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ABACUS

Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

Vol. 52 #1
September 2013

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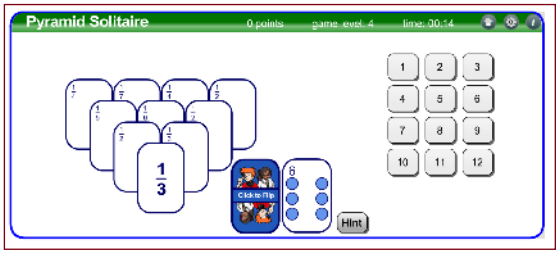
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See What's New with CLIPS? Games!



See E-Kwatee-Ons – From Mind Games to Algebra

This is the unit plan:

Day 1	E-kwatee-ons game in our heads.
Day 2	E-kwatee-ons game board on paper.
Day 3	E-kwatee-ons and writing each step as an arithmetic operation.
Day 4	Resolving E-kwatee-ons into the formal solving equations algorithm on paper.
Day …	Practice and extensions into more complicated equations as needed.
Day n	Practice, Equations Bingo, evaluation.

... we decompose terms, utilizing the Fibonacci recurrence relation and, of course, we use our assumption that: $F_5 = 5t$

$$\begin{aligned} F_{5(k+5)} &= F_{5k+4} + F_{5k+3} \\ &= (F_{5k+3} + F_{5k+2}) + (F_{5k+2} + F_{5k+1}) \\ &= (F_{5k+2} + F_{5k+1}) + (F_{5k} + F_{5k+1}) + (F_{5k} + F_{5k+1}) + F_{5k+1} \\ &= (F_{5k+1} + F_{5k}) + 4F_{5k+1} + 2F_{5k} \\ &= 5F_{5k+1} + 3F_{5k} \\ &= 5F_{5k+3} + 3(5t) \\ &= 5(F_{5k+3} + 3t), \textit{ as desired.} \end{aligned}$$

See Concatenation of Consecutive Fibonacci and Lucas Number: A Lesson in Patterns of Divisibility and Proof

The OAME 2013 Conference was a smashing success due in part to our exhibitors. TEAMS and OAME would like to thank our sponsors. THANK YOU all!

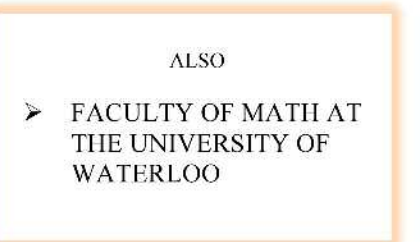










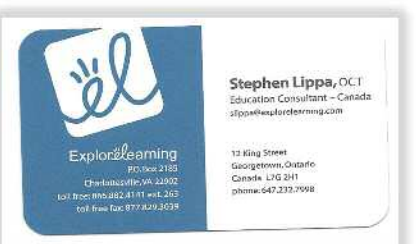










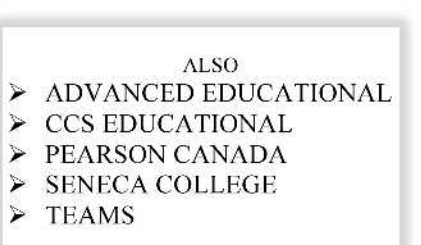
















Thank you to all the sponsors of the OAME 2013 Conference.



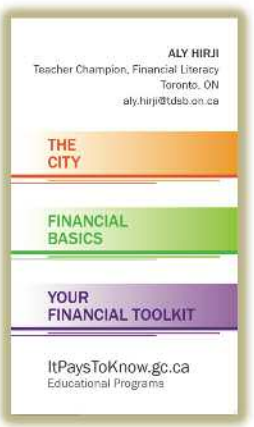


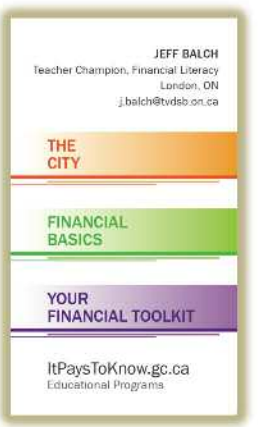


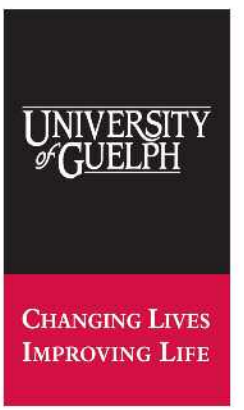
























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Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Immaculate K. Namukasa, along with **separate files for each of the text, graphics, and/or photographs**. Please e-mail all files to Immaculate K. Namukasa at **immaculate.namukasa @oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and e-mail address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME.

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The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

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Submission of Advertisements
Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** at the above address. Courier is recommended to avoid possible delays. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

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▲ Editor's Message

IMMACULATE KIZITO NAMUKASA
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Immaculate Kizito Namukasa is an associate professor at the University of Western Ontario. Her current research interests in mathematics education include mathematical thinking and activity, mathematics teacher education, non-routine problem solving and learning environments, complexity research framework, and critical mathematics education. Namukasa is also interested in globalization and internationalization of schools and schooling in general.

I trust that you all had a revitalizing summer and are set for a new academic year! In this issue, we welcome a new OAME Board, as well as congratulate the OAME Awards of Excellence winners. I am delighted to present in this edition the first message from the new OAME president, Sonia Ellison! I very much look forward to her leadership and, more specifically, to her subsequent *Gazette* messages. New in this issue is a special column featuring short articles from OAME 2014 Champions4Change speakers. The first article in the series, written by Irene McEvoy, with Marcia Charest and Jeremey Labrie, is about "Student Thinking." Mirela Ciobanu has joined Connie Quadrini in writing the "In the Middle" column, which was previously authored by Caroline Rosenbloom. In this issue, Mirela's column is entitled "From Blogging to Belonging to a Math Community." Many thanks, Mirela. With four continuous special features, "What's New with CLIPS?" is now a regular column. Many thanks to its four dedicated authors—Greg Clarke, Agnes Grafton, Ross Iseneger, and Markus Wolski, under Greg's leadership—for bringing to us, in every issue, new features added in CLIPS. In this *Gazette* edition, they share CLIPS games.

It is a great experience for an editor to work with dedicated authors. I would like to take this opportunity to thank both regular and occasional contributors for their continued contributions to the *Gazette*. Thanks also to the reviewers for regularly reviewing and evaluating articles to ensure that the *Gazette* is currently receiving a steady stream of quality articles. Several authors have always written back to express their appreciation of your

careful reading and critical suggestions you offer. While everything is terrific with respect to regular columns and articles, let me take this opportunity to encourage the *Gazette* readership to continue the engagement by contributing some shorter pieces, such as book reviews, letters to the editor and their responses, and reactions to articles published in past issues. In this issue, the OAME President encourages "each of you to share your learning by writing an article for the *Gazette*" (p. 4). This would be an outstanding way for this publication's readers and authors to engage in multi-way conversations.

With this first issue of the 52nd volume of the *Gazette*, I very much look forward to the coming issues in the new academic year. I wish you all a wonderful and productive school year! ▲

▲ Call for Manuscripts

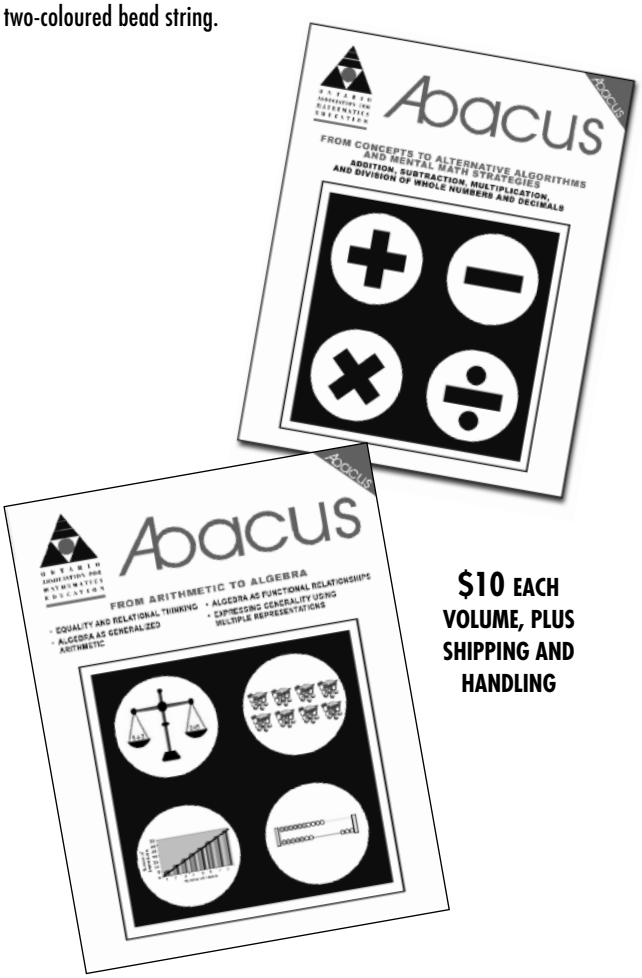
The *Ontario Mathematics Gazette* is inviting manuscripts for all grade levels.

Instructions for submission of manuscripts are found on page 1 of any *OMG*.

Contact the Editor for further details.

Abacus: A Compilation

Volume 46 – This resource for teachers includes the concepts of addition, subtraction, multiplication, and division of whole numbers and decimals, alternative algorithms, mental math strategies, manipulatives, and links to literature. It will be useful for exploring these aspects of teaching and learning personally or with a team of teachers. Includes one two-coloured bead string.



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▲ President's Message

SONIA ELLISON
E-MAIL: sonia.ellison@oame.on.ca



When I was President-Elect, the 2013–2014 school year seemed so far away, and yet we find ourselves here already. As I was preparing for the role of President of OAME, I began to look more closely at our organization. I knew that OAME is known for its outstanding support of mathematics educators around the province. I knew that OAME has created and continues to create amazing resources for classroom teachers. I knew that OAME's strength can be found in each and every one of its members. As an organization, we are strong and moving forward because we continue to strive for excellence in mathematics education.

We envision a learning environment where all students do, see, hear, and touch mathematics in profound and meaningful ways. In order for this to happen, we need to learn as much as we can about how students learn math and how we can better teach our students. Our learning never stops. This learning continues to happen at the Leadership Conference, the Annual Conference, and Chapter events throughout the year. I encourage each of you to share your learning by writing an article for the *Gazette* or leading a session at the conference. We learn more when we learn together.

As we look ahead to the future of mathematics education, let us never forget that we are not alone in our awesome responsibility of bringing quality mathematics education to all students. We look to our mission statement to remind us that we **strive to promote excellence in mathematics education throughout the Province of Ontario**. In schools and classrooms around the province, students are benefitting from the great work of our members. We believe that learning is enhanced when students experience a variety of instructional approaches, ranging from direct instruction to inquiry-based learning.

Twenty-first-century fluencies in communication, collaboration, critical thinking, and problem solving can be easily connected to our numeracy goals. According to *EQAO's Provincial Elementary School Report* (2012), students need to:

> execute mathematical operations accurately; decide what needs to be measured or calculated; use relevant mathematical knowledge and skills to explain or clarify ideas. Students are expected to apply several cognitive skills and mathematical processes or procedures in five strands. Students are also expected to select and apply a variety of procedures, connect mathematical ideas, select and use a variety of types of reasoning, and communicate their thinking in writing using mathematical language and conventions (p. 97). (**www.eqao.com/Provincial Report/.%5CFiles%5C12%5CPDF/EQAO_PR_PJe_21_CenturySkills_0912_Web.pdf**)

As we endeavour to equip our students with the skills necessary for success in their current grade and beyond, OAME will continue to look at engaging teachers and students in developing twenty-first-century skills for teaching and learning.

This year's Leadership Conference will have Karen Hume as the keynote and workshop facilitator. Karen will be using her book, *Tuned Out*, to help math educator engage their students by highlighting twenty-first-century teaching and learning. In *Tuned Out*, Karen discusses five aspects of schooling where teachers can make a positive difference to student engagement. They are: Competence, Creativity, Community, Context, and Challenge. A central premise of *Tuned Out* is that if something is important to student engagement, it is equally important to teacher engagement. Therefore, some of the suggested actions are for students, some are for teachers, and some are for use by facilitators of adult learning. Please join us November 8–9 at the Leadership Conference for an opportunity to learn from and with other math educators. Other opportunities for developing teaching strategies for the twenty-first century will take place at the Annual Conference in May. Be sure to visit **oame2014.ca** for more information.

It is indeed my honour and privilege to represent each one of you throughout my term as President. I look forward to learning with you this year. If you have any comments or concerns, please contact me at **sonia.ellison@oame.on.ca**.

All the best for the 2013–2014 school year,

Sonia Ellison ▲

OAME/AOEM AWARDS FOR EXCELLENCE IN MATHEMATICS EDUCATION



The purpose of the Ontario Association for Mathematics Education's Awards is to formally recognize individuals and educators who demonstrate an outstanding contribution to mathematics education in Ontario. The successful candidate of each award enhances the learning environment, providing opportunity for students to do, see, hear, and touch mathematics in a profound and meaningful way. If you know someone who qualifies for this honour, please nominate her/him/them for one of the following awards.

OAME AWARDS

OAME/AOEM Life Membership Award recognizes an educator who has contributed in a significant way to OAME/AOEM; demonstrates outstanding leadership in mathematics education; and has accumulated ten or more years of membership in OAME/AOEM.

Award for Outstanding Contribution to OAME/AOEM and Mathematics in Ontario recognizes an individual who has made a significant contribution to OAME/AOEM, but who is not necessarily an educator or a member of OAME/AOEM.

Award for Exceptional and Creative Teaching in Elementary Mathematics recognizes an exceptional and creative elementary teacher who demonstrates excellence in mathematics education and contributes to the overall development of students.

Award for Exceptional and Creative Teaching in Secondary Mathematics recognizes an exceptional and creative secondary teacher who demonstrates excellence in mathematics education and contributes to the overall development of students.

Award for Leadership in Mathematics Education recognizes an educator who has demonstrated leadership by contributing in a significant way to the development of mathematics teachers and enhancing mathematics education in Ontario.

Secondary School Department Award for Exceptional and Collaborative Mathematics Teaching recognizes a secondary mathematics department which fosters collegiality, teamwork, and excellent classroom teaching; contributes to the overall development of students; and demonstrates leadership in the mathematics education community.

Elementary School Staff Award for Exceptional and Collaborative Mathematics Teaching recognizes an elementary school (a school accommodating any or all of Grades JK–8), whose staff fosters collaborative and excellent mathematics teaching; contributes to the overall development of student learning of mathematics; and demonstrates leadership in the mathematics community.

OAME/AOEM Awards Nomination Process

1. Complete the Award Nomination Cover Page appropriate to the specific award, found at **www.oame.on.ca**, or contact the Executive Directors at **EDs@oame.on.ca** for an electronic version. You may include a one-page letter of recommendation.
2. Provide up to three letters of support, of no more than two pages each. These supporting documents may be gathered from a variety of sources to reveal a full picture of the nominee. The support documents will provide precise information about the nominee's contributions as they relate primarily to the specific Award.
3. Include the nominee's Curriculum Vitae of up to five pages. (C.V. not required for Secondary School Department Award and Elementary School Staff award)
4. Send the completed cover page and supporting documents by January 15, 2014 to: OAME Executive Directors, 4751 Mack Avenue, Town of Plympton-Wyoming, ON N0N 1J6, or email: **EDs@oame.on.ca**

DEADLINE: Nominations are due by January 15, 2014 to the Executive Directors (contact info above).
The awards will be presented at the Annual OAME Conference in May 8, 2014.

▲ What's the Problem? A Plethora of Ones

SHAWN GODIN
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Shawn is the head of Mathematics, Business Studies, and Library at Cairine Wilson Secondary School in Orleans, Ontario. He is a past member of the OAME Board of Directors and is currently Editor-in-Chief of the Canadian Mathematical Society's problem-solving journal, Crux Mathematicorum*.*

Welcome back to another new school year, problem solvers. Last time, I left you with the following problem: Calculate: $6^2 - 5^2$, $56^2 - 45^2$, and $556^2 - 445^2$. Find and describe the pattern in the questions and answers and prove that the pattern continues. As always, see if you can find other similar patterns.

This problem is question #149 from *Five Hundred Mathematical Challenges* by Edward J. Barbeau, Murray S. Klamkin, and William O. J. Moser, published by the Mathematical Association of America. Ed Barbeau of the University of Toronto is well known to many teachers in Ontario for his outreach activities, as well as his presentations at past OAME conferences. This book is a treasure trove of nice problems.

The challenge at hand provides us with an opportunity to explore patterns, make some conjectures, and possibly find some new patterns and prove them. First, calculating the given expressions and extending the pattern, we get:

$$6^2 - 5^2 = 11,$$
$$56^2 - 45^2 = 1111,$$
$$556^2 - 445^2 = 111111,$$
$$5556^2 - 4445^2 = 11111111,$$
$$55556^2 - 44445^2 = 111111111.$$

The pattern seems obvious. The first number is several 5's followed by one 6, and the second number is several 4's followed by one five. Both of the original two numbers have the same number of digits. The result is a number with twice as many digits, all 1's, which, in recreational mathematics, is referred to as a *repunit*, which belongs to the larger class of repdigit numbers.

Examining the problem, we notice that if we ignore the exponents, then:

$$6 - 5 = 1,$$
$$56 - 45 = 11,$$
$$556 - 445 = 111,$$
$$5556 - 4445 = 1111,$$
$$55556 - 44445 = 11111.$$

This seems to relate to the original problem, yet if we pick similar numbers, such as $732 - 621 = 111$, we get $732^2 - 621^2 = 150\,183$, which doesn't follow the same pattern. It appears that we will have to dig a little deeper.

Looking at the numbers with all the 4's and 5's suggests a pattern in their sum as well. Notice that:

$$6 + 5 = 11,$$
$$56 + 45 = 101,$$
$$556 + 445 = 1001,$$
$$5556 + 4445 = 10001,$$
$$55556 + 44445 = 100001,$$

and also:

$$1 \times 11 = 11,$$
$$11 \times 101 = 1111,$$
$$111 \times 1001 = 111111,$$
$$1111 \times 10001 = 11111111,$$
$$11111 \times 100001 = 1111111111.$$

Recall, the identity for factoring a difference of squares:

$$x^2 - y^2 = (x - y)(x + y)$$

Our pattern is a consequence of this identity. Notice that, since $667 - 334 = 333$ and $667 + 334 = 1001$, then $667^2 - 334^2 = 333333$. Similarly, since $56173 - 43828 = 12345$ and $56173 + 43828 = 100001$, then $56173^2 - 43828^2 = 1234512345$.

This problem reinforces the idea that an algebraic identity is a statement that is true for *all* values of the variables involved. You can have similar fun with the identities $(x + y)^2 = x^2 + 2xy + y^2$ and $(x - y)^2 = x^2 - 2xy + y^2$ and by examining $11^2, 111^2, 1111^2$, ..., and , 99^2, 999^2, 9999^2, ..., respectively. Have fun with your algebraic identities!

Now for your homework:

Given the matrix below,

10	15	16
3	8	9
5	10	11

perform the following operations.

- Circle any number.
- Cross out all numbers in the same row or column

as the number you circled.

- Circle a number that hasn't been circled or crossed out.
- Cross out all numbers in the same row or column as the number you circled.
- Circle the last remaining number.
- Add up the three circled numbers.

Next issue, I will reveal the sum of your circled numbers! Try to figure out what is going on.

Until next time, happy problem solving! ▲

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▲ Technology Corner – Just a Few (More) Good Apps

MARY BOURASSA
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Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

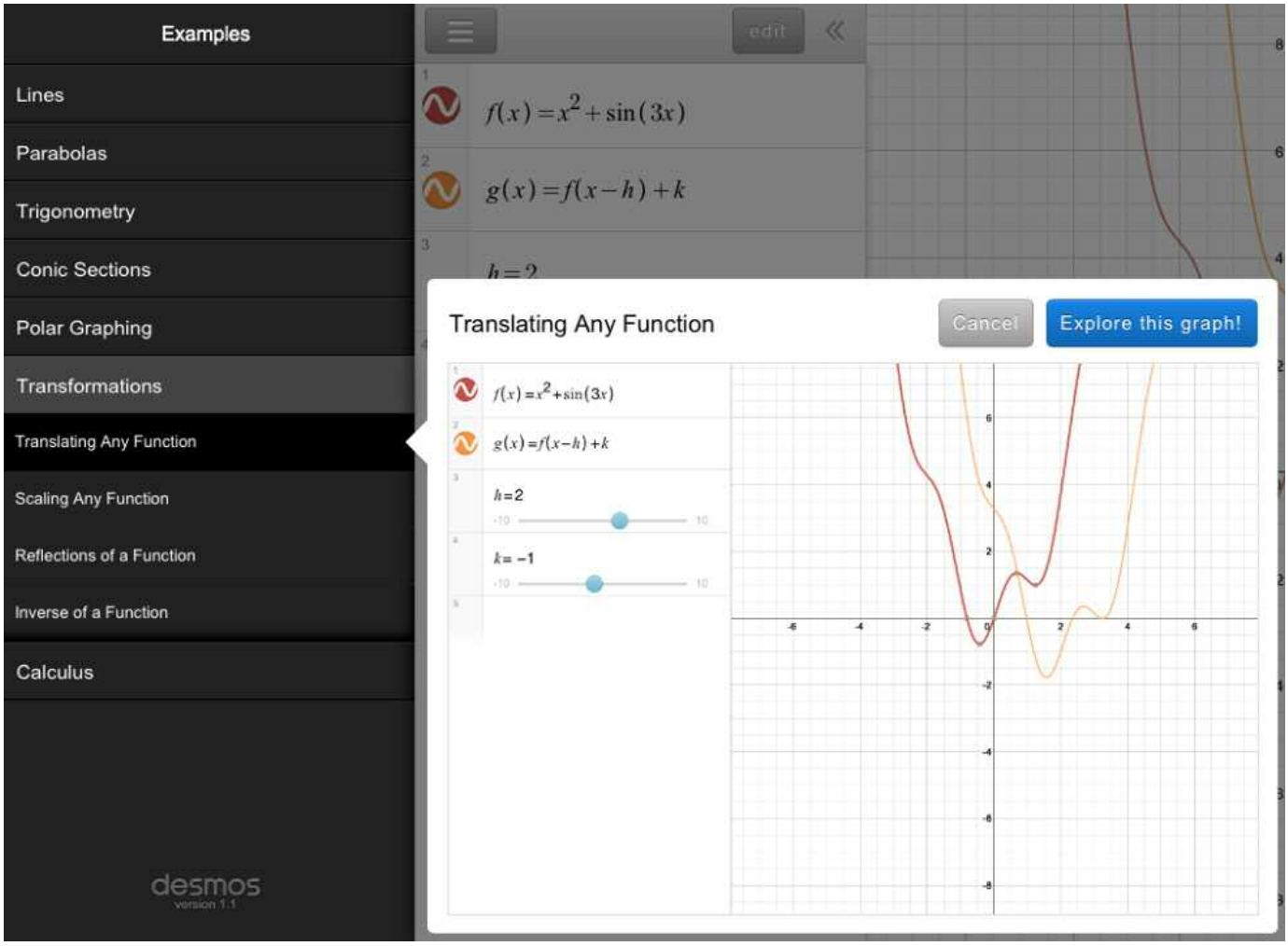
As more of our students are coming to class with their own connected devices, I thought it would be good to once again showcase a few math-related apps. If you search for "math" in the App Store or Play Store, you can spend the rest of the day looking at the results! I have tried to feature some for elementary, some for secondary, and have included some for Android as well.

I will start with an app for the very young—those learning to add and subtract: The Math Tree by Bellamon (iOS – $0.99). To play this game, you have to add items to the tree or subtract them from the tree. This involves listening to the instructions, recognizing the correct item(s), and helps develop motor control when moving the objects. You then need to determine how many items remain in the tree. The prompting is good and is provided as needed. The best quotation I can give comes from my 4-year-old: "I love this game!"



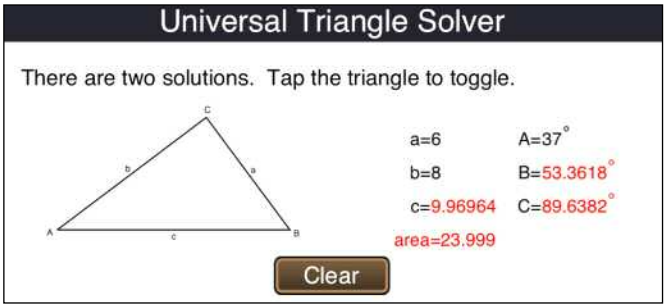
Counting Caterpillar (iOS – $1.99) has also been recommended to me along with Move the Turtle (iOS – $2.99), Motion Math (iOS – free/various prices), and Questimate! (iOS – free).

In June, Desmos (**www.desmos.com**) came out with

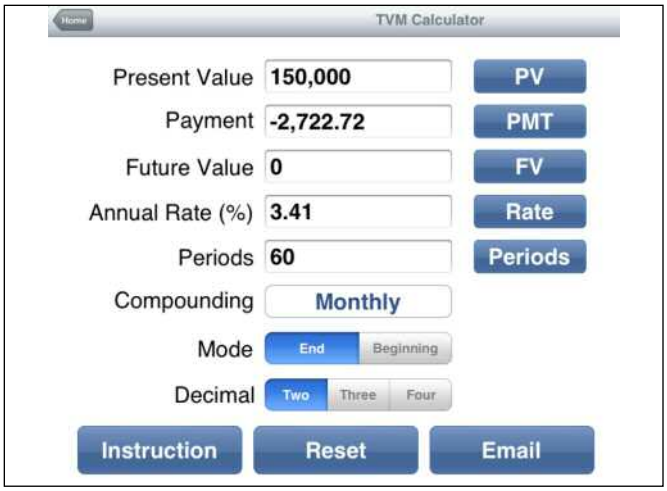


a free iOS app to accompany their free online graphing software. If you have yet to check out Desmos, do it now! This is fantastic software that is continually getting better. The app is a great tool for exploring. Be warned that there are some things that you cannot do on the app compared to the software, such as changing the colour of graphs, saving, printing, and sharing. There is an examples tab at the top left of the screen that allows you to explore graphs through the use of sliders. Everything is ready to go! These examples correspond to the "Explorations" on the online version of Desmos.

Many Triangle Solver (iOS and Android) apps exist. The one shown here can be useful when exploring properties of triangles, checking solutions, and demonstrating the ambiguous case. You must enter three pieces of data, either side lengths or angle measures, and the rest of the triangle is solved. The number of solutions, along with the triangle's area, will be displayed.



Several secondary math courses make use of a TVM (Time–Value–Money) calculator. There are many TVM apps (iOS and Android) that students can use on their phone at school and at home. This is also very handy to have when negotiating a loan!



Fluidity Software Inc. has a line of iPad apps including FluidMathFree (allows 5 minutes of use before restarting), and a free version of FluidMath 2014 ($14.99) that allows handwritten input that dynamically connects to graphs and tables and has integrated CAS capabilities. FluidMath is definitely worth a look.

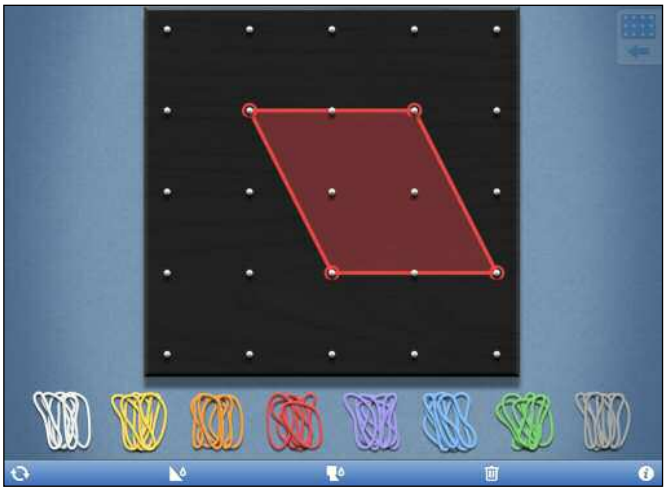


MyScript Calculator is a free handwriting calculator (iOS or Android). It converts what you have written to typeset, then evaluates it. I can see this being especially useful to students who struggle with calculator keys or with how to correctly enter an expression in a calculator.

ShowMe (iOS – free) allows you to write on your iPad and simultaneously record audio so you can explain what you are showing. You can then share the file.

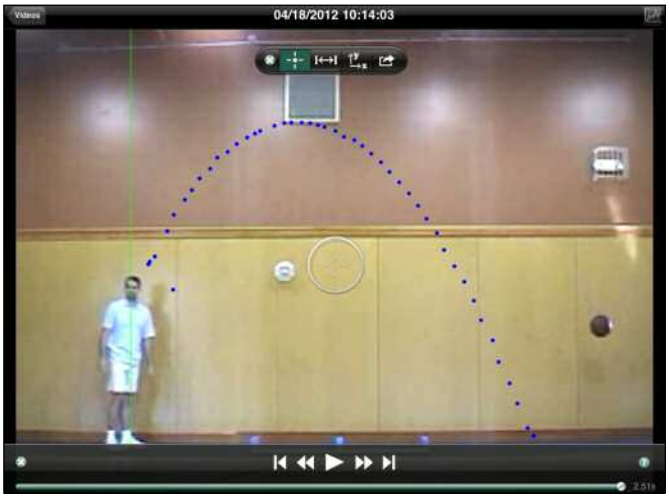
Algebra Touch (iOS – $2.99) was recommended for use with autistic students. The app description states, “Algebra Touch will refresh your skills using touch-based techniques built from ground up.” The kinesthetic learner will likely benefit from this app.

There are a number of apps that can replace their physical counterparts. For example, the Geoboard app (iOS – free) acts just like a real geoboard, and the clinometer app (iOS and Android – free) gives precise angle measurements. There are other apps that simulate a tuning fork that may also give you the frequency of the note (Tuning Fork/Pitch Perfect). This is great for one of my favourite activities—collecting sound data to model a sinusoidal.





Video Physics is an app from Vernier that allows you to capture video, then mark points along the path of interest. An example of a parabolic path is shown. Vernier has other software or Vernier Graphical Analysis app (iOS – $4.99) that can help you analyze your data, but you can also take a picture and place it in the background of other graphing software.



SmartTools (Android – free individually) is a set of apps that measure length, angle, slope, level, and thread. They also have apps for a compass, sound meter, metal detector, speed gun, vibration meter, protractor,



flashlight, magnifier, and mirror—all kinds of cool tools for your math classroom!

Physics Gizmo (Android – free) is a promising-looking app. It has an accelerometer and photogate. I am hopeful that this, or other such apps, will be able to replace CBRs in the near future.

Finally, the all-in-one graphing, geometry, and algebra TI-Nspire CAS app (iPad – $29.99) is a great tool to explore mathematics and science. It is equivalent to the TI-Nspire software and handheld with the addition of the touchscreen.

There are so many more apps out there. If you have a favourite that is not listed, please email or tweet me (**@MaryBourassa**) with your recommendation! ▲

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▲ Hey, It's Elementary – Observing Mathematics: Learning to See What WE are Looking at

LYNDA COLGAN
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Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.

It is absolutely amazing how one can *look* at something almost every day, but never really *see* it.

Think about a stop sign. How many sides does it have? Six or eight? Imagine a warning sign (alerting a driver to dangerous or unusual conditions ahead, such as a curve, turn, dip, or sideroad). What shape is it? In and around Kingston, it is not unusual to see a road sign that shows the route to a ferry service. What are the specific characteristics of this particular sign? What is the shape of a school zone sign?

Most people know that a stop sign has a red background with white text (Figure 1). Many of us pay less attention to the fact that the familiar shape is actually a polygon called a regular octagon, that is, an eight-sided figure in which all the sides are the same length.



Figure 1

A warning sign, whether about deer (Figure 2), slippery pavement, a narrow bridge or a sharp turn ahead, is typically a bright yellow four-sided polygon (commonly known as a square) that is oriented in what is commonly called the "diamond position" because it is rotated 90° to the right or the left about its centre point. On these signs, text or images are always black.



Figure 2



Figure 3

The information sign that directs travellers to a ferry (a common sight around Kingston and The Islands) is a green square in standard orientation with a white image (Figure 4).



Figure 4

To let drivers know that they need to slow down, drive with care, and watch for children, residential neighbourhoods have "school zone" signage (Figure 5). These signs are also polygons, specifically pentagons, five-sided geometric shapes, but they are not regular pentagons (as in Figure 6) because not all sides are equal in length.



Figure 5



Figure 6

You undoubtedly see manhole covers (Figure 7) every day, whether walking or driving. What shape are they? If you answered "circular," you are correct, but do you know why they are circular? A manhole cover rests on a small lip inside the hole. A square manhole cover could be picked up, rotated, and dropped diagonally through its own hole. The same is true for manhole covers made from rectangles, pentagons, or hexagons—all of them have diagonals that are longer than at least one side, which means the lid can be maneuvered down through the hole diagonally, and fall. A cast-iron cover can weigh up to 95 kilograms, and given that a sewer can be 6 metres deep, this makes for a deadly combination if a cover were to land on a worker below. However, a circular manhole cover cannot fall into the hole no matter how it is rotated because its diameter is the same all around.



Figure 7

I got to thinking about the myriad of everyday objects that are not only "invisible" because of their ubiquity, but remote from 2D representations in elementary math class. These simple objects, all of which embody and illustrate mathematical characteristics and vocabulary, represent some of the "applied mathematics" of the elementary curriculum.

As classroom teachers, we have, at one time or other, shaken our heads when children can recognize geometric shapes in a textbook, but struggle to identify those same shapes in "real world" contexts. The child who can name every 2D shape drawn in outline form is perplexed when a standard equilateral triangle is in non-standard orientation (think about the confusion caused by the great "Diamond Shreddie" cereal campaign a few years ago), and may be befuddled when asked to identify the geometric shape from a photograph of a hexagonal paving stone.

The famous physician, philosopher, and educator, Maria Montessori, once said, "We cannot create observers by saying 'observe,' but by giving them the power and the means for this observation."

I interpret this to mean that as elementary classroom teachers of mathematics, we must nurture in our students the development of strategies to become keen observers of their natural and man-made environments and support the development of mathematical language to detail the richness of open-ended observation and research exercises.

Imagine opening a class by holding a soccer ball in one's hand. Children (many of whom play soccer from a very young age) and adults have *looked* at soccer balls many times, without *seeing*—the soccer ball (Figure 8).

Children and adults alike may quickly respond that there are two kinds of patches on a soccer ball, black and white, but without a careful look at the object, they are likely not to know whether they are the same or different geometric shapes.



Figure 8

A careful examination of a real soccer ball, and a little research, can turn into a rich opportunity to learn mathematics for elementary school children. Did you know that since about 1950, soccer balls have been stitched or glued together from 32 patches of material—12 of them five-sided polygons and 20 of them six-sided polygons. These patches are arranged so that each pentagon is surrounded by hexagons. Traditionally, the pentagons are coloured black and the hexagons white. A soccer ball is modelled on a geometric shape called a "truncated icosahedron" (shown in Figure 9), which has 60 vertices, 32 faces, and 90 equal edges. The faces on a soccer ball, however, are curved rather than flat so that the ball is roughly spherical (Figure 10).



Figure 9



Figure 10

To imagine the skeleton of a soccer ball, we turn from mathematics to science. In 1996, the Nobel Prize in Chemistry was awarded to three chemists for their discovery of a carbon molecule (Carbon 60) known as the "buckyball," or *Buckminsterfullerene*, or just *fullerene*. The buckyball (Figure 11) was named after the famed architect R. Buckminster Fuller, whose unique structures, called Geodesic Domes (Figure 12), resembled… soccer balls.

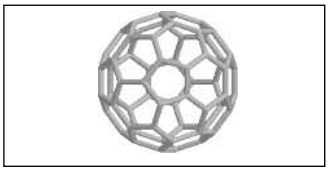


Figure 11

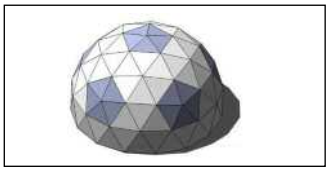


Figure 12

To understand why two distinct shapes are needed to make a soccer ball, we turn to transformational geometry… and the mathematics of tessellations. A tessellation is a tiling (e.g., of a floor that goes on forever) with geometric shapes so that there is no overlapping of shapes nor gaps between shapes. If we look at a "sheet" of hexagons (Figure 13), we can see that hexagons form a tiling that could continue infinitely. But a "sheet" of hexagons, like a piece of paper, cannot be folded up from the corners to form a sphere. On the other hand, we cannot tile or fill a "sheet" with pentagons (Figure 14) because there will always be spaces between the shapes.

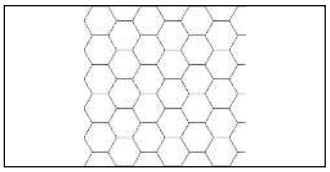


Figure 13

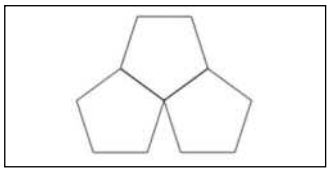


Figure 14

However, by combining the hexagons with the pentagons, we end up with the perfect configuration to fold up to form an approximate sphere… in this case, the soccer ball (Figure 15).

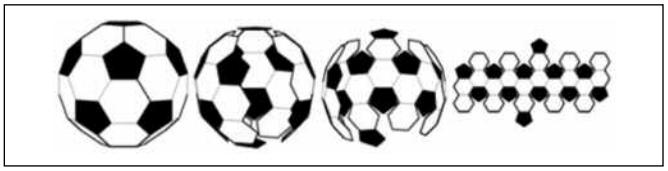


Figure 15

Imagine the positive reactions and motivation if children had the opportunity to create their own soccer ball by paper-folding, using a geometric net (Figure 16). What scale factor would be required to generate a paper soccer ball with approximately the same volume as a regulation soccer ball?

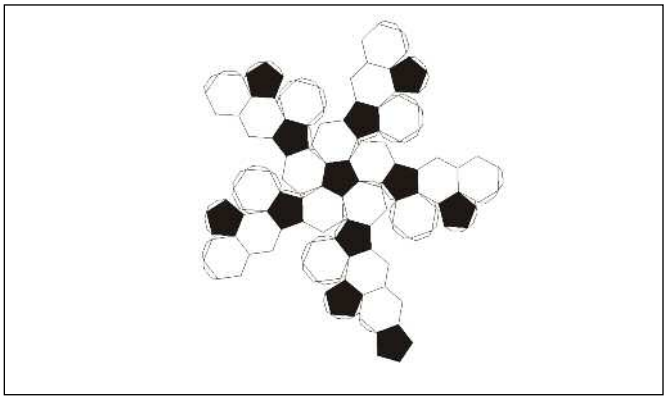


Figure 16

Observation. Open-ended problem solving. Multi-modal learning opportunities. This rich learning task—based on that most humble of objects… the soccer ball—is scalable. Given that many of today's classrooms are abundant in technology, from digital and document cameras and interactive whiteboards to Internet access and Moodles (Modular Object-Oriented Dynamic Learning Environments), similar investigations, constructions, and deconstructions can easily be generated in the form of teacher-facilitated or student-centred lessons/demonstrations/investigations/assessments.

If you are looking for a place to start to develop rich real-world learning tasks for your own students, I would strongly suggest the web resource **http://educ.queensu.ca/community/outreachcentre/resources/Recognizing_Geometry_in_Real_Life.pdf**.

This package of photographs and questions was developed by a gifted 2012 Queen's BEd grad, Samantha Laforet, who introduces her materials in this way:

> These are intended to be a starting point to help your class apply what they have learned in the Geometry and Spatial Sense strand to real-life situations. There are ten pictures included, and each one has suggestions for the teacher regarding how to use each picture in a Grade 1, 2, or 3 classroom. The pictures will offer your students a way to engage themselves further in learning by applying their mathematical knowledge about shapes, polygons, and three-dimensional figures to a new situation—recognizing geometry when they see it in the real world.

I hope that you find that Sam's stunning photographs will be the catalyst for many other inspired and inspiring observation-based lessons that will help children to open the windows of knowledge around them. ▲

▲ MB4T... The Right-Size Piece: A Key Idea in Adding Fractions (and Algebraic Expressions!)

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Ann Kajander is an experienced classroom teacher currently teaching mathematics and mathematics education at Lakehead University. Her research interests relate to teachers' enhanced learning of mathematical concepts. She and her classroom-teacher colleague, Tom Boland, are currently writing a new book for teachers called Math Without Memorization: Big Ideas in Modelling and Reasoning for Elementary Teachers and Other Learners.

Students need to be exposed to models and reasoning as a way to develop and support their deep understanding of fraction concepts, and such work has become the norm in many classrooms. In particular, a strong conceptual connection needs to be made between initial hands-on work and subsequent procedure development, rather than simply moving from one to the other. The key point is that procedures and general methods are actually the *outcomes* of our models and reasoning work, and should be developed this way—that is, as evolving from, and making use of, models and reasoning. The benefit of such an approach is that the models remain necessary and connected to the procedures, and that the procedures, as necessary outcomes of the models, can be justified. The mathematical background required to make such connections with respect to fractions procedures will be the topic of this year's column series, beginning with the concept of the common denominator.

From work with manipulatives, children may have found that some fractions need to be re-represented before the parts can be added (or subtracted). As children work concretely to solve such problems by re-representing fractions (for example, moving from halves and thirds to sixths), they might be asked, "How did you know to use sixths?" While at first, children will simply use trial and error, after a while, they will begin to recognize a connection among the denominator values. Sufficient hands-on work with well-chosen examples is the key to early recognition of the denominator relationships. Only when such connections begin to emerge should consideration be given to exploration of more formal methods.

An initial observation to encourage students to notice is that, to find a denominator size ("fraction piece") that works to add or subtract fractions with unlike denominators, we can always multiply the denominator values, which will give the numeric value of a suitable new denominator. For example, when adding $\frac{1}{2}$ and $\frac{1}{3}$, we can use sixths, and for $\frac{2}{3}$ and $\frac{3}{4}$, we can use twelfths —we observe that $2 \times 3 = 6$ and $3 \times 4 = 12$. We can reason about how this makes sense; to find a part or fraction size that works to re-represent both thirds and fourths, it makes sense that we can cut the thirds into four times as many parts and the fourths into three times as many parts. So in both cases, the total number of parts is 3×4. (Recall that the denominator value or "bottom number" in a fraction tells how many parts into which the one-whole is cut.)

Trying a similar process of subdividing with other unlike denominators allows the generalization to emerge, that multiplying the denominators together to find the new denominator will always work. Further exploring the example $\frac{2}{3}+\frac{3}{4}$, we can look for the equivalent number of twelfths. We can group twelfths into groups of three (to make fourths) or groups of four (to make thirds). Or we can "cut up" the thirds into four times as many parts, and the fourths into three times as many parts. So it is possible to multiply each of the three and the four by some number to make 12 (since 12 is exactly 3×4). We note the direct relationship to multiplication and division families; the three and the four clearly divide into 12, and 12 is the *product* of three and four. Thus, 12 is a *multiple* of both 3 and 4—and 3 and 4 are *factors* of 12. Since 12 is a multiple of *both* 3 and 4, we say it is a "common" multiple. However, I have found that early language such as "a number that both 3 and 4 divide into" can be a helpful place to start; the word "common" may be initially confusing to children. Similarly, if we try finding a new denominator for $\frac{1}{2}$ and $\frac{2}{5}$, we see that a new denominator can be generated from 2×5, which is 10. It works to re-represent both $\frac{1}{2}$ and $\frac{2}{5}$ as tenths because numbers exist that multiply each of two and five to get

10. Said differently, 10 is a *multiple* of both two and five—and we multiplied two and five to generate the 10.

The method just explored is an important and useful one. Multiplying denominators will *always* generate a unit that works, or *common denominator*. If we have three or more different fractions with unlike denominators, we can still multiply all the denominator values to generate a new denominator unit. For example, to add $\frac{1}{2}+\frac{1}{3}+\frac{3}{5}$, since 2 × 3 × 5 = 30, we know we can use thirtieths (i.e., 30) as a common denominator.

Looking ahead into algebra for a moment, such knowledge will later be important in developing techniques for solving expressions with unlike denominators. For example, for students who struggle to simplify algebraic expressions such as:

$$\frac{[\text{numerator A}]}{x(x+3)}+\frac{[\text{numerator B}]}{4(x+2)}$$

it may in fact be the case that it is the *fraction*-related concept of a common denominator, rather than other algebraic ideas, that may be the root of the difficulty.

Careful choice of further classroom examples will now allow exploring a refinement of this initial numeric method of multiplying denominators. This refinement allows us to find the best (simplest) possible value for the denominator, which the previous method of simply multiplying does not always do.

From our previous hands-on work with fraction pieces, we may have observed that, for example, $\frac{1}{6}+\frac{1}{4}=\frac{2}{12}+\frac{3}{12}=\frac{5}{12}$. Yet by using the proposed numeric method of multiplying denominator values, we might have chosen to use 24 as a common denominator instead of 12, since 6 × 4 = 24. How can we know whether a simpler common denominator value exists, other than finding the product of the denominators, and if so, how can we determine what it is?

Exploring this issue might make an excellent problem-based classroom lesson. The critical conceptual element to the notion of a common denominator, or unit that works to re-represent all fractions, is that the new denominator must be a *multiple* of *each* denominator value. But we would like to find the *lowest* such multiple because the lowest multiple is the lowest (or smallest possible) common denominator.

For example, if we want to add halves and sixths, the first or lowest number that is divisible by both 2 and 6 is in fact 6. We find we can cut the halves into *three times as many* pieces, to re-represent them as sixths:

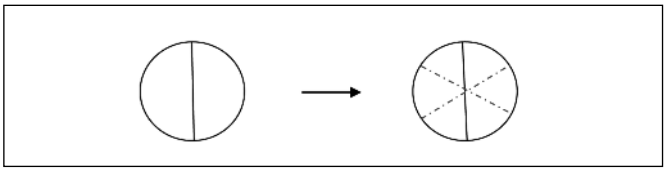


Figure 1. *2 pieces cut into* three times as many *pieces*

We can use sixths to re-represent both halves and sixths; it is a common denominator.

As a further example, if we use manipulatives and reasoning to explore the simplest denominator to add 1/2 + 1/6 + 3/4, we find twelfths work to add these fractions. If we had used the numeric method of multiplying the denominators together, we would have gotten 48 (the product of 2, 6, and 4), but the lowest common denominator is 12 because it is the smallest shared multiple. Other possible denominators are 24 and 36 because each of these numbers is a multiple of all three denominators—2, 6, and 4.

Via such examples, we now know that sometimes a new denominator exists that works as a common denominator, but that is smaller than simply using the product of all the denominators. So how can we help students to find this value efficiently?

Recall that a number that is a multiple of two or more other numbers is called a common multiple. If we have the smallest one, it is called the lowest common multiple. If the concept of lowest common multiple has already been studied when learning about whole numbers, then in fact we already have a procedural method of finding the lowest common denominator! If not, it is time to address the concept.

A visual illustration of the first shared multiple or lowest common multiple concept for the sample problem 1/4 + 1/6 is illustrated in Figure 2.

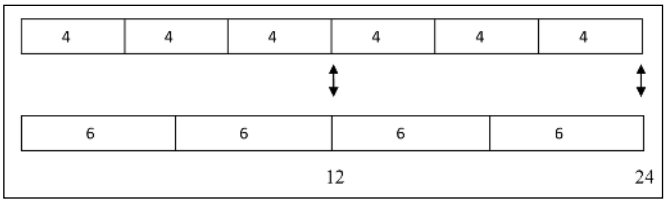


Figure 2

We might use 4- and 6-centimeter strips or rods, cubes, or lengths measured with rulers to build this model. This model (sometimes nicknamed the *trains* model, since we are building "trains" of the length of each value) illustrates the numeric lengths of multiples of 4 and 6. The trains show 12 and 24 as being multiples of *both* 4 and 6. Hence, 12 is a common multiple of 4 and

6, and is in fact the lowest or smallest one. Note that if we had simply multiplied 4 × 6, the result would have been 24. Although multiples of 4 and 6 line up at both 12 and 24, 12 is the *first* place… so 12 is the *lowest* common multiple. It is the smallest number that both 4 and 6 divide into. That means it is the smallest number that can be *grouped* into groups of 4 or groups of 6—which is the connection between 12 and these two values. Thus, 12 is the lowest common denominator.

Being able to numerically predict the best possible denominator for, say, fourths and sixths, we can reach for the manipulatives pieces from the twelfths set with confidence, and without using trial and error. So far, though, we still would need to subsequently use the manipulatives to figure out *how many* twelfths we need to use to actually add the fourths and sixths. Finding a numeric method to determine the new numerator is the last step in developing the procedure for adding and subtracting fractions.

For example, to solve $\frac{2}{3}+\frac{1}{4}$, we know to use twelfths, but not how many twelfths. Let's examine the process of changing $\frac{2}{3}$ to twelfths, shown in Figure 3.

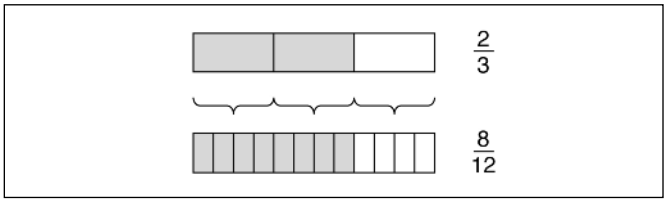


Figure 3

Recall that the denominator value of 3 tells us how many pieces the whole is cut into. We might ask students, "To get from thirds to twelfths, how many *times the number of pieces* do we need to have the whole cut into?" Examining Figure 3 shows that each third is now cut into four times as many pieces.

The "2" in the fraction numerator of the $\frac{2}{3}$ tells us that we originally had 2 pieces (of size one-third). So the next question is, "How many *times the number of pieces* do we need to shade or select, if the pieces are now size one-twelfth?" You might have noticed that as we move from thirds to twelfths, the pieces are cut or divided into four times as many parts. From the model in Figure 3, we see that both the number of pieces in the denominator, as well as the number in the numerator, are four times as great.

A similar process is observed when we re-represent $\frac{1}{4}$ as twelfths. Figure 4 shows that there are three times as many pieces in the numerator and denominator when changing fourths to twelfths. Each one-fourth piece is re-represented in three times as many pieces (which are smaller—one-third the original size).

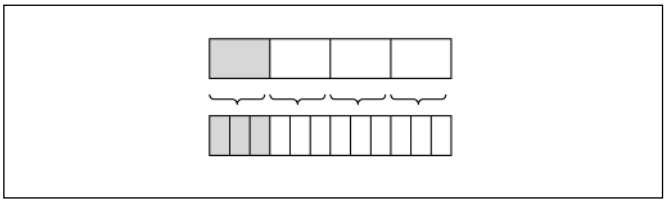


Figure 4

Note that this re-representation process does not change the overall value of the fraction; we are just making the pieces *smaller*, and thus have to use *more* of them. This thinking can be recorded more formally, using the fraction notation. It is important to remember the role of the two numbers in any fraction (bottom number tells the number of pieces in one whole, top number tells how many we have) as we do this. Multiplying the denominator value of 3 by four cuts the one-whole into four times as many pieces. Multiplying the numerator value of 2 by 4 also means we are then *taking* four times as many of these new (smaller) pieces. This reasoning falls directly from the role of the two numbers in the fraction definition. The formal method as shown in Figure 5 can now be understood via this process of subdividing. The reasoning relies *only* on the meaning already established of the numerator and denominator in the part-whole fraction notation. The only multiplication concept required here is that of whole-number multiplication, i.e., 2 × 4 and 3 × 4.

$$\frac{2}{3}=\frac{2\times4}{3\times4}=\frac{8}{12}\quad\frac{2}{3}=\frac{2}{3}\times\frac{4}{4}=\frac{8}{12}$$

Figure 5

Another (not recommended) explanation for the equivalent fraction procedure, sometimes imposed on students, involves telling them to "multiply $\frac{2}{3}$ by $\frac{4}{4}$ because that is one, and that multiplying by one doesn't change the number." Although later we might be able to see multiplying by $\frac{4}{4}$ as having the same numeric *result* as $\frac{2\times4}{3\times4}$, this alternate way of thinking has a number of problems. In fact, I advise against using or suggesting the method $\frac{2}{3}\times\frac{4}{4}$ at this developmental stage

for two important reasons. First, it is hard for children to see a conceptual basis for why we would want to multiply by something that is "just equal to 1 so it won't change the number." How would we justify this with a model? This method becomes one of those tricks students don't understand. The technique is hard to conceptually justify, so when children ask why, there may not be a good answer. Furthermore, there is a problem in the developmental progression here. Most children at this stage do not know the numeric method for multiplying two fractions because in most curricula, multiplication of fractions is *not* introduced prior to addition. So imposing, without justification, that $\frac{2}{3}=\frac{2}{3}\times\frac{4}{4}=\frac{8}{12}$ requires acceptance of a rule about multiplying fractions that children likely have no basis for, either conceptual or procedural, and must thus just memorize. Sense making is lost. Rather, I suggest using the method explored earlier in this section which relies on the definition of the fraction, in order to see that we are changing the value of the 2 and 3 in $\frac{2}{3}$ by simply cutting the $\frac{2}{3}$ into *four times as many* pieces, which we then see is (re)named $\frac{8}{12}$.

Returning to the example, suppose we want to add $\frac{2}{3}$ and $\frac{1}{4}$, using a numeric method. The completed procedure is illustrated in Figure 6.

$$\frac{2}{3}+\frac{1}{4}=\underbrace{\frac{2\times4}{3\times4}}_{\text{cut into four times as many pieces}}+\underbrace{\frac{1\times3}{4\times3}}_{\text{cut into three times as many pieces}}=\frac{8}{12}+\frac{3}{12}=\frac{11}{12}$$

Figure 6

Note that this numeric method exactly aligns with the ideas of the fractions in the previous models being cut into more pieces. In fact, it is possible to execute this procedure in parallel to creating and using a model; the steps align. A key idea in knowing what size unit will work is the idea of common multiples. Children can be encouraged to move back and forth between the model and the numeric method, strengthening the conceptual connection between the two. ▲

▲ In the Middle – From Blogging to Belonging to a Math Community

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Mirela Ciobanu is currently a middle school teacher. She has been teaching for over 12 years, 9 of which with Toronto District School Board. Mirela is a keen advocate of classroom inquiry, particularly in the area of formative assessment. Her recent work includes conducting an action research in mathematics as part of the Teacher Learning and Leadership Program for Experienced Teachers, developed and funded by the Ontario Ministry of Education in partnership with the Ontario Teachers' Federation.

"Learning is social; and the feeling of belonging to a community affects performance."

~ Annie Murphy Paul, 2013

All educators have heard the phrases "twenty-first-century teacher" or "twenty-first-century learner." They are associated with the expectations of infusing technology in the classroom and the attitudes teachers should adopt when using it. The opportunities to access content through technology or using it to model and demonstrate learning are limitless. Our goal as educators is not to keep up with the latest technological advances (it is actually impossible), but to cultivate and develop attitudes, or "habitudes" (Maiers, 2012), to help us grow along with the ever-changing complex technology landscape.

For every teacher, September brings a new set of students and parents, and along with them, the hope of turning into practice the new school year's resolutions, among which for many of us, for various reasons, is the use of technology.

The challenge is not infusing technology in the lesson, but using it with the right pedagogy to maximize the quality of learning that happens in the classroom and beyond it, to help students deepen their understanding of the task at hand or content being studied.

Cultivating a "Growth Mindset"

One of the most powerful messages we can convey

to parents from the beginning of the school year is that the mathematics we teach is no different from the one they studied, and that knowing and loving mathematics is possible for every student in our classes. There is a large body of recent research that emphasizes the importance of cultivating and developing a "growth mindset" as opposed to a "fixed" mindset, drawing on the work of Stanford University psychologist Dr. Carol Dweck.



Figure 1. *Dr. Carol Dweck and Her Growth Mindset Book*

A "growth mindset" is particularly important in mathematics, where many of our students, who have a "fixed mindset," believe that this subject is for the ones who are "born" with the ability to understand it and are more likely to give up when facing challenges and difficulty. Adopting a "growth mindset" means seeing learning as linked to effort, perseverance, and practice. This is key to developing motivation, an attitude undoubtedly linked to academic success.

A great resource for teachers is the program called "Brainology®," developed by Mindset Works® and based on Dr. Carol Dweck's research. This award-winning program, which combines instruction and interactive activities, teaches students about how their brain works, and it equips them with strategies and skills that can be applied across the curriculum. Resources to get started can be accessed from the website **www.mindsetworks.com**, which also details the Brainology program and offers resources to support teachers in emphasizing a growth mindset in their classrooms.

One of the latest studies conducted in Germany, which made headlines from *Time Magazine* to the *Wall Street Journal*, belongs to UCLA's psychology professor, Kou Murayama, who stated that it is not the IQ, but the motivation that predicts long-term success in mathematics.

Getting Started with Blogs

"What situations can you create, or help your students create for themselves, that will give them a sense that they are not numbers in a database, but members of a community (Paul, 2013)?"

Blogs are one of the most powerful social-media platforms that allow teachers and students to engage in rich conversations about mathematics. Blogging in mathematics develops the communication skills and, by making them more reflective learners, it increases their self-regulation skills. Metacognition, or "thinking about learning," blooms when mathematics is done collaboratively through solving of challenging problems. So, what do teachers need to consider before setting up the class' mathematics blog?

Author and TED talk speaker Annie Murphy Paul (2013) identifies three essential factors that lead to developing a sense of belonging:

Community: Learning collaboratively leads to more success than studying alone.

Transitions: Changing schools, class, or grades affects a student's ability to perform. Students who are transitioning to a learning environment need time to identify the "cues" in that new setting that signal that they belong.

Avoid impossible role models: When trying to motivate students by using role models, emphasizing success alone might do more damage than good, particularly for female students. The focus on the failure that one took to succeed is what inspires and motivates.

Begin Blogging

Among the most famous social learning platforms that are safe, secure, and easy to use are: Blogger, Edublog (free version offers limited options for bloggers), EdModo, GlogsterEDU (not free), and Kidblog.

A mathematics blog starts from the premise that learning is continuous and that each member of the community benefits from learning from each other. Setting up the class' blog is easy, but a few steps need to be taken, starting in September, before expecting our students to blog effectively.

1. **Set classrooms expectations** that emphasize respecting the valuable contributions each member of the mathematics community will make throughout the year. Their online interactions will mirror the interactions that happen in the classroom.

2. **Send a letter to parents/guardians,** describing the expectations and requirements for the published work and online conduct for every member of the community. Blogs will make the learning process more transparent to them. This will keep them informed about their children's learning. An explanation about how peer feedback is given, structured, and its purpose is recommended, as many parents may feel uncomfortable about their children's work being commented upon by their peers on a social media site. This introduction can also be done during the Meet the Teacher Night or Curriculum Night.
3. **Reinforce and develop the "growth mindset"** as much as possible, both in mathematics and across the curriculum. Students should perceive failure and difficulty as important opportunities for learning. Identifying difficulty and describing it is a very important step in learning.
4. **Hard work leads to success**. Before setting up the class community, students should have an opportunity to solve problems collaboratively, reflect on their learning, extend their learning, pose questions, reply to questions, provide descriptive feedback and act on it. All this effort serves one goal: exploring concepts and skills in depth in order to improve academic performance.
5. **Students need ongoing feedback and challenges**. Students need to receive descriptive feedback that is constructive and timely. Blogging will enable teachers to continue to model this skill for their students, but blogging should not be their first exposure to this.
6. **Acting on feedback needs to be modelled** before expecting students to be able to do it. Feedback is directly connected to the learning that took place in the classroom. It derives from the success criteria co-constructed by the class community during the consolidation stage of each lesson. It is a skill that develops in time and needs practice.
7. **Finally, don't start in September.** Allow for transitions. Students need time to become a community and get used to the new expectations, use of feedback, and types of interactions. When you are satisfied with the type of discourse existent in the classroom, you will know your students are ready. Until then, using a variety of instructional and assessment strategies, such as Gallery Walk, Math Congress, and Bansho, will facilitate the students' ability to communicate, listen intentionally, build on each other's ideas, and see the lesson learning as the result of the collective effort of all members of the community.

Benefits of Blogging: The Students' Perspective

I have used Kidblog for three years, and when I interviewed a group of Grade 6 students about it, the following three points were identified as its benefits:

1. **Work in Progress** – Every individual contribution consists of the initial post, revised post based on peer and teacher feedback, feedback provided to students' peers, connections, and questions they posed to further their learning, to name a few. In other words, it was not a one-time entry, much like a test. It was a rich learning experience.
2. **Belonging to a Community** – Having their work observed and commented on by their peers motivates them to go into depth and facilitate more opportunities for content clarification, unlike homework, which is done alone. All students emphasized how beneficial their peers' feedback was to their learning.
3. **The response from parents/guardians** was favourable to the use of the blog because their children are seen as part of a learning community. The students' ongoing focus on learning and exploring concepts is reassuring for parents. Additionally, it is useful to have this online portfolio as a base for discussing progress at the parent–teacher–student interviews.

Some Technical Aspects

Kidblog, used by K–12 teachers, is one of the safest, easy-to-use blogging sites. It takes very little time to set up, and initially, the teacher creates all students' individual blogs. It is recommended that the students change their password and keep this personal to avoid any problems. The teacher remains the administrator of the blog and can make any changes necessary to maintain safety and adherence to the class requirements. Once a math blog has been set up, try to separate it from other subjects' blogs for assessment considerations.

Students can personalize their individual blogs by changing their theme or avatar. An e-mail address or any use of personal information is not needed on Kidblog. As the administrator of the blog, the teacher will choose who may post and read posts, who may comment and read comments, and update information about users.

Each new post can be revised, and the students can upload videos about problems they solved, post pictures they created as models of representation (Figure 2 and Figure 3), attach documents, and insert links. Be prepared to accept the "learner" role when it comes to graphics and learn from the students. This will only make the classroom a more democratic learning environment.

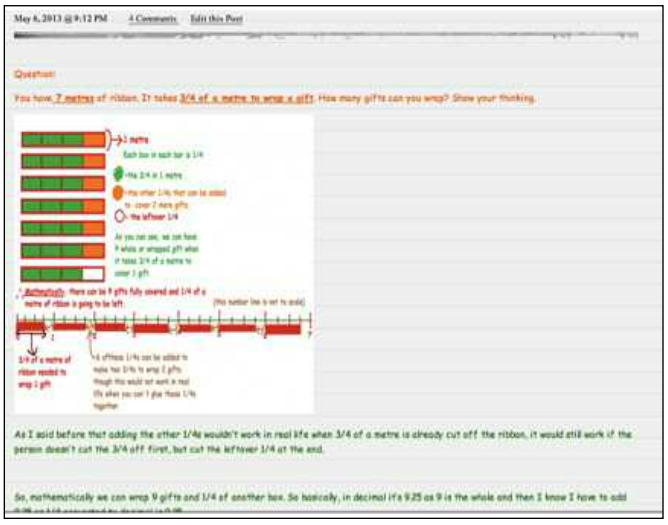


Figure 2

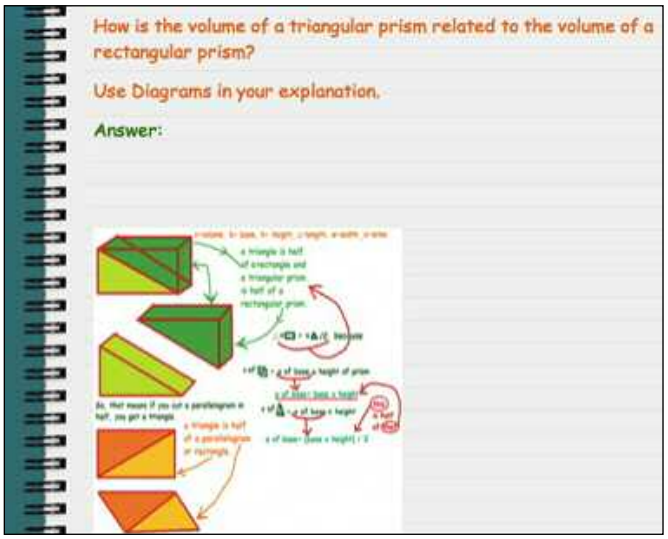


Figure 3

Teachers are more than assessors when it comes to blogging. They facilitate learning and relationship building. In your feedback to a student, refer to other students' solutions or ideas, and model building on each others' ideas. Teachers can post class notes (Figure 4) for students to review at home (such as the photos of Bansho board or the success criteria anchor chart), and recommend videos or additional materials to clarify or deepen the learning experience.

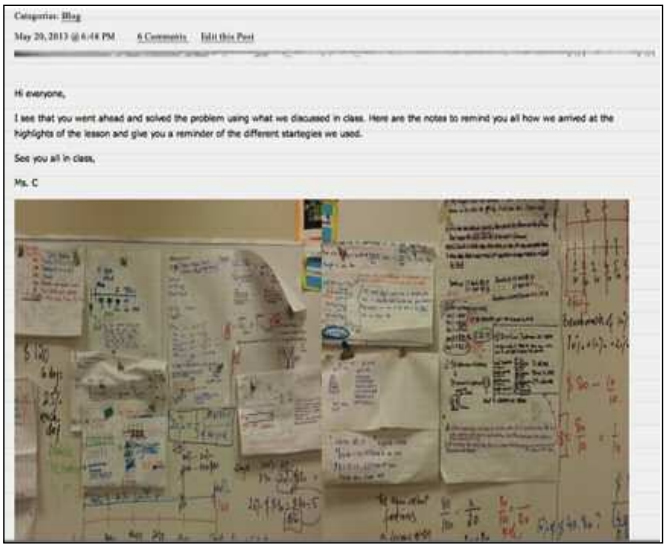


Figure 4

Students are expected to not only post their entry, but to ensure that they respond to two other peers' posts, beginning with the ones that have received the fewest comments, reply to comments, act on feedback, and attempt to challenge themselves further. Empty or negative comments, such as "Great work," "Nice pictures," emoticons, and use of jargons or abbreviations, are discouraged and dismissed.

Providing feedback and keeping the conversations flowing are very time-consuming at first, but very rewarding in the long run.

Thirteen years ago, visiting scholar and adviser to the Provost at University of Southern California, John Seely Brown (2000), published an article, in which he coined the now widely used phrase "growing up digital" to describe the generations of students who enter our classrooms. He reflected that the shift we are experiencing is "a shift between using technology to support the individual to using technology to support relationships between individuals. With that shift, we will discover new tools and social protocols for helping us help each other, which is the very essence of social learning."

It is my strong belief that using blogs is an anxiety-free, easy-to-use start to integrate technology in the mathematics program that empowers students to learn with and from each other, to fail and persevere, much like real life expects us to do.

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▲ Directors' Dialogue

ANN MICHELE STENNING
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Ann Michele Stenning is a mathematics teacher at South Collegiate with the Thames Valley District School Board. She enjoys mentoring new teachers and learning new ways of how to address the various learning styles of her students.

Minutes from the Board of Directors' Meeting

Doubletree Hotel, Toronto International Airport
Friday, June 7, 2013

OAME's president, Cathy Hall, welcomed the board of directors and reflected on the challenges of the previous year. She also thanked the many people who helped her as president, including the incoming president, Sonia Ellison. Lynda Ferneyhough thanked the outgoing chapter representatives and announced that Bart Vanslack, Melissa Black, and Lori Yee would be taking over for TEAMS, COMA, and PRMA, respectively. The representative for SAME is to be determined. Lynda noted that more members are receiving the *Gazette* digitally and that there are 300 new OAME members as a result of the annual conference last May.

Todd Romiens, the NCTM rep, announced that NCTM's membership is $78 per year. Members can access daily Smart Briefs that feature topics with links to websites and resources. The Illumination section offers lessons complete with activities, and YouTube videos show lessons being taught. There is also a large amount of print material available at discount prices.

Bill Lundy shared how the organizing committee decided to cancel the Ontario Math Olympics (OMO) for 2013. Sonia Ellison reported on her plans for the Fall Leadership Conference of 2013. The event will run on Friday night and Saturday. Connie Quadrini reported that the Math Working Group has not been meeting this year, but that there are plans to focus on Patterning and Algebra in the future. Board members were also reminded that Math Camppp will run from August 6–9. Anne Yeager announced that the Financial Literacy team

has created material for Grades 4 to 6, and 11 to 12. Judy Mendaglio provided board members with details on the Grades 11 and 12 resources, while Bill Lundy detailed some of the Smart Board™ features in the resource. In the Fall, CHAMP's virtual conference will go ahead in spite of coinciding with one of the unpaid PD days.

Before the meeting was adjourned for the evening, board members were divided into groups according to their responsibilities. Each board member discussed his or her role and responsibilities as chapter rep, director, executive, or executive director. These discussions, along with questions, were shared with all of the board members.

Saturday, June 8, 2013

The day's agenda was approved along with the minutes from February's board of director's meeting. Lynda Ferneyhough reviewed the financial report as of May 31, 2013 and then presented the current budget. The financial report was accepted.

Board members were reminded that there is still no host for OMO 2015 and OAME 2015. COMA may organize OAME 2015, but no venue or date has been set. OMO 2014 will be held at Lakehead University's Orillia Campus on May 30–31, 2014.

After meeting in various working groups, the progress within each group was shared with the entire board of directors. The Strategic Planning Committee developed a new definition for a chapter in good standing. The Outreach Committee discussed the idea of hiring a Conference Consultant who can assist with the annual OAME conference. The Curriculum Committee looked at the proportional reasoning documents. An article on this topic will contain artifacts, hopefully for the December *Gazette*. Greg Clarke will find an appropriate location to post this article on the OAME website. The Communications Committee announced that Paul Alves has been heading up our presence on Facebook and Twitter. The committee also discussed the creation of a motto for OAME.

Before the meeting adjourned, OAME's president, Cathy Hall, said goodbye to Mary Bourassa and Robert Sherk, whose terms as directors have expired, and to Angelina Glynn, who is stepping down as the chapter representative for COMA.

Chapter Reports

CHAMP has been busy planning two events: 2013 Fall eVent and OAME 2014. The Fall event will be the first ever virtual mini-conference, to held on October 10, 2013 at Humber College North Campus. Damian Cooper will be the featured speaker for the evening. The cost is free for chapter members who are too far away to attend in person. OAME 2014 will be held May 8–10, 2014 at Humber College North Campus. Mawi Asgedom, Sugata Mitra, and Jo Boaler are confirmed keynote speakers. This event will also feature an eConference. Please visit the OAME 2014 website at **www.oame2014.ca** for more information.

COMA continued to reach out to their membership. A travelling road show offering professional development was held on Saturday, April 27 at St. Michael Catholic School in Kemptville, south of Ottawa. Claire Bonner, Rebecca Black, Robin McAteer, and Anne Holness from the Ottawa-Carleton District School Board facilitated sessions that explored patterning as a support to algebraic thinking. These sessions were based on the research of Cathy Bruce and Ruth Beatty. COMA was also well represented at the annual OAME conference, with many of their members attending as delegates. Meanwhile, the executive plans to maintain their Pay-It-Forward campaign with their road show and their financial support of members wishing to attend OAME conferences.

ISOMA is wrapping up a great year after hosting two very successful events: Math Olympics and a mini-conference. Under the leadership of Brad Stephens and his colleagues at Holy Trinity School, the Math Olympics hosted 32 teams from 18 different schools. Students participated in four events: an individual contest, two team events, and a math relay. Congratulations to the top team from the University of Toronto School. In the Fall, ISOMA will host their annual mini-conference, with plans to have Holy Trinity School host the Math Olympics again in April.

MAC² has had a successful year by offering professional development workshops to its members. These workshops have had good attendance and were lovely evenings of learning and growth. The first evening, held in November, was entitled Mega Math Mystery. Members solved a mystery by following cues visible only by using QR technology. The second evening, held in April, was entitled Math Connections. During this event, participants were asked to effectively apply their mathematical skills. MAC² held its local Math Olympics at both of the local school boards. The top two teams from both boards were brought together for a day of fun-filled math activities. The day was enjoyed by the

students and their coaches. MAC^2 is eagerly looking forward to hosting OMO 2014.

NOMA has elected a new president and vice-president to lead us into the 2013–2014 school year. We are excited about NOMA hosting three satellite sites that will join the CHAMP online mini-conference in October. Each site will personalize the learning opportunity for its particular group of attendees. Plans are also underway for hosting chapter play-down for OMO in more than one city next year. While play-downs will require the creation of standardized scoring, the increased participation in OMO is a sign of growth.

NWOAME has enjoyed a busy Spring. Our annual Pi-Day celebrations happened at several local schools on March 12, with many junior and intermediate students participating. Learning and fun were had by all. Despite the unfortunate cancellation of the OMO, we scheduled our regional competition in mid-April, only to have it snowed out. It was quite a Winter in Thunder Bay. Two weeks later, a group of individuals reorganized and ran a successful event.

QSLMA enjoyed the financial rewards of hosting a successful OAME 2012 conference. As a result of this success, we decided to host OMO 2013, only to cancel the event later on. While the decision was regrettable, it was necessary because we couldn't expect full participation from each of the chapters. Our second bold move was to host a mini-conference in the Winter rather than the Fall. The event turned out to be a great success. Rather than serve lunch, we served breakfast and front-end-loaded our feature speakers, Cathy Bruce and Ruth Beatty. This set a rather laid-back tone to the entire day. Unfortunately, a mid-Winter blizzard quashed travel plans for a number of people who had intended to come to the conference. Thus, numbers were down compared to the past few years. As a consequence, we have decided to host our 2013–14 mini-conference on November 2 at Queen's University.

QSLMA has also held a couple of modest math social events. As drawing cards for the mini-conference and social events, we elected to hold a draw for two registrations for OAME 2013. Our decision was to give every chapter member one ticket, then give additional tickets to those who attended the chapter events. This past year, someone who attended all events would have had four tickets. A draw was held at the mini-conference, and two chapter members had their registrations paid for by us. We plan to repeat this in 2013–2014, with a draw most likely in February. The chapter is also motivated to use OAME's website for its online presence. Finally, our re-elected chapter president has agreed to develop a chapter constitution.

SAME hosted our annual Math Olympics at J.S. Woodsworth Senior Public School. We had 23 teams from 14 schools participating. We charged $50 per team and collected a total of $1150. From all the receipts, we spent a grand total of $1197. This includes food, loot bags, and prizes for the winners. Our gold medal went to the team from Sir Ernest MacMillan Public School.

WOMA did not offer professional development sessions this year because of the unfavourable working conditions. Now that members have agreed to a contract, WOMA will host a mini-conference next Fall.

Due to the Federation labour relations issues, the **Y4MA** executive decided that professional development opportunities and the OMO be deferred to the 2013–2014 school year. We intend to offer a mini-conference in the Fall. Details will be announced at a later date. ▲

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What's New with CLIPS? Games!

GREG CLARKE, AGNES GRAFTON,
AND ROSS ISENEGGER





Greg Clarke is from the Simcoe Muskoka Catholic District School Board, and Ross Isenegger is from the Near North District School Board. Both are currently on assignment with the Ontario Ministry of Education as Provincial Math Leads, working on the CLIPS project, in conjunction with their colleague, Agnes Grafton, from the Brant Haldimand Norfolk Catholic District School Board.

Summer is over and you are back in the classroom. Want a fun way to engage students while allowing them to consolidate their learning at the same time? Check out a selection of the games available in CLIPS.

How do I find a game?



Every cluster of CLIPS activities has a number of games that have been designed as part of the learning trajectory and hence appear in the CLIPS activity list. Usually, games were designed as consolidation activities (denoted in blue) and are often referenced in "Show What You Know." Alternatively, a list of all the games for the specific cluster you have selected can be accessed by clicking on the Games icon located in the right panel of the wrapper. Here, in addition to the electronic games, you will find a number of activities that can be done without a computer (most often in PDF format).



If you're not sure where the various games are stored in CLIPS, click on the Search link located at the top of the wrapper. Do a search on the word "game" and you will see a number of results.

In Development

The CLIPS team is currently working on developing a Games tab that will allow the user quick access to the entire collection of games. Games will be categorized by math topic (see draft image). After choosing a topic, the list will expand with details about each game, including links to start the game directly or download it to your computer for later use.

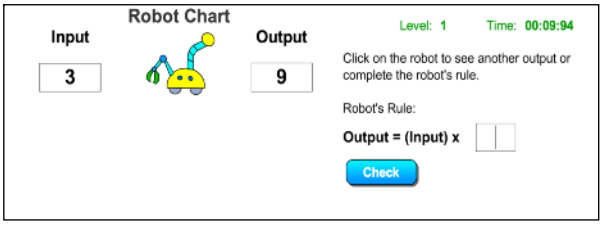


Sample Games

Here are some of our favourite CLIPS games. This is by no means an exhaustive list. Also, we expect to develop more games during this coming school year, so stay tuned!

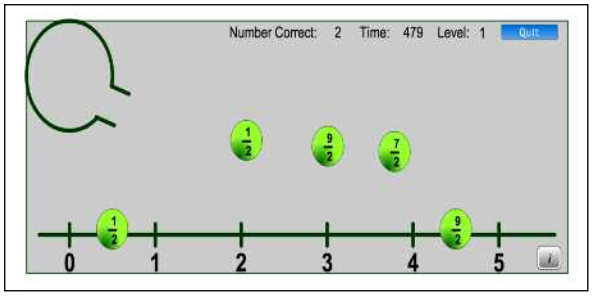
Robot Rules

Guess the linear growing pattern rule based on the input/ouput table generated by the robot. Using an unordered table of values is an excellent strategy to use to help students learn to think functionally rather than recursively which they would have learnt before.



Drop Ball

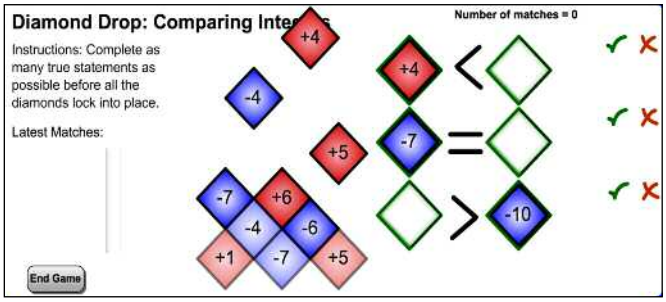
Drag fractions (proper and improper) to their appropriate place on the number line. At each level, values become more challenging and the balls move more quickly. Can you make it to level 20?



Diamond Drop

Students demonstrate their ability to compare integers by dragging diamonds to create "less than," "greater than," or "equal to" statements. How many

statements can you complete before the diamonds pile up? Stay tuned for a version of this game that deals with addition of integers.

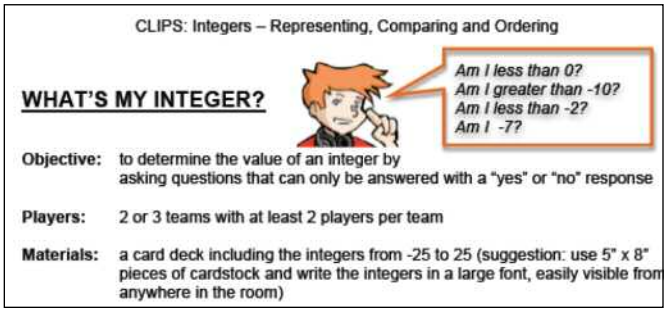


PDF Game Examples

What's My Integer?

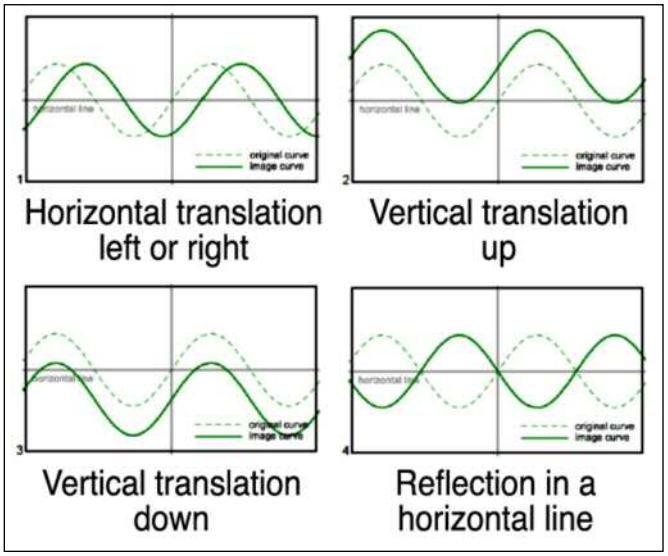
Determine the value of an integer by asking questions that can only be answered with a "yes" or "no" response (e.g., Am I less than 0? Am I greater than -10?).

Look for the electronic version of this game.



Naming Transformations Card Game

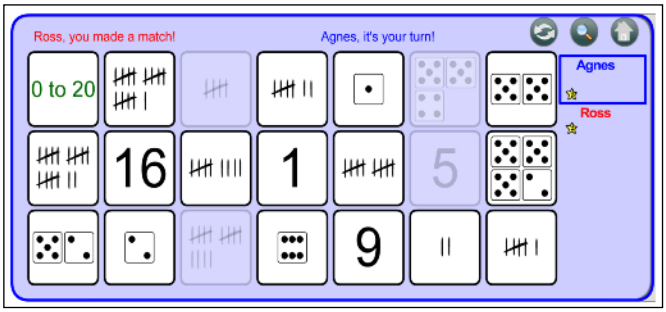
This game consists of a set of cards showing various transformations of sine graphs and descriptions of the transformation. Instructions for a number of different possible games, including a memory match game, are provided.



Coming Soon

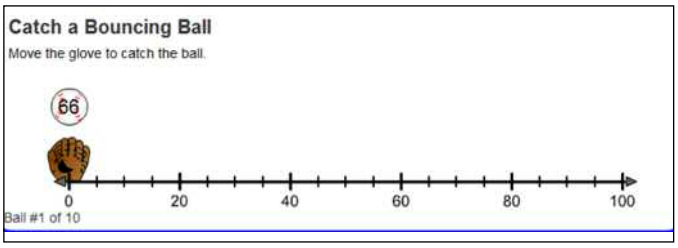
Primary "Number Match" Game

The purpose of this game is to match various representations of numbers. Representations include dice, fingers, tallies, rekenrek beads, random dots, and numerals. Values range from 0 to 20, depending on the level. The easiest level is appropriate for young students just learning to count.



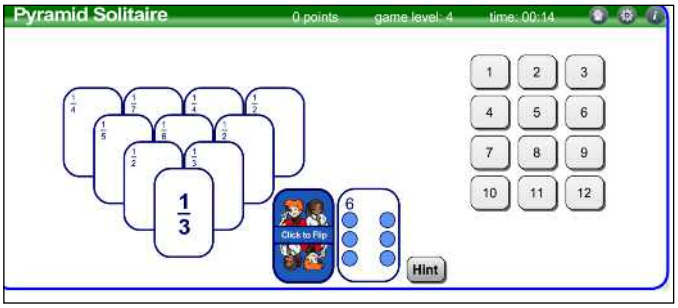
Junior "Catch a Bouncing Ball" Game

Baseballs showing various whole numbers are tossed. Students catch each ball by moving a glove along a number line to the value indicated on the ball.



Pyramid Solitaire

Students clear a pyramid of cards by finding a fraction card and a whole-number card that multiply together to make a whole-number product. Visual representations help students develop an understanding of what it means to determine products like 5 groups of $\frac{2}{5}$ or $\frac{1}{3}$ of 6 .



To find these games, and more, visit **http://MathCLIPS.ca/**. ▲

▲ The Abstraction of a Histogram

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Nico Rowinsky is the author of Sally Strange: And How She Learned to Stop Worrying and Love Grade 7 Math*, published at* **leanpub.com/Sally_Strange_Grade_7**. *You can find him tweeting* **@NicoRowinsky**.

In most cases, the word "abstract" in *The Ontario Curriculum, Grades 1–8: Mathematics (Revised)* (2005) is used in the context of an adjective:

> "…the abstract nature of mathematics in the secondary level" (p. 4)."Using manipulatives… helps to make connections between the concrete and abstract" (p. 15).
>
> "These concrete learning tools invite students to explore and represent abstract mathematical ideas in varied, concrete, tactile, and visually rich ways" (p. 25).

Inspired by several articles online, I've developed an interest in the actual process my students use to take something from the concrete and develop it into something we consider abstract.[1] This student process of abstraction will most strongly be associated with the mathematical processes we call *connecting* and *representing*. But in this article, I would like to refer to the process as *abstracting*.

My goal was to document the abstraction of a histogram in an attempt to answer the question, "What path will some students take to organize a collection of numbers?"

What follows is a map of the path taken by two of my students.

The Classroom Set-up

I teach Grade 8 mathematics to a wide range of skill levels over six classes. Class sizes range from 20 to 31. During this particular lesson, a typical class would enter the room and I asked them to tell me how long it took them (to the nearest minute) to get to school this morning. I took a marker and sticky notes, and as each student told me his or her number, I wrote it on the sticky note and asked him or her to stick it on the board.

Naturally, the students really wanted to round to the nearest 5 minutes, but I emphasized the "nearest minute" part in an attempt to get more accuracy and less repetition.

As the class took their seats, I presented the results on the board. Initially, the results were presented as sticky notes randomly scattered on the board. For the most part, as a class, we tried to organize, sort, structure, and then label our way into a graphical representation. The end product was a histogram of "travel time intervals" versus "frequency."

However, during one of my classes, I tried something a little different. I took two students aside and worked with them as the rest of the class caught up on some work from the previous day.

It is noteworthy that the two students I chose had shown specific characteristics that I wanted. They have shown in the past that they don't give up.

They have shown in the past to struggle more than average with certain acts of abstraction. So, starting at the whiteboard with random sticky notes, I wanted to guide the process and see where the students took it.

Instruction #1 from the teacher:

"Can you group these sticky notes in whatever way you want so they are more organized?"

Here's the result:

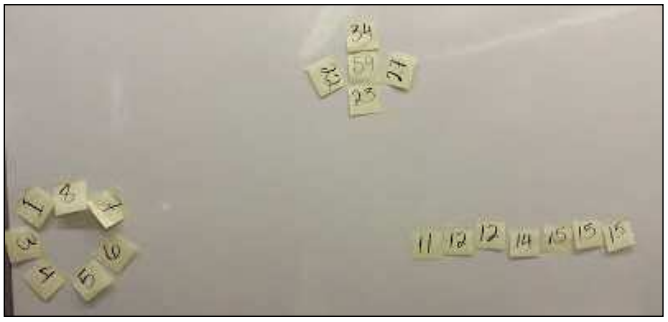


Instruction #2:

"I can't really see some of the numbers. Can you fix them?"

[1] *Namely, [LOA]* The Ladder of Abstraction, Part One of Probably a Lot *(2012) by Dan Meyer,* **http://blog.mrmeyer.com/?p=14066**

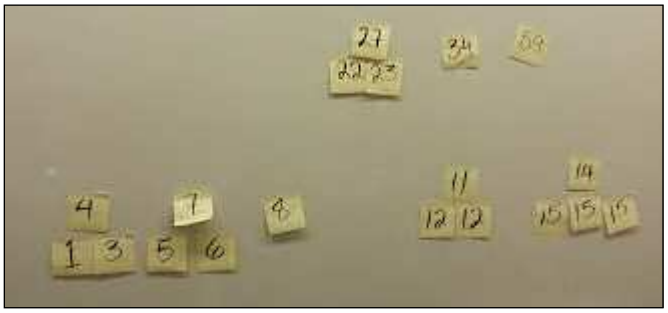
Result:



Instruction #3:

"Oh, I'm wondering why you chose those groups?" and, "I'm getting a bit distracted by the circle, the line, and the star shape. Can we fix them?"

Result:

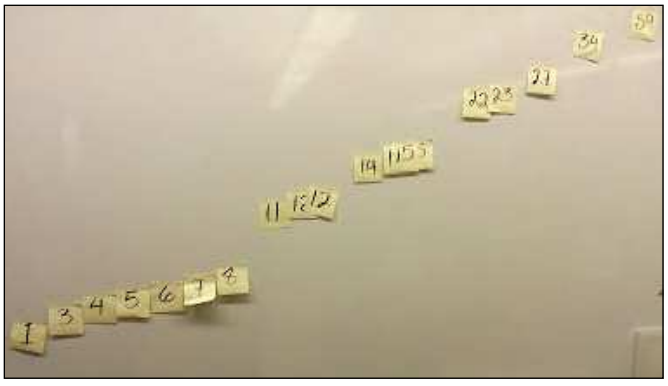


Instruction #4:

"I noticed you changed some of the groups. Why?" and, "As I'm looking at these, the order is kind of from left to right, but then I have to go up to the next level and go left to right again. I wish it were just left to right, without having to backtrack."

nb: This instruction is MY personal preference because I need that left to right for the histogram, but both students were accepting, although I felt manipulative here.

Result:

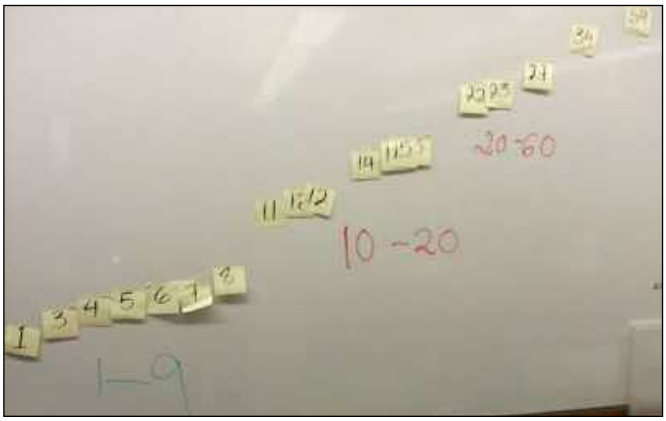


So now I'm staring at this new structure and I realize the impact of my previous instruction. Encouraging them to go left to right broke down their group stacks. But I rolled with it and gave them:

Instruction #5:

"OK, can we add something here so I know what the groups are?"

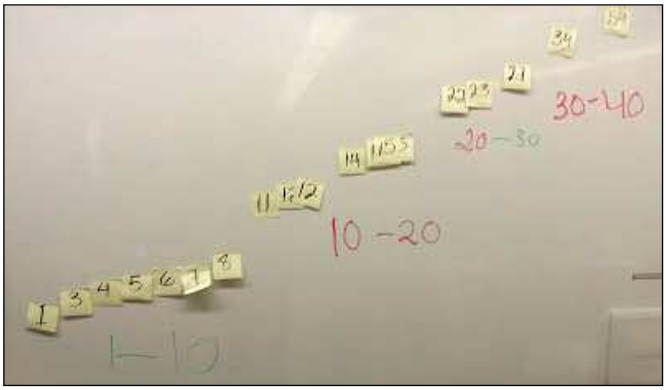
Result:



Instruction #6:

"OK, now I see all your groups. It looks like you have three groups. Is that true?" The students looked at it and changed some things because of my question. Note that they didn't answer me; they just went to work.

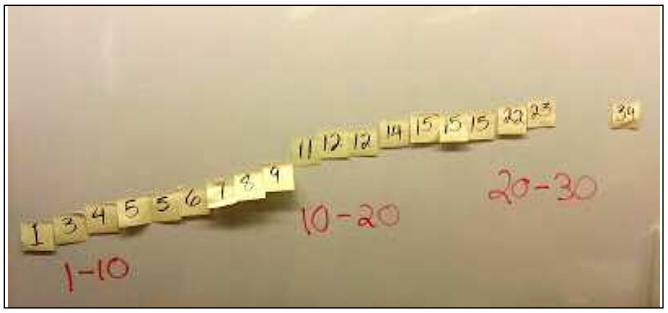
Result:



Instruction #7:

"I'm noticing you are having trouble reaching the top right of the board, both to reach the sticky note and to even label your groups. Can we fix that?"

Result:



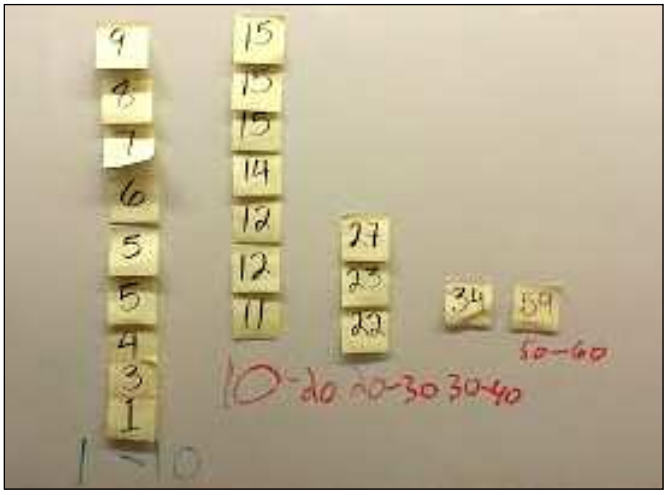
OK, here I didn't have a one-liner. So, during, Instruction #8, I indicated that the numbers look like they're just in a straight line, AND that the labels don't seem to match where the group of numbers are, AND, I asked whether there is a fix, AND, whether this is even a problem.

I'm battling the urge to tell the students exactly what to do and want to give them power over the decisions.

They started stacking sticky notes on top of (covering) each other. "Wait, I still want to see them all.

"Now they paused, thought about the "problem," and eventually came up with the following result. It is noteworthy that this seemed to be the most challenging step—perhaps the most "abstracting" they had to do?

Result:



At this point, I felt as if I could bring in the *x*-axis and *y*-axis. I had chart paper ready and asked them:

Instruction #9:

"Can you move your stacks onto the chart paper?"

Result:



Instruction #10:

"Do you mind if I label these for you? What would I put here?" (pointing to the left side of the *x*-axis). The students asked me to write "0-10"... and then "10-20." The result created the next level of organization, including a category (interval) that was empty.

Result:



To wrap things up, I added some vocabulary at the end—words like *intervals*, *frequency*, *axis*. Also, some details regarding spacing and the notable differences with respect to bar graphs were mentioned.

Some Thoughts

- It was interesting to note which instruction created the most difficulty (#8).
- Although I tried to get out of the way, I felt I was imposing myself too much in certain spots (#4).
- I don't know if getting out of the way is an intrinsic part of my goal (documenting the abstraction of a histogram).
- The difficulties students experience while abstracting are very student specific. In other words, this would have taken less instruction for some, and perhaps more for others.

Finally, given that the process was documented, I was able to use these pictures of each step and showcase them to other classes. A great discussion about space and organization was had, which provided me with another level of instruction.

Documenting the abstraction of the histogram was an exercise in witnessing how a student abstracts. As teachers, we know that math is abstract, but analyzing the process of abstraction can lead us to more insights.This article is a modification of a blog post by Nico Rowinsky.

The original article is posted at **ynaughtmath .blogspot.ca**.

If you have any questions about this article, please feel free to contact the author. ▲

▲ E-Kwatee-Ons – From Mind Games to Algebra

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I think students like to see some magic and personal power in their mathematics. I remember my Grade 2 and Grade 3 teachers and how awesome and mysterious these people were. They had great powers too! My teachers had extra-sensory perception powers… just like the Amazing Kreskin (a magician I watched on television all the time – he seemed to be able to read our minds through the television). My teachers could read my mind. It was fantastic!

There was one particular "mind" game they played with us in class. It started with the teacher saying, "Pick a number between one and ten, and keep it in your head." Then the teacher would say, "Multiply that number by two, and keep the result in your head. No… don't tell me." Then the teacher would say, "Add three to that number, and keep the result in your head. Now tell me your final number in your head. Emma?" Emma would share her number, and in a blink of an eye, the teacher would tell us the number with which Emma had started! It was magic! We were astounded! I think school children, especially high school students, still want that magic. While I am not a magician, it is fun to try to bring a little magic into the classroom.

I often teach the Grade 9 or 10 students who struggle with math. It is possible that many of these students will never "solve an equation" once they leave high school. If they do, it will likely be within such a specific setting that

the context—such as plumbing, electricity, woodworking, the computer—camouflages the technical and formal process of the algebraic solving of equations. I have endured a few years of students' rolling eyes, groans, apparent monumental suffering, and more than a few failing course marks.

The textbook treatment of solving equations often starts with equations that require only one transformation:

$$ax = c, or \frac{x}{b} = c, or\ x + b = c$$

Students almost always exclaim, "These are so easy! We can do them in our heads. Why do we have to write all these steps?"

However, emphasizing the traditional paper-and-pencil approach of solving single-step equations first, seems to lose students' interest. By the time students get to equations that could use a good algebraic, multi-step solution process, students are dismissive of equations and do not care to learn a formal algebraic method. Thinking from a differentiated instruction perspective, I decided to let students solve those one-step equations in their head, and instead, focus my teaching energies on a process that would solve (almost) any equation that involved two or more operations, such as

$$ax + b = c, or \frac{x}{b} + c = d, or \frac{x + b}{c} + d = e$$

the equations that are often more difficult for students to solve mentally just by looking at them.

What follows is a unit I use for students' learning of solving equations that considers students' attention spans, attention to details, and learning styles. The unit may run from three to six days, depending upon my students' learning needs and the order I have planned for curriculum topics in the course I am teaching. Time is relatively flexible because of an overriding belief I have—that if I do not take the time now, then I will reteach "solving equations" in every unit for the rest of the course. Students have experienced many learning opportunities for solving equations (e.g., Ontario Ministry of Education [OME] (2005a) curriculum by Grade 8), but often do not recognize they can "solve an equation" for the variable in different mathematical contexts or topics. This unit provides an engaging discovery atmosphere, where students generate a personally memorable algorithm to solve for unknown values.

The unit starts with a game I pronounce as "ee quatee ons" with three obvious syllables. Once, I tried writing "E-quati-ons" as a game title, but students figured out the play on words right away and many gave up immediately, often resorting to habitual thinking that they were unable to learn how to solve equations. The current title of the game comes from a former student's spelling of what he heard in class, "ee-kwatee-ons."

I start with a mental game of number manipulation. It is a game that all students can perform in their heads and feel comfortable speaking out loud in a "…safe and secure environment, characterized by respect for others, that allows them to participate fully and responsibly in the educational experience" (OME, 2005b, p. 26). NCTM (2000) also encourages high school teachers to use oral communication when students are learning and sharing mathematics by "…creating a climate in which all students feel safe in venturing comments, conjectures, and explanations" (p. 350).

Then, I gradually incorporate a paper format to the game, which transforms over the lessons into a formal algebraic and algorithmic process of solving equations with which we are all familiar. A game is "safe," there are established rules to games, and cheating in a game appears to be a universal foul. A game is also, usually, fun to play.

This is the unit plan:

Day 1 E-kwatee-ons game in our heads.

Day 2 E-kwatee-ons game board on paper.

Day 3 E-kwatee-ons and writing each step as an arithmetic operation.

Day 4 Resolving E-kwatee-ons into the formal solving equations algorithm on paper.

Day … Practice and extensions into more complicated equations as needed.

Day n Practice, Equations Bingo, evaluation.

Day 1

I start with a story about my elementary school teacher having ESP—Extra-Sensory Perception—she could read our minds! She played a game called "equations" (pronounced ee – kwatee – ons). "In fact, let's try it here with us!," I say to the class, and I launch into, "Pick any number between 1 and 9 and keep it in your head. Don't tell me!... Now multiply that number by 2. Keep that result in your head. Now add three to the number in your head. Ready?… Jake, what number did you end up with?"

Jake tells me "13." I say, "You started with 5." There are some laughs, some squints, some frowns, and some

smiles. The game has started! I ask for students' numbers a few more times, and then play the game with the whole class, using a different set of operations. Finally, I ask, "How do I know the number you are starting with?"

Sometimes the answer comes quickly, sometimes I provide a bit of scaffolding with well-placed questions to students, but eventually we understand that I go "backwards." I have found helpful scaffolding questions start with prompts such as, "What were you thinking when I said…?," followed with, "Why do you think you were thinking of [that] at that time?" I introduce terminology such as *inverse operations*, and help them emphasize the one-step-at-a-time nature of our thinking. I then have the students play the game in pairs, five times each, two operations each, with numbers between 1 and 9. I work the crowd and listen, comment, encourage, and enjoy the noise and laughter in the room.

There are a number of wonderful by-products with this activity at this time. Students invariably find they have fractions or decimals to work with. If they cannot or do not wish to work with fractions, I ask them to explain why their calculations involve fractions, and how they might play the game so they do not get fractions. From a teacher's perspective, this has the effect of students thinking about division rules, and that order matters. Estimation thinking processes, thinking ahead and hypothesizing as one works, and attending to order of operations are all valuable mathematical thinking and solving skills that appear naturally when playing this game.

The next phases of the day's lesson involve numbers between 1 and 20 and two operations, and then numbers between 1 and 100 and three operations, in addition to the options of the assistance of calculators and writing quick notes on paper to help memories. By the time the period comes to an end, approximately 75 minutes in the district where I teach, everyone is proficient with playing E-kwatee-ons. Homework (students groan) is to play the game five times with someone at home—a parent, guardian, older sibling, etc.—and have that person write his or her signature on a page in the student's homework section of his or her class notes (students are happy and think this might be OK). I look for that signature the next day—students cannot believe that homework is checked by looking for a signature!

Recently there was a student in my class who was in Grade 10 math for the third time. By now, she was 17 years old. The next day, when I checked homework, I saw she had signed the page herself. In my mind, I was disappointed. I thought she was trying to avoid doing the work, or to pull a fast one on me. But, I persevered, took a chance, and asked her if she had played the game for homework. She said, "yes." I inquired into the reason why she signed the page herself. She told me that she lived alone, so she phoned a friend. I was ecstatic! How responsible of her!

Pick a number ☐ ☐ Get the starting number

I asked whom she had played the game with over the phone. She said, "You know her, sir. Penny was in your class last year." Look at that! Even when students think they are done with math, someone phones them and has them review solving equations a year later!

Day 2

After checking homework, my story with the class continues. "Like any really good game, there is a game board. Think of Tic Tac Toe, Sorry, Monopoly. E-kwatee-ons has a game board too!" I have students write the game board in their notes (see Figure 1 for the lesson "note" I put on the chalkboard). I then use this game board with an example of a game we played in our heads the previous day. I speak the game, and simultaneously write each step vertically down the left side in the game boxes. I ask a student to tell us his or her answer, which I write in the bottom box on the left side. I then write that answer in the bottom box on the right side, and moving one box up at a time, speaking out loud, I write the inverse operation from the left corresponding box, and do the calculation in my head. I get to the top box and write the result. I write "? =" and the number to the right of the game boxes.

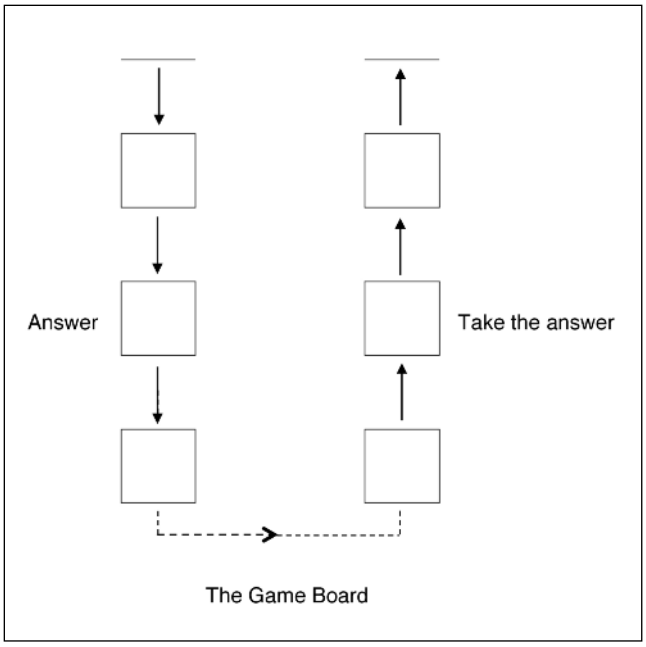


Figure 1. *The Game Board*

I use a question mark, "?," for the first box of the left column because that is the number I do not know. I do not wish to introduce the letter *x*, since a) it might tip them off that we are solving those "dreaded" equations, or b) that somehow this is more like "math" than just a fun game. Students play the game with the game board in pairs, again in a pattern of increasing size of numbers and number of operations (see Figure 2 for the lesson note I put on the chalkboard for this day). Homework for today is to play the game five times with someone at home, and have that person sign the paper on which the games are written. Students often have time at the end of this class period to prepare five empty game boards before the end-of-class bell rings.

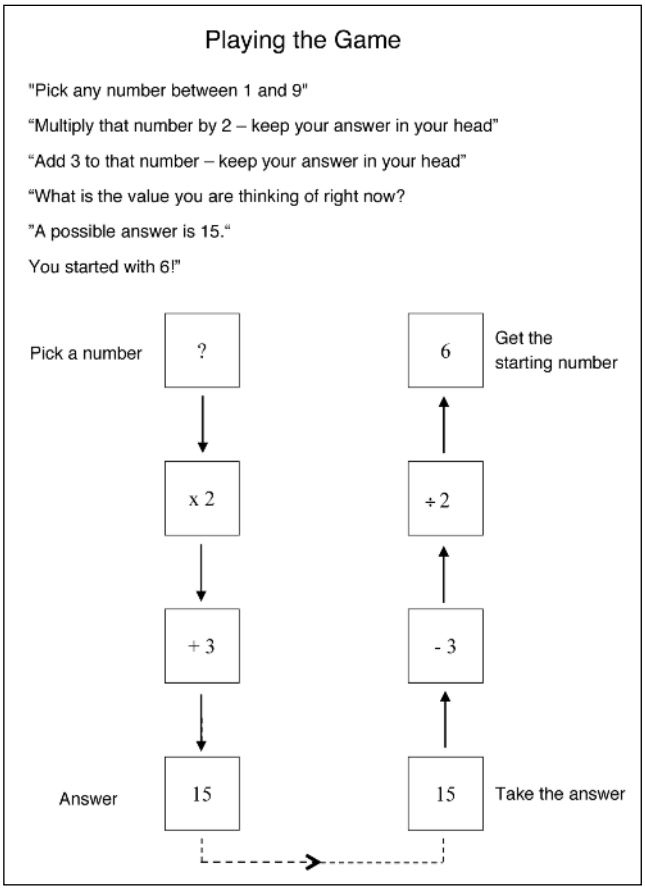


Figure 2. *Playing the Game*

Day 3

On the third day of the unit, I suggest there might be another way to write down the game, and using a game example from a previous day, I say the game and write a symbol for each step on the board. This creates an equation, for example, from Figure 2, $2x + 3 = 15$.

Another year, while I was teaching this unit, all of a sudden, one of the older students who was repeating the course, Todd, blurted out, "S**t sir, I am solving freakin' equations!!! I've never done that before!" Now, I had a "teacher moment" right then. Do I tackle the obvious swearing in the classroom, or appreciate the intent and success behind that expletive-laden celebration. It was dead silent in the class anyway, as students waited for me to respond because this was not the accepted talk in my classroom, but I took a chance and took the latter approach, and I just looked at Todd, smiled, and said "Yes! Isn't that awesome! Let's do some more!" Students need to know the reasons behind their mathematics performances. "Mathematics should make sense to students; they should see it as reasoned and reasonable" (NCTM, 2000, p. 341). As a postscript to this little classroom episode, Todd passed the course.

Day 4 and onwards

The next phase of the unit moves students into a more formal and algebraic method of solving equations. Initially we use the game and follow it backwards, moving up the right column of boxes. At each box, we write a single equation, using the operation that shows in the box. For example, from Figure 2, the first line we would write is 15 – 3 = 12. The second line we would write is 12 ÷ 2 = 6, and the third line we would write is 6 = *x*. Each successive line shows a single calculation until, ultimately, the final value matches up with the unknown "?" or "*x*." These steps align visually with the calculations that appear in a formal algebraic method of solving equations. For example, these calculations appear on both sides of the equation in the algorithm I tend to use—the "balance" method of performing the same operation on both sides of the equation in an effort to isolate the variable.

For some students, this has been a transformative experience. Some students move quickly to the formal algebraic method, while others continue to use a method that mixes a particular piece of the game (the left column) with the formal algebraic form. There are a few students who cannot grasp any part of the algebraic method and solve all their equations with the game board of Figure 1. I accept any of the three methods for solving equations on the final exam. Each method shows mathematical thinking and mathematical communication appropriate for the learner's stage of development. I find that students who use the modified game method of solving equations for the duration of this course often leave it behind and solve equations naturally with the more formal algebraic method once they start their next math course.

The upcoming Part 2

Part 2 of *E-kwatee-ons – From Mind Games to Algebra* will follow in the next publication of this journal. The second part will include the lesson-board notes showing the use of the game format and template, along with concluding comments regarding constructivism in mathematics classrooms, differentiated instruction, and assessment supportive of current thinking in the field of mathematics education, and which is also supported by Ministry of Education and NCTM directives.

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▲ Concatenation of Consecutive Fibonacci and Lucas Numbers: A Lesson in Patterns of Divisibility and Proof

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Thomas Moore has delighted in teaching courses in abstract algebra and number theory over a career that has spanned 44 years, all spent at Bridgewater State University (BSU) in southeastern Massachusetts. He has been an active member of the Mathematics Association of America (MAA) and, with his colleagues, has twice hosted the annual Fall meeting of the Northeastern Section of this organization (this section includes four Canadian provinces) at BSU in 2001 (recreational mathematics) and 2012 (the art of problem posing and problem solving). He is also the recipient of the MAA's Award for Distinguished College or University Teaching.

Abstract

The lure of the Fibonacci sequence and the related Lucas sequence for those who seek numerical patterns is well known. In this note, we discover that concatenating consecutive terms in both sequences leads to observations on the results being divisible by certain positive integers. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,… and is governed by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ $n \geq 3$, with $F_1 = 1$, $F_2 = 1$. Fibonacci (ca. 1175–1250), also known as Leonardo of Pisa, wrote a book, the *Liber Abaci*, around 1202, which has a significant place in mathematical history. In it he advocated for the use of the Hindu-Arabic decimal system (base 10) over the use of Roman numeration. And, also in the book, is a question whose answer introduces the famous sequence described above.

The sequence is a treasure trove of numerical patterns. For example, one notices $F_5 = 5$, $F_{10} = 55$ and $F_{15} = 610$ are each divisible by 5. These particular numbers are visibly multiples of 5 because they have units' digit 5 or 0. However, to see that *every* fifth Fibonacci number is divisible by 5, we must produce a proof and, in our case, a proof by mathematical induction.

We claim then that F_{5k} is divisible by 5, for all positive integers $k \geq 1$.

As shown above, this is true for $k = 1,2,3$. This begins an induction argument. Assume.

$F_{5k} = 5t$, for some positive integers k and t. We will deduce it is true for $F_{5(k+1)} = F_{5k+5}$.

To see this, we decompose terms, utilizing the Fibonacci recurrence relation and, of course, we use our assumption that: $F_{5k} = 5t$

$$\begin{aligned} F_{5(k+5)} &= F_{5k+4} + F_{5k+3} \\ &= (F_{5k+3} + F_{5k+2}) + (F_{5k+2} + F_{5k+1}) \\ &= (F_{5k+2} + F_{5k+1}) + (F_{5k} + F_{5k+1}) + (F_{5k} + F_{5k+1}) + F_{5k+1} \\ &= (F_{5k+1} + F_{5k}) + 4F_{5k+1} + 2F_{5k} \\ &= 5F_{5k+1} + 3F_{5k} \\ &= 5F_{5k+1} + 3(5t) \\ &= 5(F_{5k+1} + 3t), \text{ as desired.} \end{aligned}$$

Now, given two numbers, a and b, we may add, subtract, multiply, and divide them as usual. However, we may perform other operations "$*$" on them, such as exponentiation: $a*b = a^b$. In this paper, we look at the operation of concatenation of two positive integers x and y, in that order, which simply places x in front of y. We will write this operation as $x : y$.

For example, $53 : 126 = 53126$. Notice that this concatenation is equivalent to $53(10^3) + 126 = 53126$ and, likewise, $146 : 37 = 14637$ is equivalent to $146(10^2) + 37 = 14637$.

The result proved above that F_{5k}, that is always a multiple of 5, implies that, if we prefix F_{5k} using any of the base-10 digits 0, 1,…, 9 as many times as we want, then the resulting number is again a multiple of 5. In particular, we immediately have the kind of result we are looking for in this paper, that $F_{5k-1} : F_{5k}$ is divisible by 5, $\forall k \geq 1$.

Examples

Among these concatenated Fibonacci numbers, the first few are 35, 3455, and 377610.

Now, this concatenation result is a bit underwhelming, given that it is encompassed by the ability to prefix any digits at all in front of F_{5k}. So let us turn to a more interesting result. First, we observe that $F_4 = 3$, $F_8 = 21$ and $F_{12} = 144$ and that all these numbers are multiples of 3. Recall that there is a well-known digit test (base 10)

for divisibility by 3; namely, a given number n is divisible by 3 if, and only if, the number formed by the sum of the digits of n is divisible by 3. For example, 12234072 is divisible by 3 by this criterion. However, to see that *every* fourth Fibonacci number is divisible by 3, we again turn to mathematical induction.

Our claim is that F_{4k} is divisible by 3, $\forall k \geq 1$.

The above examples show that this is true for k = 1, 2, 3. Now assume that F_{4k} is divisible by 3, for some k, say $F_{4k} = 3t$. We must deduce the same is true for $F_{4(k+1)} = F_{4(k+4)}$.

$$\begin{aligned} F_{4k+4} &= F_{4k+2} + F_{4k+3} \\ &= (F_{4k} + F_{4k+1}) + (F_{4k+1} + F_{4k+2}) \\ &= F_{4k} + 2F_{4k+1} + (F_{4k} + F_{4k+1}) \\ &= 3F_{4k+1} + 2F_{4k} \\ &= 3F_{4k+1} + 2(3t) \\ &= 3(F_{4k+1} + 2t), \text{ as desired.} \end{aligned}$$

Now, prefixing any digits before F_{4k} will not necessarily produce a number divisible by 3. For example, 21 is a multiple of 3, but 421 is not. However, when examining the Fibonacci numbers and concatenating consecutive terms, we discover that $F_2 : F_3 = 12 = (3)(4)$ and $F_6 : F_7 = 813 = (3)(271)$ and $F_{10} : F_{11} = 5589 = (3)(1863)$.This is suggestive and, indeed, we are able to prove that a concatenation of certain consecutive Fibonacci numbers has the desired property in our next result. In its proof, we employ the notation $\equiv_3$, standing for "is congruent mod 3 to." By definition, $a \equiv_3 b$ means a and b have the same remainder when divided by 3. For example, $10 \equiv_3 1$, so also $10^q \equiv_3 1$ for any positive integer q.

We are claiming that $F_{4k+2} : F_{4k+3}$ is divisible by 3, $\forall k \geq 1$.

This follows directly from the previous result on F_{4k+4}, like this:

$$\begin{aligned} F_{4k+2} : F_{4k+3} &= F_{4k+2}(10^q) + F_{4k+3} \text{ for some positive integer } q, \\ &\equiv_3 F_{4k+2}(1) + F_{4k+3} \\ &\equiv_3 F_{4k+2} + F_{4k+3} \\ &\equiv_3 F_{4k+4} \end{aligned}$$

and since this last number is a multiple of 3, so is the number $F_{4k+2} : F_{4k+3}$.

The Lucas sequence L_n begins 2, 1, 3, 4, 7, 11, 18, 29, 47, 76,… and is governed by the same recurrence relation as the Fibonacci numbers, but with different start values. Namely, $L_n = L_{n-1} + L_{n-2}$, $n \geq 3$, with $L_1 = 2$, $L_2 = 1$. Edouard Lucas (1842–1891) was a French mathematician, perhaps best known for his invention of the *Tower of Hanoi* game, in which disks of different sizes are stacked on one of three pegs and must be restacked one disk at a time on another peg while never placing a larger disk atop a smaller one.

As before, we will consider the concatenation $L_n : L_{n+1}$ of consecutive Lucas numbers and test the result for divisibility by certain numbers, namely, 3 and 9.

We observe:

$L_1 : L_2 = 21 = (3)(7)$
$L_5 : L_6 = 711 = (9)(79)$
$L_9 : L_{10} = 4776 = (3)(1592)$

and also that 3 divides $L_3 = 3$; 3 divides $L_7 = 18$; and 3 divides $L_{11} = 123$.

These suggest firstly that $L_{4k+1} : L_{4k+2}$ is a multiple of 3, and secondly, that the theorem to which our desired concatenation is a consequence is this: L_{4k+3} is divisible by 3, $\forall k \geq 0$.

Let us prove this last statement. If k = 0, 1, or 2, this is true by the above examples. Now assume that L_{4k+3} is divisible by 3 for some k, say $L_{4k+3} = 3t$. We must deduce that $L_{4(k+1)+3} : L_{4k+7}$ is divisible by 3.

$$\begin{aligned} L_{4k+7} &= L_{4k+5} + L_{4k+6} \\ &= (L_{4k+3} + L_{4k+4}) + (L_{4k+4} + L_{4k+5}) \\ &= L_{4k+3} + L_{4k+4} + L_{4k+4} + (L_{4k+3} + L_{4k+4}) \\ &= 2L_{4k+3} + 3L_{4k+4} \\ &= 2(3t) + 3L_{4k+4} \\ &= 3(2t + L_{4k+4}), \text{ as desired.} \end{aligned}$$

And now we may deduce the concatenation result: $L_{4k+1} = L_{4k+2}$ is divisible by 3, $\forall k \geq 0$. For

$$\begin{aligned} L_{4k+1} : L_{4k+2} &= L_{4k+1}(10^q) + L_{4k+2}, \text{ for some positive integer } q, \\ &\equiv_3 L_{4k+1}(1) + L_{4k+2} \\ &\equiv_3 L_{4k+3} \\ &\equiv_3 0, \text{ by the revious result.} \end{aligned}$$

We leave the proofs of the next result and its corollary to the interested reader. Be assured that these proofs may be constructed along the lines of what we have done in each case above.

Notice that 9 divides each of $L_7 = 18$, $L_{19} = 5778$ and $L_{31} = 1860418$. This is easily seen via the digit divisibility test for 9, analogous to that for 3; namely, a given number n is divisible by 9 if, and only if, the number formed by the sum of the digits of n is divisible by 9. But to show that *every* L_{12k+7} is divisible by 9 will require an

inductive proof (again, left to the interested reader).

And so, L_{12k+7} is divisible by 9, $\forall k \geq 0$.

Our concatenation corollary follows, easily proved in the style of the previous ones. That is, $L_{12k+5} : L_{12k+6}$ is divisible by 9, $\forall k \geq 0$.

For example, with $k = 0$, we have $L_5 : L_6 = 711$ with sum of digits 9, and with $k = 1$, we have $L_{17} : L_{18} = 22073571$ with sum of digits 27.

Summary

Our look at two special sequences and the concatenation of consecutive terms in them has paid off in patterns of division and in providing the need for proofs by induction. Other special sequences can be looked at in this way. For instance, the author has considered the sequence of triangular numbers in this same way. This led to a problem which will be posed in an upcoming issue of *Mathematical Spectrum* (a journal of The Applied Probability Trust in the United Kingdom).

Moreover, the novel idea of studying concatenation of integers allows the teacher and students a new and perhaps exciting chance at pattern discovery, reasoning, and proof. This fits into the curriculum framework as noted on page 13 of *The Ontario Curriculum, Grades 9 and 10: Mathematics (Revised)* (2005):

Reasoning and Proving

> An emphasis on reasoning helps students make sense of mathematics. Classroom instruction in mathematics should always foster critical thinking – that is, an organized, analytical, well-reasoned approach to learning mathematical concepts and processes and to solving problems.
>
> As students investigate and make conjectures about mathematical concepts and relationships, they learn to employ *inductive reasoning*, making generalizations based on specific findings from their investigations. Students also learn to use counter-examples to disprove conjectures. Students can use *deductive reasoning* to assess the validity of conjectures and to formulate proofs.

▲

▲ OAME Awards 2013

Elementary School Staff Award for Exceptional and Collaborative Mathematics Teaching:

Epiphany of Our Lord Catholic Academy

Located in Scarborough and part of the Toronto Catholic District School Board, Epiphany of Our Lord Catholic Academy is a JK to Grade 8 school with a high profile for mathematics in the province. The school has played host to visitors from across Ontario, as a showcase for people to see exemplary practices at work in real classroom settings. The teachers in the school welcome other educators to collaborate and learn with them as they utilize the Three-Part Lesson in a framework of successful teaching strategies. But even when no visitors are present, the teachers engage in rich dialogue around ways to deepen their students' understanding of mathematics and try new strategies supported by positive research. The results include students who show high levels of mathematical reasoning, proving, and communication.

The teachers at Epiphany of Our Lord Catholic Academy vary in years of experience as well as in grades taught. Yet they come together with enthusiasm in Study Groups, openly sharing their questions, observations, and insights about student thinking in mathematics, gathered from both their personal teaching experiences, as well as from research articles that have been provided by the principal. It is no wonder that these teachers have been noticed by so many other educational jurisdictions, and have been the subject of a webcast for the Literacy and Numeracy Secretariat. They are to be congratulated for the commitment that has earned them this award.



Award for Leadership in Mathematics:



Dwight Stead

Dwight's demonstration of leadership within the mathematics and education communities has long been evident on numerous fronts. Whether within his school, his board, or at the provincial level, Dwight has made a positive impact on whatever sphere of contact within which he might find himself.

Dwight's leadership within the Dufferin-Peel Catholic District School Board reached to touch all areas of teaching, from the mentorship of new teachers, to the building of teams in a fast-growing demographic area of the province that needed that core of expertise to maintain a vision for mathematics teaching. He has been the forerunner in promoting technology use in his board's schools, and has authored works in conjunction with a variety of mathematics associations and on a broad variety of mathematics-related topics. Dwight's contributions to mathematics on the broader scale include the development of curriculum supports for the Ontario Ministry of Education, to name but one among a number of such projects.

A member of OAME since 1999, Dwight has served the organization at the chapter level, including numerous years as the president of CHAMP. In addition, Dwight has been a Director at the provincial level of OAME. His enthusiasm for mathematics education in Ontario has been evident in the number of times he has been a conference presenter at both the chapter and the provincial levels, and in his willingness to take on a leadership role in the Ontario Mathematics Olympics.

OAME is proud to add Dwight's name to the roster of outstanding leaders in mathematics education who have earned this award.

Award for Exceptional and Creative Teaching in Secondary Mathematics:

Carolyn Crosby

Carolyn has been described as "dedicated," "committed," and "persistent." While there is no doubt that those qualities will be at the core of a successful teacher, perhaps most notable is how Carolyn has put these qualities to work in her teaching in the Catholic District School Board of Eastern Ontario. There, Carolyn has impacted the work of her colleagues with her collaboration and teamwork. But more importantly, she has positively impacted the achievement and outlook of her students, particularly those for whom mathematics has held challenges. Carolyn believes that students in Locally Developed mathematics courses should see a pathway into post-secondary endeavours, and works with her students to uncover the math strengths that lie in each of them. Through her differentiation of instruction and her dedication to bringing a real-world context to those learners, she guides them to success in an area where success might have been elusive in their past studies. Ways in which Carolyn has made this happen include the embedding of career education in her lessons, the use of online tools and manipulatives, and by taking mathematics out of the classroom and into social justice projects such as helping those in need through growing food and donating used clothing. Carolyn found ways in which her students' numeracy skills could be enriched in these applications.



OAME is not the first organization to recognize Carolyn's teaching talents. She was awarded the *Bernard Smyth Award* in 2011, awarded to a teacher for "promoting equality, democracy, and solidarity for a just, peaceful, and compassionate society." We congratulate Carolyn as a deserving recipient of OAME's award, for her creativity in teaching has truly made a difference.

The Margaret Sinclair Memorial Award:

Kathy Kubota-Zarivnij



Abacus editor and OAME past president, Kathy Kubota-Zarivnij has been honoured as the first recipient of The Margaret Sinclair Memorial Award, Recognizing Innovation and Excellence in Mathematics Education. Congratulations Kathy! For more details on the award, visit **www.fields.utoronto.ca/honours/sinclairaward/index.html**.

▲

▲ OAME 2014 CHAMPions 4 Change May 8–10, Humber College in Toronto: STUDENT THINKING

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Irene McEvoy is a Provincial Mathematics Facilitator and Co-Chair of the OAME 2014 Promotions Committee.

The 2014 annual OAME Conference on May 8–10 at Humber College in Toronto will be hosted by the local chapter CHAMP, Credit Humber Association for Mathematics Promotion, comprising the Peel and Halton regions. The conference theme, "CHAMPions 4 Change," recognizes the many changes educators are making in their practice to support all learners of mathematics. Changes in teaching strategies and technology are having a big impact in today's mathematics classroom. As a result, the OAME 2014 Promotions Committee has identified "Student Thinking" as an area of focus for this conference.

Each month, the Committee will post mathematical problems on the conference website with the purpose of:

- promoting student thinking/reasoning in the classroom
- engaging and challenging students
- providing a more global environment for student learning
- expanding the horizon of assessment opportunities

When you visit the conference website, **www.oame2014.ca**, you will find problems in each division, and samples of student solutions in the Student Thinking menu. We invite teachers to use these problems with their students and compare their students' strategies and reasoning.

To encourage professional learning, a wiki has been created: **http://oame2014studentthinking.wikispaces.com/OAME+2014+Student+Thinking**. Educators can join this wiki, view student solutions, and share their responses to the questions posted, as well as submit samples of their students' work.

Follow-up activities with the materials will be presented at the OAME 2014 Annual Conference. The facilitators for these activities include Dr. Marian Small, Dean Emerita, Faculty of Education, University of New Brunswick; Dr. Christine Suurtamm, Associate Professor, Faculty of Education, University of Ottawa; Connie Quadrini, Mathematics Consultant, York Catholic District School Board; Shelley Yearley, Trillium Lakelands District School Board, Provincial Math Lead on assignment with the Ontario Ministry of Education; and our teacher leaders in this initiative, Marcia Charest and Jeremy Labrie from the Peel District School Board.

We hope you will join our journey examining student thinking this year by going to the conference website, participating in the Student Thinking wiki, and attending the follow-up activities at the conference.

HOW DID THE IDEA OF A "STUDENT THINKING" SUB-THEME ARISE?

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We all have students who can be given a list of ten factoring questions and can get a perfect result in a matter of five minutes or less when they know the task at hand and have studied enough. The difficulty arises when those same students are faced with an application problem where they must first model a quadratic equation and then make a choice about the tools required to solve the problem. Very often, they struggle to make the appropriate choice.

This problem forced us to ask ourselves, "How can we help those students develop the skills to recognize when to choose each tool they have mastered? What are they thinking when they reach the point where a

choice is necessary?" Our first thought was to have more math talk and discussions with the students. Trying to get individualized time for these discussions was an obstacle. So we decided that we would let students talk through a solution while simultaneously recording their work. Students would then listen to themselves and review their process. Then, when time allowed, we would sit and listen to the recording with them, and ask questions about why they were making certain decisions.

Our early results show that when students talk out loud they are discovering many of their own weaknesses, as well as how to improve their strategies and decision-making skills. Along with recording student solutions, we have encouraged students to work in pairs or small groups to discuss problems and strategies so they can hear what ideas are possible, and critique and analyze their work to discover the most efficient method. Many of our intermediate and senior students are learning how to work effectively in groups in a mutually beneficial way. In the past, the strong student told the weaker one how to complete questions. Now students are working together to investigate WHY, and to develop a deeper conceptual understanding.

Over the next several months, while we lead up to OAME 2014, we plan on building a broader collection of student work and sharing our developments with you at the conference. ▲

▲ Book Review

Review of *Our Days Are Numbered*

REVIEWED BY TIM SIBBALD



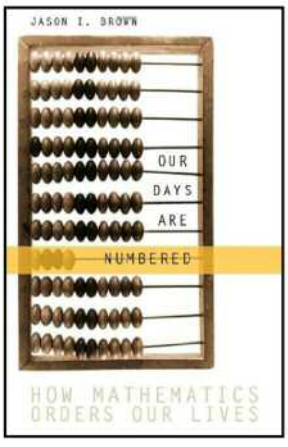
Tim Sibbald is an assistant professor at the Schulich School of Education at Nipissing.

Brown, J.I. (2009). *Our days are numbered: How mathematics orders our lives.* Toronto: McClelland & Stewart Ltd.

Jason Brown has written a very palatable read. It is an excellent book for the busy professional who may only find short snippets of reading time. Well suited to the commuter or traveller in that respect. The book has nearly 300 pages full of interesting elements of math that are presented with an assumption of little expertise. It is very well written in the sense of making topics accessible and then moving on before the weight of detail or mathematical machinery becomes too much.

The book starts with forays in numeracy and statistics. Applications to daily life situations and interpretation of the media are presented well. Along the way, there is an examination of introductory game theory ideas and how it applies to simple circumstances. Toward the end, there is musical content that deals with an examination of a long-standing music question.

What struck me most about this book was not the content. As an avid math reader, I was already familiar with most of the material, although there were a few bits and pieces I was not aware of. What was interesting was the possibility that kept coming to mind—that I could use material from this book with students. On several occasions, I thought, "This would be interesting to give to a group of students to see what they would do with it." In some sections, I felt the book could be used as a source of reading material that would reinforce classroom content or provide enrichment. ▲



▲ 2013-2014 OAME Board

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8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Abacus

Volume 52 • Number 1 • September 2013

PROPORTIONAL REASONING IN NUMBER SENSE AND NUMERATION

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A Publication of the OAME/AOEM

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KATHY KUBOTA-ZARIVNIJ currently focuses on K to 6 mathematics professional learning at the provincial level. She is a long-time OAME board member and OAME past president. In relation to her work in mathematics education, she uses complexity thinking to interpret mathematics teaching and learning for students and teachers. **kathkubo@rogers.com**

Many topics within the school mathematics curriculum require proportional reasoning, such as problem solving with fractions, decimals, percentages, scaling factors, unitizing, place value, and multiplication and division operations, to name a few. Proportional reasoning is fundamental to real-world and every-day situations, such as: determining which soccer team has the best ratio of experienced to new players; sharing submarine sandwiches among three people; determining if 5 cents less per bagel is better than a 5% discount on a dozen bagels; calculating the sale price when everything is 40% off.

In this issue, the Research Summary and problems (with multiple solutions) focus on strategies for representing proportional reasoning situations within the number sense and numeration strand only. The problems we provide are designed for use within a three-part, problem-solving lesson structure.

LINKS TO LITERATURE AND MANIPULATIVES



PAT MARGERM is an independent literacy and mathematics consultant with more that 30+ years of K to 8 teaching experience with students, in-service and pre-service teachers. She is a long-time OAME board member and works in various aspects of mathematics education. **pmargerm@edu.yorku.ca**

Although proportional reasoning first appears as a subheading in grade 4 in the Number Sense and Numeration strand of the Ontario Mathematics Curriculum (2005), some experiences with counting contribute to the development of proportional reasoning. Unitizing is one of the principles of counting. Unitizing occurs when a student is able to understand that a single object may stand for more than one thing. For example, children come to realize that one train of ten cubes has the same value as ten single cubes; one nickel is equal to five pennies; one dot pattern of four can also represented as four single dots. In this issue, let's look at the use of dot plates, ten frames, number cubes and a picture book to develop the concept of unitizing.

RESEARCH SUMMARY – PROPORTIONAL REASONING (RATIO AND PROPORTION)

About Proportional Reasoning

Proportional reasoning is about being able to make comparisons between quantities in a multiplicative, rather than additive way. For example, multiplicative comparisons can be described using terms, such as double, half, three-times greater; rather than comparing objects as being smaller or bigger. Small (2010, 2012) explains that proportional reasoning involves the deliberate use of multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another. Consequently, proportional reasoning is commonly interchanged with the term, multiplicative thinking, as an implicit multiplicative comparison is used. For example, there are 6 marbles in 1 bag. How many marbles are there in 4 bags? You are compelled to compare the number of marbles in 1 bag to the number of marbles in 4 bags in a multiplicative way. Therefore, "proportional reasoning involves a sense of co-variation, multiple comparisons and the ability to mentally store and process several pieces of information" (Post, Behr and Lesh, 1988, p. 79).

From Additive to Multiplicative Thinking

Though this Caterpillar Leaves Problem is intended to provoke students' multiplicative thinking, Kenny, Lindquist and Heffernan (2002) report that students use a range of strategies that include additive and multiplicative thinking:

Caterpillar Leaves Problem:

A fourth-grade class needs 5 leaves each day to feed its 2 caterpillars. How many leaves would they need each day for 12 caterpillars.

- 15 leaves → think the amount to be added must remain constant rather than the multiplier remaining constant; since there were 3 more leaves than caterpillars, then there should be 3 more leaves for the 12 caterpillars (15 leaves)
- 15 leaves → focused on 10 more caterpillars rather than 6 times as many caterpillars, so added 10 more leaves to original 5 leaves, keeping the amount added constant
- 30 leaves à since 5 is for 2 caterpillars, adds 2-1/2 leaves for each caterpillar, repeatedly adds 2-1/2 leaves 12 times; multiplies 2-1/2 leaves x 12 caterpillars (size of group is 2-1/2; number of groups is 12)
- 30 leaves → 5 leaves for 2 caterpillars, so groups the 12 caterpillars into 6 groups and multiplies 5 leaves (for 2 caterpillars) x 6 groups; uses a table of values, showing # of caterpillars as 2, 4, 6, 8, 10, 12 in relation to leaves, 5, 10, 15, 20, 25, 30

Siemon and Breed (2006, p. 4) say that "multiplicative thinking develops as each element (equal groups, number of groups, total) is abstracted … dealt with as a mental object."

Modelled	Abstracted
Equal Groups Makes all, counts all	Trusts the count, sees equal group as a composite unit
Number of Groups Sees in terms of each group, counts all groups, 1 group, 2 groups, 3 groups	Can deal with number of groups in terms of part-part-whole understanding, e.g. 6 groups is 3 groups and 3 groups or 5 groups and 1 group
Total Arrived at by counting all, skip counting	Total seen as a composite of composites, e.g. 18 is 2 nines, 9 twos, 6 thress, 3 sixes

Ratio and Proportion

Solve this ratio problem in 2 different ways. How do your solutions compare to the solutions below?

• part to part – clear : green → 5 : 10 → 15/10 = 3/2
(3/2 times (x) more clear than green marbles)

• part to whole – clear : total marbles → 15 : 45 → 15/45 = 1/3
(1/3 of total marbles are clear)

Marbles Problem 1:

In a bag of 45 marbles, 15 are clear, 10 are green, and 20 are solid black. What comparisons can be made?

Ratio is a multiplicative comparison of two or more quantities, which can be expressed as a fraction of two quantities of the same kind (part to part, part to whole). Also, a ratio is the division of two quantities, having the same units. The ratio has no unit. It is a numerical quantity, which indicates how many times one quantity is greater or less than the other. (Lobato, Ellis, Charles, & Zbiek, 2010; MOE, 2012; Small, 2010, 2012).

How does the solution to this problem 2 change multiplicatively from problem 1? Why?

• part to part – clear : green → 2 x 15 : 2x10
→ (2)15/(2)10 = 3:2 (meaning that 3/2 times (x) more clear than green marbles)

• part to whole – clear : total marbles → 2 x 15 : 2 x 45
→ (2)15/(2)45 = 1/3 (meaning that 1/3 of the total marbles are clear)

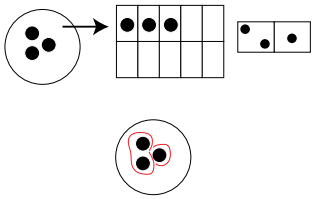
Marbles Problem 2:

What is the ratio of clear marbles to green marbles and total marbles if there are double the bags of marbles?

A ***proportion*** is a relationship of equality between two ratios. In a proportion, the ratio of 2 quantities remains constant as the corresponding values of the quantities change. A ***continued ratio*** is a comparison of more than 2 quantities having the same units.

LINKS TO MANIPULATIVES: DOT PLATES, TEN FRAMES AND NUMBER CUBES

Unitizing can be developed by flashing familiar dot configurations, from number cubes /dice, dominoes, dot plates, and five and ten frames. Ask students, “What number did you see?” Students can respond by holding up fingers to show the number seen. Providing each student with materials such as a set of number cards, ten frame cards or ten frames and counters provides students with experiences to represent the number in a variety of ways.



- Flash this dot plate. Students might choose this ten frame or a domino to respond to what they saw.
- Ask students, “How did you see the number?” They may be seeing a dot pattern as one unit as well as a collection of single units. (i.e., thinking of the dot pattern as a single unit of three as well as a collection of three ones).
- A child who cannot unitize will begin at one and count by ones to determine sums of basic combinations; count all the objects in a set by ones instead of seeing smaller sets of twos, threes, and fours imbedded in the set.

These actions support students in beginning to think of a quantity.

Unitizing is a precursor to counting on, using ten as a unit, part/part/whole and place and value.

LESSON LEARNING GOALS

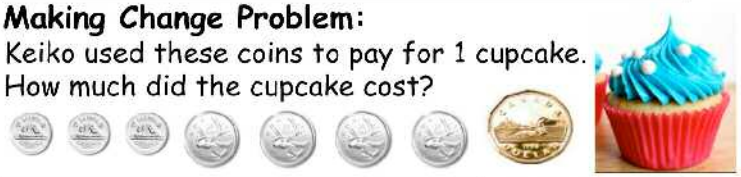
Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (lesson), and After (Practice) parts of the lesson

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Making Change Problem:

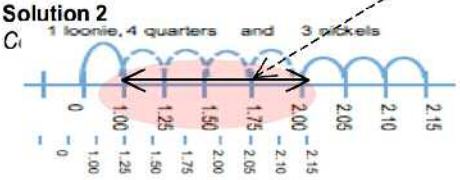
Keiko used these coins to pay for 1 cupcake.
How much did the cupcake cost?



Solution 1

3 nickels: 5, 10, 15 ..... 15 cents
4 quarters: 25, 50, 75, 100 ... 1 dollar
1 loonie: 100 cents ..... 1 dollar
Keiko paid $2.15 for the muffin.

Solution 2



4 quarters for 100¢
25¢ x 4 = 100¢

4 quarters for 1 dollar ($1.00)
$0.25 x 4 = $1.00

100¢ for 1 dollar
$0.01 x 100 = $1.00

size of group *- 25¢, $0.25, or $0.01*
# of groups *- 4 or 100*

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to make a plan to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Coin Box Problem 1:

A grade 4 class store started with a coin box with $30.00. After 1 week, they had 24 nickels, 22 quarters and 35 loonies.
How much money did they make in their first week?

What information are we going to use to solve to solve the problem?
- *$30.00 at the start*
- *24 nickels, 22 quarters, 35 loonies after 1 week*
- *How much more money than $30 is in the coin box now?*

AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria for the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas,

multiplicand *- size of group (e.g., nickel, 5 cents, or $0.05)*
multiplier *- number of groups (e.g., 24 or 4 groups of 6 nickels)*
product *- result of multiplication ($1.20 = 4x6x$0.05 = 24 x $0.05)*

Proportional Relationships are multiplicative
1 x 4 x $0.25 = 1 x $1.00 (1 group of 4 quarters)
5 x 4 x $0.25 = 5 x $1.00 (5 groups of 4 quarters)
1 group -> $1.00; 5 groups -> $5.00 has a multiplicative relationship of X 5

Solution 1
(a) 35 loonies is $35.00

(b) 22 quarters
-4 quarters make $1.00
-5 groups of 4 quarters
(20) makes $5.00
-2 more quarters (22) makes $5.50

(c) 24 nickels
-6 nickels make $0.30
- 4 groups of 6 nickels
(24) makes $1.20

$41.70 -$30.00=$11.70
(money made)

Solution 2
1 nickel →$0.05
10 nickels →$0.50 (x 10)
20 nickels →$1.00 (x 10, x 20)
24 nickels →$1.20 (x 20)

1 quarter →$0.25
10 quarters →$2.50
20 quarters → $5.00 (x 22)
22 quarters →$5.50 (x 22)

$1.20 + $5.50 + $35.00
= $41.70
Subtract start of $30.00
$41.70 - $30.00= $11.70
They made $11.70.

Making Groups of Same Value Coins Near to $1.00 Unit

Scaling Up (Multiplying by Larger # of Groups) Starting at Unit of 1

Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of students' mathematical discussion.

Grade 5	Grade 6	Grade 7
-explain concept of equivalent fractions using concrete materials -demonstrate … simple multiplicative relationships involving whole number rates	-represent ratios found in real-life contexts, using concrete materials, drawings and standard fractional notation	-determine through investigation the relationships among fractions, decimals, percents and ratios -demonstrate an understanding of rate as a comparison or ratio of two measurements with different units

Ratio

- multiplicative comparison of two or more quantities of the same kind (part to whole - 1 nickel to total 24 nickels)
- has no unit
- is a numerical quantity - how many times (multiplier) one quantity is greater than the other quantity.

Equivalent Ratios *- are equivalent fractions*

$\frac{35 \text{ loonies}}{1 \text{ loonie}} - \frac{3500 \text{ cents}}{100 \text{ cents}}$

24 nickels = 120 cents
1 nickel 5 cents
X 25
22 quarters = 550 cents ÷ 22
1 quarter 25 cents
X 25
35 loonies = 3500 cents
1 loonie 100 cents

- ratio as fraction
- doesn't matter which quantity is numerator or denominator (X or ÷)

Values in ratio table added together (2+20=22) quarters corresponds with $0.50+$5.00 = $5.50 (value of quarters)

2 values changing together
x 4 top and bottom
x 20 top and bottom

A proportion *is a relationship of equality between two ratios. The ratio of 2 quantities remains constant as the corresponding values of the quantities change.*

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria

Highlights/Summary:

Ratios are multiplicative (proportional) comparisons (part to whole) of 2 or more quantities of the same unit.
Represent (equivalent) ratios as:
- a numerical quantity (not unit, a constant) stays the same as corresponding values change

X 25
X 22 (1 quarter = 25 cents
22 quarters 550 cents) X 22
X 25

- showing how many times (multiplier) one quantity is greater than the other quantity
- equivalent fractions (1/22 = 25/550 or 22/1=550/25)

AFTER (Practice) – 5 to 10 minutes

- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Solution 3

Nickels	1	4	20	24
Value	0.05	0.20	1.00	1.20

Quarters	1	2	20	22
Value	0.25	0.50	5.00	5.50

Loonies	1	35
Value	1.00	35.00

$1 + $5 + $35 + $0.20 + $0.50 - $30 = $41.70 - $30 = $11.70 (1 week earned)

Solution 4

X 24 (1 nickel = 5 cents
24 nickels 120 cents.) X 24
X 25
X 22 (1 quarter = 25 cents
22 quarters 550 cents) X 22
X 25
X 35 (1 loonie = 100 cents
35 loonies 3500 cents.) X 35

(120 + 550 + 3500) cents
4170 cents = $41.70
less $30 → $11.70
earned in 1 week

Ratio Table
Multiplicative Relationships

Ratio (Fraction)
# of Coins to Coin Values

Coin Box Problem 2:



1 roll of coins has either 25 loonies, 40 quarters or 40 nickels. These coins were deposited at the bank:
15 rolls of loonies; 5 rolls of quarters; 84 nickels. How much money was deposited?

Solution 1

25 loonies →$25
1 roll → $25
15 rolls → 15 x $25 = $375

40 quarters -> $10
1 roll → $10
5 rolls → $50

40 nickels → $2
80 nickels → $4
84 nickels → $4.20

Total:
$375.00
50.00
+ 4.20
$429.20

Solution 2

loonies
$25 x 15 = $375.00

quarters
$0.25 x 40 x 5 = $50.00

nickels
$0.05 x 84 = $4.20

$375 + $25 + $25 + $4.20
= $400 + $25 + $4.20
= $429.20

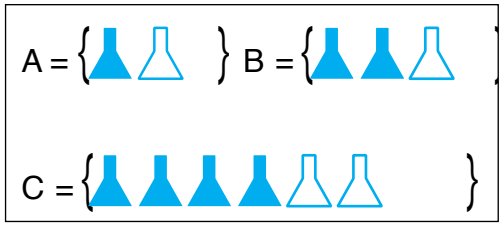
Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.

LET'S DO MATH

Before (Getting Started)

Blueberry and Water Mixtures Problem:

How do the mixtures compare to one another?



Key

represents 1 part blue

represents 1 part clear

Solution 1

Mixture A

part to whole → blue to clear
→ 1 to 1 → 1 : 1
→ 1/1 blue to clear
1 part blue to 1 part clear water

Mixture B

part to part → blue to clear
→ 2 to 1 → 2/1 blue to clear
2 parts blue to 1 part clear water

2 >1 → Mixture B is bluer than Mixture A.

Solution 2

Mixture A

part to whole → blue to total → 1 to 2 → 1 : 2
1/2 of the mixture is blue

Mixture C

part to whole → blue to total → 4 to 6 → 4 : 6
4/6 of the mixture is blue

Mixture B

part to whole → blue to total → 2 to 3 → 2/3 = 4/6
2/3 or 4/6 of the mixture is blue

Mixtures B and C are 2/3 blue; Mixture A is 1/2 blue
1/2 < 2/3 → Mixture A is lighter blue than mixtures B and C



During (Working On It

Ladybugs and Grasshoppers Problem

The ratio of ladybugs to grasshoppers is 12:11 in a garden. If there are 1955 insects, then how many ladybugs and grasshoppers are there?

After (Consolidation)

Solution 1

Part + part = 12 + 11 = 23 insects in one group

$\frac{1955}{23}$ = 85 size of each group

85 groups of 23 insects = 1955 insects
12 x 85 = 1020
11 x 85 = 935
12 to 11 = 1020 to 935 = ladybugs to grasshoppers

Solution 2

12 to 11 means part to whole is 12 to 23.

23 x 50 = 1150
23 x 30 = 690 (1150 + 690 = 1840)
23 x 5 = 115
23 x 85 = 1955

$\frac{\textit{lady bugs}}{\textit{insects}} = \frac{12}{23}$

$\frac{\textit{lady bugs}}{1955} = \frac{12 \times 85}{23 \times 85} = \frac{1020}{1955}$

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solutions 2?

- *Solution 1* – shows comparison of 23 insects in one group to 1955 altogether; determines 85 groups and multiplies once by 12 and once by 11.
- *Solution 2* – proportions are used to solve the problem, using fractions and equivalent fractions.

LINKS TO LITERATURE: *ONLY ONE*



Is it math or is it language? The picture book, *Only One* is a counting book that includes collective nouns.

"There can be 11 cows. But there is only one herd."	"There may be 12 eggs, But there is only one dozen."

- Read the book to the students and ask, "What did you learn about the number one?"
- Sample response: "One can mean one like one sun in the sky, but sometimes is means more than one."
- Review the book and look for examples of when one refers to a specific number e.g., one dozen is 12 and when one refers to a quantity that can change like, the number of cows in one herd.
- Collect and record examples of one as a unit and the specific quantity referenced, e.g., one hundred, one ten frame, one nickel, one hand, one baseball team, one hour.

LETS DO MATH – BUTTERFLY PROBLEMS

Before (Getting Started)

Butterfly Problem 1:

Each butterfly has 4 wings.

How many wings are on 26 butterflies?

Solution 1

BF	1	2	3	5	10	20	25	26
Wings	4	8	12	20	40	80	100	104

25 butterflies have 100 wings

26 butterflies have 104 wings

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

During (Working On It)

Butterfly Building Problem 2:

Students are building butterfly models at a science fair. Each butterfly needs 4 wing pieces, 1 body and 2 antennae. If Jan has 29 wings, 8 bodies and 13 antennae, how many complete butterfly models can be built?

After (Consolidation)

Anticipating Student Responses:

Solution 1

Wings (4)	4	8	12	16	20	24	28	Only 29
Body (1)	1	2	3	4	5	6	7	8
Antennae (2)	2	4	6	8	10	12	Only 13	
Number of models	1	2	3	4	5	6		

We skip counted by the size of each group until we reached the total number of each part available. Then we knew they could make 6 complete butterflies.

Solution 2

Wings: 4 x 5 = 20
4 x 2 = 8
28 wings → 7 butterfly models.
29 wings not enough for 1 more model.

Antennae: 2 x 7 = 14 but we only have 13 → so only 6 butterfly models can be made. With 8 bodies, 6 models can be made.

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solutions 1?

- *Solution 2* – Students used guess and test with multiples of 4 to determine 28 wings would make 7 butterflies. They went on to reason the same way with antennae.
- *Solution 1* – Students skip counted, using a table to record until they got to the total number of parts available at the science fair. When they got to the maximum number for that part they stopped and saw the answer in the bottom row. They would use 24 wings, 6 bodies, and 12 antennae to make the 6 butterfly models.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Try these proportional reasoning problems yourself first and then offer them to your students.
- To suit your students' learning needs, vary the ways that you have students engage with noticing and using multiplicative (proportional) relationships inherent in multiplication and division operations, in comparing part-whole relationships and equivalence between fractions, decimals and percentages, and in representing situations using ratios. Use different models of representation, such as open number lines, polygon grids, and sets of objects, visible in concrete materials and visuals, such as counters, base ten blocks, square tiles, pattern blocks, ten frame, dot plates and fraction bars to model proportionalities.
- Practise noticing the different strategies and the breadth of the mathematics your students use to describe, show and make sense of multiplicative thinking (proportional reasoning) in relation to solving multiplication, division, fraction and ratio problems. Take note of the kinds of proportional reasoning language students' use from primary to junior years (e.g., equal size groups, size of group, number of groups, fractions as part- to-part and part-to-whole ratios).
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit key mathematical ideas, strategies for transitioning from additive to multiplicative thinking (e.g., partitioning a whole into equal size groups, doubling using an array grid, multiplicand (size of group), multiplier (number of groups), multiples, halving,) as the students discuss and analyze each other's solutions to co-construct success criteria for the lesson learning goal.

Suggested Readings

Readings used in the Research Summary and design of the problems and solutions include:

- Karplus, R., & Peterson, R. (1970). Intellectual development beyond elementary school II: Ratio, a survey. *School Science and Mathematics, 70*(9), 813-820.
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CLOSING

Our next issue will focus on proportional reasoning within the measurement strand. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a problem solving-based teaching and learning lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or mkestell@sympatico.ca