8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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GEOMETRIC TRANSFORMATIONS

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A Publication of the OAME/AOEM

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Much of our cultural life is visual. Aesthetic appreciation of art, architecture, music, and cultural artifacts (e.g., photographs, pottery, tiling patterns, artwork) include geometric objects. These can be analysed using geometric principles such as symmetry, perspective, scale, and spatial orientation. Geometric transformations enable students to extend notions of congruence and similarity and apply them to shapes in different orientations. So, to develop students' understanding of and reasoning about the geometric transformations of 2-D shapes, what kinds of tasks and problems should teachers use and in what sequence, from grade 1 to grade 6?

In this issue, the Research Summary and problems (with multiple solutions) focus on strategies for translations, rotations, reflections, and dilatations of 2-D shapes. The problems we provide are designed for use within a three-part, problem-solving lesson. Consider solving these geometric transformation problems yourself first, before examining the solutions we provide, in order to deepen your conceptual understanding of the mathematics you teach and to deepen your noticing of the range of mathematical thinking possible.

LINKS TO LITERATURE AND MANIPULATIVES



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Links can be made between literature, mathematics, and other curricular areas; for example, location and movement are related mapping skills developed in the Social Studies curriculum. Pop-up books can offer concrete experiences with transformations. For example, a picture slides when a tab is pulled or a picture turns when a wheel is rotated. If you look at the science and technology curriculum expectations for movement, you'll see more curricular connections.

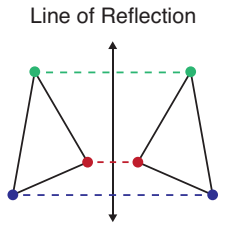
RESEARCH SUMMARY – ANALYSING AND CONSTRUCTING TRANSFORMATIONS

Randall (2005) explains that 2-D figures and 3-D shapes in space can be oriented an infinite number of ways. Only reflections result in a different orientation of an image compared to its pre-image. But translations, rotations, and reflections do not change the other attributes of a pre-image. The pre-image and image remain congruent and the same shape (similar). Shapes can be transformed to larger or smaller shapes (similarity) with proportional corresponding sides and congruent corresponding angles. Finally, shapes can be rotated around a point in less than one complete turn and land exactly on top of themselves (rotational symmetry).

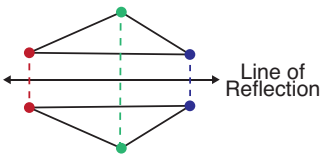
Some Key Concepts in Elementary Transformational Geometry

Transformational geometry is a general term to describe four specific ways to manipulate the shape of a point, a line, or figure. The original figure is called the pre-image and the final is the image under the transformations. Reflections, rotations, and translation are *isometries*; that is, the image and pre-image remain congruent (same side lengths and angle measurements). On the other hand, a dilatation is not an *isometry*.

A **reflection** is a correspondence between points and their image points so that each is transformed as a mirror image over a *line of reflection*. Every point is the same distance from the central line (line of reflection) and the reflected image has the same size and shape as the pre-image.

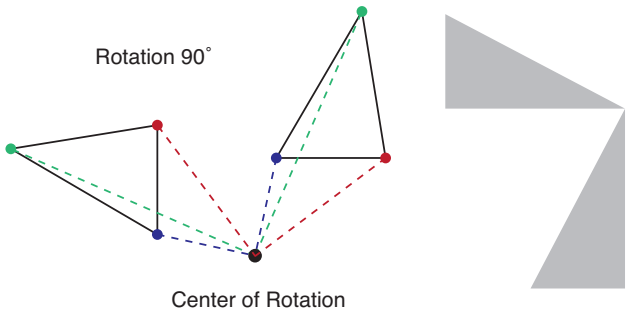


Horizontal Reflection (flips across)



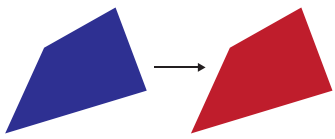
VerticalReflection (flips up/down)

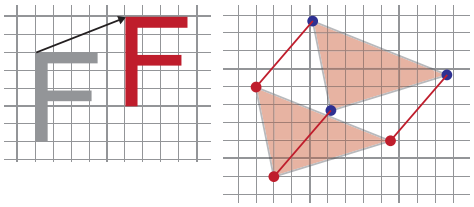
A **rotation** is a correspondence between points and their image points where one point is fixed and the image points are transformed to a new angle position. A rotation means turning a pre-image around a centre, where the distance from the centre to any point on the image stays the same. In fact, every image point lies on a circle around the centre. To rotate a figure you need a centre of rotation and an angle representing the measure of the rotation.



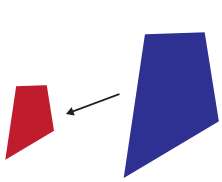
By convention, positive rotations go counter-clockwise (CCW) and negative rotations go clockwise (CW). The *centre of rotation* is the point around which the rotation is performed.

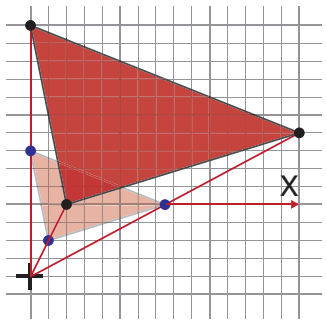
A **translation** is a correspondence between pre-image points and their image points so that each is the same distance in the same direction from its original pre-image point.





A **dilatation** (or **dilation**) produces an image that is the same shape as the pre-image, but results in either a stretch or shrink of the original figure, creating similar (not congruent) 2-D figures. The centre of dilation is a fixed point in the plane about which all points are expanded or contracted, in relation to a scale factor (or ratio).





Spatial Orientation, Visualization, and Imagery

According to Cross, Woods, and Schweingruber (2009), spatial thinking includes two main skills: spatial orientation and spatial visualization. Other important competencies include knowing how to represent spatial ideas and how and when to apply such abilities in solving problems. *Spatial orientation* involves knowing where one is and how to get around in the world. Children have cognitive systems, based on their own position and their movements through space and involve the use of external referents. *Spatial visualization* is understanding and performing imagined movements of 2-D figures and 3-D objects. To do this, you need to create a mental image and manipulate it, signifying the close relationship between the two cognitive abilities.

Common Errors, Misconceptions, and Instructional Strategies

Xistouri and Pitta-Pantazi (2011) report that while students' understanding of translations and reflections are equally difficult, rotations seem to be more difficult. Small (2008) identifies the following common errors. Students:

- confuse horizontal and vertical reflections; for example, they think that a vertical reflection is to be made across a vertical line of reflection, rather than for a horizontal reflection, the shape moves horizontally across a vertical line of reflection
- recognize only horizontal and vertical reflections and not consider reflections across a diagonal line of reflection
- confuse reflections and rotations, in a context that a reflection and a 180˚ rotation look similar at first glance, but are not in the same position.

2-D models, geoboards, grid paper, dot paper, mirrors and Miras™, and protractors are effective learning tools for developing students' understanding of geometric transformations. Drawing programs (found in word processing software), dynamic geometry software (Geometer's Sketchpad™) and geometry applets found on the Internet are useful technological tools.

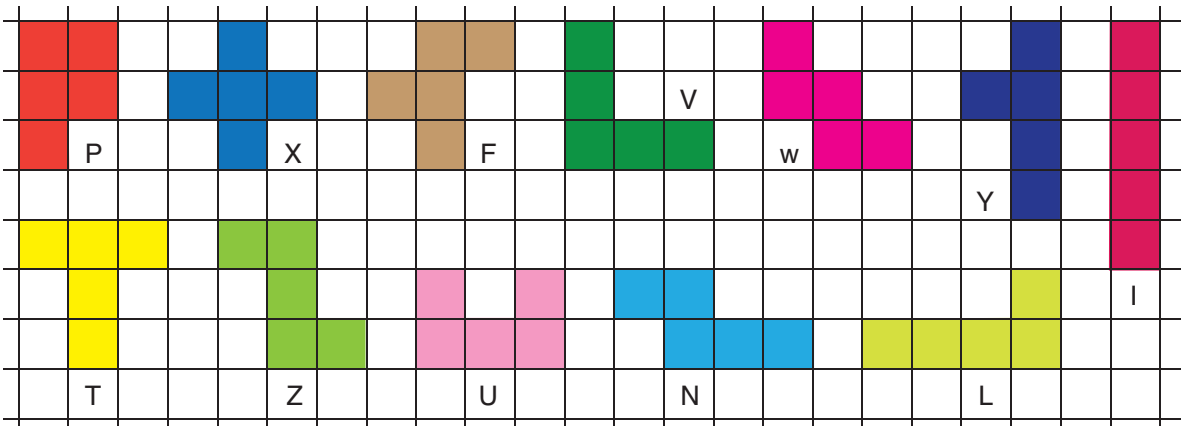
LINKS TO MANIPULATIVES: PENTOMINOES

Geometric puzzles and games like Tetris™, which uses tetrominoes (4 square unit shapes) provide experiences in visualizing geometric transformations.

A pentomino is 5 square unit shape formed by adjoining five squares with one another edge to edge. There are 12 different pentomino shapes.

Some pentominoes activities are as follows:

- Create a set of pentominoes by creating all the possible arrangements for polygons that have an area of 5 square tiles. Use square grid paper.
- How do you know that you have created all of the possible arrangements?
- How would you determine if a polygon is a transformation of another pentomino? (e.g., Do you already have this shape? How do you know?)
- Visualize which polygons or pentominoes fold into open boxes.



LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal:
- to see the development of mathematical knowledge, skills, and strategies across grade levels
- to determine the mathematical focus of the problems chosen for Before (activation), During (lesson), and After (Practice) parts of the lesson

BEFORE (Getting Started) – 5 to 10 minutes

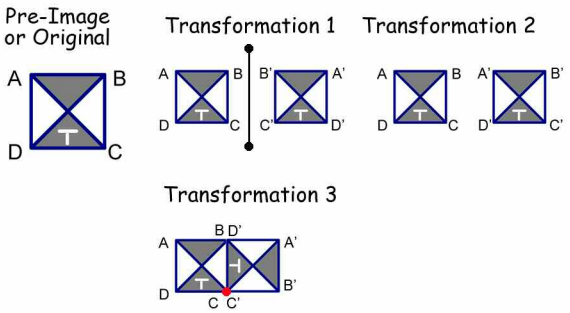
- Activating students' mathematical knowledge and experience using a prompt or problem that is directly relates to the mathematics in the lesson problem.
- Includes student responses to prompt or problem to highlight ideas, strategies, and tools that will provide access to lesson problem

AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates.
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit to students.

Comparing Transformations Problem:

Name and compare the transformations. What stays the same and what changes under each type of transformation?



Solution 1

#1 reflection; #2 translation; #3 rotation

Solution 2

All transformations result in an image that is the same size and shape as the pre-image. #2 and #3 have the same orientation – read labels CW ABCD and #1 has a different orientation because you read ABCD clockwise and A'B'C'D' is read counter-clockwise.

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to make a plan to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for whole class discussion) in pairs or in small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Transformations Doubles Problem:

Cory says, "If I double transformations of one kind, I can make it look like I did a single transformation of a different kind." True or false?

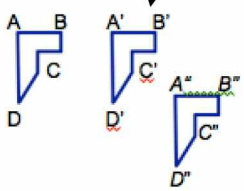
What information will we use to solve this problem?

Compare a double transformation to a single one (e.g., do two translations of a shape so it looks like it was reflected.)

-one translation followed by another -results in a translation by an amount that looks like it might be the sum of the two translations. (e.g., 2 cm right then 1 cm right and 1 cm down results in one translation 3 cm right and 1 down)

-a reflection through a vertical line followed by another reflection through a parallel vertical line -results in a translation but, at this point, without a grid, we cannot identify the details of the translation.

Solution 1

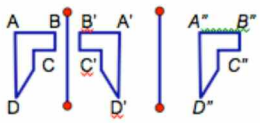


Comparing ABCD to A"B"C"D".
- *Same size*
- *Same shape*
- *ABCD clockwise and A"B"C"D" same*

Double translations are like a single translation.

2 Translations

Solution 2



Comparing ABCD to A"B"C"D".
- *Same size*
- *Same shape*
- *ABCD clockwise and A"B"C"D" same*

Double reflections in parallel lines are like a single translation.

2 Reflections

Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All annotations are records of the students' mathematical discussion.

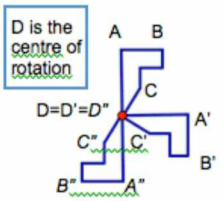
Grade 4	Grade 5	Grade 6
-identify, perform and describe reflections using a variety of tools -create and analyse symmetrical designs by reflecting a shape or shapes using a variety of tools	-identify, perform and describe translations, using a variety of tools -create and analyse designs by translating and/or reflecting a shape or shapes using a variety of tools	-identify, perform and describe rotations of 180^0 and clockwise and counter clockwise rotations of 90^0 with the centre of rotation inside or outside the shape -create and analyse designs made by reflecting, translating and/or rotating a shape or shapes by 90^0 or 180^0

We marked the vertices on the original ABCD, the first image A'B'C'D' and the second image A"B"C"D".

-a 90 degree rotation clockwise (centre on the figure) followed by another 90 degree rotation clockwise (same centre)
-results in a rotation through that centre of 180 degrees

-a reflection through a vertical line followed by another reflection through a line perpendicular to that first line
-results in a rotation of 180 degrees through a centre located at the intersection of the two reflection lines

Solution 3

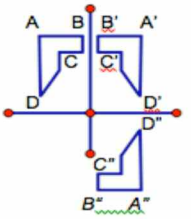


Comparing ABCD to A"B"C"D".
- *Same size*
- *Same shape*
- *ABCD clockwise and A"B"C"D" is the same*

Double rotations in the same centre of rotation is like a single rotation through a greater angle.

2 Rotations

Solution 4



Comparing ABCD to A"B"C"D".
- *Same size*
- *Same shape*
- *ABCD clockwise and A"B"C"D" same*

Double reflection in perpendicular lines looks like a single rotation in with centre at the intersection of the intersecting lines.

2 Reflections

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal.
- Teacher lists key ideas, strategies, and models of representation separately, so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal.

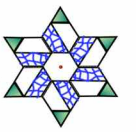
Highlights/Summary:

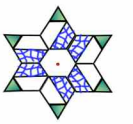
- Reflection - a shape is flipped across a line of reflection, facing the opposite direction, like being reflected in a mirror
- Rotation - a circular movement of a shape around a center (or point) of rotation, CW or CCW
- Translation - every point of the pre-image is moved the same distance in the same direction to form the image

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

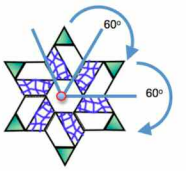
Designs Problem:

What transformations were used to make each design? Describe the lines of reflection and centres of rotation.



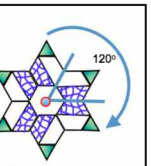


Solution 1



Design 1 - rotations around the centre shown as an open circle in the middle; 6 rotations of 60^o which makes a full rotation of 360^o.

Solution 2



Design 2 - reflection through the first slant line (at 1:00 on a clock); 2nd reflection through the horizontal line (about 3:00); 1st point (at 12:00;) 3^{rd} point (at 4:30), 5^{th} point (at 7:30) are rotations of 120^0

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.

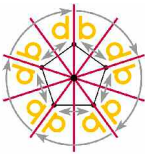
LET'S DO MATH – DETECTING SYMMETRY

Before (Getting Started)

Detecting Symmetry Problem:

Choose one of the two images and describe its symmetries.





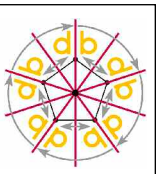
Solution 1

No line symmetry. Rotational symmetry of order 4 with rotation centre in the centre of the circle.



Solution 2

This image has 5 lines of symmetry that allow the figure to be folded onto itself. The lines go through each vertex of the pentagon and the midpoint of its opposite side.

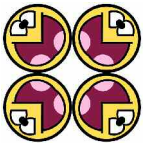


The image also has rotational symmetry of order 5. It can be rotated 5 times through angles of 72˚ to land back on top of itself.

During (Working On It)

Goofy Face Symmetry Problem:

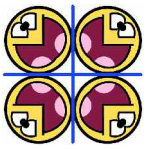
What symmetries are there in the logo? Justify your responses.



After (Consolidation)

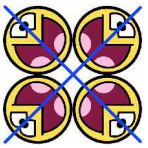
Solution 1

The image can be folded in a vertical line so it has horizontal symmetry. It also can be reflected in a horizontal line so it has vertical symmetry.



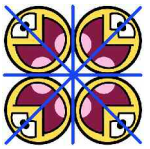
Solution 2

When the pre-image (in the top left) is rotated clockwise through 4 quarter-turns (90˚) through the dot in the centre it lands on the original image again. It has rotational symmetry of order 4.



Solution 3

I think the image can be folded across 4 different lines so it has line symmetry through vertical, horizontal, and two diagonal lines.



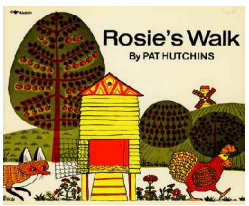
Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solution 3, and then solution 2?

- *Solution 1* - shows two lines of symmetry and names the symmetry appropriately.
- *Solution 3* - shows four lines of symmetry and names the symmetry appropriately.
- *Solution 2* - shows quarter-turn rotational symmetry of order 4. It takes four quarter turns to make the original figure rotate back onto itself.

LINKS TO LITERATURE:

Rosie's Walk by Pat Hutchins describes Rosie's walk around a farmyard (e.g., across the yard, around the pound, over the haycock, past the mill, through the fence, under the beehives).



- Ask the students to use cut outs from the book **http://www.kizclub.com/storypatterns/rosie.pdf** and retell the story using positional language in order to describe the relative locations of objects.
- Ask students to use the same or other positional language to create a classroom walk past the rocking chair, around the table, over the _____.
- Ask the students to make a map for a walk for Rosie in another setting such as the schoolyard.

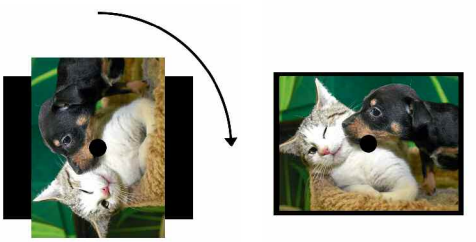
Before (Getting Started)

Problem A:

How can you rotate the picture around the black dot marking the centre so the image fits on top of the black mat?

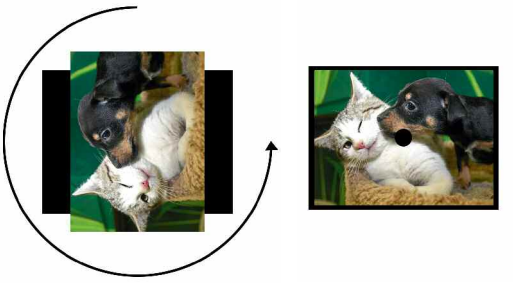


Solution 1



90˚ rotation clockwise or – 90˚.

Solution 2



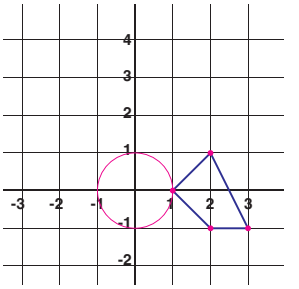
270˚ rotation counter-clockwise or +90˚.

During (Working On It)

Tracking Points During Rotations Problem B:

What does a rotation of 90˚ counter-clockwise with the rotation centre at the origin (0,0) of this quadrilateral pre-image look like?

Show how you know.



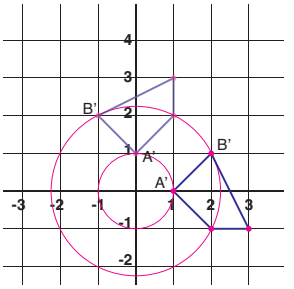
After (Consolidation)

Anticipating Student Responses:

Solution 1

I drew a circle through A (with centre at the origin) and marked a point for A' on the vertical axis as that represents a quarter turn to A'.

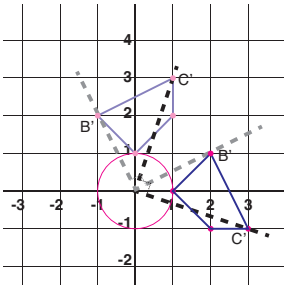
I then drew a concentric circle (same centre (0, 0)) through B and located B' up one unit and one unit left from A'.



Solution 2

I used 90˚ angles to track where B' and C' would be located. I could do the same for A' and D'.

My right angle must be centred at the origin every time.



Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solution 2?

- *Solution 1* - illustrates the property that every image point lies on a circle around the centre.
- *Solution 2* - illustrates the property that every image point is the same distance from the centre of rotation and is 90˚ from its pre-image.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Try these geometric transformation problems yourself first and then with your students.
- To suit your students' learning needs, vary the ways that you have students engage with transformations (e.g., use of concrete materials, such as pattern blocks, Power PolygonsTM; use of technology, such as Geometer's SketchpadTM dynamic geometry software; use of paper models).
- Think about how the context of the problem evokes visual images and serves as a tool for understanding and determining and using geometric transformations (e.g., mapping our dance steps; showing the movement of objects from one location to another; game-playing moves like in the game, TetrisTM).
- Provide students with non-examples of geometric transformations (e.g., Teri asks Nathan why he thinks that a shape was moved by a translation followed by a rotation. What would be a convincing argument?).
- Practise noticing the different strategies and the breadth of the mathematics your students use in their solutions. Take note of the number and kinds of solutions that show strategies identifying and showing geometric transformations. Which geometric transformations do students use more often? (e.g., translations, reflections, rotations, dilatations)
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit key mathematical ideas, strategies for geometric transformations when the students discuss and analyze each other's solutions.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below:

Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education Leadership, 7*(3), 9-24.

Cross, C. T., Woods, T. A., & Schweingruber, H. (2009). *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity.* National Academy Press.

Small, M. (2008). *Making math meaningful to Canadian students, K-8*. Toronto, ON: Nelson Education.

Small, M. (2007). *Geometry: Background and strategies*. Toronto, ON: Nelson Education.

Xistouri, X. & Pitta-Pantazi, D. (2011). Elementary students' transformational geometry abilities and cognitive style. Proceedings from CERME 7, Poland.

CLOSING

Our next issue will focus on key notions of data management and probability, how teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a problem solving-based teaching and learning lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or mkestell@sympatico.ca