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Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

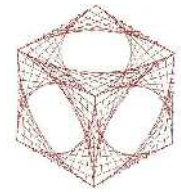
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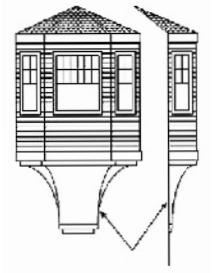
Vol. 52 #2
December 2013

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See Hey, It's Elementary – Using Found Materials and Anthropometry to Construct, Measure, and Create Mathemagic





See What's the Problem? Gardner's Grid

A sequence of functions is defined recursively by $f_n(x) = f_0(f_{n-1}(x))$, *where* $f_0(x)$ is some function. For example, if $f_0(x) = x^2 - 1$,

then $f_1(x) = f_0(f_0(x))$

$= f_0(x^2 - 1)$

$= (x^2 - 1)^2 - 1$

$= x^4 - 2x^2$

and $f_2(x) = f_0(f_1(x))$

$= f_0(x^4 - 2x^2)$

$= (x^4 - 2x^2)^2 - 1$

$= x^8 - 4x^6 + 4x^4 - 1$

and so on.

1. Given $f_0(x) = \frac{1}{1-x}$, determine $f_{2013}(x)$.
2. Determine another sequence of functions, defined by $g_n(x) = g_0(g_{n-1}(x))$ where $g_0(x)$ is some function that behaves similarly to the sequence of functions in 1.

See What's New with CLIPS? mathies.ca!



Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Immaculate K. Namukasa, along with **separate files for each of the text, graphics, and/or photographs**. Please e-mail all files to Immaculate K. Namukasa at **immaculate.namukasa @oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and e-mail address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME.

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The views expressed or implied in this publication, unless otherwise noted, should not be interpreted as official positions of OAME.

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The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

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Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** at the above address. Courier is recommended to avoid possible delays. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

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▲ Editor's Message

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Immaculate Kizito Namukasa is an associate professor at the University of Western Ontario. Her research projects include "Mathematical thinking and activity of elementary and middle school learners," "New teachers learning to teach mathematics," "Expert teachers implementing innovative mathematics methods and environments in their classrooms," "Parents doing mathematics with their children in community settings," and "Complexity research inquiry and critical mathematics education in Canadian and international contexts."

Holiday season is a great time to catch up on professional and personal reading. In this *Gazette* edition, three books are reviewed: Catherine Heideman reviews *Sally Strange*, while Tim Sibbald reviews *The Mathematics Teachers Need to Know, and Don't Count Me Out!* Two articles and one feature are about investigations in mathematics, three articles share ideas about tools and technologies in mathematics, and two articles report on activities and new developments in the mathematics education community.

Thomas Moore shares a mathematics investigation for middle and high school students. He bases the investigation on the eighteenth-century mathematics of Fermat, Gauss, Lagrange, Bolyai, and others. But what stories from modern mathematics can be shared with young learners? The Ontario Ministry's *Patterning and Algebra, Grades 4 to 6: A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6* (2008, p. 29) provides an example of extending a number pattern problem, in which students explore visual proofs for sums of numbers in a pattern. A visual representation of the sum of the first 10 natural numbers 1 + 2 + 3 + 5 + 5 + … + 10 is given. On the same page, this sum is visually related to that of square with dimension 10 by 10. The sum is also visually related to the area of a triangle with base 10 and height 11. Elsewhere I have related it to the area of a rectangle with dimensions 10 by 11 (Namukasa, 2011). This leads to an expression for the sum of the first 10 natural numbers as (10 x 11)/2. The next section of *The guide to effective instruction*

connects to understanding expressions, variables, and equations. The general rule for the summation of numbers is attributed to the work of Carl Friedrich Gauss (1777–1855) when he was a schoolboy. The visual solution and proof of the sum of a numerical series can be illustrated by the classic "handshake problem" (Namukasa; see also, Ciobanu in this issue).

I cannot help but recall a theorem recorded by Pierre de Fermat (1601–1665) in the margins of a mathematics book he was reading. Fermat claimed to have solved the theorem, but had not written the solution down due to limited space. Thus, mathematicians joined in the amazing mathematics race to uncover Fermat's original proof. Fermat's famous last theorem puzzled mathematicians for three centuries, until it was eventually solved by a current mathematician, Sir Andrew John Wiles (1953–), in the 1990s. The theorem, put into plain language, states that, whereas it is possible to separate a line into two lines and a square into two squares, "It is not possible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers." That is, you could find values a, b, c, which satisfy $a^n + b^n = c^n$ for $n = 1$ and $n = 2$. There are solutions for $a^1 + b^1 = c^1$, the specific case for separating a line into two lines; and for $a^2 + b^2 = c^2$, the specific case for the Pythagorean theorem. However, it is not possible to find values of a, b, c for $n = 3, 4, 5\ldots$ and so on. The proof that Fermat's theorem was true for the specific case for $n = 3$, that is to say, that it was impossible to separate cubes into cubes, or, no solutions existed for $a^3 + b^3 = c^3$ was produced by Leonhard Euler (1707–1783). Legend has it that from the age of 10, Wiles was determined to solve Fermat's last theorem. By building on the work of several other mathematicians, his first proofs were imperfect and were recalled. In 1994, however, together with his university student, he proved the impossibility of finding values a, b, c for the equation $a^n + b^n = c^n$ for all n greater or equal to 3. Tales of Gauss and Wiles are great stories about, and for, young mathematics learners. I will leave it to the reader, especially the middle and high school mathematics teacher, to find some stories or mathematics contributions of other eighteenth-century mathematicians that are relevant to the current mathematics curriculum.

Moore, in his article, nicely connects eighteenth-century European mathematics back to the ancient mathematics of Greece, and then forward to the mathematics investigations enabled by computer programing. If you have not heard about the *n*-gonal numbers and their relations, I encourage you to read on to see how this concept relates to triangular and square numbers. Moore uses visual proofs, software and algebra to prove, by induction, relations among other *n*-gonal numbers and among their functions, thus illustrating a connection between elementary school investigations of number patterns and secondary school algebraic functions and proofs by induction.

Like Moore's article, Ciobanu and Shawn share mathematics investigations. Ciobanu's regular column fits with this exploration of investigations in mathematics and shares ways of modifying problems to make them engaging investigative activities. Shawn's problem in this issue is one of mathematics magic, with a trick he is delighted to reveal via algebra. In another article, also on algebra, "From Mind Games to Algebra – Part 2," a continuation of Part 1 in the September issue, Pyper shares a method of teaching equations, the E-kwatee-ons game, a method of teaching equations that promotes student interest, even in solving multi-step equations.

Bourassa, Colgan, and Kajander's articles share innovative ideas on the use of tools and technologies in mathematics learning. Bourassa outlines examples of online resources for *three-act mathematics* inspired by Dan Meyer. Colgan's piece considers, "while often there is not much time in the busy school schedule to fit in the more inspirational and enriching side of maths, many of which could invite children to see the history and diverse application of the domain," there is a way "to make in our classrooms a place in which one important goal is to restore children's love of mathematics and bring back some joy to their mathematical learning journey" (p. 15 in this issue). She offers examples of real-life tools that can be used in learning mathematics. Kajander's feature is on modelling problems, using manipulatives, paper-and-pencil models, as well as procedure, and how it is important to show clear links among these three dimensions of mathematics. In *What's New with CLIPS? **mathies.ca**!*, Clarke, Grafton, and Isenegger share a new Ministry website, **mathies.ca**.

Craven and Romiens report on Fall activities and new developments that have taken place at the Fields Institute and NCTM, respectively. Craven reports on a Field's forum, at which teacher beliefs were discussed by two authors, Priscilla Bengo and Martha Mavor, as well as a discussion entitled "Moving Toward a Two-Year Teacher Education Program in Ontario," which was led

by a panel of university professors. Also included in this issue is an obituary for Paul Pogue (1940–2013), who served as OAME president in 1984 and 1985.

I end my message with a brief reflection on the recent conversations around EQAO 2012 results. I know that, while the general public, especially the media, has many reactive responses to any perceived successes or failures of mathematics teaching, mathematics teachers, educators, researchers, and policy makers are well equipped with proactive responses. For me, proactive responses are the ongoing innovative instruction, curricula, and assessment practices that are based on research; supported by policy and publications on reform in mathematics teaching; implemented in classrooms by teachers; reflected upon in teacher collectives; and shared among teachers at conferences, in journals, and elsewhere. To quote the NCTM (2013) position statement on recent reform curricula:

> [Reform curricula] offer a foundation for development of more rigorous, focused, and coherent mathematics curricula, instruction, and assessments that promote conceptual understanding and reasoning as well as skill fluency. This foundation will help to ensure that all students are ready for college and careers when they graduate from high school and that they are prepared to take their place as productive, full participants in society (p. 1).

Proactive responses and support for implementing curricula, instruction, and assessment needed to promote conceptual understanding and reasoning, as well as skill fluency, are shared in this and past issues of the *Gazette*.

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▲ Call for Manuscripts

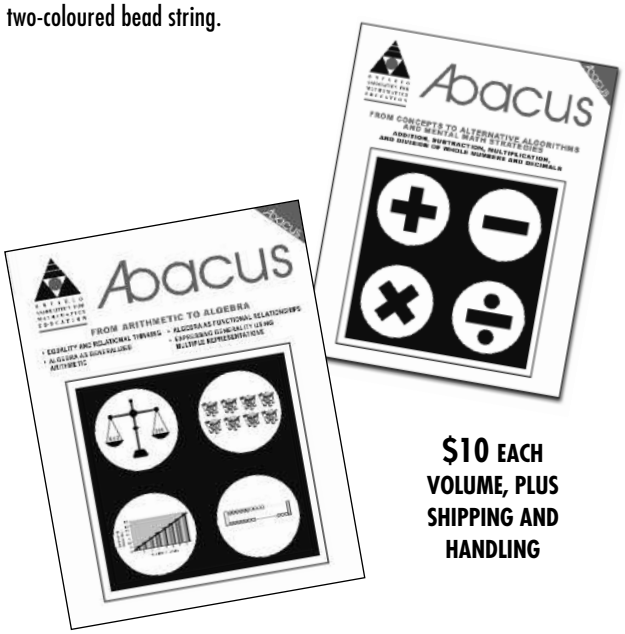
The *Ontario Mathematics Gazette* is inviting manuscripts for all grade levels.

Instructions for submission of manuscripts are found on page 1 of any *OMG*.

Contact the Editor for further details.

Abacus: A Compilation

Volume 46 – This resource for teachers includes the concepts of addition, subtraction, multiplication, and division of whole numbers and decimals, alternative algorithms, mental math strategies, manipulatives, and links to literature. It will be useful for exploring these aspects of teaching and learning personally or with a team of teachers. Includes one two-coloured bead string.



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▲ President's Message

SONIA ELLISON
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In my first column, I remarked that it was hard to believe summer was over and we were back to school. Now here we are in December. 2013 is almost over and 2014 is around the corner. Let's stop and take a look back at the first four months of school. Have you participated in any new learning? Have you used any OAME resources in your work? Have you attended a Chapter Conference? Did you attend the Leadership Conference? What changes have you made to your practice based on your new learning? We are all very fortunate to belong to an organization that seeks out ways to support its members. For many of us, our passion for teaching math is enhanced by the opportunities to learn with and from each other.

As part of my job responsibilities, I am very fortunate to learn with and from beginning teachers through the New Teacher Induction Program (NTIP). The program focuses on improved teacher confidence, efficacy, practice, and continuous learning, all focusing on the student. As a way to achieve the goals of the program, each NTIP teacher works with a mentor.

The role of the mentor teacher for each NTIP teacher has proven to be a very important one. The relationship between the beginning teacher and his or her mentor is one that can take on different roles, depending on the needs of each person. The mentor can act as a consultant, coach, or collaborator. As a consultant, the mentor is simply providing information, answering questions, or attending to logistics. As a coach, the mentor is questioning the beginning teacher, supporting him or her in developing in the practice of teaching. As a collaborator, the mentor and beginning teacher are learning and working together to enhance each other's teaching practice. In each case, the mentor is acting as a teacher leader. As collaborators, both participants are teacher leaders.

Teacher leadership is a topic that I have been reading about recently, and it is so clear to me that an organization like OAME is made up of teacher leaders. Helterbran (2010) suggests that, even though teacher leadership theory exists in academia, teachers need to move away from the idea that they are "just a teacher" and recognize their potential for leadership. He continues to suggest that "(t)eacher leadership rises from within the teaching ranks and expresses itself in a myriad of ways for the betterment of students, specifically, and school in general" (p. 364). Is this not who we are in OAME? I believe that we are the teacher leaders who look for opportunities to further our own understanding of what it means to be a teacher of mathematics.

Our organization is full of teacher leaders—those members who are looking for ways to positively influence teaching and learning of mathematics. As members of OAME, we look to our mission statement to remind us that we *promote excellence in mathematics education throughout the Province of Ontario*. **We work for** *excellence for students by striving for a curriculum that will prepare them to meet the mathematical challenges in their lives, by fostering familiarity with the methods of mathematics and an admiration for its principles, and by giving recognition and respect to their own mathematical achievements.* **We work for** *excellence for educators by promoting their role as facilitators and resource persons, by supporting a variety of teaching methods to meet the varied learning styles and needs of students, by encouraging their students to gain facility in learning both co-operation and independence, and by celebrating the mathematical excellence of both individuals and groups.* Let us continue to develop ourselves as teacher leaders, recognizing that leadership does not come with a title and power. But rather, Helterbran (2010) reminds us that teacher leadership is emergent, self-generated, and holds the promise of continuous learning and innovation for the improvement of self, student, and school (p. 365).

I hope that each one of you sees yourself as a teacher leader. I encourage you to reach out to those educators who need your support and insight. We know that OAME is driven by the hard work of its Board of Directors, Chapter Representatives, Executive, and you, the members. I thank you for your work and hope that as we move into 2014, there is a renewed motivation for making a difference in mathematics education for our students, our colleagues, and our organization.

May the New Year bring peace, prosperity and a world filled with laughter and love.

Sonia Ellison

Citations:

Helterbran, V.R. (2010). Teacher Leadership: Overcoming 'I Am Just a Teacher' Syndrome. *Education, 131*(2), 363–371.

Ontario Ministry of Education. (2010). *The New Teacher Induction Program (NTIP)*. Retrieved from **www.edu.gov.on.ca/eng/teacher/induction.html** ▲

▲ Technology Corner – Three-Act Math Tasks

MARY BOURASSA
E-MAIL: mary.bourassa@oame.on.ca



Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

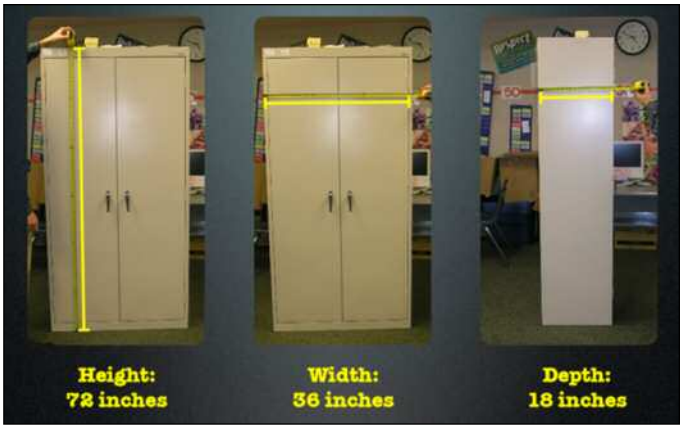
Dan Meyer (**@ddmeyer**) has brought the three-act math task to the forefront of the mathematics community. He perplexes students by showing them a picture or video that makes them need to know the answer and therefore need to do the math. It connects math to a real situation, has students working toward a common goal, and then, instead of telling them if their answer is correct, they watch the video to find out the answer. Dan Meyer's blog can be found at **http://blog.mrmeyer.com/**, and his three-acts can be found under "My Curricula" on the right-hand side.

The idea is that in Act I, students are presented a video that clearly and quickly shows them the situation. For example, Andrew Stadel's (**@mr_stadel**) Filing Cabinet three-act. The video shows a filing cabinet and a lot of sticky notes. Andrew numbers the sticky notes as he affixes them on the cabinet.



Students then ask questions. They should all converge on one question. In this case, how many sticky notes will it take to cover the filing cabinet? What information do students require to answer that question? Act II gives them that information. In this case, the dimensions of the filing cabinet and of the sticky notes are shown.

Students then do the mathematics necessary to answer the question. Act III is the full video with the dramatic conclusion. Students cheer or groan, depending on whether their solution is correct. I won't show you Act III for the filing cabinet, as I know you'll want to figure it out yourself! You can find Andrew's tasks at **http://mr-stadel.blogspot.ca/**.





Kyle Pearce (**@MathletePearce**) has gone one step further and has organized many three-act math tasks by grade for Ontario's curriculum. You can find these at **http://tapintoteenminds.com/3act-math/**. You can search by grade and by course and even submit your own math task. Timon Piccini (**@MrPicc112**) also has a collection of three-act math tasks at **http://mrpiccmath.weebly.com/3-acts.html** that are great for Grades 7–9. John Scammell (**@thescamdog**) has three-act math tasks for elementary students at **http://thescamdog.wordpress.com/**.

Here are a few examples.

Dan Meyer's Stacking Cups three-act is a favourite of many. Students have to determine how many Styrofoam cups it will take to reach the top of their teacher's head. Students work in groups and are given ten cups. They have to determine a model relating the number of cups to the height of the cups and extrapolate, given the teacher's height. Act III is

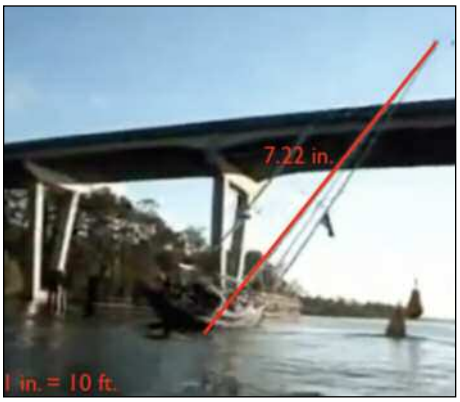


great, as students actually stack up the cups and count them to see if their calculations were correct. There are thought-provoking extensions to this three-act math task that I encourage you to check out.

I use Andrew Stadel's Boat on the River when I introduce primary trigonometric ratios. Act I is a video (with great music) of a boat approaching a bridge. Students quickly ask, "Will the boat make it under the bridge?"



On to Act II and a little trigonometry to see if the captain got it right or if there will be a catastrophe!





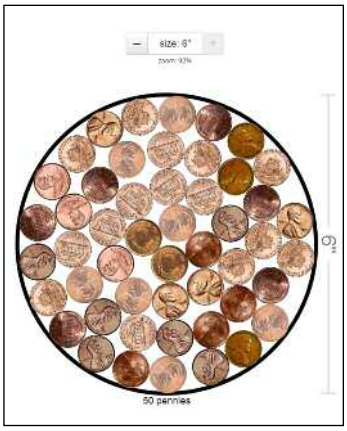
Dan Meyer's Bean Counting is not the most exciting three-act, but it gives those shared work, rational equation problems meaning. My students have always struggled with these types of questions, but having done this three-act task with my MHF4U class this year, the results were much better. Act I shows Dan counting beans alone, then Chris counting the same number of beans alone. Each of their times is displayed. Then Dan and Chris sit down to count them together. How long will it take them? There is also an extension to this task. Annie joins Dan and Chris and you see their time together. You need to determine how long it would take Annie if she counted on her own. If you have enough time, you could get your students to actually count the beans themselves. There are metronome apps that they can use to keep a steady beat.



Dan Meyer's Penny Circle three-act has gotten exponentially better as a result of his collaboration with Desmos. It is now an online math activity that you can do with your students, as long as they can connect to the Internet. It is available at **https://labs.desmos.com/pennies**. Act I is a video of Dan putting as many pennies as possible in a circle, and then he starts placing them onto a large (22") circular floor tile. How many will fit? You get to guess and then give an answer that you know is too high and one that is too low. Then you start modelling. You actually place pennies in circles, up to 6" in diameter, and record your data, as shown below. The great thing is that if you run this as a class, you see everyone's data, not just your own.



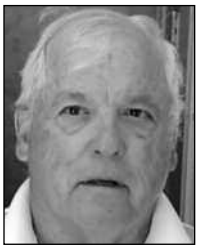
Then choose a linear, quadratic, or exponential model for the scatter plot shown, and adjust the parameters to get the best possible model for the data. You can change your model and adjust it even after it extrapolates to the 22'' circle. Act III is the video of Dan filling the real circle.



There are so many more fantastic three-act math tasks. Students learn by doing these tasks and they are fun! Explore and see what would work in your classroom. And tweet me (**@MaryBourassa**) if you find a good one!▲

▲ OAME/NCTM Report

TODD ROMIENS
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Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a past president of OAME and a life member, as well as a previous editor of the Gazette.

"MATH SKILLS DECLINE IN ONTARIO ELEMENTARY SCHOOLS." This was the headline in an article published in the *Globe and Mail* in mid-September. Data reflecting decreases in EQAO scores from previous years in Grade 6 were cited. An interesting paragraph stated, "Critics say the math curriculum is to blame because it places more emphasis on real-world concepts and applications, rather than building on the basics." It sounds like the "critics" assume that the two are mutually exclusive.

NCTM feels, as I do, that putting children in places that connect to their reality is a better solution to having them learn and understand the "basics," as opposed to the drill activities that the critics seem to be pushing. There are a number of programs started and supported by NCTM and teachers who reflect this philosophy.

A while ago, the STEM (Science, Technology, Engineering, and Mathematics) project was launched in the United States and is gaining support at all levels of teaching. This program is also being introduced to Ontario schools. The focus is on engaging students in activities that broaden their understanding in these areas, as well as showing where the math they are taking in school is relevant to their futures and their understanding.

The following are some examples:

A Florida high school has launched a curriculum that is part of the New Tech Network and focuses largely on hands-on learning and the development of problem-solving skills. The curriculum is currently in 130 schools in 23 states and in Australia.

A new math program, called "Go Math," is helping Pennsylvania elementary students learn how math connects to their daily life. During a recent lesson, a chef showed students how math helps in measuring ingredients for a recipe. The teachers said, "We are dealing with manipulatives in real-life situations." NCTM has another new program called "MOTO" (Math of Tomorrow), which is a Response to Intervention digital series for Kindergarten to Grade 2. MOTO comprises seven student books and a teacher website called "Teacher Connect," which is available exclusively through the NCTM website. The series is designed to meet and support the Common Core Standards and can be accessed at **www.nctm.org/moto**.

A growing number of high school districts are offering students more career-focused courses. These include aerospace; aviation; and other science, technology, engineering, and math programs. Students are learning how math applies in these areas.

A high school in Kansas has started a student-operated bank that will give students in the banking and finance classes an opportunity to apply what they learned in the classroom to a real-life setting. School staff members and students are able to open accounts at the bank.

Students in some public schools in Sante Fe, New Mexico are learning math through a story-driven curriculum. Students become characters in the stories and use critical thinking to solve problems created by the stories.

All of the prior examples illustrate what I see as involving students in the "stories" of mathematics. To me, the stories are where the math exists and how it relates to a student's reality. I call these stories "environments." To illustrate why these are important in mathematics, I would ask my students if they ever told stories to children, and of course, they all indicated that they had. I asked them when they started to tell stories to children and they responded that it was at a very early age. Then I asked them why they told children stories, and they gave many reasons including: children learned vocabulary; storytelling inspired imagination and creativity; and the stories were fun to hear. So I asked the students whether they taught the children about nouns, adjectives, or adverbs before relating stories. They all felt that would have been silly because these would be learned through the stories. Then I suggested that in math, we teach the nouns, adjectives, and adverbs of math (addition, subtraction, multiplication, etc.) before we tell the stories.

The stories of mathematics are embedded in places like McDonald's, the grocery store, banks, supermarkets, and pretty well everywhere students go. Finding ways to

connect the math to these environments and children's everyday lives will be significantly more effective in teaching them about math.

Relating the stories of mathematics will demonstrate to children that mathematics is useful and important in their lives, while teachers concern themselves with the practice of computation.

"The best educated human being is the one who understands most about the life in which he is placed." ~Helen Keller

▲

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▲ What's the Problem? Gardner's Grid

SHAWN GODIN
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Shawn is the head of Mathematics, Business Studies, and Library at Cairine Wilson Secondary School in Orleans, Ontario. He is a past member of the OAME Board of Directors and is currently Editor-in-Chief of the Canadian Mathematical Society's problem-solving journal, Crux Mathematicorum.

Welcome back, problem solvers. Last time, I left you with the following task:

Given the matrix in figure 1,

10	15	16
3	8	9
5	10	11

Figure 1

perform the following operations:

- Circle any number.
- Cross out all numbers in the same row or column as the number you circled.
- Circle a number that hasn't been circled or crossed out.
- Cross out all numbers in the same row or column as the number you circled.
- Circle the last remaining number.
- Add up the three circled numbers.

This problem is inspired by Chapter 2, "Magic with a Matrix," from *Hexaflexagons, Probability, Paradoxes, and the Tower of Hanoi* by Martin Gardner, volume 1 (of 15) of *The New Martin Gardner Mathematical Library*, published by Cambridge University Press. Martin Gardner was the author of the "Mathematical Games" column in *Scientific American* for over 25 years. Although not a mathematician by training, his well-written columns inspired a diverse group of people: teachers, educators, parents, researchers, policy makers, professional mathematicians, and other professionals. His columns have been collected into several books. These books are being reissued by Cambridge University Press with

updated information and expanded bibliographies. If you have never read anything by Martin Gardner, put this journal down, go to your nearest bookstore or library, and get one of his books now!

Wasn't it everything I told you it would be? Now, let us return to last month's problem. I told you that I would predict the sum of your three numbers. I need you to stare at my picture at the top of this column and concentrate on your number… I see… I see a two-digit number… one of the digits is a smaller number, the other a larger number… it is getting clearer… it is a prime number… oh, you sneaky sneak, isn't that a coincidence? This just happens to be my 29th column, I was born on the 29th day, my childhood hero wore number 29, and 29 is the 10th prime; hence, 29 is my favourite number… and it is also the sum of your three numbers! Thank you, thank you! Be generous when the hat comes around. I'm here all week, doing shows at 7 and 10.

Now, we show business types don't usually reveal our tricks, but I have grown fond of this group. Let's take a closer look at that matrix. Take a second and compare the rows and columns.

Hopefully you would have noticed that in each column, the difference between corresponding entries in a row is a constant:

10 – 3 = 7	5 – 8 = 7	16 – 7 = 7
10 – 5 = 5	15 – 10 = 5	16 – 11 = 5
5 – 3 = 2	10 – 8 = 2	11 – 9 = 2

Figure 2

Similarly, the differences between corresponding entries for any pair of columns are a constant. We can use these facts to show that you would get 29, no matter how you picked the numbers.

Suppose, for the sake of argument, the first number you chose was the 8 in the centre. According to the rules, you will have to pick numbers so that exactly one number is from each row, and similarly, exactly one number is from each column. Thus, I know I will have to pick a number from the first and third rows, and similarly, one from the first and third columns. We know that entries in the first row are 7 larger than the corresponding entries in the middle row, and the entries in the third row are 2 larger than the middle row. Similarly, entries in the first column are 5 smaller than those in the middle column, and the entries in the third column are 1 more than those in the middle column. Each number that I pick will be related to your original number by two of these factors (one from the row and one from the column. Thus, we can total them all without knowing which ones we pick. That is, the total must be: 8 + 8 + 8 + 7 + 2 – 5 + 1 = 29.

You can make the same argument for any starting point. For completeness, we could look at any row and column and show that for each entry, the sum is 29; hence, since each group of three must contain one entry from each row or column, you have looked at all possible choices.

Since the elements have an additive relation with corresponding elements in the same row or column, it shouldn't be surprising that the matrix itself is just an addition table. For instance, since we know the relations between the rows and columns, I will label to the left of the first row with x and above the first column with y. Using the relationships we know, we get the set-up in figure 3.

	y	$y+5$	$y+6$
x	10	15	16
$x-7$	3	8	9
$x-5$	5	10	11

Figure 3

Thus, each entry is the sum of the numbers on the outside above the column and to the left of the row. That is, we can pick any two numbers x and y, and as long as they add to 10, we will get our grid with our desired sum. For example, if we choose $x = 8$ and $y = 2$ we get figure 4.

	2	7	8
8	10	15	16
1	3	8	9
3	5	10	11

Figure 4

Now we get a clearer vision of where 29 came from. Each entry is the sum of one number above and one to the left. Since the method of picking the three numbers guarantees that we have one number from each row and column, each of the numbers on the outside is used only once. Thus, the total of the three numbers should be 2 + 7 + 8 + 8 + 1 + 3 = 29. Now you and your students have a method of creating similar matrices, of any size, with any number you like as the sum of the numbers. You can also experiment: What if the three numbers

were to be multiplied instead of added? Could you create a matrix with similar "magical" properties? Have fun!

Now for your homework.

A sequence of functions is defined recursively by $f_n(x) = f_0(f_{n-1}(x))$, where $f_0(x)$ is some function. For example, if $f_0(x) = x^2 - 1$,

then $f_1(x) = f_0(f_0(x))$

$= f_0(x^2 - 1)$

$= (x^2 - 1)^2 - 1$

$= x^4 - 2x^2$

and $f_2(x) = f_0(f_1(x))$

$= f_0(x^4 - 2x^2)$

$= (x^4 - 2x^2)^2 - 1$

$= x^8 - 4x^6 + 4x^4 - 1$

and so on.

1. Given $f_0(x) = \frac{1}{1-x}$, determine $f_{2003}(x)$.
2. Determine another sequence of functions, defined by $g_n(x) = g_0(g_{n-1}(x))$, where $g_0(x)$ is some function that behaves similarly to the sequence of functions in 1.

Until next time, happy problem solving! ▲



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▲ Hey, It's Elementary – Using Found Materials and Anthropometry to Construct, Measure, and Create Mathemagic

LYNDA COLGAN
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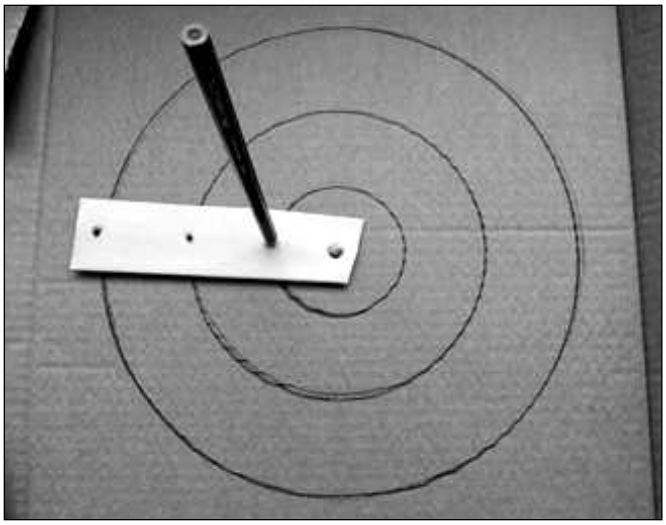
Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.

During my Junior/Intermediate Mathematics Curriculum course at the Faculty of Education, here at Queen's a few weeks ago, a few teacher candidates were lamenting the fact that in their field-placement schools, few students brought math sets, and there were no classroom sets of tools such as safety compasses (shown below).

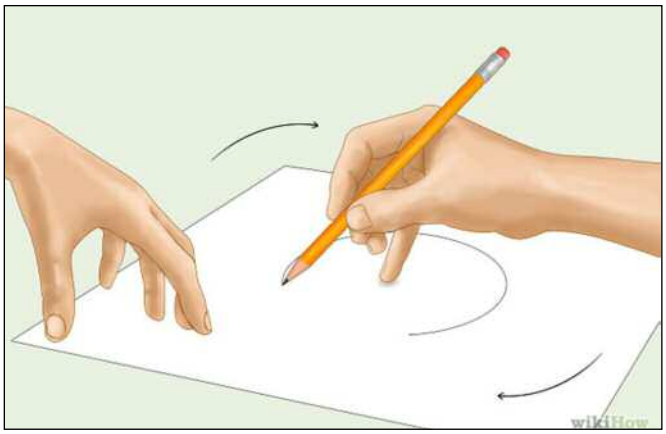


When I replied that they needed to use impromptu strategies and available resources in such situations so that the effectiveness of the lesson is not compromised, there were a few blank expressions in the room. Thus began my own spontaneous lesson about how to draw circles with everyday objects… including just one's hand.

There were a lot of doubtful looks as we used found materials in the classroom, first to craft an effective safety compass from a strip of card stock, a hole punch, and a brass fastener, and then to draw a series of concentric circles, but the satisfied smiles at the end of the short exercise told a different story.



Feeling encouraged by the enthusiasm for so low-tech a solution to a simple problem—drawing a circle—I decided to demonstrate how to use one's own hand as a compass. The following diagram from wikiHow demonstrates the technique simply and effectively. Once again, there were surprised looks from the teacher candidates as they recognized not only that every teacher and student has a built-in compass, but sloppy circles on chalkboards, Smart™ Boards, and notebooks could be a thing of the past.



Source: wikiHow (Image by the author, Chris Hadley, used with permission)

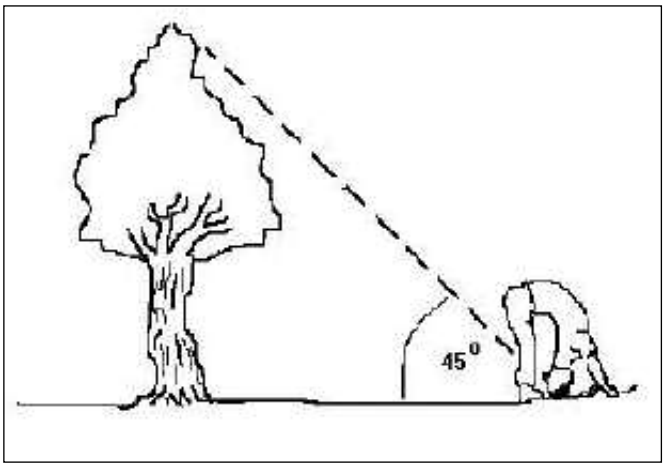
That brief experience was the catalyst for this column, one in which I describe some everyday materials that not only have a long history as useful tools and resources for the kitchen, bathroom, and garage: they have long served important roles in mathematics and in applications of mathematics, from carpentry to art. In many cases, all you need is yourself, given that there are many useful measures built into our bodies. Sometimes you might need to sort through the junk drawer in the kitchen to find some tacks or string, but that is about as fancy as the specialized equipment in these next examples gets, yet the products can be mathemagical indeed.

Did you know that gardeners often use their hand spans (hand held with thumb and little finger spread as far apart as possible) as a quick way to estimate lengths of things in the garden, such as how much room to leave between plants, stepping stones, and garden beds? Hand spans can make a trip to the building centre a piece of cake. Since a brick's length is approximately one hand span, it is easy to determine how many are needed to frame a small garden. For reference sake, most adults' hand span measures approximately 8 inches.

If sewing, not gardening, is your hobby, you will be in need of finer digital instruments for measurement. Did you know that the width of your little finger is approximately 1 cm and that your hand is about 4 inches, or 10 cm, wide? Did you know that the distance from the tip of your thumb to the first thumb joint is about 1 inch? Your thumb is probably about 1 inch wide as well. This equivalence is well documented: in many countries, the word for "inch" is also the name for "thumb."

If you have bigger things to measure, there is still no need to dig out the tape measure or invest in a trundle wheel: your own two feet will get the job done.

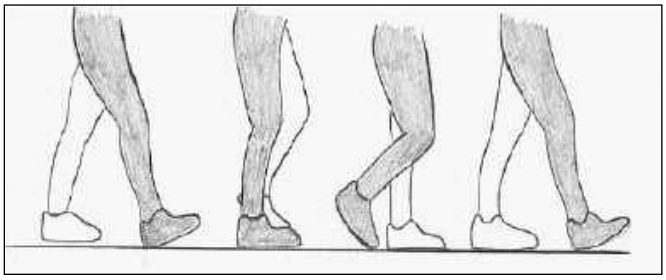
Need to measure the tree in your yard? No problem.



Walk away from the tree, bend over, and sight its top between your legs. When you can see the top of the tree while in this position, stop. The approximate height of the tree will equal your distance from the tree.

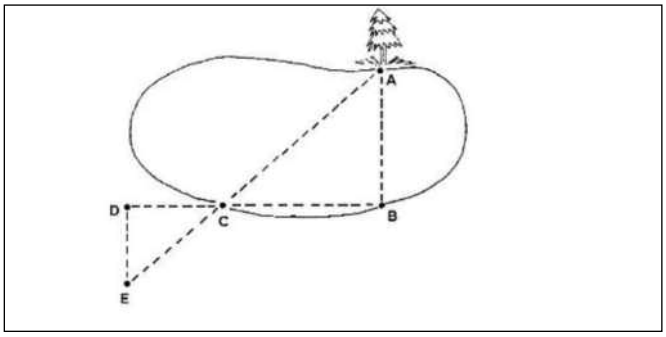
You can then measure your distance away from the tree, using paces (or strides). A pace is the distance

between two successive falls of the same foot; thus, one pace equals two steps. In ancient Roman times, a mile was 1000 paces, or, in Latin, *mille passuum*—which is the origin of our word "mile." For most people, a pace is equal to about 5 feet, or 1.5 metres: important information that you require to be able to compute the height of the tree after you have paced it out by walking the equivalent distance on the ground.



Are you wondering why this works? For a normal, healthy (limber) adult, the angle formed by looking under one's legs is approximately 45°, meaning that we have created a right isosceles triangle—one in which the two sides opposite the right angle are equal in length. Hence, the distance to the tree must be around the same as the height of the tree.

Paces can also be useful if you need to measure the dimensions of something even bigger than a tree—like the width of a river or pond.

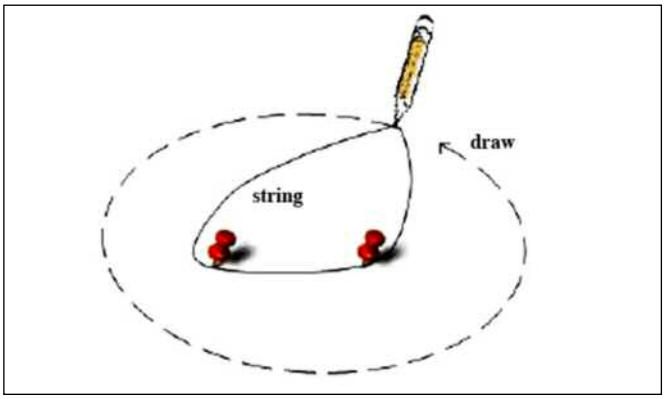


Refer to the diagram above and follow these simple steps:

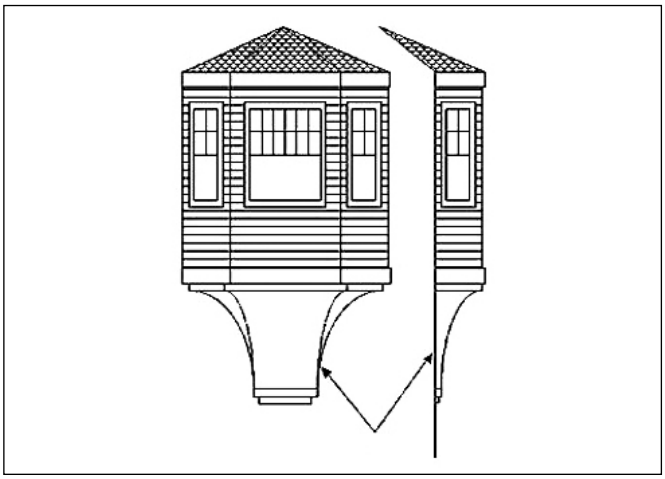
1. Note a landmark (A) on the other side of pond—like a tree. Place a stick (B) where you are, exactly opposite the landmark.
2. Stand at stick and pace off 100 feet (remember that one pace is approximately 5 feet) at a right angle to line (AB). At this point, place another stick (C).
3. Continue pacing along this line for half as much distance as before (in this case, 50 feet, or 10 paces). Place another stick (D).
4. Make another right angle and walk until you can sight stick (C) and the landmark (A) in a straight line, then stop. With another stick, mark this point (E).
5. Now line (DE) is half the distance across the pond. Pace line (DE). Multiply by two. This is the approximate distance across the pond in paces. Use the fact that one pace is about 1.5 metres to compute the distance.

As these examples illustrate, when it comes to linear measurement, just use the ruler you always have with you—yourself. For more sophisticated challenges, start digging in the kitchen drawer to unearth the string and some tacks.

To draw an oval (ellipse), you just need a few pushpins and a piece of string—a classic procedure that woodworkers have used for years and years. As they say, necessity is the mother of invention. This tack and string ellipse construction was probably the discovery of a carpenter or craftsperson, who needed an elliptical shape for some practical purpose when no compass, protractor, or computer-based drawing tool was at hand.

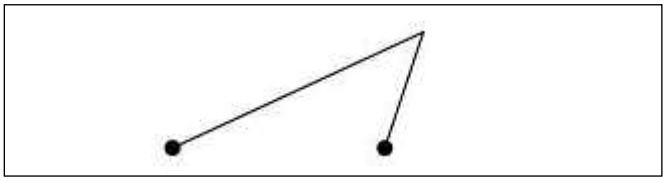


If you are saying to yourself that you cannot possibly conceive of a situation in which you ***need*** to construct an ellipse, then consider this: imagine that you are a do-it-yourselfer who wants to construct an ornamental base for a bay window.

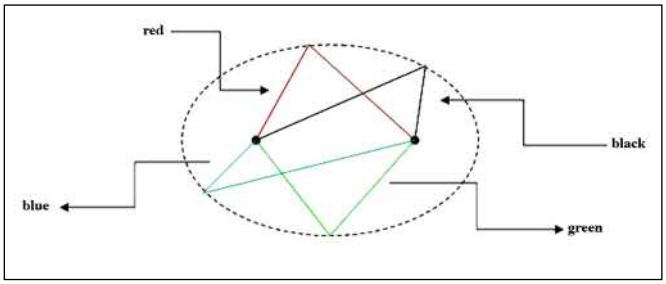


Close inspection of the diagram above suggests that the proposed pedestal base is 1/4 of an ellipse. There is no doubt about it—not only do you *need* to construct an ellipse, you *need* to do so accurately.

To the mathematician, an ellipse is a carefully defined curve. It resembles an oval, but must comply with strict mathematical rules. Start with two fixed points in space and draw a straight line, which we shall call a "leg," leading away from each point. Orient the legs so that their ends meet:

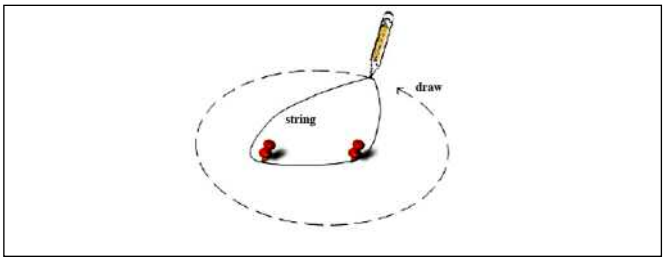


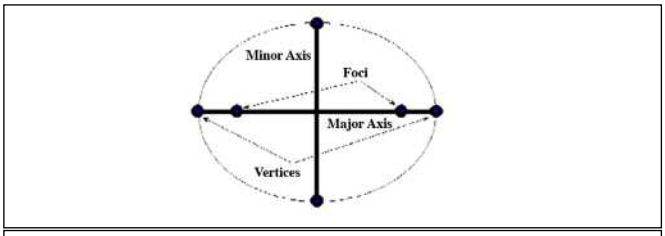
Keep the sum of the length of the two legs constant. Under these conditions, the tips of the legs always meet on a curve that is an ellipse:

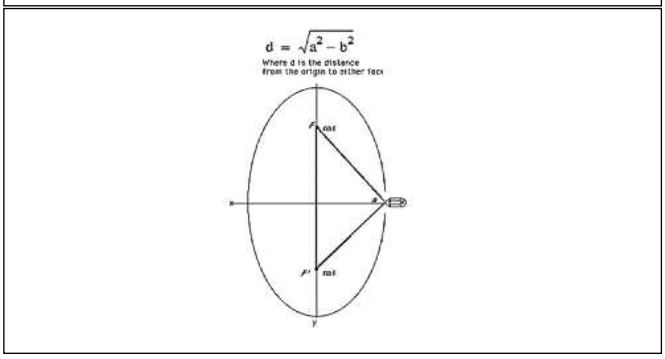


Each colour represents a set of legs in the ellipse above. The combined length of the two green legs is the same as the combined length of the two red legs or the two black legs. Imagine that the two black points are tacks that are hammered into a board. A string of a definite length is cut and its ends are tied together so that the string now exists as a loop of string of constant length. The loop of string is placed around the tacks on the board and is pulled taut by a pencil to result in a curved shape pencilled onto the board.

The beauty of this string and tack ellipse is that it is mathematically valid. It fulfills the strict mathematical definition of an ellipse. The two tacks in the board correspond to the two fixed points in space. Each of the two points in space that is used to generate an ellipse is designated a "focus" of the ellipse.

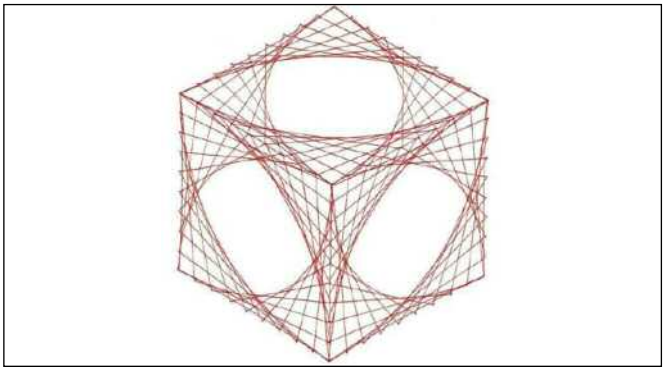


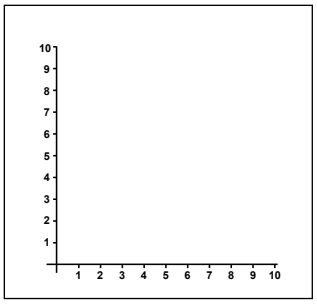




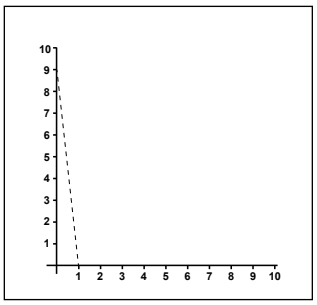
If drawing an ellipse for construction purposes is not your thing, you may still find a creative use for some coloured string (or thread) and a few tacks, since they are the raw materials for beautiful geometric string art designs.

String art, also called curve stitching, is a technique invented by Englishwoman Mary Everest Boole (1832–1916) in 1904, which produces apparent curves by means of perfectly straight lines. Mary Boole created her original pieces with coloured thread and card stock, although you can simply use coloured pens or pencils and a straightedge to draw the lines on paper. Either way, the results will be spectacular. If you don't believe me, then look carefully at the intricate creation below. It began with a drawing of a cube.

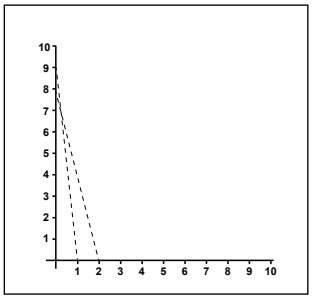




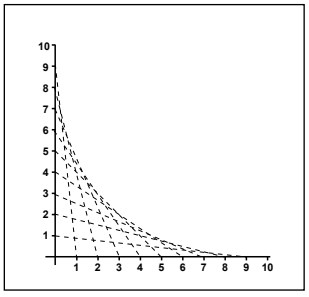
Using a ruler and pencil, draw a set of axes on a piece of scrap paper numbered from 0–10 in both directions, like the one shown here.



Join the 1 on the horizontal axis to the 9 on the vertical axis with a straight line. Your diagram should now look like this.

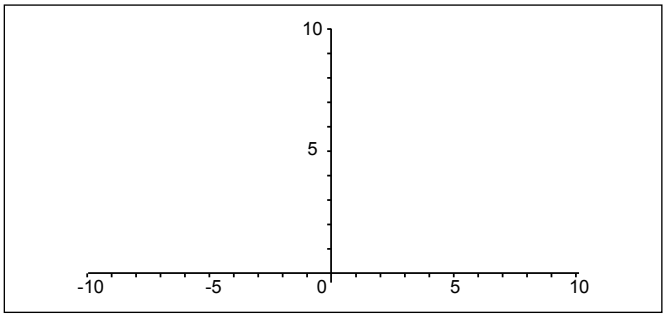


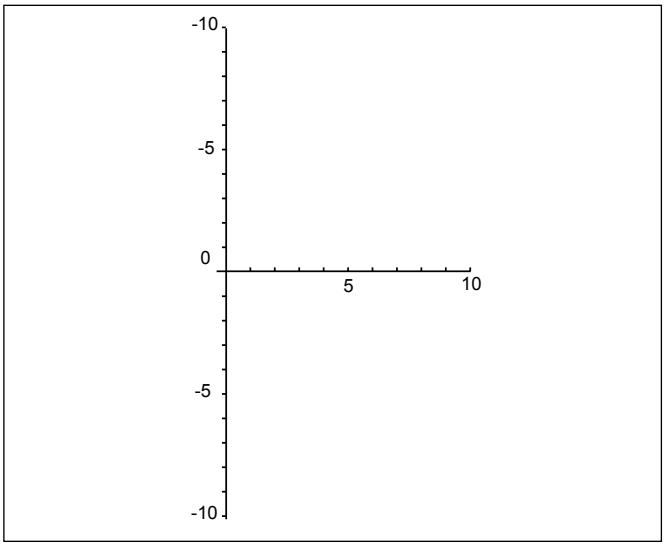
Next, join the 2 on the horizontal axis to the 8 on the vertical axis with a straight line.

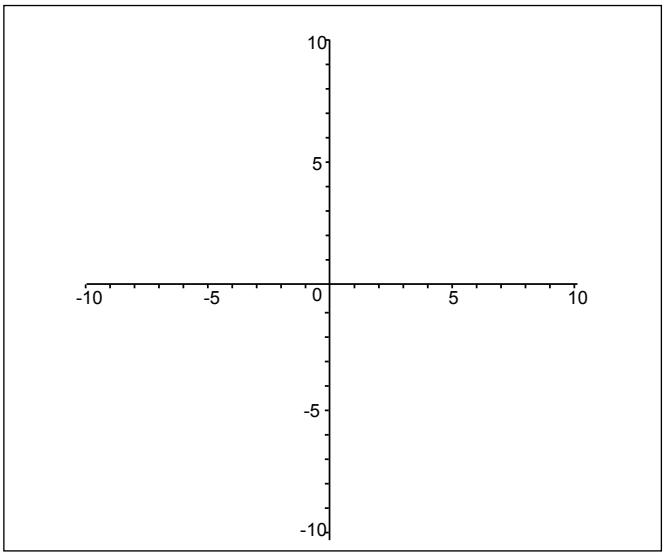


Continue the pattern (making a sum of 10 each time), joining numbers on the horizontal axis to numbers on the vertical axis. The final product should look like this.

Now that you have the hang of it, try using different axes to make other beautiful designs.







We have known for some years now… that most children's mathematical journeys are in vain because they never arrive anywhere, and what is perhaps worse is that they do not even enjoy the journey (Whitcombe, 1988).

We need to make in our classrooms a place in which one important goal is to restore children's love of mathematics and bring back some joy to their mathematical learning journey. While often there is not much time in the busy school schedule to fit in the more inspirational and enriching side of maths, many of which could invite children to see the history and diverse application of the domain, I hope that these simple ideas can bring smiles to your students' faces, the way a holey strip of paper brought smiles to mine! ▲

▲ In the Middle – Taking a Closer Look at Mathematical Investigations

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Mirela Ciobanu is currently a middle school teacher. She has been teaching for over 12 years, 9 of which have been with the Toronto District School Board. Mirela is a keen advocate of classroom inquiry, particularly in the area of formative assessment. Her recent work includes conducting an action research in mathematics as part of the Teacher Learning and Leadership Program for Experienced Teachers, developed and funded by the Ontario Ministry of Education in partnership with the Ontario Teachers' Federation.

> *"A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. There remains always something to do; with sufficient study and penetration, we could improve any solution, and, in any case, we can improve our understanding of the solution."* ~ George Pólya, *How to Solve It*, 1945

One of the most popular problem-solving tasks among middle school teachers is the "Handshake Problem." Let's take a look at two versions of this task.

Task 1: At a workshop, 100 participants shake hands once with each of the other participants. Find the total number of handshakes (Yeo & Yeap, 2009, p. 5).

Task 2: At a workshop, 100 participants shake hands once with each of the other participants. Investigate (Yeo & Yeap, 2009, p. 5).

Which one of the above is an investigation and which one is a problem? If you consider the first one the problem and the second an investigation, you might be only partially right, as the difference between the two concepts lies in the cognitive processes employed by the students when solving each tasks.

What is the relationship between problem-solving tasks and mathematical investigations? The Ontario Mathematics Curriculum (2005) links mathematical investigations closely to the seven mathematical process expectations. Students are expected to conduct investigations to develop their mathematical understanding. Additionally, investigations constitute key formative and summative assessment pieces for teachers. Investigations shape students' love for math in real life and develop their conviction that mathematics is more about the process, developing ideas, looking at the world, than it is about getting the right answer.

As I searched for the answers to the question above, I realized that it is not even an accurately worded question and that any teacher who uses the term "investigation," merely meaning an "open" task, disregards its most important characteristic. Investigation is a process entailing four cognitive processes: specialization, conjecture, justification, and generalization.

As opposed to problem-solving tasks, mathematical investigations are mostly perceived by educators as open, rich tasks, often characterized as divergent in their search for the solution, involving a more systematic exploration of a math situation (Yeo & Yeap, 2007).

Joseph Yeo and Dr. Bar Har Yeap, two researchers from the National Institute of Education, Nanyang University in Singapore, argue that, in fact, it is wrong to compare the two terms on the basis of the openness and closeness of a problem-solving situation. Additionally, it is actually possible for closed problem-solving tasks to involve investigation as a process. Their work aims at distinguishing between open investigative tasks, investigation as a process, and investigation as an activity involving open investigative tasks.

What does this distinction mean for educators? Does knowing the process of investigation guarantee the quality of the investigation? How can we maximize on teaching problem solving that includes the process of investigation and forming habits of mind for a lifetime? And finally, the question on every educator's mind: "Where can we find rich problem-solving tasks that will allow us to use the process of investigation, and subsequently will lead to the development of the four cognitive processes?

Knowing the Difference

Let's return to the first task. It looks like the problem has a closed goal, and it can be solved using a variety of strategies, for example, such as drawing or acting it out. If the students engage in specializing and try to find the number of handshakes for a small number of people,

recording their answers systematically, moving from specific to general (induction), then they would have started the process of investigation. This is why, Yeo and Yeap argue, even close problems can involve investigation (Figure 1). This is particularly important for teachers to know when choosing problem-solving tasks that may seem to have a closed goal, but they can be opened and they may also be solved through the process of investigation.

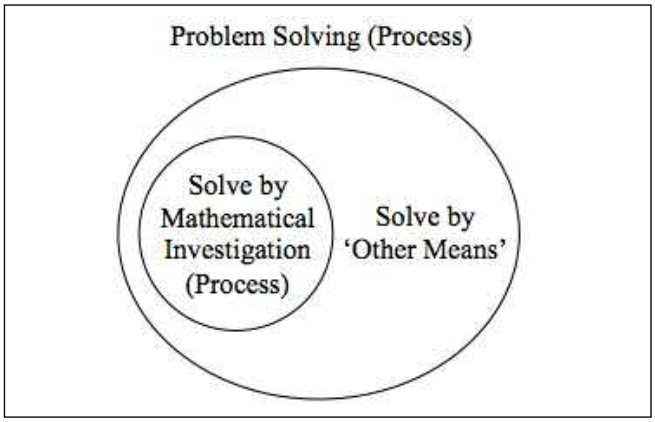


Figure 1: *Relationship between Problem Solving and Investigations as Processes*

The second task includes the word "investigate," but it allows the students to decide what to look for. They engage now in the process of posing questions, such as trying to find out the number of handshakes for any n number of participants. This is an open investigative task. In Figure 2, you can see the relationship between investigation as a process, problem solving, and investigations as an activity, as envisioned by Yeo and Yeap. The two researchers have also clarified the difference between "task" and "activity," as the former is chosen by the teacher and the latter is what the student does to solve the task.

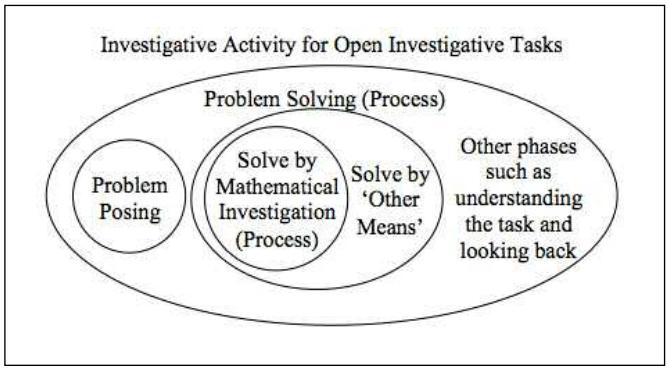


Figure 2: *Relationship between Investigation as an Activity and as a Process.*

THE FOUR COGNITIVE INVESTIGATIVE PROCESSES

1. The Entry Phase of the investigation process

STUDENT ACTIONS: They try to understand the problem, get started, and get involved in designing a plan of action and selecting tools.

The students have not yet started their investigation.

Some sample questions that teachers might use to guide students through this stage:

a. What do you know that is helpful for you to solve this problem?

b. What do you already know that might help you?

c. What is "the problem"?

d. Are there any restrictions?

e. What similar problems does this remind you of?

f. What do you want to find out?

g. How are you going to record your findings? (e.g., systematic listing using a table)

h. How will you represent your thinking?

2. The Attack Phase of the investigation process

STUDENT ACTIONS: They work on the problem, look for patterns, draw preliminary conclusions, and test them.

This is Pólya's combination of "devise a plan and carry it out" steps (Yeo & Yeap, 2007, 2009).

2.1 SPECIALIZATION entails examining special cases, specific examples. It is the beginning of engaging in the investigation. For instance, calculating how many handshakes are at a workshop, where 10, 11, 12 people are present, by using a systematic list, means that students have begun their investigation, but this alone does not constitute an investigation. It becomes an investigation only when the students have posed a question and have established a sub-goal (Yeo & Yeap, 2010).

2.2 MAKING CONJECTURES

STUDENT ACTIONS: As they observe patterns, they begin to formulate preliminary conjectures and test them.

Sample questions to guide the students through this stage:

a. What patterns do you see?

b. What do you notice?

c. What are your findings so far?

More questions of this nature can be accessed at

EduGAINS, in the math processes section: **www.edugains.ca/resources/LearningMaterials/MathProcesses/MathProcessesPackage.pdf**

2.3 JUSTIFYING

STUDENT ACTIONS: They prepare to defend their conjectures to a peer.

Sample questions teachers might ask:

a. Can you explain what you have done?

b. What have you found out so far?

c. Does this make sense? Why?

3. The Review Phase of the investigation process

GENERALIZATION: Finding a pattern that works in a more generalized setting than that of the problem

STUDENT ACTIONS: They check calculations and reflect on the solution.

Sample questions the teacher might ask:

a. Is there another way to solve this problem?

b. How do we know it works for any *n* number of people?

Some students will engage in extensions and will open the question by changing some of its features, or they can try to apply it to the real world.

HOW TO OPEN UP A PROBLEM

So far, we have seen that even a problem that has a closed goal can involve the use of investigation as a process. In Chapter 1 of his book, *Ways to Think About Mathematics: Activities and Investigations for Grade 6–12 Teachers,* Steve Benson (2004) suggests a few ways in which a problem can be modified and opened to turn it into an opportunity for engaging in investigative activities.

Problem-posing strategies

1. "**Make a feature of the problem more restrictive** (finding special cases)"

   A restriction example in the Handshake problem is that each person is shaking hands exactly once with each of the others.

   At eduGAINS at teacher may find opening up questions: **http://www.edugains.ca/newsite/math2/effectivequestioning.html**

2. "**Relax a feature** (If the problem is about right triangles, see how it changes for all kinds of triangles…)"

   An extension in the Handshake problem might be either that there were 200 people (which seems awfully unrealistic, I know) or simply asking to find the number of handshakes for any *n* given number of people attending.

3. "**Alter the details of a feature:** If the problem is about right triangles, see how it changes for acute triangles."

   Changing the feature might require students to find out how many handshakes would be there if every person will shake hands with every second one or every third one at the workshop.

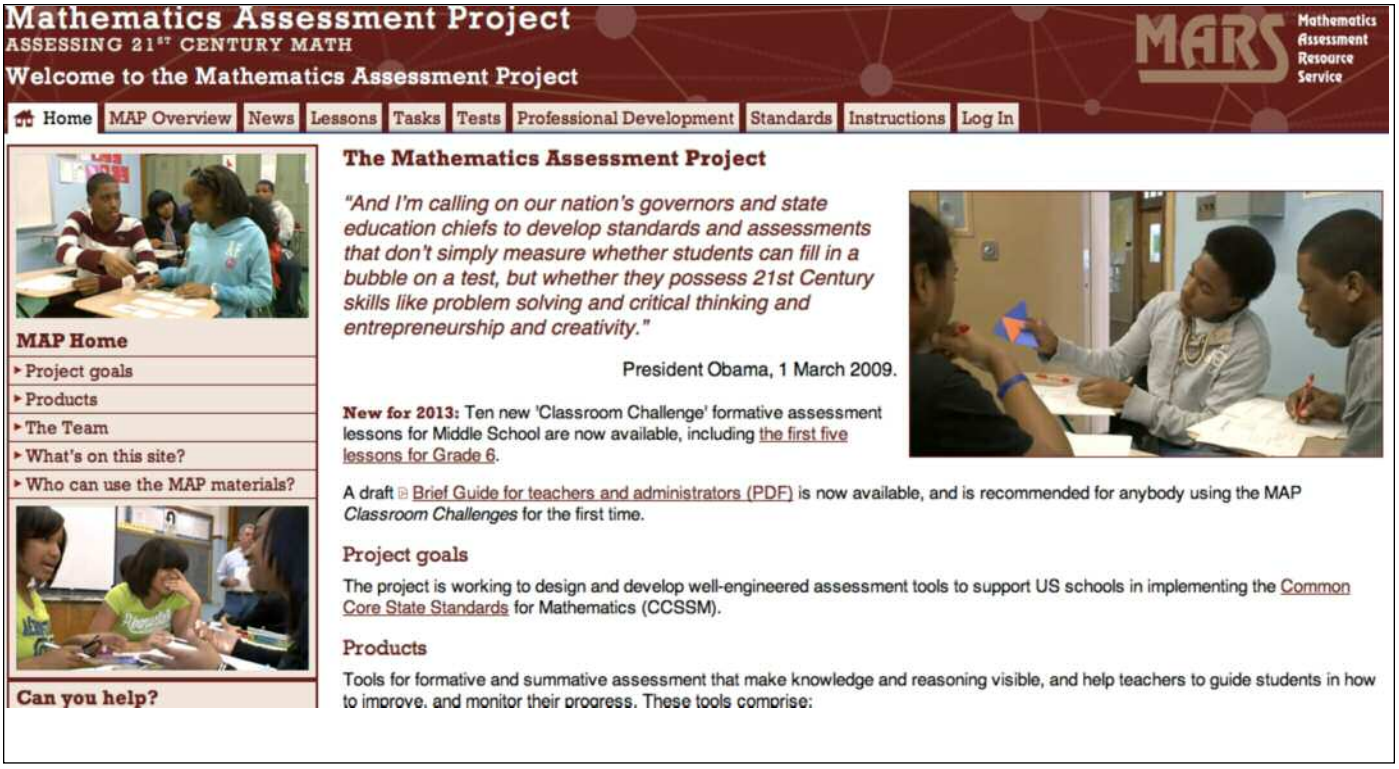
4. "**Check for uniqueness:** If the problem only asks for something to be done, ask if (or when) it can be done in *only one way*."

As you can see, one of my greatest findings is that teachers do not have to rely merely on open investigative tasks to expose students to the investigation process. Closed problems, sometimes found in textbooks, might have the potential of being transformed into great opportunities for investigative activities. I also believe that teachers must explicitly teach the four cognitive thinking processes that make up an investigation. Further research seems to be needed (Yeo & Yeap, 2009) to find out how the quality of learning differs during investigations as part of problem-solving tasks from that during open investigative tasks.

ADDITIONAL CONSIDERATIONS

Most of the teaching in the mathematics classroom revolves around problem solving, and it involves the use of the problem-solving steps suggested by Pólya. It is by no means a simple task and it is not a linear, sequential process. Oversimplified problem-solving models that imply that one stage of the process follows another do not reflect the reality. Such unidirectional models might have a negative impact on our students' perception that once they started solving the problem, there is no looking back, and that they should keep moving in one direction, sequentially, toward finding the solution. To develop persistence and good problem-solving habits, we should emphasize the importance of revisiting the problem and of constantly looking for reasonableness in their thinking. This, however, requires continuous modelling of metacognitive habits in the classroom. Teaching students to constantly wonder whether their findings make sense or why they make sense is something that researchers like Jo Boaler believe we should do in our classrooms in order to develop conceptual understanding and an ability to make deep connections.

Investigation as a process begins only after the students have understood the problem and have a plan of "attacking" it.



GREAT ONLINE INVESTIGATION RESOURCES

MAP: The Mathematics Assessment Project. (2012). Retrieved from **http://map.mathshell.org/materials/tasks.php**

This new resource is developed by the Shell Center team at the University of Nottingham, in collaboration with the University of California at Berkeley. This is a wonderful site filled with middle school and high school challenge lessons, summative assessment task collections and professional development modules. For middle school problems, search "expert tasks."

NRICH. Retrieved from **http://nrich.maths.org**

This website contains a wealth of resources for math educators of all grade levels.

Nuffield Foundation. Retrieved from **www.nuffieldfoundation.org/applying-mathematical-processes/nuffield-amp-investigations**

This is a charitable foundation funding research projects for young people who study STEM projects. The site contains great educational resources, including open investigative activities involving APM (Applying Mathematical Processes).

Similar resources to help with assessment, evaluation and reporting may be found at Ontario EduGAINS.

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Yeo, J. B.W., & Yeap, B.H. (2007). Solving mathematical problems by investigation. In B. Kaur, B.H. Yeap, & M. Kapur (Eds.), *Mathematical problem solving* (pp. 118–136). Singapore: World Scientific. ▲

▲ Fields Institute – MathEd Forum Notes

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Stewart Craven presently teaches courses for the Continuing Education Department and the Transition Years Program at York University. He is an active participant in the Fields Mathematics Education Forum.

Our first meeting was held in September. In the morning, presentations and discussions were led by Priscilla Bengo (OISE) and Martha Mavor (University of Western Ontario and the Lambton Kent District School Board).

Priscilla's theme was "Mathematics coaching to improve teaching practice: The experiences of mathematics teachers and coaches." Her abstract stated:

> *The study explores the relationship between teachers' specific emotions, teacher learning, and teacher coaching in secondary mathematics classrooms. Using a case study approach, it shows that: a) mathematics reforms produce negative and positive emotions; b) emotions are a result of not knowing how to implement the mathematics reforms, beliefs about teaching and learning mathematics, the nature of coaching, gains in student achievement and engagement and positive in-school factors; c) coaching may not help teachers build their professional self-understanding when it fails to address their self-image issues; d) teacher learning can occur even when teacher beliefs are inconsistent with reform initiatives; and e) even when teacher learning results from coaching, reform strategies are modified by teachers.*

Martha spoke about "Mathematics Therapy." Her abstract stated:

> *When confronted with the question: "How do you feel about mathematics?," most adults openly share a dislike, aversion, or negative opinion about the subject, which is a challenge faced by faculties educating the province's newest teachers. So, what experiences do pre-service students need to have in order to develop positive feelings and opinions about math, and in turn help create those in their own students? Based on the work of Gadanidis and Namukasa (2005), this presentation will discuss Math Therapy, an approach to generalist elementary school teacher preparation that allows participants an opportunity to experience mathematics in a markedly different way than they have in their own education. For this presentation, the majority of the time will be spent highlighting and discussing the main themes and images of mathematics teaching (the role and nature of the teacher, student, math, and environment) that emerge from the writing of pre-service teachers after engaging in therapeutic math experiences, and the implications of these findings in our own professional positions.*

The afternoon was organized as a panel discussion about "Moving Towards a Two-Year Teacher Education Program in Ontario." The panel included Christine Suurtamm (University of Ottawa), Ann Kajander (Lakehead University), Donna Kotsopoulos (Wilfrid Laurier University), Dragana Martinovic (University of Windsor), George Gadanidis (Western University), Walter Whiteley (York University), and Daniel Jarvis (Nipissing University). The panelists were challenged to respond to the following questions:

- To what extent will extension of the program (including teaching practice) resolve the issues we have with preparation of mathematics educators?
- What opportunities for math education community are opening with these changes?
- What challenges remain?
- What else can we (as the MathEd Forum) do to support mathematics education faculty in exceptional preparation of pre-service teachers?

The next meeting will have been held at Humber college on November 26. The topic for our exploration was "Innovative Mathematics Teaching at Ontario Colleges," with a speaker list that included Rebecca Milburn (Associate Dean), Karen Lawrence (Mohawk College), Cameron Redsell-Montgomerie (Humber College), and Anand Karat (Vretta).

Educators from all levels are welcome to attend our Saturday conversations at the Fields Institute (University of Toronto, 222 College Street) from 10:00 a.m. to 2:00 p.m. Meetings dates include November 30 and January 26, with future dates TBD. ▲

▲ MB4T... The Standard Procedure for Fraction Multiplication: Generalizing the Area Model

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Ann Kajander is an experienced classroom teacher currently teaching mathematics and mathematics education at Lakehead University. Her research interests relate to teachers' enhanced learning of mathematical concepts. She and colleague Tom Boland have written a new book for teachers called Mathematical models for teaching: Reasoning without memorization *available January 2014 from Canadian Scholars' Press.*

When providing concrete materials to children, the question "When should we stop using the manipulatives?" sometimes arises. Perhaps this is because there is the pervading sense that using manipulatives is not "math," due to a feeling that the "math" is the computation aspect. I beg to differ! Mathematics involves being able to generalize, and I believe that the thinking and reasoning that connects, and in fact necessarily establishes, a general method or numeric procedure based on models and reasoning is indeed the most deeply "mathematical" aspect of the development of a new numeric method.

A child who is told when to "stop" using manipulatives and "start" using a procedure may not be ready to understand a procedure well enough to use it reliably or understand when to apply it. Children will tend to gravitate toward a provided procedure as soon as they are aware of it—even if they don't fully understand it. Hence, keeping the manipulatives around as long as possible, and encouraging their use, is a good thing; at times children may even "check" their procedural work by pulling out the manipulatives—which is even better!

When the numbers get harder, however, at some point, there is a need for use of a procedure. With fractions, for example, the plastic manipulatives only exist in sizes for simple fractions. The key point of the current article is to emphasize that the use of manipulatives and models is not a precursor to doing the real "math"; rather, their proper use can exactly provide the mathematical reasoning needed to generalize and thus find a new procedural method. Such reasoning *is* the math! One of my very favourite examples of such reasoning relates to fraction multiplication. With the proper use of models, the traditional fraction multiplication procedure literally falls into our lap.

I want to emphasize that the description to follow applies to the type of modelling work that follows a great deal of prior work with manipulatives, models, and contexts. For example, prior to providing problems related to the idea to follow, children should have been given ample opportunity to work through problems such as *If there is 3/4 of a chocolate bar left and you eat 2/3 of it, how much of the whole bar did you eat?*

The problem just given can be modelled with three pieces of size one-fourth from a fraction kit. The idea of "two-thirds of" these three pieces can be thought of as "two of the three", which fairly easily yields 2/4 as the answer. This answer might then be re-represented as 1/2, although 2/4 is a perfectly correct answer also.

We can even model problems involving calculations, such as "three-fourths of two-fifths," by recognizing that if the 2/5 is re-represented as 4/10, it is then possible to take three-fourths of these four (one-tenth) pieces, resulting in 3/10.

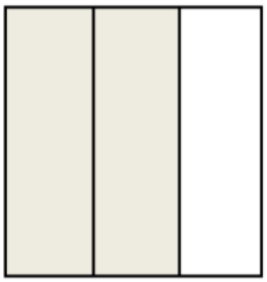
But as problems get more complicated, the plastic manipulatives no longer encompass all examples, and paper-and-pencil models become more helpful. For example, modelling the problem 4/5 × 2/3 cannot be done directly with the manipulatives because we can't get an exact answer to "four of every five pieces" of the 2/3. We might even try to change the order of the numbers and look for a model of "two of every three pieces" of four-fifths, but this doesn't help, nor does re-representing to other units in the fraction kit make things any better. Instead, this choice of numbers requires a paper-and-pencil model to be drawn and subdivided.

Consider the following problem, which might contextualize this situation:

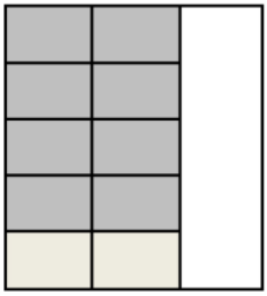
> *A square garden of 1 m by 1 m is divided into three parallel strips, two of them planted with parsley, as follows:*

Today, you will be harvesting 4/5 of the region planted with parsley.

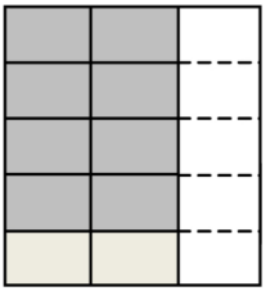
Illustrate today's parsley harvest on the diagram of the entire garden. What fraction of the entire 1 m by 1 m garden will you be harvesting today?



After some experimenting, one possible way of dividing the regions is to imagine taking 4/5 of each one-third strip, as follows:

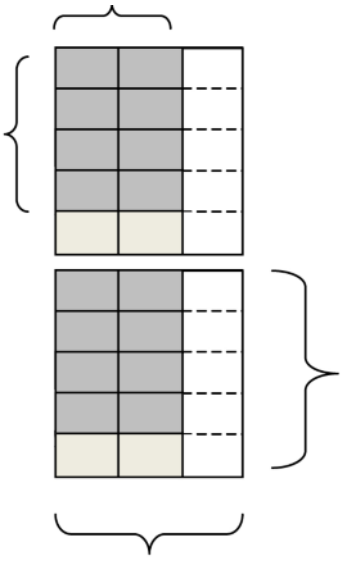


While the darker shaded region may be seen visually as the answer—today's parsley harvest—we can't give the region a fraction name without making sure all the pieces in the original one-whole are the same size. So the model can be refined as follows:



The subdivision of the rest of the area into parts of the same size allows us to find the darker region as a fraction of the one-whole.

Now comes the exciting part!

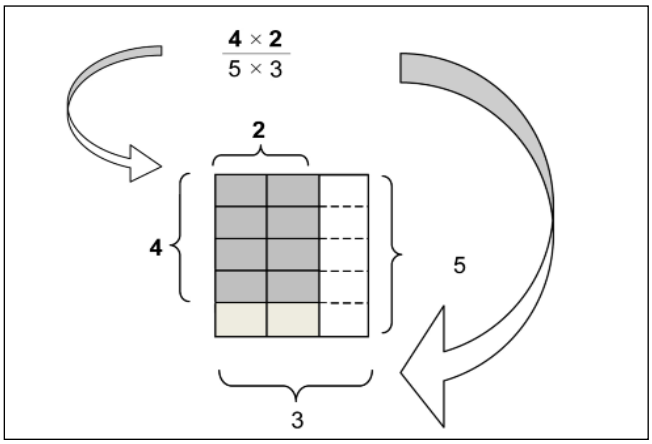


The harvested region is shown as 4 × 2 small rectangles out of a total region of 5 × 3 small rectangles.

So, in total, the answer can be calculated by taking the number of darker (harvested) small rectangles (a region of 4 × 2 of them) out of the total number of small rectangles in the entire garden (5 × 3), as follows:

$$\frac{4 \times 2}{5 \times 3}$$

The key point is that the values in the numeric calculation exactly and necessarily derive from the values in the model:



We see that the calculation

$$\frac{4 \times 2}{5 \times 3}$$

yields the final answer of 8/15. Indeed, the calculation as just done from the model is exactly the standard numeric method or procedure! A few more examples will soon convince us—and our students—that this numeric method always works because the numbers in the initial two fractions always generate an analogous model.

While textbooks typically show both the model and the procedure, I have found that the key idea of necessarily linking the two is often less clear. It is not accidental that the model aligns with the procedure—the related reasoning when we create the model to solve such problems ensures this will always happen, and this is the important aspect to notice.

In summary, the key idea is that the reasoning via a model is what creates the procedure for students. No longer is the procedure something that we "tell" students because it is the "rule." Students can and should be given enough experiences and carefully chosen problems and contexts to work with, the solving of which encourages them to create models with the potential for this reasoning to occur. The reasoning, followed by discussion of how we notice the connection of the model to the numeric steps, as well as experimentation to further notice that the reasoning always applies in the same way, is what allows our students to make the generalization themselves. And this is the mathematics! ▲

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▲ What's New with CLIPS? mathies.ca!

GREG CLARKE, AGNES GRAFTON, AND ROSS ISENEGGER





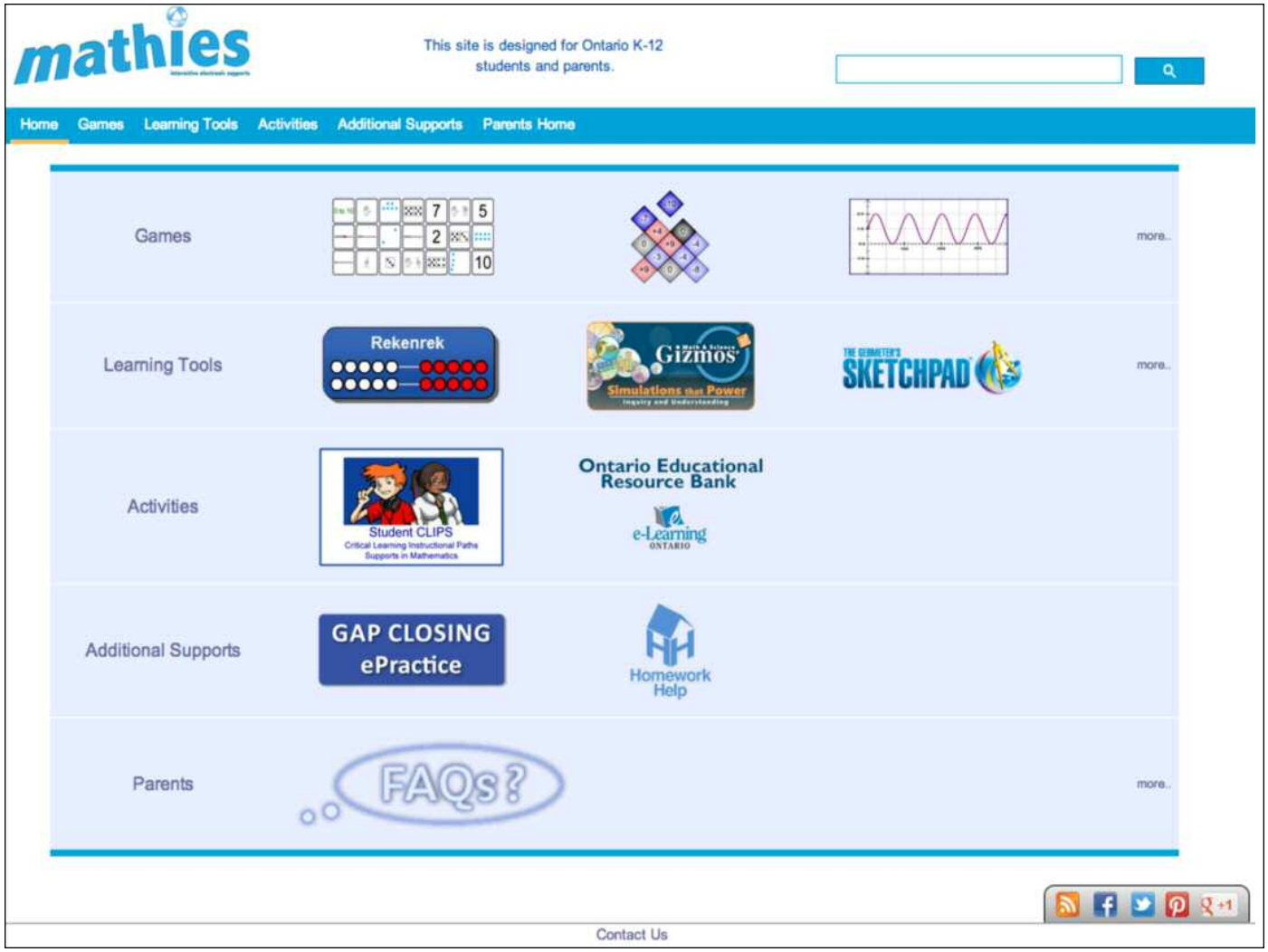
Greg Clarke is from the Simcoe Muskoka Catholic District School Board, and Ross Isenegger is from the Near North District School Board. Both are currently on assignment with the Ontario Ministry of Education as Provincial Math Leads, working on the CLIPS project, in conjunction with their colleague, Agnes Grafton, from the Brant Haldimand Norfolk Catholic District School Board.

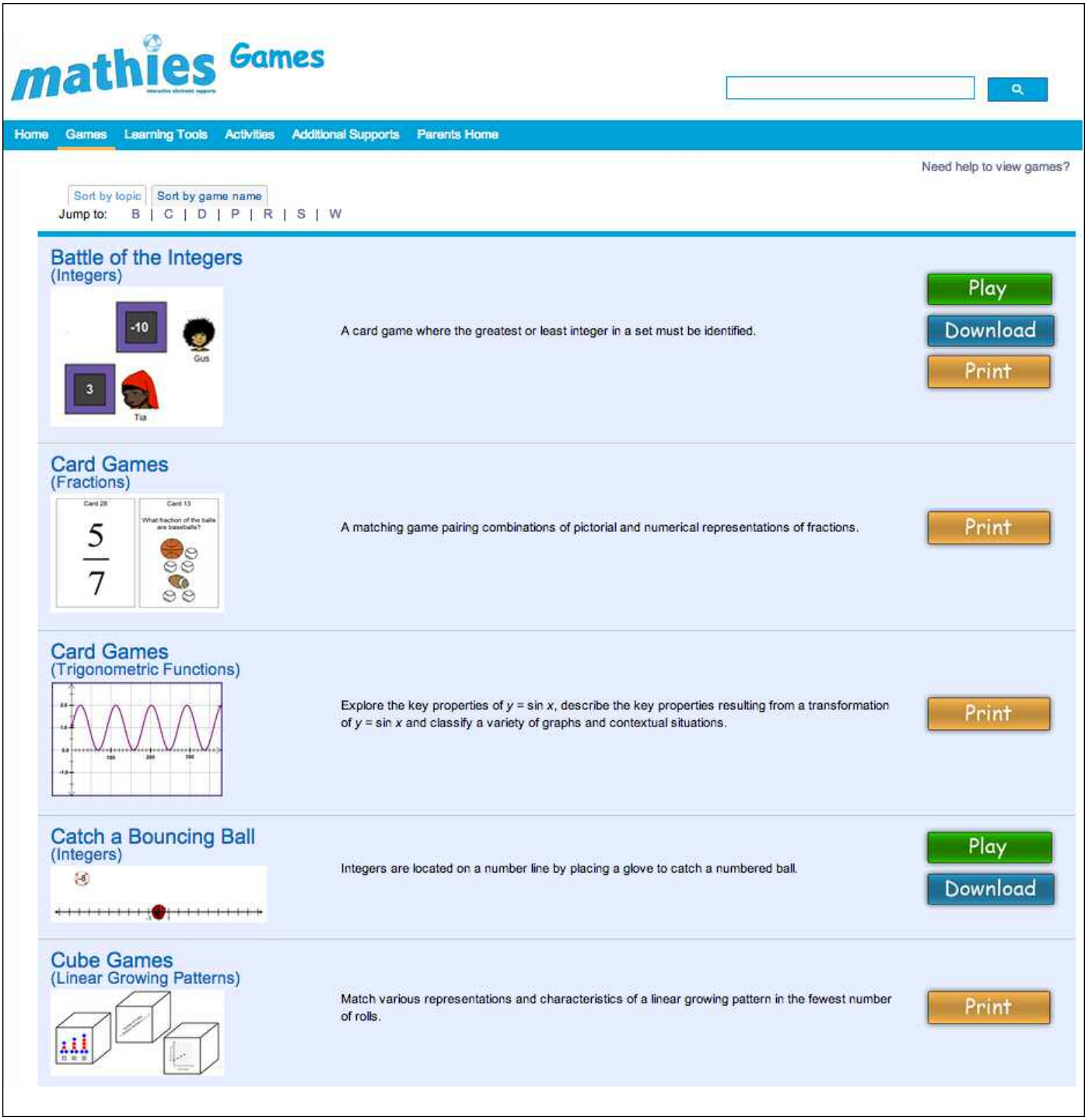
This Fall, a new website is being launched, **mathies.ca**, which has been designed for students and their parents or guardians. Mathies provides easy access to a variety of Ministry-approved resources, including CLIPS.

NOTE: *The following screenshots have been taken when this article was submitted, which was before the official launch. Some changes will have been made since then, but the general organization and look should be similar.*

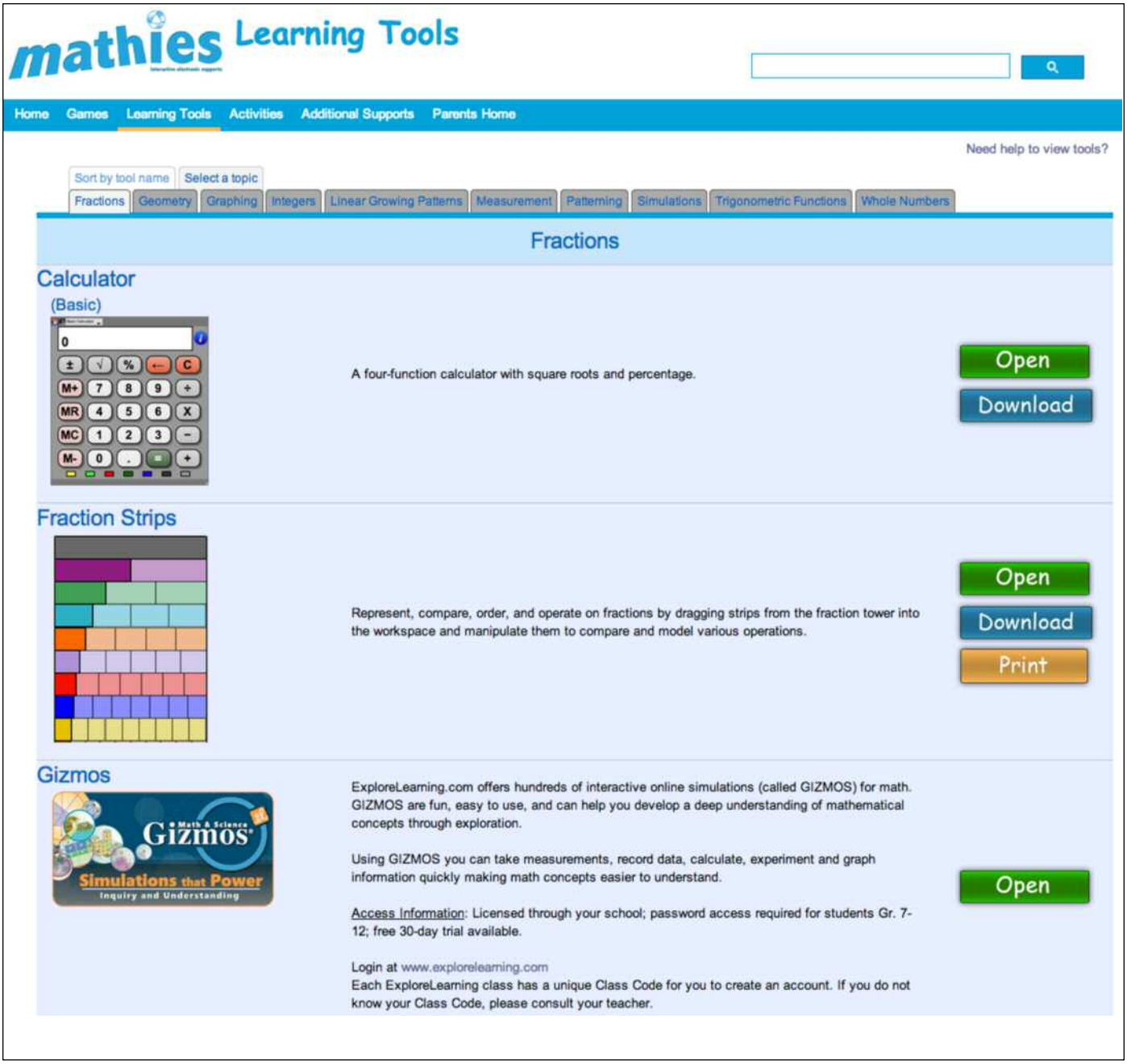
What does it look like?

The home page for **mathies.ca** gives quick access to the main types of online supports and resources for which students and parents/guardians are likely to be searching.





The **Games** section lists all the games available from **MathCLIPS.ca** (including the PDF version of many games that can be used offline), as well as other games developed by the CLIPS team that are not currently part of a specific learning trajectory. The games are listed by topic (Fractions, Integers, etc.), but can also be sorted alphabetically. When is the last time you played a version of Drop Ball or Representation Match? How about the Robot Rule game or Sinesweeper? Look for the soon-to-be-released Representation Match and Catch a Bouncing Ball games, which focus on whole-number representations, ideal for primary and junior students.



The **Learning Tools** section gives access to the electronic tools that are part of the CLIPS Tool Container (refer to What's New with CLIPS column from Volume 51 Issue 4 – June 2013 for more details). It also lists other learning tools licensed by the Ontario Ministry of Education for students in publicly funded K–12 schools, including The Geometer's Sketchpad™ and Gizmos (**explore learning.com**).

The **Activities** section provides access to all the CLIPS activities and also has a link to the OERB (Ontario Educational Resource Bank) set of online activities. A compilation of select OERB activities by grade level is also available to help narrow the search for quality math eLearning objects.

Additional Supports highlights the Gap Closing resources (both print and online epractice.ca activities), as well as a link to the Homework Help service for students in Grades 7–10.

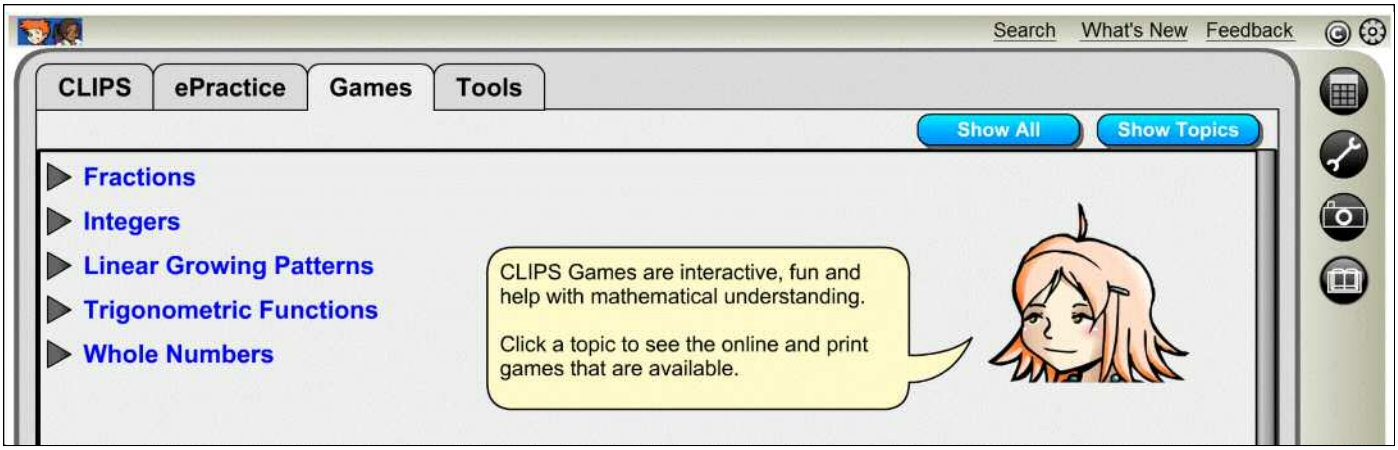


The final section of **mathies.ca** is designed for **Parents**, and is organized around a series of frequently asked questions.

- How can I help with the learning of mathematics?
- What mathematics activities can we do together?
- What digital supports are available?
- What additional support is available?

The **Parents** section features overview videos of many of the resources listed on the student pages, as well as videos showcasing various sample activities, including CLIPS, Gizmos, and the OERB. There are also links to Ministry resources that have been developed specifically for parents/guardians, some of which have been translated into multiple languages. Additionally, you will find links to activities that can be enjoyed at home.

All the great work cataloguing games and tools on **mathies.ca** inspired the CLIPS team to make CLIPS games and tools more prominent when you visit **mathclips.ca**.



Games tab joins the CLIPS and ePractice tabs, giving access to all the games in CLIPS, without having to navigate to a cluster of activities and clicking the games (rook) icon. The Tools tab provides a comprehensive list of tools, which includes a helpful description. Tools remain available from the wrench icon, which is the best way to launch a learning tool to help support a specific CLIPS activity.

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▲ E-Kwatee-Ons – From Mind Games to Algebra Part 2

JAMIE PYPER
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www.educ.queensu.ca/research/collaborations/mste.html



Jamie is an Assistant Professor of Mathematics Education, and the Coordinator of a research and curriculum development team, the Mathematics, Science, Technology Education Group (MSTE), at the Faculty of Education, Queen's University, he spent 20 years teaching high school mathematics in two school boards in Ontario and mathematics education at Western University. He has been a member of OAME since 1995 and is often a workshop presenter at OAME and local chapter conferences.

I often teach the Grade 9 or 10 students who struggle with math. An often-seen treatment of solving equations starts with equations that require only one transformation: $ax = c$, or $\frac{x}{b}$, or $x + b = c$. Students almost always exclaim, "These are so easy! We can do them in our heads. Why do we have to write all these steps?" Brilliantly, the Ontario Ministry of Education (OME) (2005b) Grades 9 and 10 curriculum and the NCTM Standards (2000) support and encourage a mental ability. For example, "[t]hey should become fluent in performing such manipulations by appropriate *means—mentally*, by hand, or by machine—to solve equations …" (NCTM, 2000, p. 296, italic added), and

> Mental computation involves calculations done in the mind, with little or no use of paper and pencil. Students who have developed the ability to calculate mentally can select from and use a variety of procedures that take advantage of their knowledge and understanding of numbers, the operations, and their properties (OME, 2005b, p. 15).

However, emphasizing the traditional paper-and-pencil approach of solving single-step equations first, seems to lose students' interest. By the time students get to equations that could use a good algebraic, multi-step solution process, students are dismissive of equations and do not care to learn a formal algebraic method. Thinking from a differentiated instruction perspective, I decided to let students solve those one-step equations in their head, and instead, focus my teaching energies on a process that would solve (almost) any equation that involved two or more operations, such as $ax + b = c$, or $\frac{x}{b} + c = d$, or, $\frac{x+b}{c} + d = e$, the equations that are often more difficult for students to solve mentally just by looking at them.

What follows is a unit I use for students' learning of solving equations that considers students' attention spans, attention to details, and learning styles. The unit may run from three to six days, depending upon my students' learning needs and the order I have planned for curriculum topics in the course I am teaching. Time is relatively flexible because of an overriding belief I have-that if I do not take the time now, then I will reteach "solving equations" in every unit for the rest of the course. Students have experienced learning opportunities for solving equations (e.g., OME (2005a) curriculum by Grade 8), but often do not recognize they can "solve an equation" for the variable in different mathematical contexts or topics. This unit provides an engaging discovery atmosphere, where students generate a personally memorable algorithm to solve for unknown values.

I start with a mental game of number manipulation that all students can perform in their heads and feel comfortable speaking out loud. Then, I gradually incorporate a paper format to the game, which transforms over the lessons into a formal algebraic and algorithmic process of solving equations with which we are all familiar.

This is the unit plan:

Day 1 E-kwatee-ons game in our heads.
Day 2 E-kwatee-ons game board on paper.
Day 3 E-kwatee-ons and writing each step as an arithmetic operation.
Day 4 Resolving E-kwatee-ons into the formal solving equations algorithm on paper.
Day … Practice and extensions into more complicated equations as needed.
Day n Practice, Equations Bingo, evaluation.

Day 1

This whole first lesson is verbal and conversational with students in the class as a whole group and with students in pairs. Whether it is learning styles or students' perceptions of how they learn best, Day 1 appeals to many students because of its verbal and

conversational emphasis. It can be challenging to incorporate all learning styles simultaneously, and I find an emphasis on one learning style on one day, and an emphasis on another learning style on another day, often allows students to engage when an activity appeals to a strength in a particular learning style, and is then supported by peers and teachers when students work on an activity that emphasizes a learning style in which they are weaker. Details of the verbal lesson process of the game appear in the *Gazette 51*(2), 29–31 for **E-kwatee-ons – From Mind Games to Algebra**.

Day 2

Like any really good game, there is a game board. For example, think of Tic Tac Toe, Sorry, and Monopoly. E-kwatee-ons has a game board too. I have students write the game board in their notes (see Figure 1 for the lesson "note" I put on the chalkboard). I then use this game board with an example of a game we played in our heads the previous day.

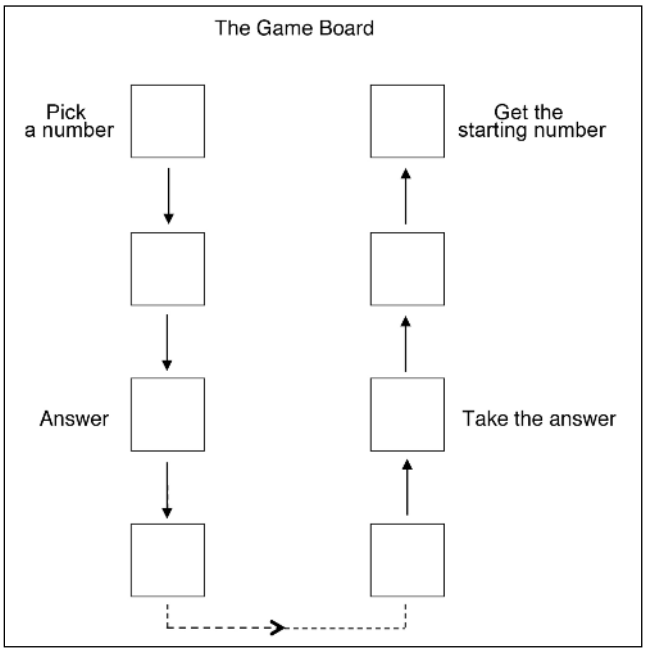


Fig. 1. *The Game Board*

Students play the game with the game board in pairs, again in a pattern of increasing size of numbers and number of operations (see Figure 2 for how the lesson note on the chalkboard develops for this day).

Playing the Game

"Pick any number between 1 and 9." "Multiply that number by 2 – keep your answer in your head." "Add 3 to that number – keep your answer in your head."

"What is the value you are thinking of right now?"

A possible answer is 15.

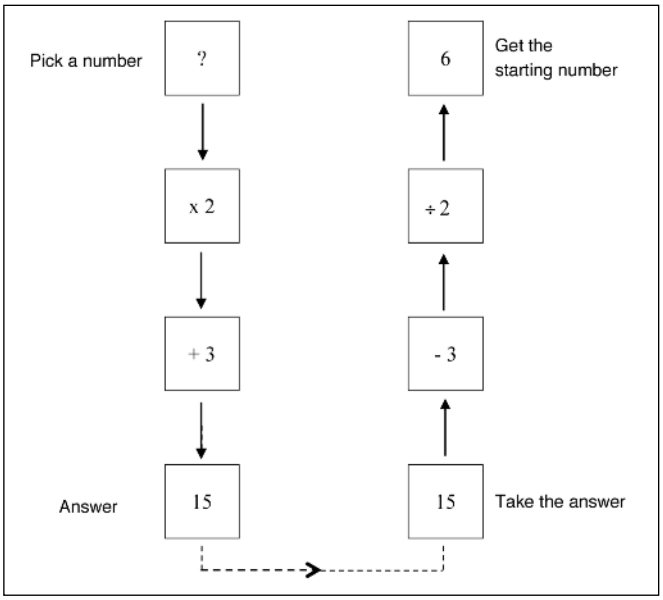


Fig. 2. *Playing the Game*

Day 3

On the third day of the unit, I suggest there might be another way to write down the game, and using a game example from a previous day, I say the game and write each step on the board. Figure 3 illustrates the step-by-step nature of this development.

Words I would speak to the class	What I would write on the board
"Pick any number between one and nine."	?
"Multiply that by 2."	2 x ?
"Add 3 to that."	2 x ? + 3
"What is your answer? Brent? 11"	2 x ? + 3 *is* 11
"Now, what is another way to write 'is'?"	2 x ? + 3= 11
"And what do we usually write instead of a question mark when we don't know the value?"	2 x *x* + 3= 11
"And now remember, the multiplication sign often doesn't get written between a number and *x*, so it would look like this"	2*x* + 3= 11

Fig. 3. *Writing the spoken game on paper.*

Figure 4 is the lesson note I put on the chalkboard. Using the acronym "BEDMAS," we work through the note, do some examples, and homework comes from the textbook-questions from a chapter on solving equations. An option I have used is to have students make up their own equations, solve them, and quiz the person beside them, or do this at home and have the person at home solve the equation, using the game. This has the advantage of providing an opportunity for the students to teach this skill and the game-an excellent assessment as learning (Ontario Ministry of Education, 2010) strategy.

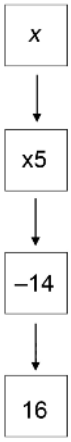
Playing the game with a textbook

Solving means to look for a value for the variable that makes the equation true.

Use BEDMAS to help you write the boxes for the game. Then go backwards through the boxes to find x.

Example: Solve $5x - 14 = 16$

Solution:



$16 + 14 = 30$

$30 \div 5 = 6$

$6 = x$

Check: L.S. $= 5x - 14$

$= 5(6) - 14$

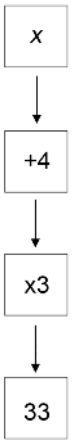
$= 30 - 14$

$= 16$

$= R.S.$

Example: Solve $3(x + 4) = 33$

Solution:



$33 \div 3 = 11$

$11 - 4 = 7$

$7 = x$

Check: L.S. $= 3(x + 4)$

$= 3(7) + 4$

$= 3(11)$

$= 33$

$= R.S.$

Fig. 4. *Playing the game with a textbook*

Day 4 and onwards.

The next phase of the unit moves students into a more formal and algebraic method of solving equations. The algorithm I tend to use is the "balance" method of performing the same operation on both sides of the equation in an effort to isolate the variable. Figure 5 is the chalkboard note for this lesson. The left side of the note illustrates the game solution to solving an equation, and the right side of the note illustrates a more formal algebraic method. Once both methods are written on the chalkboard, we explore and emphasize the similarity of the steps, operations, and results of the game and the formal algebraic method.

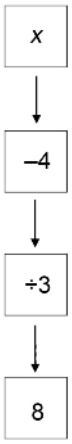
Solving Equations More Formally

Use BEDMAS to help you think or write the boxes for the game.

Then use the inverse operations as you go backwards through the boxes to find *x*.

Example: Solve $\frac{x-4}{3} = 8$

Game Solution: $\frac{x-4}{3} = 8$



$8 \times 3 = 24$

$28 = x$

More formal method:

$\frac{x-4}{3} = 8$

$3\left(\frac{x-4}{3}\right) = 3(8)$

$(x-4) = 24$

$x - 4 + 4 = 24 + 4$

$x = 28$

Fig. 5. *Solving equations more formally*

For some students, this has been a transformative experience. Some students move quickly to the formal algebraic method, while others continue to use the modified game method of Figure 4. There are a few students who cannot grasp any part of the algebraic method and solve all their equations with the game board of Figure 1. I accept any of the three methods for solving equations on the final exam. Each method shows mathematical thinking and mathematical communication appropriate for the learner's stage of development. I find that students who use the modified game method of solving equations for the duration of this course often

leave it behind and solve equations naturally with the more formal algebraic method once they start their next math course.

Solving Equations When You Cannot Play the Game at First

Step 1 Simplify each side.
Step 2 Move variables to one side.
Step 3 Find the value. (Repeat steps, if necessary.)

Example: Solve $5x - 10 = 3x - 3x + 6$ ("collecting variables to one side)

Solution:
$$5x - 10 = 3x + 6$$
$$5x - 3x - 10 = 3x - 3x + 6$$
$$2x - 10 = 6$$ (play the game, or solve more formally)
...

Remember: Keep the equation balanced do the same thing on both sides of the equation.

Fig. 6. *Solving more complicated equations*

There are many developments to pedagogical conceptions and related teaching and learning classroom practice, such as constructivism and differentiated instruction. From a constructivist perspective, teaching strategies in our classrooms should provide students with opportunities to connect prior knowledge and experiences to current knowledge and experiences. Pelech (2010) suggests a "bridging question strategy" to connect "everyday, manifest knowledge to an academic concept the students are learning" (p. 121). The E-kwatee-ons game is the bridging strategy between the prior knowledge and experiences of thinking in our heads of operations with numbers and games and magic, and the academic concept of solving equations algebraically.

From a differentiated instruction perspective, the E-kwatee-ons game provides an "appropriate challenge" to attain the "essential understandings" with a continuous "evidence base" (Hume, 2008, p. 9). The *evidence base* appears naturally on paper because it is generally understood that games with a defined game board have a set of rules that must be followed. Otherwise, one is "cheating" and is called out loudly and immediately by fellow players. There are three distinct phases to using the game board of E-kwatee-ons on paper to solve equations: mental arithmetic and the game board (Figure 2), then the modified game board (Figure 4), then a more formal algebraic method. This strategy provides students with multiple representations of the same skill, and students have a choice of solution method, another fundamental element of differentiated instruction stated in the Ontario mathematics curriculum (OME, 2010), and NCTM's Standards (2000).

The *essential understanding* is the use, and emphasis, of the order of operations. It is used to simplify numerical expressions, and now, it can be seen to be used with algebraic expressions and solving equations. The nature of the *appropriate challenge* follows from the three phases of using the game board. Students learn one step at a time, with lots of practice (Jones, 1987). As students gain facility and confidence with solving equations, they develop paper-and-pencil-solving strategies to answer the question of "What do I do next?" When a student shows he or she is ready to take the next step, that is, to use a more sophisticated manner of solving equations, another more algebraic solution method is available (for example, see Figure 6).

The Ontario (2005b) curriculum's mathematical processes and NCTM's (2000) Connections standard suggest there is value in making connections among various mathematics topics and investigating the interplay of these topics. The ability to solve simple equations is a skill that transfers into the learning and investigating of other mathematics concepts such as similar triangles, angle measures in 2-D geometric figures, optimizing surface area and volumes, and develops into more complex solving processes and algorithms. It may be the success felt with initial concepts such as solving simple linear equations that offers possible opportunities for students to persevere and continue to experience and enjoy mathematics.

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Paul Pogue
January 30, 1940 – October 7, 2013

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Paul Pogue died as a result of a tragic accident on October 7.

He is survived by his wife Linda, son Greg, and daughter Natalie (Tony) Bedard.

Paul was born and grew up in Barrie.

He started his teaching career at King Edward Public School in Barrie in 1960 and taught there until 1964. (His principal was Jim MacLean—author of *Freedom to Learn*—who helped introduce the Nuffield approach to learning in Canada.) From 1964 until 1966, he taught at Silverthorn Public School in Mississauga.

With Jim's encouragement, Paul decided to become a secondary school teacher and began that phase of his career at Barrie District North Collegiate Institute in 1966. He taught there until his retirement in 1996.

In addition to his stellar work with the more able and less able students, his elementary experience had taught him that the students enrolled in Basic (Workplace/Locally Developed) courses could learn meaningful mathematics. He became known provincially for his advocacy for, and success with, this concept.

When graphing calculators appeared, Paul became aware of their capabilities and of the effect that they could have on the curriculum. For example, instead of the usual proving of trigonometric identities, Paul would give the students a series of possible identities and ask them to identify which were probable identities and then prove or disprove the probables. (He expected his students to graph the Left Side and Right Side as functions and see if the graphs appeared to be identical and proceed from there.) Paul was a leader in the use of technology in the classroom and wrote The Graphic Calculator as a Teaching Tool. In a feature article about Paul in the *Barrie Examiner* in 1994, entitled "He's Not Just a Teacher," Paul was quoted as follows: "People who are going to survive in the next 25 or 30 years are going to have to be competent in technology."

While at Barrie District North, he presented at many mathematics teacher conferences in Ontario, other provinces, and in the United States.

In 1994, Paul became a Descartes Fellow at the University of Waterloo, and he won the Prime Minister's Award for Teaching Excellence in Mathematics.

He was on the executive of OAME from 1982 to 1986 and was President in 1984–1985.

In 1978, Paul was the co-founder of the Barrie North Matholympics, which led to the Ontario Mathematics Olympics in 1995. He was also a co-chair of the 1997 OMO hosted by our chapter, MACC.

While at Barrie District North, Paul was the senior author of six secondary school mathematics textbooks intended for students at all levels of learning. As well, he was a co-author of four mathematics textbooks for elementary school students. In addition, he was a writer of Ministry curriculum guidelines and several documents for use by teachers.

In 1974 and 1975, Paul was seconded to the Department of Education/University of Ottawa as Project Director, Contractual Research Project entitled *Recommendation for Intermediate Division Mathematics*

in Ontario, Based Upon a *Study of Local and Intermediate Programs.*

Here is a testimonial for Paul that was submitted by a former student, John Knapp, as part of Paul's nomination for The Prime Minister's Awards for Teaching Excellence in Science, Technology, and Mathematics:

> *Our class entered North Collegiate in 1967, and stayed largely intact until graduation in 1971. Along that academic trail, we were fortunate to have Paul Pogue for four of our six math courses. In Grades 9 and 10, the base and fuel of enthusiasm was laid; in Grade 11, the ignition sequence was set; and by Grade 13, the fire was roaring. One-third of that class went on to major in math at university, and at least another one-third entered math-related areas such as science and computer studies. How does this evolution occur? The teacher is the key. Enthusiasm, variety, innovation, and fun are not always easy conditions for teachers to achieve and maintain. Paul's love for and joy of mathematics was obvious to students. Hence, the conditions of enthusiasm and fun were easily taken care of. Mr. Pogue's research and development of numerous math textbooks, as well as being a favoured presenter at math conferences, allowed him to be constantly fresh and innovative in his teaching. There is nothing so contagious as example, and as students, we were infected with the enthusiasm, the fun, the joy, the knowledge.*

After his retirement, Paul remained active as a tutor, who was in constant demand, and he took pride in the number of students he was able to help. On October 5, Paul attended the 30th reunion of the Barrie North class of 1983. When a former colleague asked about his tutoring, Paul responded "I love it!"

Paul was a gifted mathematician, an outstanding teacher of mathematics, and a leader in mathematics education.

He will be missed by his family, friends, and colleagues (and we count ourselves in the latter two categories). ▲

▲ Book Reviews

Review of: *Sally Strange and How She Learned to Stop Worrying and Love Grade 7 Math*

REVIEWED BY CATHERINE HEIDEMAN
E-MAIL: theheidemans@gmail.com



Catherine Heideman was a high school Math teacher with the Upper Grand District School Board prior to her retirement. She is now a Grade 6 Math teacher and Math Department Chair at the International Christian School in Hong Kong.

"Why do people write?" asks Ms Foster, a teacher from Winona Senior Public School. ..."To tell a story?" ventures Chin, one of Sally Strange's Grade 7 classmates.

Author Nico Rowinsky draws on his experience as a Toronto middle school teacher to tell a story about Sally Strange, a student who excels in language arts, but struggles with math. The story unfolds through her eyes as she describes the emotions and drama that an 11-year-old girl experiences in middle school.

Though the title indicates it is about her overcoming math anxiety, many other themes are developed that form the fabric of middle school students' lives. There are the math topics which elude her ability to understand, but are easy for others; the novels she studies that bring understanding to her own situations; and her participation on the volleyball team that exposes the intricacies of teamwork. Conflict and care for friends cause distractions that impact her ability to focus in class.

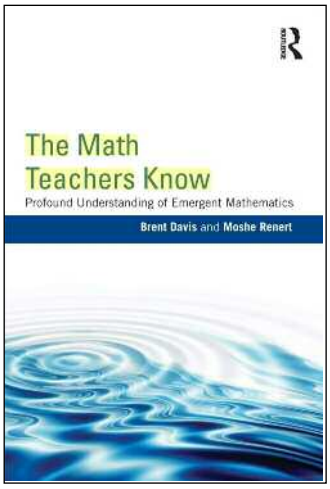
For a teacher, the story reminds one of the ricocheting emotions of middle school students. It reinforces the importance any teacher can have on the development of a student as words of encouragement are spoken, a sense of humour is used, and purposefully planned lessons develop interest. Throughout the story, one can find ideas to use in one's own classroom.

For a student, the story will describe situations that may describe one's own experience or those of friends. There are no scintillating, deeply detailed horror stories, just the usual experiences of school that make for an engaging story. As Sally learns how to overcome her fear

of mathematics, she shares useful strategies for others to apply. As she seeks to find her way through various relationships, she discovers how to be a friend and develop into a confident young person.

As Sally reflects on one piece of advice she was given, "Learning is not what the teacher teaches you, but what you choose to learn," she realizes how true this is.





Review of: *The Math Teachers Know*

REVIEWED BY TIM SIBBALD



Tim is an assistant professor at the Schulich School of Education at Nipissing University and a vice-president of OAME.

Davis, B., & Renert, M. (2014). *The math teachers know: Profound understanding of emergent mathematics.* New York: Routledge.

In the initial stages of this short book, there is a good historical accounting of many ideas about the mathematics that teachers require in order to be effective teachers. The approach used by the authors shows how our understanding of mathematics needed by teachers has emerged through the years and now demonstrates a rich diversity of interpretations. Particular attention is given to multiplication with multiple interpretations and how they relate to the curriculum. A table is also provided, highlighting different interpretations of a function. The initial historical focus explains how the approaches used to explore the terrain have changed, and a large portion of the book is about mathematics that emerged from concept study groups of teachers who were in a masters program.

There are several groups who will be interested in this book. It is of interest to the math specialist who is trying to gain a better grasp of the "big picture." It addresses both elementary and secondary math, but the latter is portrayed as being somewhat concept constrained by the amount of curriculum content and use of exams. While I was not convinced that the statements about secondary math apply universally, I did recognize the point of view and felt mildly provoked (which is a good thing). The ideas presented are inspiring for the interpretation of classroom events or differences between classrooms, but it does not focus on practical elements of implementation. The overall picture of diversity in content knowledge, and the circumstances that reveal it, will be of interest to Master's students because of the way it informs theoretical considerations in the book. Pre-service instructors will find examples that they can adopt, and approaches that they might utilize to reveal diversity of thinking in their classes, which will lead to richer conversations. It is a book for thinking about teaching, and this is why it is suited to math specialists, Master's students, and pre-service instructors.

With respect to training teachers, there are wonderful examples of being able to draw out different approaches to math concepts. The extent that one would find the various approaches in single classes is questionable, and, to the extent that one might, there is no suggestion as to whether one should incorporate a comprehensive variety of approaches. This also points to the high level focus of the book, where the teacher can consider issues that emerge from the ideas the book presents. It is not a "how to" book; rather, it is a good, readable book to challenge one's practices and to contemplate the mathematics we use to be effective math education instructors.

Review of: *Don't Count Me Out!*

REVIEWED BY TIM SIBBALD

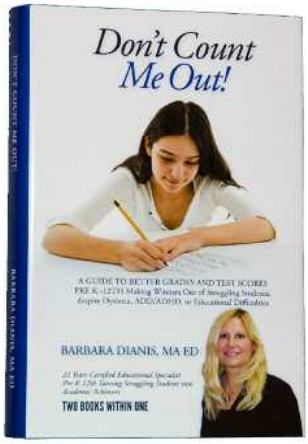
(Dianis, B. (2013). *Don't count me out!* A guide to better grades and test scores pre K-12th. Barbara Dianis (pub.). Retrieved from **www.dianiseducation.com**)

This was an unusual book from the outset. It appears to be self-published and is available at **www.dianis education.com** and through **www.lulu.com**. It claims to

be two books in one, yet they are not published separately, and chapter numbering is continuous between the "books." The details that arrived with the book (and, as given at **www.edutopia.org/user/407516**) mention a masters degree and yet the "MA Ed.D" on the cover seems to suggest a doctorate—the Masters degree "M.A. Ed." was earned from Oral Roberts University. There is no doubt that Ms. Dianis has a wealth of experience, and she clearly has a track record with respect to speaking on the topic of learning disabilities. Edutopia gives the following information about the author: She herself overcame dyslexia; she used self-taught techniques.

The strangest aspect of the book is a distinct change in the writing style that occurs after the first 45 to 50 pages. I remain unclear whether the initial chapters were intentionally designed to be very short because of the difficulty parents might have digesting details that may yield the conclusion that their child has learning difficulties. This certainly is consistent with the focal audience being the parents of students, and not their teachers. However, I remain unclear if the one-sentence paragraph or one-page chapter would have the desired effect. For teachers, there is practical advice, but few specific references to acquire more information if you question some element. An overall reference list for the book contains 11 references, with the most recent being 9 years old, and more than half being over 20 years old. References certainly would be useful to follow up on some details. For example, there is mention of using "genetic coding" (p. 2) when diagnosing a learning issue, but the reader is subsequently told: "All struggling students have the genetic ability to learn" (p. 16). The book also promotes college and university as leading to a career, to the exclusion of other viable options or schooling outcomes. In the end, I concluded that the book is better suited to parents than teachers because of the lack of support beyond references to Ms. Dianis' personal experience.

For parents, there is a dearth of ideas that carry a practical value. In some places, the book discusses elements that parents may need to hear. This led to my wondering if the slow initial pace of the book was to help parents adjust to hearing what may be perceived as a difficult message. The book promotes the use of positive words, a strong work ethic, practice through the summer, and speaks frankly about non-medical techniques that may address learning issues. I found many suggestions that parents would benefit from or, at the very least, that would give them food for thought. There is also some content that teachers will find controversial, and I respect the fact that there may be benefits in parents seeing controversial content. For example, "Remember, most children do not learn the basic math facts at school, and students will need to be drilled on them at home" (pp. 199–120). This is controversial in the sense that it is not consistent with the Ontario curriculum; however, for the parent of a struggling student, it may provide a starting point that may be helpful.



As a general resource that may benefit parents who do not know how to help their child, I believe this book may prove useful. As a teacher, I found much that I already knew from the classroom, as well as some elements lacked the support I require as a professional.

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▲ An Investigation Relating Square and Triangular Numbers

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> *It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment* (Gauss, letter to Farkas Bolyai, 2 September, 1808).
>
> *Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime* (Pólya, *How to Solve It,* 1945).

In yet another marginal note penned in his copy of Diophantus' Arithmetica, Fermat wrote, in 1636, speaking of the positive integers, that:

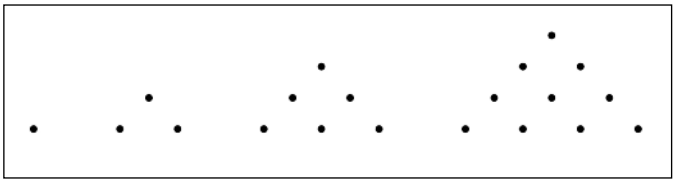
> *…every number is either a triangular number or the sum of two or three triangular numbers; every number is a square or the sum of two, three, or four squares; every number is a pentagonal number or the sum of two, three, four, or five pentagonal numbers; and so on ad infinitum, for hexagons, heptagons, and any polygons whatever….*

He called it a beautiful result, a wonderful theorem. Now the case for four squares was proven correct by Lagrange in 1770. Later, on the day of 10 July, 1796, Gauss famously wrote:

"Eureka, num = $\Delta+\Delta+\Delta$" in his diary. He shared his proof of this in 1801 in his famous and impactful book Disquisitiones Arithmeticae. More exactly, we have:

> Theorem (Gauss). Every positive integer is the sum of at most, three triangular numbers.

Now the notion of a polygonal or figurate or *n*-gonal number, such as the triangular and square numbers, goes back to ancient Greece and the Pythagoreans. Since unity, the number 1 had such special significance to these ancients, they did not count it as a polygonal number. However, it is now customary, as it was for Gauss, to do so and, in particular, 1 is an *n*-gonal number for all $n \geq 3$. Triangular numbers are so called because they can be interpreted as the total number of dots in the pictures below, which images can be extended indefinitely.



Thus, the sequence of triangular numbers begins 1, 3, 6, 10, 15, 21, 28,…. Notice that each picture incorporates the previous picture, with a new row of *n* dots below it.

The phrase "at most" appears in our statement of Gauss' theorem because, after all, the only representations of 1, 2, and 4 in this way are as 1 = 1, 2 = 1 + 1, and 4 = 1 + 3, each of which uses fewer than three triangular numbers. The number 5 = 1 + 1 + 3 is the first to require three such numbers.

In general, we let T_n be the *n*th triangular number, $n \geq 1$, so that $T_1 = 1$, $T_2 = 3$, $T_3 = 6$ and so on. We will refer to *n* as the index of T_n. If we add the number of dots from top to bottom in each triangular array of dots in the picture above, we can reveal that $T_n = 1 + 2 + 3 + \ldots + n$.

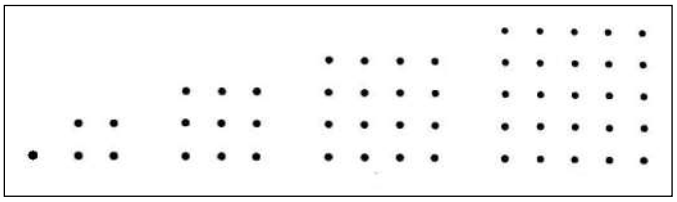
This expression can be manipulated to show that $T_n = \dfrac{n(n+1)}{2}$. Indeed there is a well-known anecdote about Gauss as a young boy applying this formula. Perhaps he calculated like this:

$$2T_n = T_n + T_n$$
$$= (1 + 2 + 3 + \ldots + n) + (n + (n-1) + (n-2) + \ldots + 1)$$

$= (n + 1) + (n + 1) + \ldots + (n + 1)$

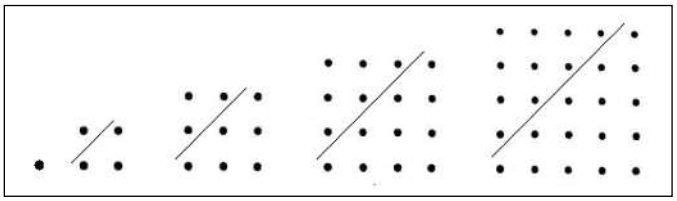
$= n(n + 1)$

Of course, the square numbers are the number of dots in each of the square arrays pictured below, which can be extended indefinitely.



So the squares are the familiar sequence 1, 4, 9, 16, … with general term $S_n = n^2$.

There is a very nice result connecting these two sequences. It is suggested by these images:



Evidently each square number 4, 9, 16, … is the sum of two consecutive triangular numbers. In fact, it is easy to verify algebraically that $S_n = T_n + T_{n-1}$, but in this case, the pictures convince us!

To the teacher: This activity we are engaging in fits under the Ontario curriculum guidelines of Mathematical Process Expectations [1]. There we find the statements that students will develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding. Students will also develop and apply problem-solving skills (e.g., the use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures.

Even if you only take the time in class to introduce the polygonal numbers and to allow some experimentation with them on worksheets, perhaps based on this paper, your students will benefit from the mathematical inductive process. For example, coins are easy to use to construct the triangular numbers. Here is $T_4 = 10$.



I became curious as to which square numbers were sums of exactly three triangular numbers. This prompted me to write a very simple Maple (© University of Waterloo) software program to generate some data. (All programs referred to in the article are in an Appendix at the end of it.)

Here is the output:

{9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089, 1156, 1225, 1296, 1369, 1444, 1521, 1600, 1681, 1764, 1849, 1936, 2025, 2116, 2209, 2304, 2401, 2500, 2601, 2704, 2809, 2916, 3025, 3136, 3249, 3364, 3481, 3600, 3721, 3844, 3969, 4096, 4225, 4356, 4624}

This set contains the squares of all the numbers from 3 to 66 as well as the square of 68. The omission of $67^2 = 4489$ is due to the range of values for a, b, and c (see Appendix, program #1). Indeed, we have $4489 = T_{37} + T_{59} + T_{63} = 703 + 1770 + 2016$. Thus, surprisingly, the data suggests that *every square number beyond 4 is the sum of exactly three triangular numbers!* Naturally I wanted to prove this conjecture. In order to get a handle on it, I wrote another program (Appendix, program #2) to see the actual T_a, T_b and T_c involved in representing each square. Here, summarized in a table, is some of that output.

a	b	c	T_a	T_b	T_c	d
2	2	2	3	3	3	9
2	2	4	3	3	10	16
1	2	6	1	3	21	25
3	5	5	6	15	15	36
1	2	9	1	3	45	49
2	4	8	3	10	36	49
3	5	7	6	15	28	49
2	3	10	3	6	55	64
5	6	7	15	21	28	64
5	6	9	15	21	45	81
4	9	9	10	45	45	100
6	9	10	21	45	55	121
2	6	15	3	21	120	144
8	10	12	36	55	78	169
6	10	15	21	55	120	196
3	11	17	6	66	153	225
5	14	14	15	105	105	225
9	13	15	45	91	120	256
9	13	17	45	91	153	289
8	12	20	36	78	210	324
14	15	16	105	120	136	361

The output showed that many squares have different representations as sums of three triangular numbers. I have included only a few examples of this in the table above, namely for the squares d equal to 49 and 64 and 225. In fact, I was struck by the proliferation of the different ways that a square number could be so written. These multiple representations presented a difficulty for me as I sought to examine the data for a pattern or patterns that would supply a direction of proof of the conjecture. Alas, I did not succeed in that grand endeavour. Consequently, I sought to prove a lesser result:

There are infinitely many squares each the sum of exactly three triangular numbers.

Combing through the data again and again, I eventually had my own EUREKA moment. I looked at the squares of just the triangular numbers and saw how they were expressible. The table above shows that:

$$9 = T_2^2 = 3 + 3 + 3 = T_2 + T_2 + T_2$$
$$36 = T_3^2 = 15 + 15 + 6 = T_5 + T_5 + T_3$$
$$100 = T_4^2 = 45 + 45 + 10 = T_9 + T_9 + T_4$$
$$225 = T_5^2 = 105 + 105 + 15 = T_{14} + T_{14} + T_5$$

Each expression is in the form $T_n^2 = T_a + T_a + T_x$, where the repeated subscript a is itself one less than a triangular number! In fact, looking closely at these four equations, a beautiful pattern emerges from the data. Our new claim is this:

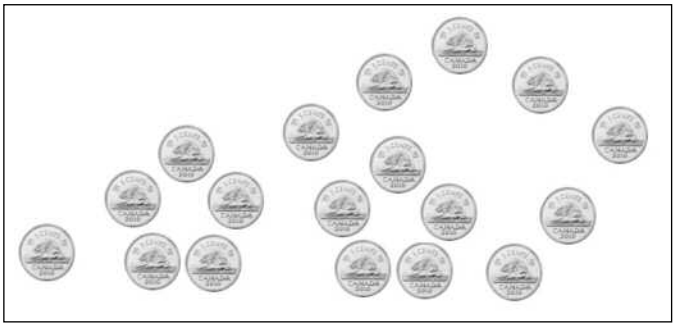
$$(1)\ T_n^2 = T_{T_n - 1} + T_{T_n - 1} + T_n,\ n \geq 2$$

The actual verification of this equation is easy, just straightforward algebra:

$$\begin{aligned} & T_{T_n-1} + T_{T_n-1} + T_n \\ &= \frac{(T_n - 1)T_n}{2} + \frac{(T_n - 1)T_n}{2} + T_n \\ &= (T_n - 1)T_n + T_n \\ &= T_n^2 - T_n + T_n \\ &= T_n^2 \end{aligned}$$

To my eye, it is certainly a beautiful relationship, and perhaps a new discovery on the triangular numbers. The equation (1) proves the claim that there are infinitely many squares that are the sum of exactly three triangular numbers.

This discovery led me to experiment with variations involving the square numbers $S_n = n^2$ and the pentagonal numbers $P_n = \frac{n(3n-1)}{2}$. Once again, the pentagonal numbers may be introduced in the classroom, using coins or counters, even candies: for example, here are $P_1 = 1$, $P_2 = 5$ and $P_3 = 12$:



To my surprise, I found that:

$$(2)\ S_n^2 = T_{S_n - 1} + T_{S_n - 1} + S_n,\ n \geq 2$$

and

$$(3)\ P_n^2 = T_{P_n - 1} + T_{P_n - 1} + P_n,\ n \geq 2$$

These equations (1), (2), and (3) are just like each other and suggest an obvious generalization. If we let N_n be the nth N-gonal number, that is,

$$N_n = \left(\frac{N}{2} - 1\right)n^2 - \left(\frac{N}{2} - 2\right)n \text{ then}$$
$$(4)\ N_n^2 = T_{N_n - 1} + T_{N_n - 1} + N_n,\ n \geq 2$$

and, in fact, this generalization is true, algebraically verifiable, as was the verification of equation (1). Here is that verification:

$$\begin{aligned} & T_{N_n-1} + T_{N_n-1} + N_n \\ &= \frac{(N_n-1)N_n}{2} + \frac{(N_n-1)N_n}{2} + N_n \\ &= (N_n-1)N_n + N_n \\ &= N_n^2 - N_n + N_n = N_n^2 \end{aligned}$$

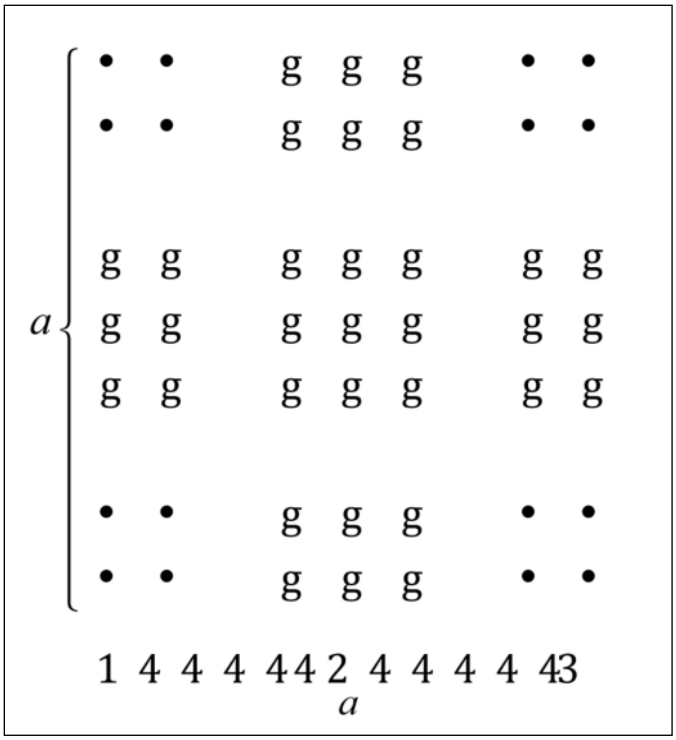
These relationships on polygonal numbers, while delightful, seemed to me a bit too flexible. I began to suspect that the truths expressed by these equations were in some way stating the obvious. And so I asked myself whether the relation (4) is peculiar to the squares of polygonal numbers, or whether it is more widely applicable.

Could it be that, for every positive integer $a > 1$, we have:

$$(5)\quad a^2 = T_{a-1} + T_{a-1} + a$$

Well, yes! And this is easily proved algebraically, as above, but also geometrically, in the spirit of the Pythagoreans, as follows.

Consider this diagram:



If we section off the last column of a dots, then the figure decomposes into an oblong, that is, a rectangle a by $a - 1$ along with the column of a dots. The Pythagorean, Theon of Smyrna, circa 100 CE, knew that an oblong number is the sum of two equal triangular numbers. So we have:

$$\begin{aligned} a^2 &= (a-1)\,a + a \\ &= \frac{(a-1)a}{2} + \frac{(a-1)a}{2} + a \\ &= T_{a-1} + T_{a-1} + a \end{aligned}$$

This investigation was a tale of numerical exploration, easily duplicated in the classroom, and emphasizes the inductive process of gathering data, looking for a pattern and proving a result that may, indeed, be very pleasing, as the result (1) seems to the author, and may also lead to a generalization, as here we have in (4), although somewhat removed from the original question. The fact that (4) is a simple consequence of the last equation (5) above does not lessen the joy of this discovery process.

Reference:

Ontario Ministry of Education. (2007). *The Ontario curriculum, grades 11 and 12: Mathematics, 2007*. Toronto: Queen's Printer for Ontario, p. 112.

Appendix

Maple program #1.

```
> S:=NULL: for a from 1 to 60 do
> for b from 1 to 60 do
> for c from 1 to 60 do
> d:=a*(a+1)/2 + b*(b+1)/2 + c*(c+1)/2:
> if floor(sqrt(d)) = sqrt(d) then S:= S union {d}: fi: od: od:
  od: print(S):
```

The last line of this code is checking to see if the square root of the sum d of three triangular numbers, indexed by a, b, and c, is an integer. If so, then d is a square and we include it in the set S. At the end, we print out the square numbers that have landed in S.

Maple program #2.

```
> for a from 1 to 20 do
> for b from 1 to 20 do
> for c from 1 to 20 do
> d:=a*(a+1)/2 + b*(b+1)/2 + c*(c+1)/2:
> if floor(sqrt(d )) = sqrt(d) then print
  (a, b, c, a*(a+1)/2, b*(b+1)/2, c*(c+1)/2,d):
  fi: od: od: od:  ▲
```

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COMING JANUARY 2014

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ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Abacus

Volume 52 • Number 2 • December 2013

PROPORTIONAL REASONING – RATE

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A Publication of the OAME/AOEM

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The development of proportional reasoning is fostered by intentional learning experiences where students explore, discuss, and experiment with situations that require proportional reasoning. These situations include real-world and every-day situations, such as: determining the better buy when 10 bags costs $9.50 and 24 bags costs $22, adjusting calculations for distance travel time when the speed fluctuates, or determining if 6 cents per litre is better than a 5% discount on your fuel charges. Imagine the impact of underdeveloped proportional reasoning in real-world situations; it could have life-threatening or disastrous consequences, such as incorrect doses of medicine being administered, miscalculation of time when expecting the arrival of a tornado or tsunami given anticipated speed in relation to time needed for safe evacuation!

In this issue, the Research Summary and problems (with multiple solutions) focus on strategies for representing proportional reasoning situations within measurement contexts. The problems we provide are designed for use within a three-part, problem-solving lesson structure.

LINKS TO LITERATURE AND MANIPULATIVES



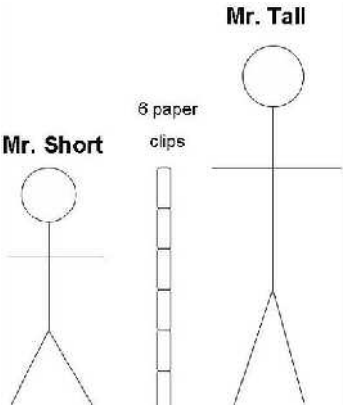
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It is interesting to consider how many rate problems involve time and money – two areas of math that are often identified as challenging to both teach and learn. The passage of time (duration of an event) is difficult to experience in the classroom, since it cannot be felt, seen, or heard. The picture book for this issue, *Just a Second* provides many examples of rates by comparing units of time to a variety of quantities such as distance, mass, volume, heartbeats, population changes, to name a few. This book provides interesting ways to think about the idea of speed by making comparisons in real life contexts. What visuals enable students to model relationships in rate problems? For this issue, the double number line is used as a tool for representing details in rate problems.

RESEARCH SUMMARY – PROPORTIONAL REASONING – FROM RATIO TO RATE

From Additive to Multiplicative Thinking

According to Piaget's theory of intellectual development, a child acquires proportional reasoning when progressing from the stage of concrete operations to the stage of formal operations. Karplus and Peterson (1970) who are pioneers in the research of proportional and rational number reasoning designed "The Mr. Tall/Mr. Short" problem to assess and enhance students' levels of proportional thinking. Since their study, variations in using this problem have been used by other researchers.



Mr. Short is six paper clips in height. If he is measured in large buttons, he is four large buttons in height. Mr. Tall is similar to Mr. Short, but is six large buttons in height. Predict the height of Mr. Tall in paper clips. Explain.

Reasoning	Explanation
Multiplicative Reasoning 1	He is nine paper clips tall. Each button is equal to one and a half paper clips. If he is six buttons tall you multiply six times one and a half to get nine paper clips.
Multiplicative Reasoning 2	Mr. Tall is 1 ½ times as high as Mr. Short. Since Mr. Short is 6 clips high, Mr. Tall must be 6 * 1 ½ = 9 clips high.
Multiplicative Reasoning using addition	For every two buttons there are three paper clips. Mr. Tall is 2 buttons taller than Mr. Short so he must be 3 paper clips taller. 6 + 3 = 9 paper clips.
Additive Reasoning 1	Mr. Tall is 8 paper clips high. Mr. Short is 4 large buttons high and 6 paper clips high. So the buttons are 2 less than the paper clips. Since Mr. Tall and Mr. Short are similar, and Mr. Tall is 6 buttons high, he must be 8 paper clips high.
Additive Reasoning 2	Mr. Tall is two more buttons taller than Mr. Short so he will also be two more paper clips taller than Mr. Short resulting in 8 paper clips.
Estimate	Nine, I figured he would be a bit taller.
Haphazard	Since Mr. Tall is 2 more buttons than Mr. Short, I took the 6 paper clips and multiplied by 2 to get 12 paper clips.

According to Lamon (2006), being able to make comparisons in additive and multiplicative terms is also referred to as absolute and relative thinking respectively. Being able to see these various equivalent relationships is a key to proportional reasoning.

Tree Growth Problem: In a schoolyard, a pine tree was 4 m high and the maple tree was 5 m high. Five years later, the height of both trees was measured. The pine tree is now 7 m high, while the maple tree is 8 m high. Which tree has grown the most?

- From an additive (absolute) perspective, both trees have grown taller by the same amount of 3 m. From a multiplicative (relative) perspective, the pine tree has grown 3⁄4 of his original height and the maple tree has grown 3/5 of his original height. The pine tree has grown more than the maple tree because 3/4 is greater than 3/5.
- What would this relative change look like on a number line?
- If the trees both doubled, the pine tree would be 8 m high and the maple tree would be 10 m high.
- The pine tree had a 75% increase, which means it is 1- 3/4 or 175% of its original size. While the maple tree had a 60% increase, which means it is 1- 3/5 or 160% of its original size.

Rate

- The growth of a pine tree and maple tree over 5 years is compared.
- A car travels 540 km and uses 60 L of fuel, so 540 km / 60 L = 9 km / 1 L.
- A student works 5 hours and is paid $60, so $60 / 5 h = $12 / 1 h.
- 1 dozen eggs costs $3.00, so $3.00 / 12 = $0.25 /1 egg.

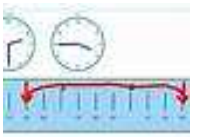
These examples show the comparison of two quantities (or 2 terms) that have different measuring units using division, which is called a rate. A rate is unlike a ratio where both quantities have same units.

- A rate can be represented as a fraction that includes 2 different units. For example, a puppy grows 2 kg per 1 week (2 kg / 1 week) or 8 kg / 4 weeks.
- A **unit rate** is a rate where the second quantity or term is a unit of 1. For example, 60 km/hour really means 60 km is travelled in 1 hour.
- If two rates are different but equivalent, they are proportional. For example, 2 km / 3 h = 6 km / 9 h. These 2 rates are proportional by a factor of 3.

$$\frac{2\text{ km}}{3\text{ h}} \overset{\times 3}{=} \frac{6\text{ km}}{9\text{ h}}$$

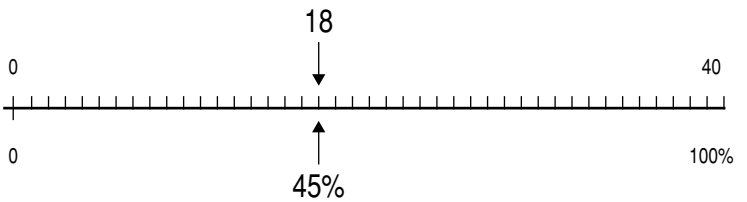
LINKS TO MANIPULATIVES:

The number line is a powerful tool for visualizing number relationships. The scale on a number line can be adapted for different mathematical contexts, such as fractions, money and elapsed time.

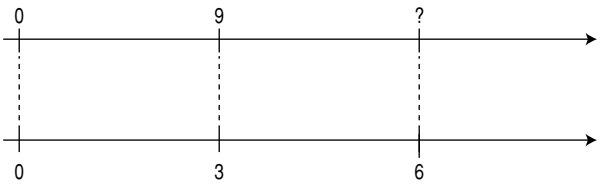


A double number line models ratio comparisons like percent.

What is 18 out of 40 as a percent? In this representation students see the “stretching” of 40 to 100, so that 18/40 is equivalent to 45%.



The double number line also models rates. For example, Jim can walk 9 km in 3 hours. If he maintained the same speed, how far can he walk in 6 hours? In this representation students can visualize the relationship between the two units in the rate.



LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Hats Problem:

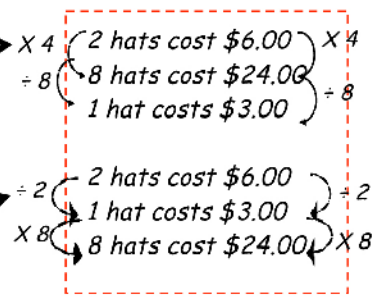
If 2 hats cost $6.00, then how much do 8 hats cost?

Solution 1

2 hats x 4 = 8 hats
$6 x 4 = $24

Solution 2

If 2 hats cost $6, then
half of 2 hats is 1 hat, so
half of $6 is $3
1 hat costs $3
8 hats cost $3 x 8 = $24



"half of" is the same as "÷ 2"

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to make a plan to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Iced Tea Problem 1:

Two stores have a sale on ice tea:
- Price Chopper 12 cans for $15.00
- Food Basics 20 cans for $23.00

Which is the better buy?

What information are we going to use to solve the problem?
- *PC – 12 cans cost $15.00*
- *FB – 20 cans cost $23.00*
- *Cans have same amount of ice tea (same capacity)*
- *Which is cheaper?*

AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

***Rate** is a special kind of ratio; a multiplicative comparison of two or more quantities of different units (e.g., price of 12 cans, number of cans)*

- *ratio as fraction*
- *doesn't matter which quantity is numerator or denominator*

What is the cost of 1 can if divide or fair share:
- *$15.00 equally among 12 cans?*
- *$23.00 equally among 12 cans?*

Solution 1

PC FB
12/15 < 20/23
12 ÷ 15 = 0.80 < 0.87 = 20 ÷ 23
FB get more of a can per $1

15/12 > 23/20
1 - 3/12 > 1 - 3/20
3/12 > 3/20
At FB 1 can cost less

Solution 2

Price Chopper ->
$15.00 ÷ 12 cans
= $1.25 per 1 can

Food Basics ->
$23.00 ÷ 20cans
= $1.15 per 1 can

Food Basics price is cheaper per can $1.25 > $1.15 per 1 can

Fraction Comparison
Ratio of 2 Quantities

Division - Unit Rate
$ Cost for 1 Can

Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of students' mathematical discussion.

Grade 5	Grade 6	Grade 7
- demonstrate an understanding of simple multiplicative relationships involving whole-number rates	- represent ratios found in real-life contexts using concrete materials, drawings and standard fractional notation	- demonstrate an understanding of rate as a comparison or ratio of two measurements with different units; solve problems involving the calculation of unit rates

Unit Rate *- describes how many units of the first type of quantity corresponds to one unit of the second type of quantity (order implied)*

quantity 1 (how many units)	*quantity 2 (one unit)*
$ cost	*of 1 can*
how much of can	*for $1.00*

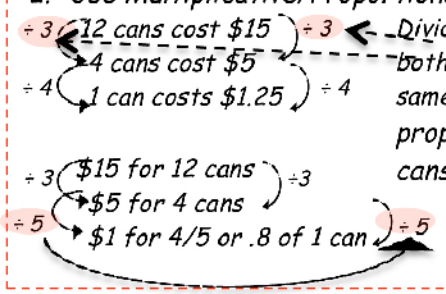
Different Ways to Calculate Unit Rate

1. Divide (or fair share) 1st quantity among 2nd quantity = number of units of 1st quantity for 1 unit of 2nd quantity

$15.00 ÷ 12 cans = $1.25 per 1 can (1 unit)

12 cans ÷ $15.00 = .80 of 1 can for $1.00 (1 unit)

2. Use Multiplicative/Proportional Relationships



Divide (or multiply) both quantities by the same amount keeps proportional ratio of cans to cost

Solution 3

Price Chopper
12 cans cost $15
4 cans cost $15÷3 = $5.00
1 can costs $5÷4 = $1.25

Food Basics
20 cans cost $23
4 cans cost $23÷5 = $4.60
1 can costs $4.60÷4 = $1.15

Food Basic is cheaper

Solution 4

Price Chopper ->
12 cans ÷ $15.00
= 0.80 of 1 can for $1

Food Basics ->
20 cans ÷ $23.00
= 0.87 of 1 can for $1

Food Basics price is better - get more of 1 can for $1

Division - Proportionality
Unit Rate - $ Cost for 1 Can

Division - Unit Rate
Amount of Can for $1

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed

Highlights/Summary:

*A **rate** is a multiplicative comparison or ratio between two quantities with different units (e.g., money, amount of iced tea), in a particular order (e.g., 20 cans for $23.00 is not the same as 23 cans for $20.00)*

Compare rates using multiplicative relationships by:
*a) **fraction comparison** (15/12 › 23/20 or 12/15 ‹ 20/23)*
*b) **unit rate** (number of units to one unit*
ice tea - total price divided by number of cans
(e.g., $15.00 ÷12 cans = $1.25 per can; 23.00 ÷20cans = $1.15 per can)
cost ($1.00) - total cans divided by total cost
(e.g., 12 cans ÷ $15.00= 0.80 of 1 can for $1; 20 cans ÷ $23.00 = 0.87 of 1 can for $1)

AFTER (Practice) – 5 to 10 minutes

- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Ice Tea Problem 2:

Which is the better buy for ice tea cans?
- T & T - 24 cans for $30.00
- No Frills - 40 cans for $46.00

Compare using fraction comparison or unit rate.

Solution 1

T&T NF
30/24 > 46/40
15/12 > 23/20

No Frills cost less (proportional to the price of Food Basics – doubled the price and # of cans)

Solution 2

24 cans for $30 is 0.80 of 1 can for $1 (24÷30= 0.8)

40 cans for $46 is 0.87 of 1 can for $1 (40÷46 = 0.87)

Get more of 1 can for $1 at No Frills

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.

LET'S DO MATH – DETERMINING RATES IN DIFFERENT WAYS

Before (Getting Started)

Rate of Growth Problem 1:

Examine each picture and each rate of growth. Match a picture to a rate of growth. Explain the rate of growth represented.

			
0.5 cm / 1 month	10 m / 1 pedal cycle	4000 drips / 1 L	3 mm / 1 week

Solution 1:

The plant might grow at 3 mm / 1 week. The plant would grow (3, 6, 9. 12) 12 cm in a month because there are about 4 weeks in a month.

If the boy is growing 0.5 cm / month then he will grow 6 cm in a year. 12 x 0.5 = 6)

Solution 2:

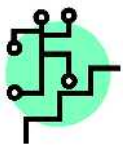
Drips from a tap might add up to 4000 drips / 1 L. If a tap drips 3 drops / min then it would be 180 drips / h (3 drops x 60 min or 1 h) or 4320 drips / day (180 x 24 h or 1 day) which means more than 1 L of water a day is lost.

A girl going 10 m / 1 pedal cycle would have to pedal 100 times to go 1000 m or 1 km. (10 m / 1 cycle = 100 m / 10 cycles = 1000 m / 100 cycles)

During (Working On It)

Climbing Rates Problem 2:

There are 1776 stairs in the CN Tower. For last year's United Way Climb, the median rate of participants; climb was 1 h 20 min to make the climb. What was the climbing rate?





After (Consolidation)

Anticipating Student Responses:

Solution 1:

1776 stairs takes
1 h 20 min or 80 min
-> 177.6 stairs in 8 min
-> 22.2 stairs in 1 min.

This person climbed about 22 stairs /min.

Solution 2:

Stairs	1776	888	444	1332	222	22.2
Time (min)	80	40	20	60 min or 1 h	10	1

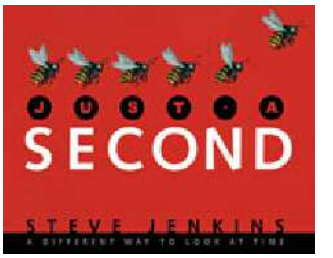
This person climbed at a rate of 1332 stairs / 1 h.

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solution1?

- *Solution 2* – shows a ratio table being used to multiplicatively compare the number of stairs and the time taken to climb them; showing a rate for 1 h and a rate for 1 min. This is like a double number line.
- *Solution 1* – shows a rate calculation using a unit rate or a rate for 1 min; comparing number of stairs to time, reducing the time to one unit or 1 minute.

LINKS TO LITERATURE:



"A second goes by pretty quickly. In fact, several seconds have passed since you started reading this sentence. And they just keep going by. There are 60 seconds in every minute, 3 600 second in every hour, and on and on. But a lot can happen in a second. And that's what this book is about," writes author, Steven Jenkins. "Just a Second: A Different Way to Look at Time," is a picture book that has a range of facts that can be used as contexts for developing rate problems

> *In one second . . . a black mamba slithers a frightening 7 m. How far can the snake slither in 4 s?*

Students might solve this problem by counting, using repeated addition (7+ 7+ 7+ 7), doubling (7+ 7 = 14; 14 + 14 = 28) or multiplying (4 x 7 = 28).

> *In one minute . . . A giant tortoise lumbers about 4 ½ m. This fact from the book can be adapted to pose a problem in which students identify the unit rate. (e.g., a giant tortoise can lumber about 18 m in 4 min. How far can it travel in one minute?*

Students might solve this problem using strategies such as halving (9 m in 2 min, 4 ½ m in 1 min), using a known fact to estimate (4 x 4 = 16, 16 is close to 18 so 2/4 left over so 4 2/4).

LETS DO MATH –



Before (Getting Started)

Lemonade Problem:

One can of lemonade concentrate requires 4 cans of water to make juice. For a community picnic, 100 cans of juice are required. How many cans of lemonade concentrate are needed?



Solution 1:

1 can lemonade + 4 cans water
= 5 cans juice

$$\frac{\text{1 can concentrated}}{\text{5 cans juice}} = \frac{\text{? cans}}{\text{100 cans juice}}$$

$$\frac{\text{1 can concentrated}}{\text{5 cans juice}} = \frac{\text{20 cans}}{\text{100 cans juice}}$$

Solution 2:

Need to compare combined parts to whole

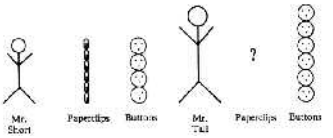
Concentrate	1	2	4	20
Water	4	8	16	80
Juice	5	10	20	100

Need 20 cans of concentrate for 100 cans of juice.

During (Working On It)

Comparing Heights Problem:

How many paperclips tall is Mr. Tall?



After (Consolidation)

Solution 1:

6 buttons for 4 buttons
so 6 : 4 = 3 : 2
So Mr. Tall is 3/2 the height of Mr. Short.

Mr. Tall is 3/2 the height of Mr. Short whose height measures 4 buttons or 6 paperclips.

3/2 x 4 buttons = 1.5 groups of 4 buttons
= 6 buttons
3/2 x 6 paperclips
= 1.5 groups of 6 paperclips

Solution 2:

Mr. Short → 6 paperclips to 4 buttons to Mr. Short. This is the same as 3 : 2
Mr. Tall → paperclips : buttons = ? : 6
3 groups of half of Mr. Short will match Mr. Tall.
We need 3 groups of half of 6, which is 9.



Mr. Short is 6 paper clips tall. Three groups of 3 is 9. Mr. Tall is 9 paperclips tall.

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solutions 2 and 3?

- *Solution 1* – These are part-to-part relationships. Mr. Tall is 3/2 the height of Mr. Small. 3/2 x 4 = 6 buttons and 3/2 x 6 = 9 paperclips. The ratio of buttons (Mr Small) and buttons (Mr Tall) must be compared or the rates (a kind of ratio) of buttons to paperclips (Mr Small) must equal buttons to paperclips (Mr Tall).
- *Solution 2* – The parts in this question are broken up and students count 3 paperclips as 1/2 of Mr. Small. Because there are 3/2 as many buttons, we need 3/2 paperclips – that is 3 groups of 3 or 9.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Try the proportional reasoning problems in this issue yourself first so that you can anticipate possible mathematical responses and then offer them to your students. Compare your anticipated responses to the students' responses. How are the solutions similar? How are they different?
- To suit your students' learning needs, vary the ways that you have students engage with noticing and using multiplicative (proportional) relationships inherent in multiplication and division operations, in representing and comparing fractions and rates. Use different models of representation, such as sets of objects and concrete materials and visuals, such as counters, base ten blocks, square tiles and fraction bars to model proportionalities.
- Practise noticing the different strategies and the breadth of the mathematics your students use in order to describe, show and make sense of multiplicative thinking (proportional reasoning) in relation to solving rate problems. Take note of the kinds of proportional reasoning language students' use in the junior years (e.g., number of units of one quantity in relation to a unit of 1 for the second quantity; a fraction can represent a rate (a ratio between two quantities of different units)).
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit key mathematical ideas, strategies for representing and determining rate (e.g., fractional comparison, unit 1 (quantity 1 compared to quantity 2; quantity 2 compared to quantity 1) as the students discuss and analyze each other's solutions to co-construct success criteria for the lesson learning goal.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below:

- Ben-Chain, D., Keret, Y., and Ilany, B. (2012). *Ratio and proportion: Research and teaching in mathematics teachers' education.* Rotterdam, The Netherlands: Sense Publishers.
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CLOSING

Our next issue will focus on proportional reasoning within the measurement, geometry and spatial sense strand. Also, we will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a problem solving-based teaching and learning lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email

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