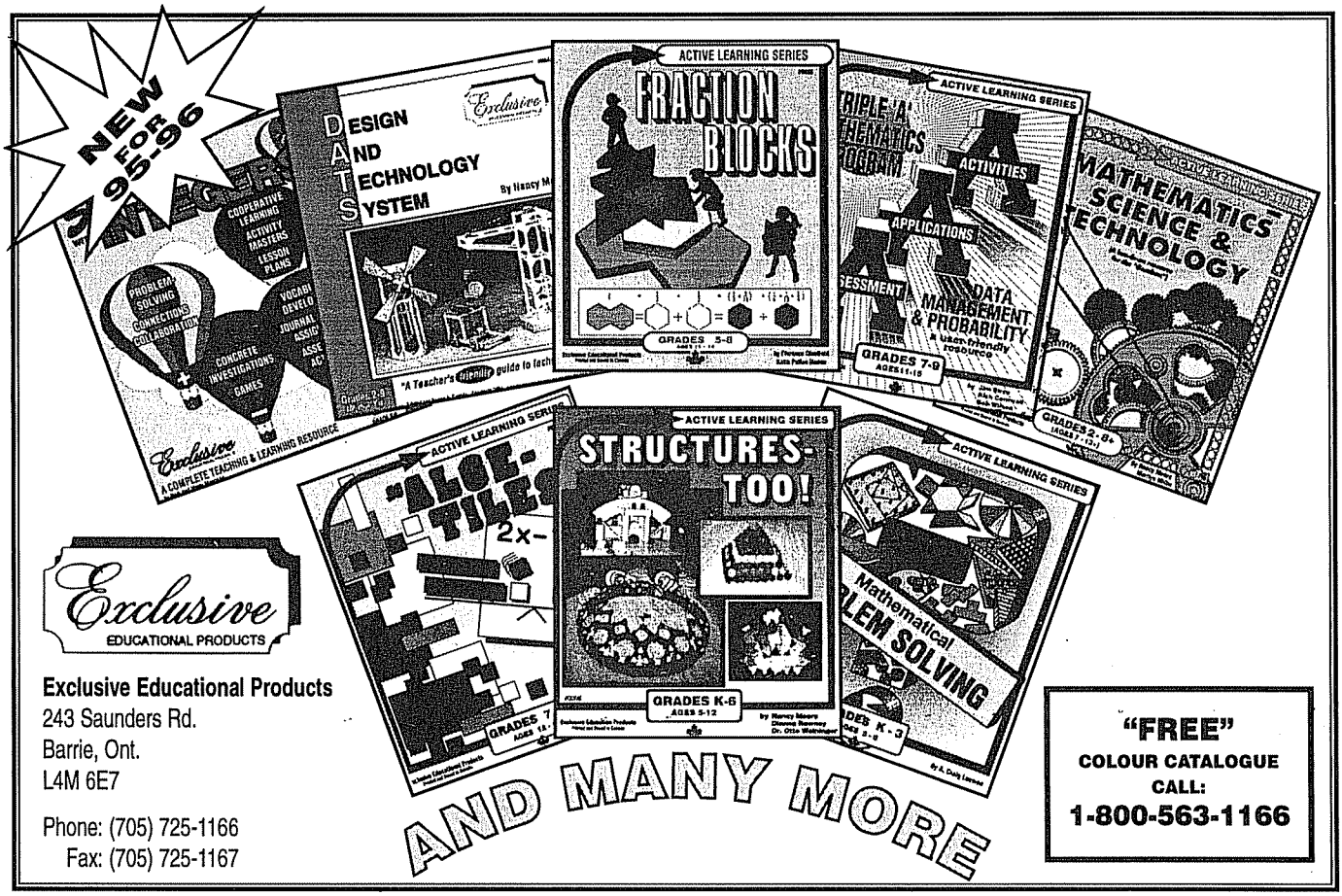
The committee presented a start to describing the graduation outcomes for students in mathematics. This early draft was sent to the Ministry as part of our on-going involvement in the Secondary Education Reform.

Promotions

Sample letters were prepared for OAME reps to send to lapsed members to entice them back, or see what changes are needed. A plan to target elementary schools will be presented in March.

Publications

The new editors for 1996-98 will be introduced to you in the June issue. Their tenure as editors will begin in September, 1996.



General

A voice from the past, Bob Robinson, a long time contributor to OAME and my predecessor for this column, has invited interested members to correspond with him at his new residence. You can contact me for his mail address for e-mail him at bobrob3333@aol.com.

Chapters

One regret I have is that most of the wonderful conferences planned by chapters may be over by the time you read this. Each chapter puts on some marvellous Professional Development. Keep in touch with your local chapter to be aware of their plans, before they happen.

Champ

Mathfest was a winner with over 180 educators attending. The eighth annual math contests for grades 7 to 11 involved almost 10 000 students from over 180 schools across the four boards. Fantastic! Spring Event is scheduled for April 11. The chapter is also holding a qualifying tournament to select their representatives at the Ontario Math Olympics and have loaned money to OAME to assist in the development of the Junior Assessment package. Thank you.

COMA

The first issue of their newsletter has been produced with another on the way. COMA has sponsored several forums which have been well attended, despite raging snowstorms. Their annual mini-conference took place Feb. 7 with Don Fraser speaking. Work is continuing with local Olympic competitions and a new logo is coming!

Golden Section

Plans were announced for the chapter's 25th anniversary spring conference in March at Brock University, a long and illustrious record.

GVMA

The latest publication, *"An Outcomes Based Grade 9 Mathematics Program"*, has been greeted with enthusiasm. The spring will see packages with new exams for OAC Finite and grade 9 destreamed. From the 1995 conference proceeds, GVMA has contributed money to: sponsor a GVMA Descartes Scholarship at U of W, support the August Problem Solving conference, and several writing projects (including OAME). The spring conference will be April 11. Applause, applause to all!

ISOMA

A well-received conference in the fall attracted more participants than ISOMA has members! Congratulations to the organizers who have to work with a very far flung crowd. In preparation is a mid-winter newsletter, the annual Euclid/Descartes preparation night and a mini-math Olympics to select their teams. Look for their web page, arriving at a network near you soon.

MAC²

Presenting a fall conference, developing a constitution, selecting some students for the Ontario Math Olympics from over 100 in the Barrie North Math Olympics (running since 1978), and surveying ways to network... have not kept this busy bunch from also planning their spring conference for April 18, 1996.

TEY²MS

The Chapter is getting fully geared up for the 1997 Annual, but still has been able to run a fall conference (160 attending), issue a second newsletter and prepare to select representatives for the Student Math Olympics.

NOMA

The first NOMA *Gazette* de IUAMNO came out in January and was simply full of articles, activities and good news. The fall conference is already in the planning stages for September 27/28, 1996 at E.S. Macdonald-Cartier in Sudbury. Reports ranged across NOMA 1995, the Trading Post , L'algébre avec les mains, a Theorist workshop and math web sites on the Internet. Well done!!

NWOMA

If you are looking for a dinnerlude, this is the chapter to contact. After a successful program (15 new members) in December presenting the new OAME Assessment booklet, another is planned for March on graphing calculators.

SAME

Following a small Murder Mystery evening in the fall, a spring conference entitled *"Chances are...?"* is in the offing for April 15. Two teachers at the conference will be lucky winners and have their way paid to London for the annual conference. SAME has also provided a bursary to add ten deserving elementary students to attend Saturday morning enrichment classes.

WOMA

This chapter has been working hard preparing for the annual conference in London, but has had time to loan money to the OAME for the Formative Years project.

Y⁴MA

A spring workshop for first year teachers is being hosted by past presidents and is called *"Believe Me It Will Work"*. April 30'th will see the Junior Olympics Regional Competition with four activities, dinner, door prizes and awards. The AGM will be May 30 and *"The Exponent"* is being revived.

You may send your comments to me on the Electronic Village (if it is working) to petesaar@enoreo.on.ca. Otherwise try me at bb304@freenet.toronto.on.ca. Cheers, Pierre.

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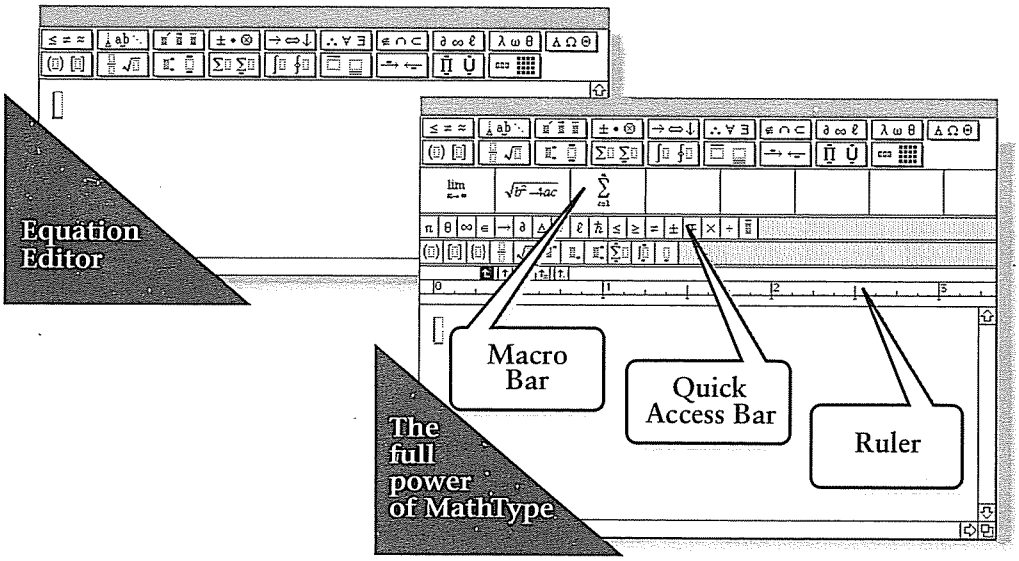
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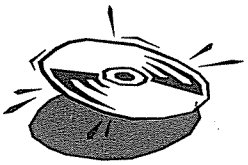
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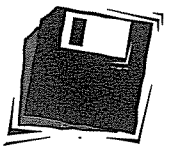
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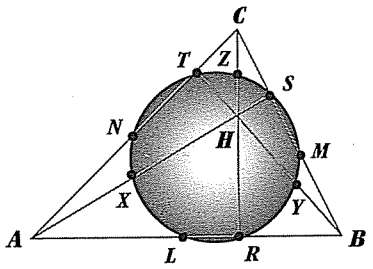


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