8-PAGE INSERT ELEMENTARY MATH

Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

Vol. 52 #3
March 2014

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See Data Rocks – Another R for Public School Students to Learn

1. A Taste of R

To reproduce the following examples, simply type the command (shown in blue) into the R window. Execution of the command is accomplished by hitting the "Enter" key. The "answer" appears in red following the [1].

Arithmetic operations are as expected.

> 23 + 49

[1] 72

*> 11111*11111 - 21*

[1] 123454300

> 1234567654321/1111111

[1] 1111111

Exponents and fractional exponents are available.

> 9^3

[1] 729

> 12345678987654321^.5

[1] 111111111

Trigonometric and other special functions are available as well. Special constants such as π and *e* are also built in.

> pi

[1] 3.141593

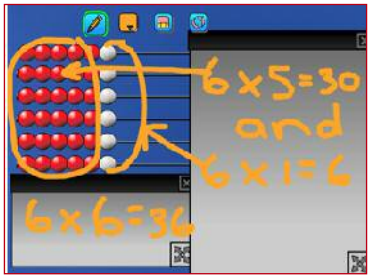
> sin(pi/4)

[1] 0.7071068

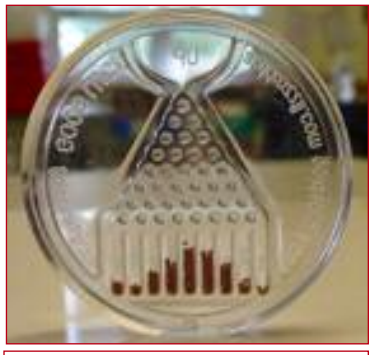
> tan(pi/3)

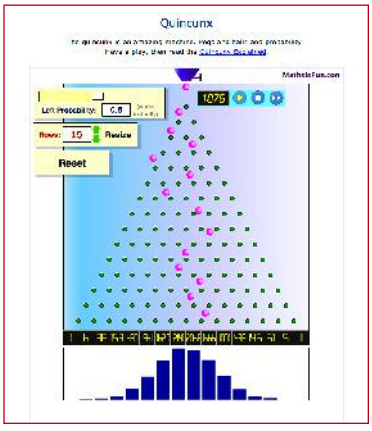
[1] 1.732051

See What's New with CLIPS? There's an App for That! (The Rekenrek Tool App!)



See Really Big Ideas: Adventures in Data







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The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Immaculate K. Namukasa, along with **separate files for each of the text, graphics, and/or photographs**. Please e-mail all files to Immaculate K. Namukasa at **immaculate.namukasa @oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and e-mail address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

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The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

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Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** at the above address. Courier is recommended to avoid possible delays. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

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▲ Editor's Message

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Immaculate Kizito Namukasa is an associate professor at the University of Western Ontario. Her research projects include "Mathematical thinking and activity of elementary and middle school learners," "New teachers learning to teach mathematics," "Expert teachers implementing innovative mathematics methods and environments in their classrooms," "Parents doing mathematics with their children in community settings," and "Complexity research inquiry and critical mathematics education in Canadian and international contexts."

Spring is about imagining new projects and programs. This March issue is about *big mathematics ideas*, what NCTM (**www.nctm.org/standards/content.aspx?id=270**) refers to as *curriculum focal points*. For me, this focus on big ideas, even for young learners (Kajander, 2007), affirms that mathematics education has made and continues to make big strides to embrace mathematics learning beyond what an increasing number of adults experienced when they went to school. Learning school mathematics with understanding was only a vision at the turn of the century, as stated in the NCTM Principles and Standards (NCTM, 2000). (A reader who is unfamiliar with this vision statement might find an executive summary of this vision at **www.nctm.org/uploadedFiles/Math_Standards/12752_exec_pssm.pdf**.) Learning with understanding is now closer to reality in an increasing number of classrooms. Increasingly, more mathematics teachers and school principals are creating ideal conditions for students to engage in broader skills, including mathematics processes of, for example, communication, representing, flexibility in tools and strategies, and even skills of imagination and ingenuity, in addition to the all basic school mathematics skills of mastering procedures and memorizing facts, formulae, and definitions. This is evinced in teaching innovations such as those shared by teachers and educators in professional publications such as the *Gazette* and at professional forums such as OAME and OAME chapter conferences.

One might argue that whereas the *processes of reasoning and proving, thinking, solving problems, communication,* and *understanding* are commonly evoked in contemporary conversations about school mathematics, skills of imagination, creativeness, attentiveness, and ingenuity are still rarely, if ever, evoked in conversations about school mathematics. I wondered about this and so set out to explore how often reference is made to these terms in the Ontario elementary and secondary mathematics curriculum documents. I also examined the curriculum documents for Alberta, another province whose curriculum with which I am fairly familiar. And sure enough, close to 1500, 500, and 200 references were made in the Ontario curricula to *problem solving, understanding,* and *thinking*, respectively. Some reference was made to images, for example, but no reference was made to *imagination* and *ingenuity*. I further pondered as to what this uncommon use of a term such as *imagination* or *imaginative* might imply? Is it that this process of learning has never been a focus of school mathematics?

Before jumping to any conclusion, I set out to explore whether any other variants and synonyms were used to advance mathematical imagination in classrooms, at least as planned in the curriculum documents. I thought to myself that there must be some aspects of imagination outlined by curriculum developers. Or better still, there might be a lot in the current curriculum documents to do with utilizing children and youth's imagination when learning mathematics. Curriculum theorists, such as those who teach curriculum courses at faculties of education, assert that examination of curricula, both as a program of study and broadly as they are implemented and evaluated, is at the core of reforming teaching and learning (Valverde, 2004). I examined curriculum planning and policy documents. Further, I set out to explore what teachers, educators, researchers, and the general public are saying about mathematics imagination by looking for citations on imagination in periodical magazines, both professional, academic, and in the public media. My initial search for imagination in school mathematics did not yield any relevant results. I needed to refine the search query. For *imagination*, for instance, I used the general root/stem imag, adding the truncation symbol "*". The search query *imag** found all articles from academic education and psychology library databases, such as ERIC and ProQuest, with terms *image, imagine, imaginative, imagination, images,* and *imagery*. This worked well when using databases, based on the logic of using the general *roots* together with a truncation symbol to find all variations of a term. The search was powered by both open-source databases and those databases available through faculty of education libraries. Google and Google scholar web databases are built on logic that differs from that of library databases, so I needed a different logic to successfully search these two databases. Further, I limited my search to mathematics publications by using Boolean set logic of combining sets—in the case of ProQuest, using the *and* operator, *math**, and *imag**. As with any search, however, the two root terms *imag** and *math** had a high recall, but low precision. They also yielded columns and articles that used the terms in a literal sense, and that had nothing to do with imagination in school mathematics. Sandieson, Kirkpatrick, Sandieson, and Zimmerman (2009) encourage the use of terms and their combinations that are specific and directly relevant. For publications about school mathematics, the narrowing of the search was easier using additional terms such as *school**, *learn**, and specific math strands and topics of mathematics.

To generate more search terms that were specific to *imagination*, I had to utilize an online visual thesaurus tool, referred to as *Thinkmap visual thesaurus* (**https://www.visual thesaurus.com/**). The beauty of *Thinkmap* is that it generated synonyms for *imagination* in English as well as in five other languages: Dutch, French, German, Italian, and Spanish. The synonyms of imagination fell within several categories including *visual*, concept*/menta*, fancy/fantasy/fantasize, invent, apply, problem-solve, perform/play, present/symbolize,* and *resourcefulness*. Each of these categories had more variants, synonyms, and meanings, depending on the language considered, thus further broadening the list of synonyms to include *estimate, suppose, idea*, concoct, gestate, fantas*, invent*, construct, formulate, underst*, think*, present, creat*, ingen*, engineer,* and *tools*. For the term *ingenuity*, *Thinkmap* generated fewer synonyms: *inventiveness, cleverness, resourcefulness, dexterity, creativity, gizmo, talent, workmanship, skillfulness,* and *handiness*. Many of the synonyms for ingenuity overlapped with those for the term *imagination*.

With this much broader list of meanings for *imagination*, I then returned to the analysis of the curriculum documents and to an advanced search for publications. In the databases, I searched for the generic terms *imag** and *math**, as well as synonyms for *imagination*, generated by the multilingual thesaurus. In

the curriculum documents, much was mentioned about the meanings—*visualizing, thinking, understanding, applying, conceptualizing, estimating,* and *problem solving*, all aspects of imagination. *Visual imagery, visualizations, concepts, mental math,* and *geometrical/transformation images* were the most dominant meanings of *imagination*, which appeared in both curriculum documents and in publications. I only cite a few of the results from the publications on mathematics imagination here.

In Brettingham (2007), the use of drama to teach mathematics is discussed. The author described a workshop put on by a theatre group who performed "A Disappearing Number," and recommended "The Music of the Primes," by Marcus Du Sautoy, also director of the film series, *The Story of Maths*. She shared the names of other theatre groups, who provide drama-based mathematics workshops, incorporation of games into teaching, and use of imagination to bridge gaps in knowledge. Neyland (2004) encouraged incorporating imagination and artistic improvisation into mathematics teaching.

> Students must be encouraged to think for themselves by tapping into their imaginations. Only when mathematical concepts are presented in an imaginative way will students fully benefit from their experience. Both teachers and students must engage themselves in this creative realm of thinking, where imagination plays a vital part in learning and teaching (Jarret, 2006, p. 16).

Generally, several documents focused on *imagery, mental images, visualization,* and *imagine*. Dominant understandings of imagination included those with sources in psychology constructs, such as mental images and use of imagination in relation to the senses and those with referring to mathematics objects, specifically geometry and algebra connotations such as geometry images, inventing images, and imaginary negative and complex numbers. Rare understandings and more recent understandings include digital images, utilization of performative arts in learning mathematics, and use of tools; however, fantasy, design, inventiveness, and manual dexterity were among the non-existent understandings of mathematical imagination. I leave it to you as a reader to list more publications you have reviewed or presentations you have attended, in which rare and more recent understandings of imagination in school mathematics are promoted.

So while the results of my search showed that it is true that *imagination* and *imaginative* are still rarely used variants of *imag** in the mathematics curriculum documents, it neatly showed that some meanings of *imagination* were already part of the conversations in mathematics education policy and periodical documents. The very basic meanings of *imagination*, which focus mainly on mental memorization and mental visualization in mathematics, were at work when most of today's adults went to school. As well, verb meanings referring to mathematics objects of *imagination* and *ingenuity*, such as symbols, images, and representations, are more pervasive than noun meanings referring to processes, such as *symbolize, present,* and *form* images. Fantasy, play, performances, and inventiveness are more recent meanings of imagination seen in mathematics education documents, which have yet to become part of public conversations about school mathematics.

In the *Growing Success* document (Ontario Ministry of Education, 2010), creating, imagining, and innovating; thinking flexibly; responding with wonder and awe are outlined as habits of mind that contribute to success in school and in life. Certain Ontario teachers, educators, and researchers (e.g., Gadanidis & Sinclair, 2008; Kotsopoulos & Cordy, 2009) share how to engage learners in mathematics imagination. The next time you are teaching a new lesson, adopting an idea from a colleague or from a publication, or planning to teaching an expectation differently, you may wish to question the extent to which you are focusing on broader skills of learning mathematics that go beyond a singular focus on mathematical objects of formulae, procedures, answers, and facts to be memorized mentally, to processes that are all at once mental and bodily, and those that extensively utilize tools and learning collectives.

The columns and articles in this March issue are ingenious and encompass the broader meanings of imagination, beginning with those that share utilization of digital technologies in learning: the *President's Message* on blended learning, the *Technology Corner*, the *Data Rocks* columns on the R software, and the *What's New with Clips* column on the Rekenrek tool. Three pieces focus on big mathematics ideas: the equal sign in the *In the Middle* column, dividing by a unit fraction in the *Math by and for Teachers* column, and the bell curve in the article on *Adventures in Data* by Ian Brodie. As is the usual practice, two columns report on recent activities and innovations at NCTM and from the MathEd Forum. Also included is an obituary for Dr. Alexander Laird

Norrie, once an OAME president. And if you have never imagined the mathematics of skeletons and bones, I encourage you to begin by reviewing the *Hey, It's Elementary* column before you proceed to exercise your mathematical imagination with a fractional functions problem in the *What's the Problem?* column. This issue concludes with a position statement by the National Council of Supervisors of Mathematics (NCSM) on the use of manipulatives.

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▲ Call for Manuscripts

The *Ontario Mathematics Gazette* is inviting manuscripts for all grade levels.

Instructions for submission of manuscripts are found on page 1 of any *OMG*.

Contact the Editor for further details.

▲ In Memoriam Dr. Alexander Laird Norrie



On January 31, Dr. Alexander Laird Norrie passed away in Burlington, Ontario at the age of 73. Alex leaves his wife Gloria, his three children James, Alyson, and Andrew, his sister Jean, and numerous grandchildren. An accomplished educator, author and respected mentor, Alex worked for the Peel Board of Education until his retirement in 1995. As Math Coordinator, he had a profound impact on teachers and the way in which mathematics was taught. He served as president of OAME and was very involved with the OAME Board. His successful teaching career in math education continued at Brock University, the Ontario Institute for Studies in Education, and Florida International University. Alex was immensely proud to celebrate 50 years of marriage to his loving wife Gloria this past December, and spoke fondly of their early days spent teaching at Lorne Avenue Public School in London, Ontario, where their story began. During his final days, Alex met the challenges of his medical conditions with great courage and dignity. He will be missed and remembered with much love.

▲ President's Message

SONIA ELLISON
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Even though it is March, it is my first opportunity to wish many of you a "Happy New Year." Have you set any resolutions? Are you sticking to them or are they already a distant memory? I set two New Year's resolutions for myself this year. The first resolution was to eat better and exercise more. Well, I am definitely eating better, but the cold winter has made it difficult to exercise more (and my children's activity schedule!). The second resolution was to think more about math and blended learning. This one I am keeping!

According to the Ministry of Education website, "(B)lended learning uses the tools of the provincial learning management system (LMS) to teach and support learning in a face-to-face class." Blended learning looks different in each class. For example, some blended learning classes have lessons delivered in a traditional manner and make use of the learning management system (LMS) to find due dates, reminders, homework assignments, class notes, and enrichment or remediation materials. In this model, students access the LMS mainly outside of class—before and after school, at lunch, on weekends and holidays, and while absent. Another example of blended learning occurs when teachers have a small bank of computers in their classrooms so that students may take turns using the computers, or they may work in pairs/small groups on some assignments. Some blended-learning teachers have access to enough computers for their entire class. In this case, teachers may deliver the majority of their lessons online and make use of class time to make observations, confer with students, and offer face-to-face instruction to small groups of students.

The benefits of learning math in an online environment are many. As OAME members, we are already aware of electronic resources that support mathematics learning, including CLIPS, www.mathies.ca, Gizmos, and Geometer's Sketchpad®. For me, the power of blended learning is that teachers can increase the usage of these resources and tools in one click. Teachers can provide links to activities and often embed the activities right in the LMS.

I am sure that, like me, many of you are looking for ways to enhance the learning experience for your students. Blended learning provides a vehicle for doing just that. By creating a virtual classroom, we can provide access to resources and learning activities for all students. Blended learning is naturally connected to differentiated instruction. One of the many features of the Desire2Learn (D2L) LMS is the creation of groups. By placing students into groups, teachers can assign developmentally appropriate tasks. But even better than doing work that makes students feel successful, only the students assigned to the group can see what their task is. As a blended-learning teacher, I have created opportunities for my students to be successful, to increase their confidence, and to share their learning in different ways. Blended learning allows me to meet with my students in small groups or individually to ensure concept attainment and promote problem solving.

My students are benefitting from access to learning online. They participate in our class discussions via the discussion tool. They submit their assignments to me via the dropbox tool, receive feedback from me electronically, and can access that feedback whenever they need to do so. My students are also benefitting from the math learning supports provided through the Ontario Education Resource Bank (OERB), which are available for use through D2L. The OERB includes CLIPS activities, OAME-created Financial Literacy lessons, as well as multimedia activities to support student learning for Kindergarten to Grade 12. Homework Help and Gizmos are now directly linked to the class inside D2L. One click provides my intermediate students with access to free online math help.

Blended learning is one way of enhancing the learning experience of my students. I am looking forward to increasing my learning about blended learning, particularly as a way to differentiate instruction to provide enrichment and support for students in both online and face-to-face classrooms. I encourage you to explore ways to enhance student learning in a virtual classroom. As we approach the OAME 2014 Conference, we are reminded to be Champions for Change; I think this includes the classroom locations we use to engage our students in learning.

Citations

Ontario Ministry of Education. (2014). Defining blended learning. Retrieved from **www.edu.gov.on.ca/elearning/blend.html** ▲

▲ Technology Corner – Math Munch, Daily Desmos, and Ontario Math Link

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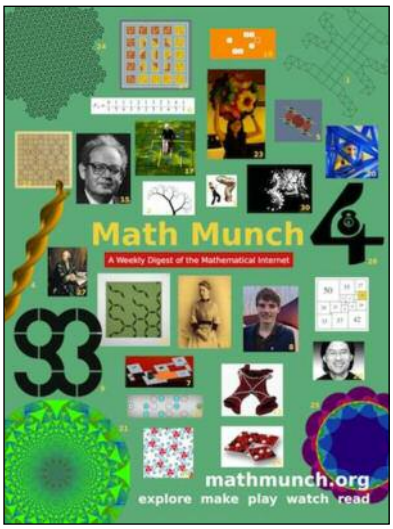


Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

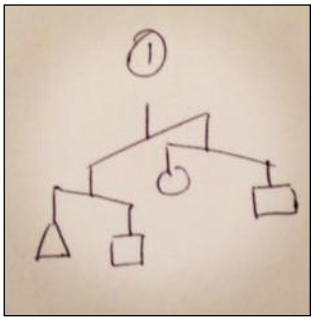
Math Munch

Math Munch (**www.mathmunch.org**) is a free weekly digest, by teachers in the United States of America of meaningful, amazing, and inspiring mathematical finds from the Internet. These include websites, videos, games, articles, profiles of mathematicians, mathematical constructions, and more. The mathematics is accessible to students in middle school and high school, but is of interest to all! You can follow them via Twitter (**@mathmunch**) or sign up to receive the weekly post via e-mail or Facebook.

Math Munch began in October 2011 and has been sharing mathematical gems ever since. It is curated by three math ambassadors, who happen to be math teachers—Justin Lanier, Paul Salomon, and Anna Weltman. They are passionate about mathematics and about ensuring that their students have a positive relationship with math. They wanted their students to find something about mathematics to love, and through Math Munch, they share that vision for your students too.



All posts are tagged by topic so you can easily search for something in particular. For example, if I search for "geometry," a number of interesting posts come up. Tags representing favourite posts are also shown on the right sidebar.



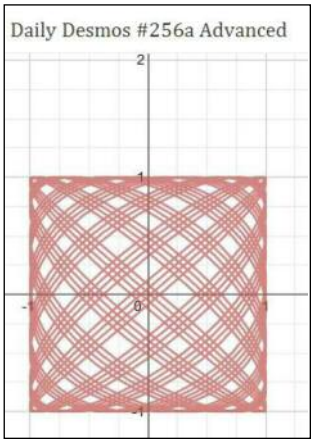
What got me hooked on Math Munch was Paul Salomon's weight puzzles featured in the post "Sam Loyd, Weight Problems, and Exercises." He was inspired by puzzles where the weights on either side must be equal. However, Paul created imbalance puzzles, where the goal is to order the shapes by weight. A simple example is shown. This post includes a link to Paul's website that has many more of these fun puzzles.

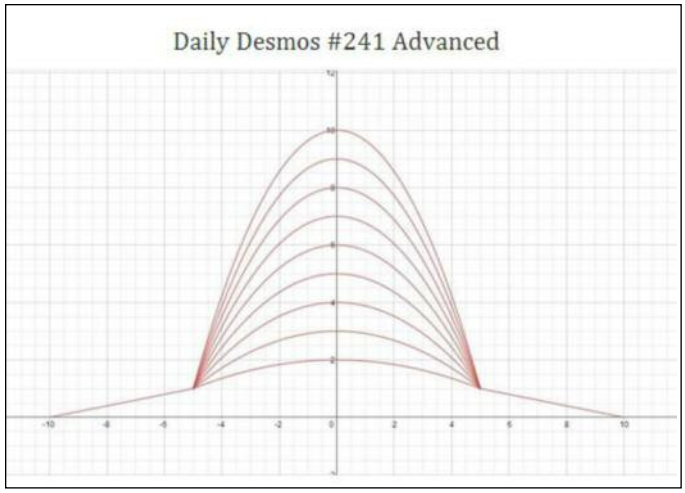
A section of the Math Munch website is dedicated to teachers. They want you to share Math Munch with your students and provide resources to help you. Students can submit their questions after reading a post or watching a video. There are reflection sheets to help students get more out of Math Munch. There are also beautiful posters for you to print and post in your classroom.

I encourage you to take some time to check out all the great information, games, and videos on Math Munch, and to share what you learn with your students.

Daily Desmos

If you read my column regularly, you know that I'm a huge fan of Desmos graphing software (**www.desmos.com**). (By the way, you can now insert a picture as the background of your graph window!) A number of teachers who also love Desmos started Daily Desmos. Two graphing challenges are posted every weekday. One is a basic graph (blue), the other a more advanced graph (red). Users, students or teachers, are challenged to see if they can recreate the graphs, using

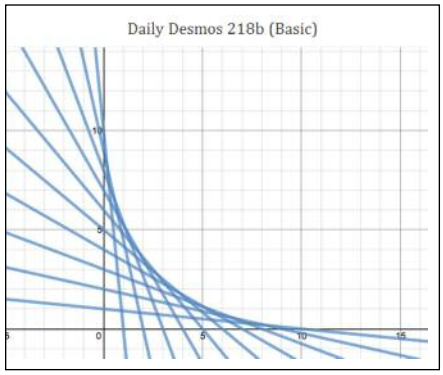




Desmos or some other graphing tool. In the Fall, most of the basic graphs were linear, to allow a wider group of students to attempt to match the graphs. This is a great class starter activity or even one to leave as an emergency plan for a supply teacher. Some of the graphs are quite challenging and provide a launching pad for learning new mathematics. Shown are a parametric example, a linear example that my students really enjoyed, and a third one with piecewise linear and quadratic functions. Other graphs also include animation, thanks to sliders! Daily Desmos is fun and rewarding—try it out!

Ontario Math Links

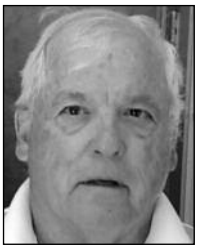
Every week, David Petro (**@davidpetro314**) publishes a summary of Ontario Math Links—web links that pertain to Grades 7–12 mathematics in Ontario. He tweets out the link to his blog



(**www.ontariomath.blogspot.ca**) for his Twitter followers, but you can also follow by e-mail. Each post has interesting information that is all tagged by Ontario course code. This means that you can search for a particular course or topic. These posts represent a fantastic collection of great activities and materials you can use in your classes. Be sure to check it out! ▲

▲ OAME/NCTM Report

TODD ROMIENS
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Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a past president of OAME and a life member, as well as a previous editor of the Gazette.

NCTM's *Illuminations*, funded by the Verizon Foundation is a very large resource for teaching mathematics. It can be accessed at **Illuminations@nctm.org**. Illuminations also has a new website that provides lesson plans, brainteasers, games for teaching math, and stories about teacher success in the use of NCTM materials. Illuminations provides access to thousands of free digital resources for K–12 teaching, using Thinkfinity (**www.thinkfinity.org**), a Verizon-generated resource. This website contains lessons on most mathematics topics at all grade levels.

NCTM has also just launched a series of 19 videos that demonstrate classroom procedures for handling students who are unco-operative in class and who are not prepared to learn. These teaching channel videos may be found at **https://www.teachingchannel.org/blog/2014/01/24/video-playlist-classroom-management**. In these videos, experienced teachers provide tips on how to handle such situations, ideas for classroom discipline, and ways to assist other students. Titles include:

- *Staying Calm: A Classroom Management Strategy*
- *Classroom Management Strategies to Take Control of Noisy Students*
- *Time Management*
- *Classroom Management Strategies – Consequences as an Effective Classroom Management Strategy*

Using the Illuminations website, teachers can also access over 40 new lesson plans covering most of the current curriculum. Each lesson provides a context, a plan, an activity sheet with answers, and ideas for expanding the lesson.

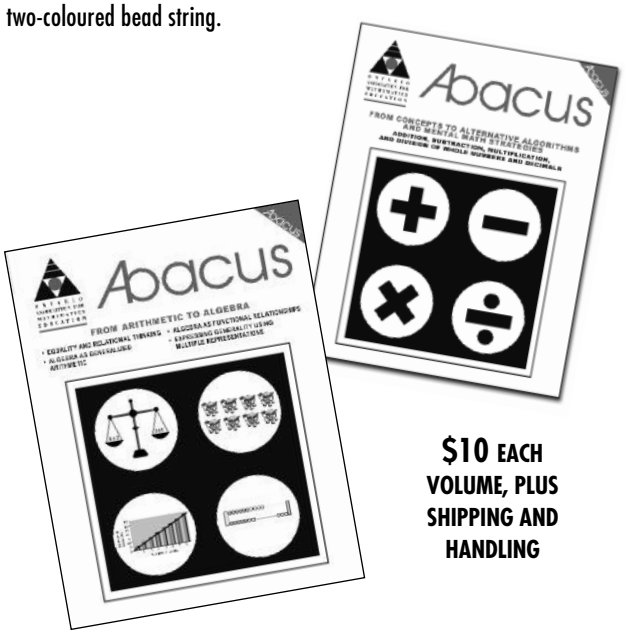
For example, one lesson, called "The Crow and the Pitcher: Investigating Linear Functions Using a Literature-Based Model," is based on Aesop's fable, "The

Crow and the Pitcher." It involves students making predictions and conducting experiments to determine the number of pebbles the crow would need to add to the pitcher of water in order for him to be able to drink it. Reference Aesop's fable for the exact story and moral. The activity sheet with answers can be found at **http://illuminations.nctm.org/Lesson.aspx?id=3667**.

▲

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Volume 46 – This resource for teachers includes the concepts of addition, subtraction, multiplication, and division of whole numbers and decimals, alternative algorithms, mental math strategies, manipulatives, and links to literature. It will be useful for exploring these aspects of teaching and learning personally or with a team of teachers. Includes one two-coloured bead string.



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▲ WHAT'S THE PROBLEM? FRACTIONAL FUNCTIONS

SHAWN GODIN
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Shawn is the head of Mathematics, Business Studies, and Library at Cairine Wilson Secondary School in Orleans, Ontario. He is a past member of the OAME Board of Directors and is currently Editor-in-Chief of the Canadian Mathematical Society's problem-solving journal, Crux Mathematicorum.

Welcome back, problem solvers. Last time, I left you with the following task:

A sequence of functions is defined recursively by $f_n(x) = f_0(f_{n-1}(x))$, where $f_0(x)$ is some function. For example, if $f_0(x) = x^2 - 1$, then,

$f_1(x) = f_0(f_0(x))$
$= f_0(x^2 - 1)$
$= (x^2 - 1)^2 - 1$
$= x^4 - 2x^2$

and

$f_2(x) = f_0(f_1(x))$
$= f_0(x^4 - 2x^2)$
$= (x^4 - 2x^2)^2 - 1$
$= x^8 - 4x^6 + 4x^4 - 1$

and so on.

and so on.

1. Given, $f_0(x) = \dfrac{1}{1-x}$ determine $f_{2003}(x)$.

2. Determine another sequence of functions, defined by $g_n(x) = g_0(g_{n-1}(x))$, where $g_0(x)$ is some function, that behaves similarly to the sequence of functions in 1.

This problem was developed as a problem-solving exercise for Grade 11 students investigating recursive sequences. It was inspired by a number of problems from different mathematics contests.

Let's deal with part 1 first. Starting with the original function, we get

$$f_1(x) = f_0\left(\frac{1}{1-x}\right) = \frac{1}{1-\dfrac{1}{1-x}} = \frac{x-1}{x}$$

Continuing the process, we get

$$f_2(x) = f_0\left(\frac{x-1}{x}\right) = \frac{1}{1-\dfrac{x-1}{x}} = x$$

Hence, $f_3(x) = f_0(x)$ and the sequence becomes

periodic with $f_0(x) = f_3(x) = f_6(x) = f_9(x) = \cdots$, $f_1(x) = f_4(x) = f_7(x) = f_{10}(x) = \cdots$, and $f_2(x) = f_5(x) = f_8(x) = f_{11}(x) = \cdots$. Note that $2003 = 3 \times 667 + 2$; hence, $f_{2003}(x) = f_2(x) = x$.

We can describe this relationship in another way. Since $f_0(f_0(f_0(x))) = f_2(x) = x$, each three copies of the function "undo each other," so by definition, f_{2003} is made up of 2004 copies of the original function. When we "eliminate" copies of the original function in threes, they all are eliminated, and we are left with the identity function.

For the second part, we could try the function $g_0(x) = \dfrac{1}{ax+b}$, for some constants a and b. After a little bit of algebra, we would discover that when we set $g_2(x) = x$, we get $a = -b^2$. Thus, every choice of b yields a function with the desired property (note that $b = 1$ gives us the original function f_0).

If we had done a bit of thinking, we could have solved the problem without doing any calculations. If we let $g_0(x) = f_1(x) = \dfrac{x-1}{x} = f_0(f_0(x))$, then $g_1(x) = f_0(f_0(f_0(f_0(x)))) = f_0(x)$ and, finally, $g_2(x) = f_0(f_0(f_0(x))) = x$.

The problem can be extended in many different ways: you can explore sequences with different periods, or you can examine the sequence generated by the starting function $h_0(x) = \dfrac{ax+b}{cx+d}$. Have fun!

Now it is time for your homework:

A man was visiting his friend and noticed that there were a number of children playing in the backyard. When he asked his friend if they were his, he stated that his children were playing with children from three other families. The friend explained that the numbers of children satisfied the following conditions:

- There were fewer children from family A than from his family.
- There were fewer children from family B than from family A.
- There were fewer children from family C than from family B.
- There were fewer than 18 children in total.
- The product of the number of children from each family is equal to the friend's house number.

The man took to calculating, but he didn't have enough information. He asked his friend if family C had more than one child. As soon as his friend answered the question, he easily determined how many children from each family were present.

Using only the information given above, it is possible for you to also determine the number of children in each family.

Until next time, happy problem solving! ▲

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▲ Hey, It's Elementary – "Watson," said Sherlock, "Can you believe this is Math?"

LYNDA COLGAN
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Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.

> *"Instead of trying to make children love math they hate, make a math they'll love."*
>
> ~ Seymour Papert, 2002

The recent release of student achievement data by the EQAO and the OECD has generated widespread discussion about the decline in provincial and national mathematics scores, respectively. Vociferous concerns have been raised about the quality and appropriateness of current curriculum, contemporary instructional methods, and the level of teachers' mathematical-content competence. While the achievement scores should be of concern, the attitudinal scores—which show an equally disturbing negative trajectory—unfortunately, have garnered less attention and raised fewer alarm bells.

There is a wealth of research to suggest that students' positivity about, engagement with, and appreciation for a subject can impact their feelings of self-efficacy, confidence, and perseverance. Given that "new advances in understanding the human brain [point out that it is] not a purely cognitive information processing system, but a system in which affective functions and cognitive ones are inextricably integrated with one another" (Picard, Papert, Bender, Blumberg, Breazeal, Cavallo, Machover, Resnick, Roy, & Strohecker, 2004, p. 253), it would seem that a response to declining achievement scores should attend to students' motivation, emotion, interest, and attention, as well as the relationship that the learner has with the content and process of learning.

The most obvious starting point to improve student engagement and investment is to embed mathematics within an appropriate and fascinating context: making the tasks realistic, relevant, and stimulating, and the focus, fresh and purposeful.

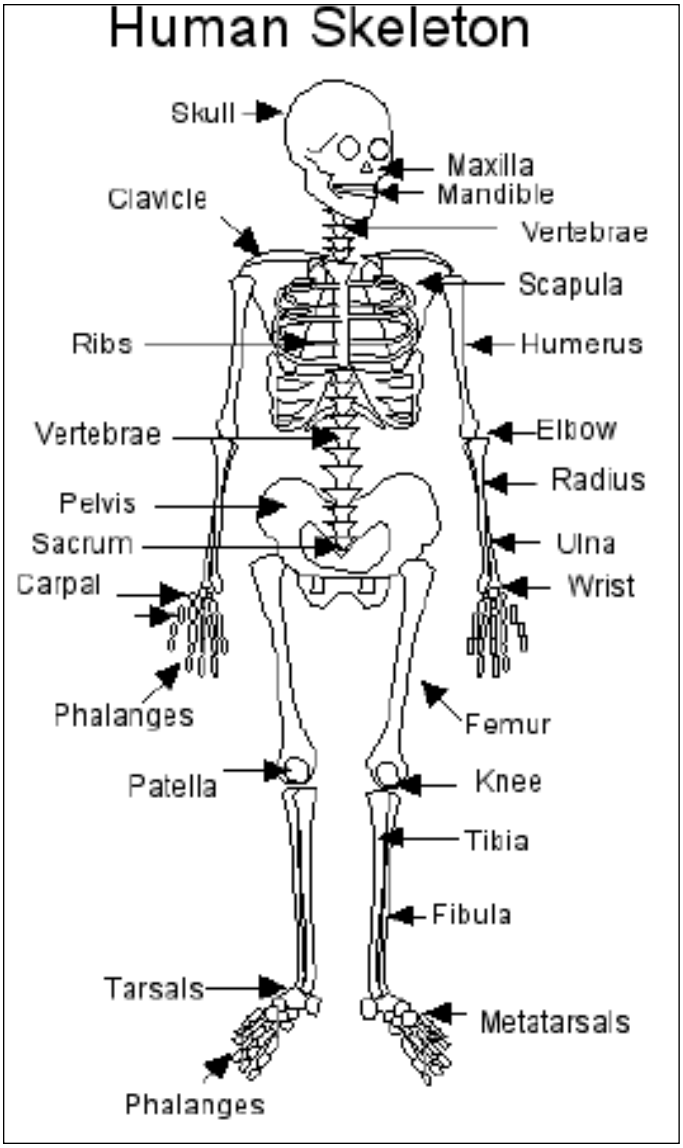
As an example of a mathematics-rich context, I present some anthropometric examples. Anthropometry, the "measurement of people," is a word derived from the Greek "anthros," meaning *man*, and "metron," meaning *measure*. Anthropometry is the study of the size, shape, and strength of the human body and includes measures of mass, volume, and proportions of both the whole body and body parts.

Why anthropometry? Because it is at the heart of popular culture and a gold mine in terms of authentic exercises that require fractions, percents, proportions, and measurement.

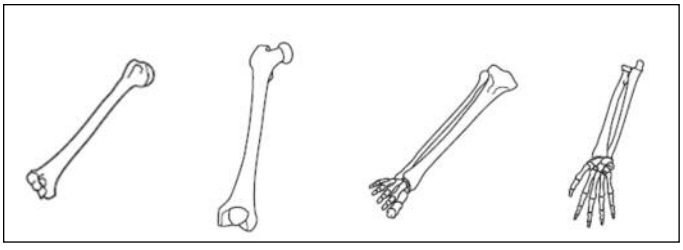
Television programs like *CSI* and *Bones* often begin with the discovery of skeletal remains in a remote location. The first expert to be called by the police is usually a Forensic Anthropologist trained in Osteology (the study of bones), who needs to determine whether the evidence is human and, if so, the gender, age, physique, and race of the person. Because our bones are living, growing tissues that respond to the factors affecting our life—nutrition, health, degree of physical activity, age, and genetic factors such as sex and race—they can be considered to be an unwritten diary of our life. In fact, it is precisely because there is such an abundance of information about each of us recorded in our bones that a large amount of forensic evidence can be obtained from skeletal remains found at the scene of a crime.

There is an enormous body of scientific knowledge required to be a skeleton sleuth, and these few examples cannot prepare you to be the next Sherlock Bones—but they do provide some insight into the tales that dead men (when assisted by math) can tell.

As an adult, your skeleton is made up of 206 bones, whether you are male or female; however, the size and weight of bones of female skeletons is generally smaller and lighter than those of males. In general, the weight of your skeleton accounts for 20 percent of your total body weight.



Determining the exact physical dimensions of a victim of a crime is one key to unlocking the identification of the victim; therefore, when a skeleton is found, one of the first things a forensic scientist does is to use the lengths of certain bones to calculate the height of the living person. The bones that are used are, from left to right, the humerus (H), the femur (F), the tibia (T), and the radius (R).



For example, your upper leg contains a large, single bone called the femur. This long bone stretches from the hip (pelvis) socket to the kneecap (patella) and is generally considered to be the longest, strongest, and heaviest bone in the body. The length of this bone can be used to roughly estimate a person's height—the femur is approximately 1/4 of a person's adult height.

Forensic scientists require more accurate methods to calculate height; therefore, they use tables and formulas that have been developed using thousands of samples to arrive at a precise relationship between bone length and height, taking into account many factors, including gender and race. For example, if you measure the length of your femur in centimetres, multiply that measure by 2.6, then add 65 to this number, you should arrive at your approximate height in centimetres. If you prefer the Imperial System of measure, you can divide your final answer by 2.54 to get an approximation in inches. How close was the calculation of your approximate height to your exact height to your first approximation, using the 1/4 rule?

To get a better approximation, it helps to add your gender to the equation. If you are female, you multiply the length of your femur by 2.47, then add 54.1 to the answer to get your height in centimetres. If you are male, you multiply the length of the femur by 2.32, then add 65.53 to get your height in centimetres.

It is also possible to infer a person's height from a measure of the humerus, the single large bone that extends from the elbow to the shoulder socket. If you are female, multiply the measured length of your humerus (in centimeters) by 3.06, then add 64.26 to this number. This final number is your approximate height based upon the length of your humerus. If you are male, multiply the measured length in centimetres by 3.279, then add 59.41 to this number to calculate your approximate height based upon humerus length.

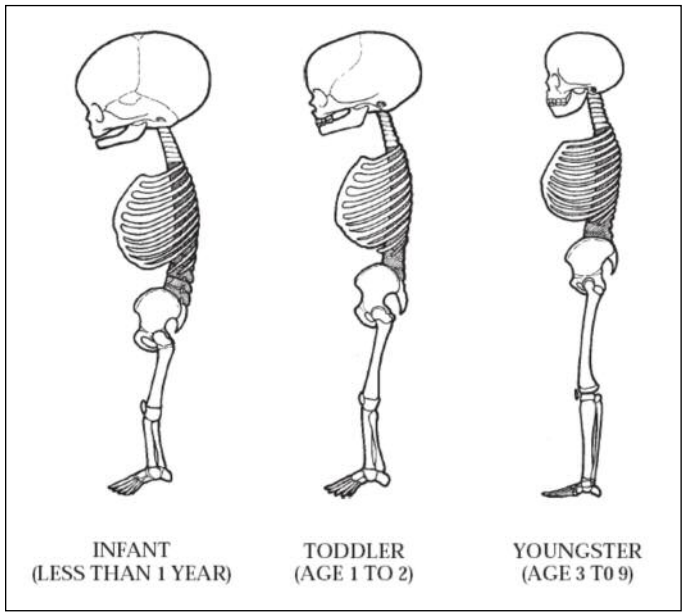
Scientists have discovered that the length of the tibia (the larger central bone of the lower leg that extends from just below the kneecap to the ankle) may also be used to calculate a person's height. For males, multiply the tibia length by 2.41, then add 81.93 to get an approximate height in centimetres. For females, multiply tibia length by 2.9 and add 61.53.

And since two bones are better than one, you can combine the measures for two of the long bones into one equation in order to infer a person's height: Height = 1.31 x (length of femur in centimeters + length of fibula in centimeters) + 63.05.

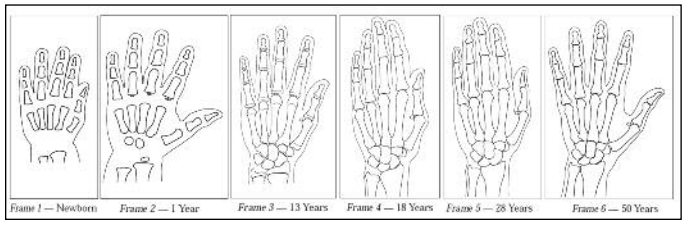
There are also growth and development changes in the skeletal system between infancy and adulthood. It is known that bones grow in length and thickness during

part of our life, and that we lose bone in other phases. And while both heredity (genetics) and environmental factors (health and nutrition) contribute to the overall development of bone, there are some general facts about skeletons that appear to be true to everyone.

For example, as we age, our head becomes proportionately smaller. As an adult, the length of our head is about 1/7 of our body height, whereas in infants, the head is about 1/4 of the body length.

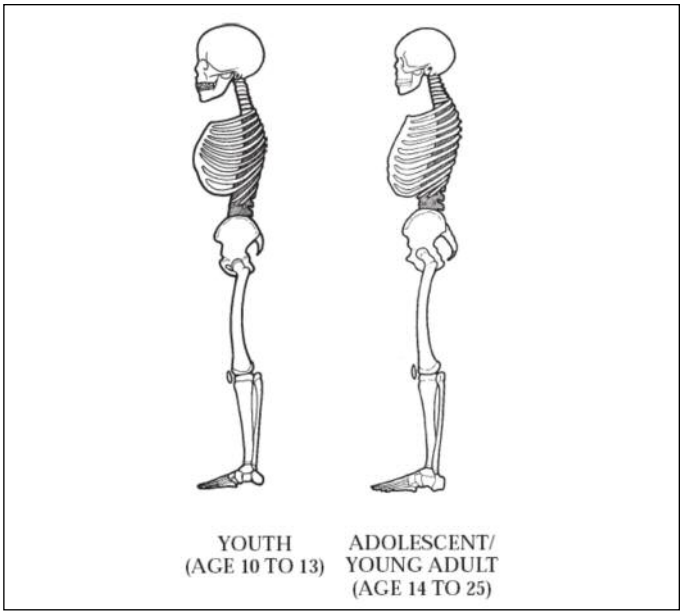


As we can see from the sequence of drawing of the hand skeleton, there are a group of bones that are completely absent at birth, but gradually appear as the hand ages.



Throughout childhood and adolescence, there is a dramatic growth spurt in the length of long bones. New bone is added to skeleton faster than old bone is removed. Therefore, bone grows larger, heavier, and denser. Although long bone length growth ends at about 18 years of age in females and 21 years of age in males, bones are still continuing to grow in width.

An examination of the progression in female hand skeleton X-rays clearly illustrates how spaces (where growth plates of cartilage are found) gradually change to bone (a process called *ossification*). Over time,

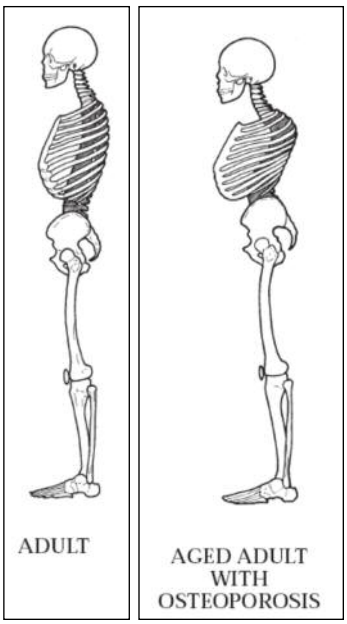


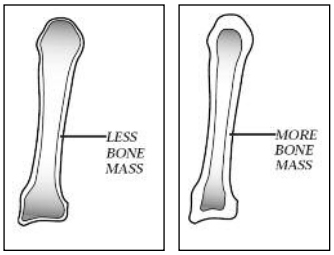
ossification centres appear in the spaces at the ends of the phalanges, carpals, and metacarpals, which will fuse or join with the shaft of long parts of the bone. This fusion will be complete by the age of 18 and growth in length of the bone will end. Can you determine the approximate age of the mystery hand by examining the other skeletons?



When we reach adulthood, we also reach a period of peak bone mass—a time when we enjoy maximum bone density and strength.

After the age of 30, the height of a person begins to decrease at the rate of approximately 0.06 cm per year. Over time, the process of bone removal begins to overtake bone replacement; however, if the rate of bone removal is too quick, our bones become weaker and weaker—a situation that may cause progressive spinal deformity. In the case of the aged adult skeleton shown below, it is





clear that extreme bone loss may cause the lower ribs eventually to rest on top of the pelvis. In fact, skeletal mass may be reduced to half what it was at age 30.

Now that you are so well informed about bones, perhaps the next time someone tells you that they may have dug up a skeleton, try running toward it, not away from it, and test out your new knowledge to determine its age and height. As you now know,

> *"…there is a brief but very useful and informative biography of an individual contained within the skeleton, if you know how to read it."* ~ Clyde Snow

Answer: The X-ray drawing is from a 6-year-old and 10-month-old female. You should have placed this photo between frame 2 and frame 3.

(All illustrations used with permission from the *Positively Aging* unit, produced by the University of Texas Health Science Center at San Antonio)

Reference

Picard, R.W., Papert, S., Bender, W., Blumberg, B. Breazeal, C., Cavallo, D., Machover, T., Resnick, M., Roy, D., & Strohecker, C. (2004). Affective learning – a manifesto. *BT Technology Journal, 22*(4), 353. Retrieved from **www.media.mit.edu/publications/bttj/Paper26Pages253-269.pdf** ▲

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▲ In the Middle – Misconceptions About The Equal Sign in Middle School

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Mirela Ciobanu is currently a middle school teacher. She has been teaching for over 12 years, 9 of which have been with the Toronto District School Board. Mirela is a keen advocate of classroom inquiry, particularly in the area of formative assessment. Her recent work includes conducting an action research in mathematics as part of the Teacher Learning and Leadership Program for Experienced Teachers, developed and funded by the Ontario Ministry of Education in partnership with the Ontario Teachers' Federation.

As a middle school mathematics educator, I have witnessed students making the following types of errors on numerous occasions, year after year:

a. Performing calculations which are connected by equal signs in a string, e.g., $21 \div 7 = 3 \times 5 = 15 + 25 = 40$ and
b. Solving an equation such as $6 + 4 = y + 5$ and finding that $y = 10$

Helping students clarify their understanding and use of the equal sign seems like an easy task at first, but any teacher who had a similar opinion soon came to an understanding that this is a much harder endeavour than initially believed. Research studies about the misuse of the equal sign among middle school students indicate that it takes consistent dialogue and explicit instruction over an extended period of time to help students understand the meaning of the equal sign (e.g., Carpenter, 1999; Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012).

Both types of errors above indicate that the student sees the equal sign as an operator, merely having the purpose of announcing the answer. The first example of misuse is one that I see constantly displayed by my students every year, mostly when solving problems or performing multiple operations. It shows a clear disregard for the equal sign's meaning and functionality. The second one is more straightforward and is a

potential cause of the first one. The student's inability to solve the equation is caused, just like in the first example, by the view that the equal sign is considered to be the symbol "following the calculation to the left" or the symbol "announcing the answer." Therefore, the student is used to performing that calculation and recording the answer *after* the equal sign.

Knowledge of the Equal Sign in School

Rittle-Johnson (2011, 2012) has created a construct map for knowledge of the equal sign, detailing four levels of understating of the equal sign among school-age children, ranging from the basic *rigid-operational* (level 1) to the *comparative relational* (level 4).

Students who display a *rigid operational* level of knowledge can only see the equal sign as an operator, preceding the calculation, often presented to the left. They are only successful with equations when the calculation is on the left.

Level 2 describes the flexible operational type of knowledge. These are students who still believe that the meaning of the equal sign is "get the answer to the question," but they can successfully calculate equations where the calculation is either to the left or to the right of the equal sign, e.g., $21 = 8 + x$ or $8 + x = 21$.

Level 3 is the *basic relational*. These are students who can solve equations involving operations on both sides of the equal sign and recognize the equal sign's relational meaning.

Level 4 describes the *comparative relational*. These students recognize the relationship of equivalence between the two sides of the equation, and can employ various strategies to solve tasks such as, "Without performing the calculations, can you explain why $56 + 80 = 55 + 81$?"

What Causes the Misconceptions about the Equal Sign?

Exposing students mainly to operations on the left, followed by the answer to the right of the equal sign, contributes to establishing an *operational* view of the equal sign. Some researchers have even tried to look closely at the formats of equations students are exposed to in mathematics textbooks (e.g., Capraro & Capraro, 2010; Powell, 2012) and even link the use of modern calculators (Capraro, 2010) to the development of the operational view of the equal sign. The *equal* key is the one pressed to perform the action of the mathematical operation, and thus, it contributes to engraining in students' minds that the equal sign is the one responsible for performing the operation and not the actual operator key.

Implications of the Equal Sign Misuse by Middle School Students

There is no doubt that understanding the equal sign is key to the development of algebraic thinking. Researchers have discovered that the format of the equations students are exposed to the most in primary school account for the difficulties they have later on solving equations and using algebraic thinking and reasoning. When given the standard *calculation-equals-answer* equations, students are capable to solve them 90 percent of the time (Rittle-Johnson, 2011, 2012; Falkner, Levi, & Carpenter, 1999). Once the format becomes *answer-equal-operation*, students have more trouble solving them correctly. When given equations with *operations on both sides of the equal sign*, the students found the most difficulty with this. In general, when variables replace blanks, students have considerably more difficulty solving the equations correctly, even though they could solve the same equations with blanks.

It is obvious, therefore, that because students struggle with understanding the meaning of the equal sign, they will have a hard time using it when trying to solve problems using algebraic thinking. Formulating equations as statements in certain problem-solving contexts becomes an impossible endeavour.

What Can Teachers Do to Help Their Students Understand and Solidify the Relational Meaning of the Equal Sign?

Knuth, Alibali, Hattikudur, McNeil, and Stephens (2008) suggest the following actions that a teacher can take to develop students' understanding of the equal sign:

1. Consistent dialogue and explicit instruction around the use of the equal sign
2. Use of equation strings to emphasize equality between both sides of the equal sign. Example: $9 + 9 = 1 + 17 = 2 + 16 = 10 + 8$
3. Equivalent equations (e.g., if $3 + 5 = 6 + 2$, then $3 + 5 – 2 = 6 + 2 – 2$). This type of equation emphasizes the idea of preserving the equality on both sides of the equal sign.
4. Equations of the format *variable-equals-calculation* or with *operations* on *both sides* (non-standard equations), such as $10 + 5 = 8 + 7$

(Falkner, Levi, & Carpenter, 1999)

Researchers agree that developing the relational understanding of the equal sign has to be reinforced throughout the elementary school years. Establishing the relational understanding starts as early as Kindergarten and primary school, where students use concrete materials such as **balances** or drawings and pictures of a balance as a visual model of an equation (Small, 2013; Ontario Ministry of Education, 2005) to compare equivalent quantities. Here are two of the Grade 1 "Expressions and Equality" specific expectations and suggested problems that involve the use of the balance to develop the relational meaning of the equal sign:

- Demonstrate examples of equality, through investigation, using a "balance" model. (*Sample problem:* Demonstrate, using a pan balance, that a train of 7 attached cubes on one side balances a train of 3 cubes and a train of 4 cubes on the other side.)
- Determine, through investigation using a "balance" model and whole numbers to 10, the number of identical objects that must be added or subtracted to establish equality. (*Sample problem:* On a pan balance, 5 cubes are placed on the left side and 8 cubes are placed on the right side. How many cubes should you take off the right side so that both sides balance?)

One of my favourite teaching resources, and a reference that does not leave my desk, is Randall Charles' (2005) article about the importance of teaching students about the big ideas in mathematics. Emphasizing these with consistency helps build a strong foundation of the concept of equality. Here is the big idea about *equivalence* (p. 14):

"Big Idea #3 EQUIVALENCE: Any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value."

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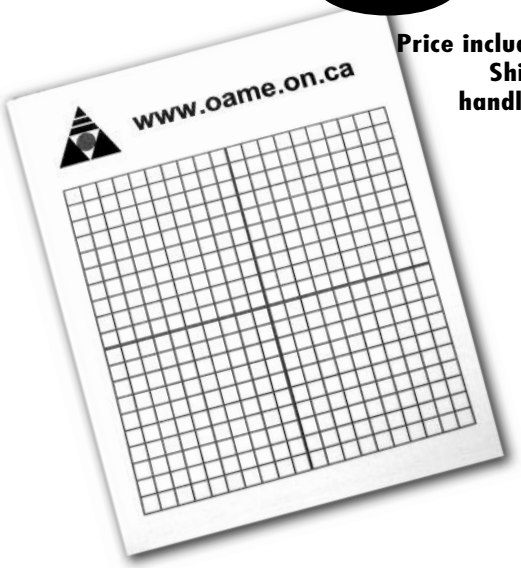
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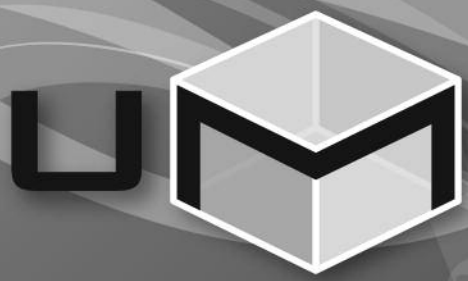
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▲ Data Rocks – Another R for Public School Students to Learn

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John Braun is a professor of statistics and has taught courses in the area since 1992. His research interests are in the modelling of environmental phenomena as well as statistical education, particularly as it relates to the R programming language.



Bethany J.G. White is an Assistant Professor in the Statistical & Actuarial Science department at the University of Western Ontario, the President-Elect of the Statistical Society of Canada Education Section, and an Associate Editor for the Journal of Statistics Education.

What do wildfire, coin flips, and Monopoly have in common? Each exhibits randomness, and each is featured in several school classroom demonstrations conducted by the authors in the London area (Braun, White, & Craig, 2013). One objective of the demonstrations was to pique student interest in data science, which is quickly gaining prominence in science and industry; another objective was to introduce students to a statistical program, R, which to date has been exclusively in the domain of university researchers.

The recent Ontario Government announcement, highlighting the need to improve student math skills in the province, included a recommendation to explore "how technology may be used to improve teacher instruction and student understanding of mathematics." R is a powerful technological tool that could have a substantial effect on student understanding of data-management concepts and calculations, and that is already available to everyone free of charge. The only cost involved is a learning curve, which initially appears steep, but which flattens out quickly. Furthermore, recent enhancements to the program have led to the possibility of simple-to-use apps that can be used on any kind of web-accessing device. A small, but enthusiastic, team of researchers at the Universities of Toronto and Western Ontario are working on app development and are creating a website that may virtually eliminate the need to learn the R program. The web page and the apps will be described in a future article. The purpose of this article is to provide a quick introduction to R and a sense of its potential in Math education.

R (R Core Team, 2013) is an open-source computer program, designed to do statistical calculations. Originally conceived as a vehicle for statistical methodological development exclusively by university researchers, R has evolved into a general-purpose, highly flexible statistical program that is used in industry and government. It is also now a standard part of undergraduate statistical programs and is spreading to other disciplines. It's certainly not limited to use in higher education. It can also be used to do elementary and high school math calculations. Although most obviously useful in data management, it can also be used as a calculator and graphing tool.

R is available online from the CRAN (Comprehensive R Archive Network). In order to use R, you first need to install it.

1. Use your Internet browser to link to **http://probability.ca/cran/bin/windows/base/**. (Mac users should remove bin/windows/base/ or replace with Mac OS X.
2. Click on the link to download the program, and choose "Run" when asked.
3. Normally, there is no reason to customize the installation, so click "Next" each time you are prompted. This might happen six or seven times.

When the program is installed, an "R" icon appears on your desktop. Clicking on it starts the program, and a window will open with a terse verbal description of the program and ending with the symbol >.

The > sign tells you that R is ready for you to type in a command. There are thousands of possible commands, which explains why the learning curve appears steep. Fortunately, many of the commands are very intuitive. For the beginner, it is best to start with R as a calculator.

1. A Taste of R

To reproduce the following examples, simply type the command (shown in blue) into the R window. Execution of the command is accomplished by hitting the "Enter"

key. The "answer" appears in red following the [1].

Arithmetic operations are as expected.

> 23 + 49

[1] 72

*> 11111*11111 - 21*

[1] 123454300

> 1234567654321/1111111

[1] 1111111

Exponents and fractional exponents are available.

> 9^3

[1] 729

> 12345678987654321^.5

[1] 111111111

Trigonometric and other special functions are available as well. Special constants such as π and *e* are also built in.

> pi

[1] 3.141593

> sin(pi/4)

[1] 0.7071068

> tan(pi/3)

[1] 1.732051

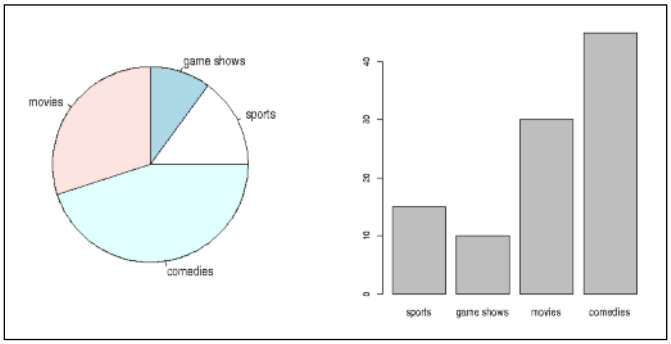
2. Graphics

We will demonstrate pie and bar charts for data on the amount of time spent by a person watching four different types of TV shows. Fifteen percent of the time was spent on sports, 10 percent on game shows, 30 percent on movies, and 45 percent on comedies.

The charts are constructed in two steps. First, the data are stored in R and called "tv." Then the pie function is applied, followed by the bar-plot function.

> tv <- c("sports" = 15, "game shows"= 10, "movies" = 30, "comedies" = 45)

> pie(tv)

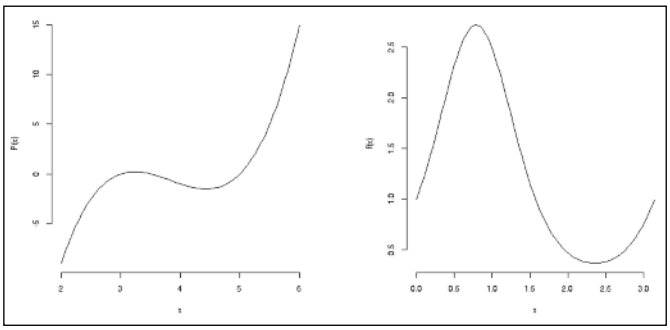


Curves of mathematical functions can also be drawn. For example, sketching the graph of the polynomial $P(x) = (x - 5)(x - 3)(2x - 7)$ on the interval [2, 6] can be accomplished using

> curve((x-5)(x-3)*(2*x-7), 2, 6)*

Trigonometric and exponential functions can also be graphed easily. For example, the following plots the function $f(x) = e^{\sin(2x)}$ on the interval [0, π]:

*> curve(exp(sin(2*x)), 0, pi)*



3. Final Comments

In this brief article, we have barely begun to scratch the surface with R. The program can be used to do almost any kind of numerical mathematics or statistics imagineable.

As stated earlier, it is used increasingly in industry, even by technology giants such as Google. It is being adopted by engineers, scientists, financial analysts, and educators. Therefore, exposure to R at the school level introduces students to a program that they could very likely use later in their careers.

To learn more about R, there is a lot of information on the Web. Unfortunately, not all is easy to read, so if something looks incomprehensible, leave it and look for something else. One useful site is **www.burns-stat.com/pages/Tutor/hints_R_begin.html**.

For those who find the command-line structure intimidating, our research team will be introducing R Tricks, a collection of apps that will exploit the power of R, while providing an easy-to-use interface. For those who can't wait for the launch of the official site, there is an experimental site with several apps for simulation and graphing at **www.stats.uwo.ca/faculty/braun/RTricks /RTrixApps.php**. We have tried some of these out in our classroom visits with great success. Use of the apps facilitated fun and meaningful activities about data and probability, and teachers have been very impressed with how easy they are to use. It is our hope that these successes can be built upon and that all students in the Ontario secondary education system can soon enjoy the benefits of R.

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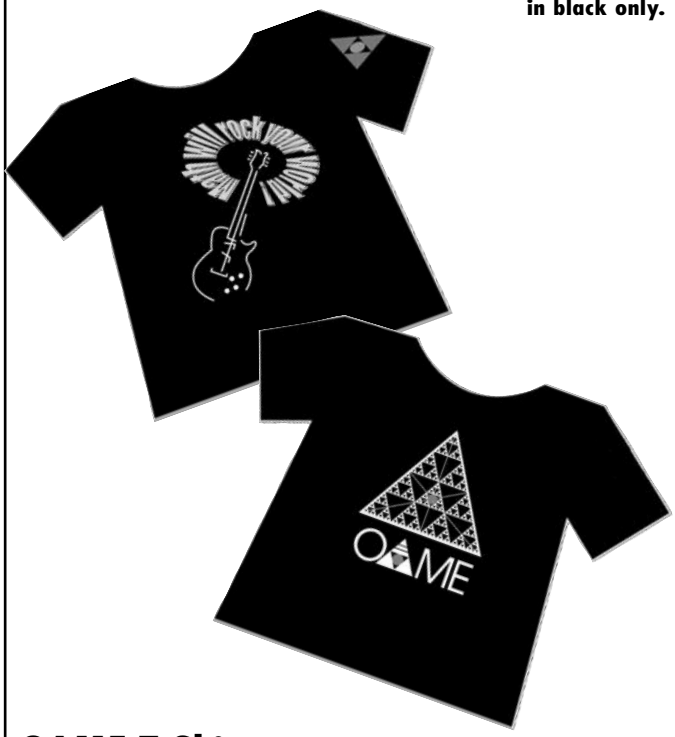
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▲ FIELDS INSTITUTE – MATHED FORUM NOTES

STEWART CRAVEN
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Stewart Craven presently teaches courses for the Continuing Education Department and the Transition Years Program at York University. He is an active participant in the Fields Mathematics Education Forum.

On January 25, our first meeting of the year was devoted to the annual "research" day. The research day was introduced to highlight the depth and breadth of the work done by education professors and graduate students across Ontario and in some cases, beyond our provincial borders. The following is the list of speakers, their affiliations, and their topics.

Long Presentations:

1. Chris Suurtamm (University of Ottawa) and Martha Koch (University of Manitoba) – "Reconceptualizing the mathematics curriculum as a dynamic, co-created network"
2. Tina Rapke (York University) – "Students and their instructor co-developing a final closed book math exam"

Short Presentations:

3. Ariel Wan-Hua Ho & Joanne Lee (Wilfrid Laurier University) – "Promoting purposeful play through LittleCounters program"
4. Shelley Yearley, Catherine Bruce, Tara Flynn, & Rich MacPherson (Trent University) – "Exploring learning and differentiated instruction of fractions through collaborative action research"
5. Ann Kajander & Jennifer Holm (Lakehead University) – "Reluctant gate-keepers: Outcomes and consequences of high-stakes mathematics content requirements for pre-service teachers"

Poster Presentations:

6. Yasmine Abtahi (University of Ottawa) – "Mathematics manipulatives: Things kids think with"
7. Malcolm Cunningham (OISE, University of Toronto) – "A socio-behavioural definition for mathematics learning"
8. Jeff Irvine (Brock University) – "From description to prescription: A theoretical model of teacher behaviour"
9. Tim Sibbald (Nipissing University), Edward Chapman (Thames Valley District School Board), & Mark Fletcher (Thames Valley District School Board) – "Comparing instructional designs in Grade 10 applied math"
10. Diane Tepylo, Joan Moss, & Carol Stevenson (OISE, University of Toronto) – "Exploring a rigorous play-based block building program"
11. Kristina Wamboldt (Brock University) – "Are you the right type for mathematics? Personality type, mathematics anxiety, and achievement in undergraduate mathematics students"
12. Zhaoyun Wang (OISE, University of Toronto) – "Creating effective approaches to upgrade in-service teachers' knowledge"
13. Joanna Zambrzycka & Joanne Lee (Wilfred Laurier University) – "The nature of spatial talk in childcare centres"

At the time of writing, the topics for the next two meetings on February 22 and March 29 have not been confirmed. These themes are being considered:

a. **Math and Fiction** – Proposed panel speakers: Karl Schroeder, Tony Pi, Suzanne Church, and John Mighton.
b. **Assessment** – Teachers are teaching to expectations and are asked to triangulate assessment. While co-operative work is encouraged, student assessment is based on the individual work. How to meet these criteria? Possible speakers: Iain Brodie, Susan Drake (Brock University), Chris Suurtamm (University of Ottawa).

Please remember that all are invited. Meeting agendas (March 29 and April 26) will be posted at **www.fields.utoronto.ca/programs/mathed/forum/index.html**. ▲

▲ MB4T... Conceptual Building Blocks for Traditional Fraction Division Procedure: Division By a Unit Fraction

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Ann Kajander is an experienced classroom teacher currently teaching mathematics and mathematics education at Lakehead University. Her research interests relate to teachers' enhanced learning of mathematical concepts. She and her classroom-teacher colleague, Tom Boland, are currently writing a new book for teachers called Math Without Memorization: Big Ideas in Modelling and Reasoning for Elementary Teachers and Other Learners.

As with multiplication of fractions, it is very important that we spend as much time as possible working with concrete materials, models, and reasoning when developing an understanding of the meaning of operations. In this article, we will explore the interpretation and modelling of one component of the topic of dividing by fractions, namely dividing by a *unit* fraction; simply put, a fraction with a *one* in the numerator.

The *measurement* or repeated subtraction interpretation of division, which can be explored with whole numbers, is an important previous concept for dividing by any fraction. For example, in problems such as *Cookies are packed in packages of 4 for the bake sale; how many packages could be made from 20 cookies?* lends itself to the measurement model of division. Students might think, *How many groups of 4 can I make from 20?*; this can also be solved by repeatedly subtracting 4 from 20, and counting these groups of 4. Such an interpretation generalizes easily to division by a fraction. For example, the following problem—*The cafeteria sells servings of pizza that are one-half of a small pizza each. If there were six complete pizzas left, how many lunch servings could be made?*—can be solved by repeatedly subtracting one-half from 3 (we can do it six times before we run out of pizza), which

is like "measuring" or counting how many one-halves are in 3. Division is one way to count how many servings or amounts of a certain size are in a given quantity. For example, division of an amount by 1/2 is like asking, *How many pieces of size one-half do we have in this amount?*

It is important that children are given many opportunities to explore division by a fraction, using concrete materials. As they explore examples such as *It takes 3/4 of an hour for Sarah to shovel each driveway in her block. If she works for 3 hours, how many driveways can she complete?*, children come to understand what it means to interpret division by a fraction. This problem can be modelled by drawing "3" wholes, subdividing each into fourths, and counting the number of groups or sets of three of these fourths. Because the problem results in exactly four groups of three-fourths, it is relatively easy to solve with a model.

As the problems and contexts get harder, the models become more complex, and at this point, we need to start thinking about the development of a numeric method for dividing by fractions. As was found in previous columns that division procedures can in fact be developed using the very models with which we have been working. This is an important point; rather than "switching" from models to procedures, we are *using* the models to actually develop the procedures. The models and reasoning aspects become necessary components of the procedural understanding.

It should be noted that there are in fact a number of different numeric methods for dividing by a fraction; some of these less common methods will be discussed in a future column. However, the procedure often nicknamed *invert and multiply* remains a frequently used method. While its importance in elementary fraction calculations may be over-rated, the concept remains relevant to the manipulation of algebraic expressions, and hence, needs to remain with us. Understanding the effect of dividing by a *unit* fraction is the key conceptual idea needed to develop this formal procedure.

Returning to the lunch-counter example of pizza sales, the measurement model interprets division by 1/2 as *counting the number of one-halves*. Completing the model illustrates that there are 6 pieces of size 1/2 in 3 wholes. Modelling 5 ÷ 1/4 as another example could involve drawing five circles and then counting the number of one-fourth circles, which is 20. Note that when counting halves, the number of pieces is *double* the number we started with, and when counting one-fourths, the number of pieces is *four times* the initial number! Will this always be true? In other words, expressed more generally, if we divide an amount by 1/*n*, will we always get n times as many pieces as we started with?

As we think about the generalization, it may help to reason with a concrete example, such as 2 ÷ 1/3, and think about the fraction size. Using the measurement interpretation of division, we think of dividing 2 by 1/3 as like asking, *How many one-thirds are in two wholes?* We see six of the one-thirds in our model, three in each one-whole. The idea of three one-thirds in *each* of the two wholes is the same as 2 × 3, or *three* times as many as two. We also recall that when we divided by 1/2, we got *twice* as many pieces. It seems reasonable that if there are *n* equal parts or pieces in each whole, determining the total number of pieces can be done by taking the number of one-wholes multiplied by *n*.

We will explore further to see if this method of counting the number of pieces in each one-whole will work if the first number is a fraction too—in other words, if the quantity available isn't a whole number.

As an example, consider a model of 5/6 ÷ 1/3. It might be helpful to sketch the model and determine the answer, *using your model*, before reading on. You should have been able to find 2 1/2 pieces of size 1/3 in your picture of 5/6. According to our observations so far, this should be the same result as we obtain by calculating 5/6 × 3. Multiplying and simplifying shows that 5/6 × 3 is 15/6. If we group these sixths in groups of six to find the number of one-wholes in 15/6, we see that we have 2 3/6 as our answer—which is the same as 2 1/2. We could also perform a further check by calculating the inverse operation; calculating 2 1/2 × 1/3 is indeed 5/6.

It makes sense that dividing an amount by a unit fraction of the form 1/*n* (remembering that dividing by 1/*n* is like counting the number of 1/*n*-sized pieces) gives *n* times as many pieces. If we are starting with less than one-whole, then we only have *that fraction* of the *n* pieces. For example, to find how many 1/4's there are in five, we could multiply five by four, and to find out how many 1/6ths there are in 2/3, we can multiply 2/3 by 6... there will be two-thirds of the 6 pieces that there would have been in one-whole, so there are 4 one-sixths in the 2/3.

Using models, reasoning, and generic examples, we can conclude that as a numeric method to divide by 1/*n*, we can simply multiply by *n*! Such reasoning is the basis of the *multiply by the reciprocal* method; in the next column, the method will be generalized to division by more general fractions. ▲

What's New with CLIPS? There's an App for That! (The Rekenrek Tool App!)

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Greg Clarke is from the Simcoe Muskoka Catholic District School Board, Agnes Grafton is from the Brant Haldimand Norfolk Catholic District School Board, and Ross Isenegger is from the Near North District School Board. Together they are currently on assignment with the Ontario Ministry of Education as Provincial Math Leads, working on the CLIPS project developing electronic supports.



Kathleen Corrigan has recently retired from the Simcoe County District School Board and is currently keeping herself busy with numerous projects, including consulting with Agnes, Ross and Greg on resources for CLIPS and mathies.ca. She is also the President-Elect for MAC2, and slated to be President in the 2014-2015 school year.



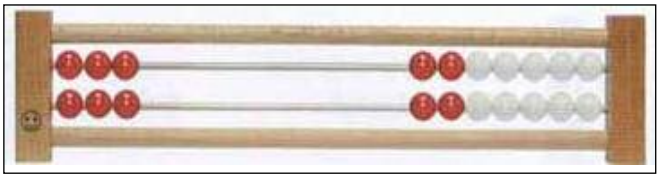
Pat Margerm is an independent literacy and mathematics consultant with more that 30 years of K to 8 teaching experience with students, in-service and pre-service teachers. She is a long-time OAME board member and works in various aspects of mathematics education.

The CLIPS team continues to expand into new areas. Our first foray into the app world is a version of the virtual Rekenrek tool, available on the App Store for iOS devices, and from Google Play for Android devices. The Rekenrek continues to be available from the Tool Container inside CLIPS (**www.MathCLIPS.ca**) and from the Tools page of the **www.MATHIES.ca** website, but now it is also available for mobile devices.

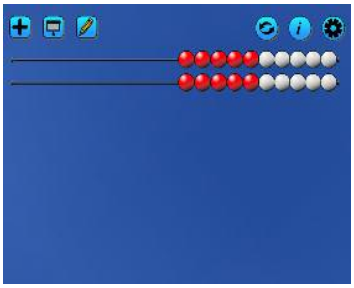
(NOTE: At the time of writing this article, the app has been submitted to both the App Store and Google Play store. The team is hopeful that the approval process timeline is shorter than the publication timeline so that by the time you are reading this, it is indeed available as promised! If not, rest assured it is indeed coming soon!)

What is a Rekenrek?

A Rekenrek is a manipulative tool made of two rods of ten beads. Some Rekenreks have ten strings of beads. Each set of ten beads has five red beads and five white beads, thus encouraging students to work with anchors of five and ten. The student enters a number by sliding beads from the right to the left.



The virtual Rekenrek, available as an app or online, allows the student to determine the number of wires, as well as the colours of the beads and background. The virtual tool provides two shades that can be sized in various ways so the students can be challenged to consider number flexibly. Further, the app offers a writing tool that allows students to annotate their screen and communicate their thinking.



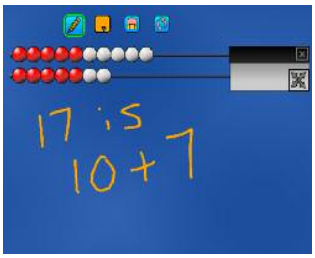
What kind of math can be done with a Rekenrek?

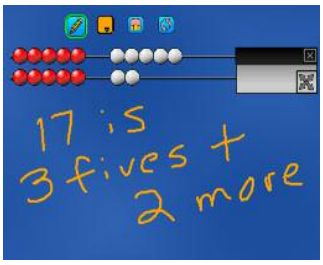
The Rekenrek is a versatile tool that combines the attributes of several familiar math tools, such as ten frames, number lines, base-ten blocks, and counters.

This tool is most commonly used in the primary grades; however, it can be used in junior grades and beyond.

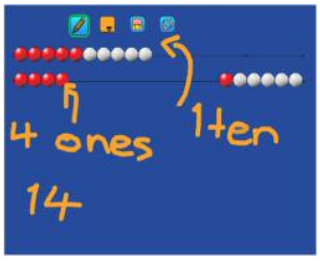
Students are able to explore mathematical thinking, such as:

- subitizing ("instantly seeing how many")
- counting principles, such as one-to-one correspondence, cardinality, and the "movement is magnitude" principle
- number relationships, using anchors of 5 and 10
- understanding addition and subtraction in terms of joining and taking away
- understanding addition and subtraction in terms of missing addends
- order of operations
- decimals
- composing and decomposing numbers

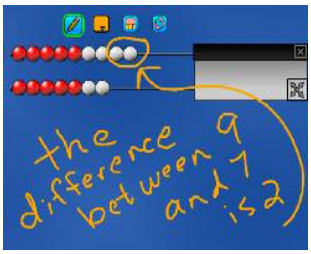




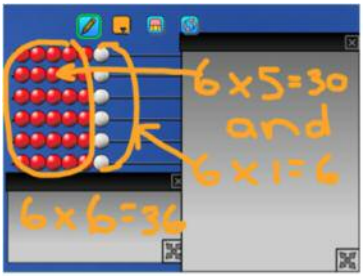
- understanding place value



- understanding subtraction in terms of differences or comparisons

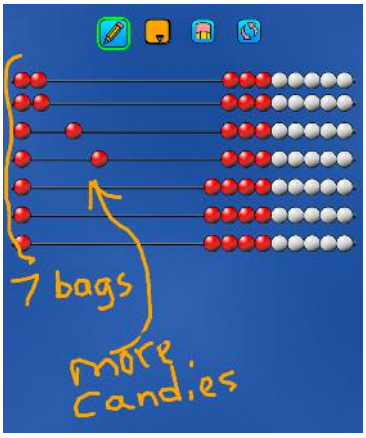


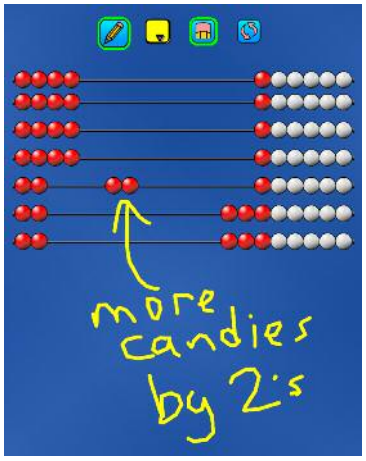
- developing multiplicative thinking

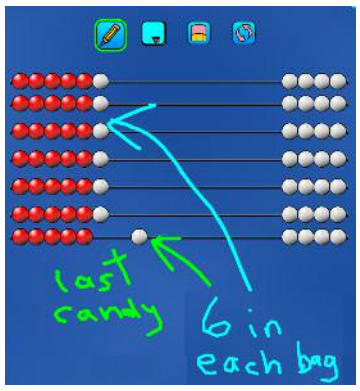


- visualizing division facts (e.g., of partitive division)

  Example: I have 42 candies for the loot bags. I have 7 friends. How many candies do I put into each bag?



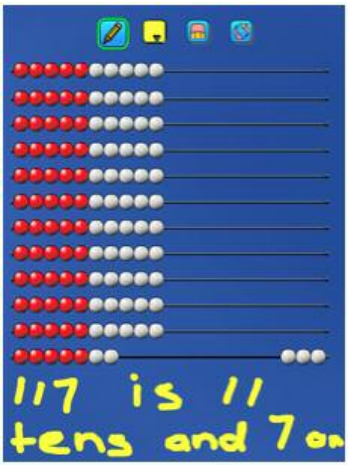




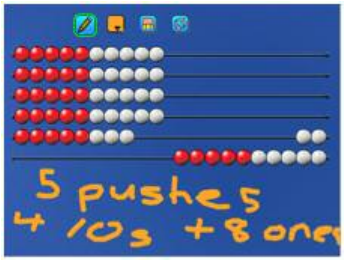
- etc.

Students are also encouraged to think quantitatively (and more and more abstractly), model mathematics, and look for patterns and structures.

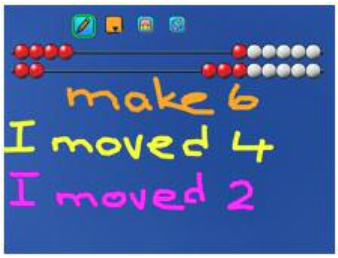
- Make 117.



- Make 48, using the fewest possible pushes.



- Game idea:
  - Students turn over a number card (e.g., 6).
  - First player uses the top wire to make part of 6.
  - Partner has to finish making 6.



How could a teacher use a Rekenrek with students?

Rekenreks are flexible tools that can be used in a variety of ways.

Initially, students should freely explore the Rekenrek and share what they try and the math that they notice. The teacher can record students' ideas about the tool, using math notation and vocabulary. Later, exploring number within a three-part math lesson format will soon result in students who confidently use the Rekenrek to solve problems and communicate their thinking.

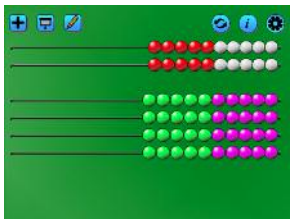
A few tips:

- Initially, encourage the students to explore for themselves. Later you can use a large Rekenrek, or project the virtual Rekenrek, so students and adults can share ideas with the whole class.
- Encourage multiple ways to demonstrate the same concept or quantities.
- Use shades (on the app, or made of cardboard) to widen the possible uses of the Rekenrek.
- Encourage students to articulate their thinking as they manipulate the beads.
- Connect Rekenrek representations of operations and number with other manipulatives, as well as traditional notations.
- Link the math actions with stories that can illuminate the ideas in other contexts.
- Encourage students to consider the Rekenrek as a tool when they are determining the best tool for a situation.
- Model thinking and language to support metacognition, so students begin to recognize the visual models they are making in order to discover number relationships and develop computational skills.

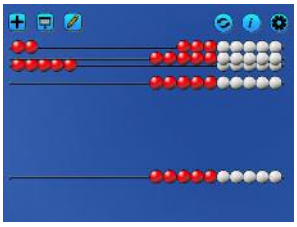
What features does the virtual Rekenrek app have?

The virtual Rekenrek provides many features to enhance students' use of the tool.

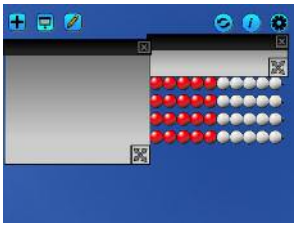
Colour and wire spacing can be adjusted to better meet the visual and fine-motor needs of students.



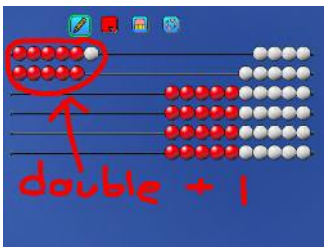
Wires can be overlapped to highlight relationships between numbers.

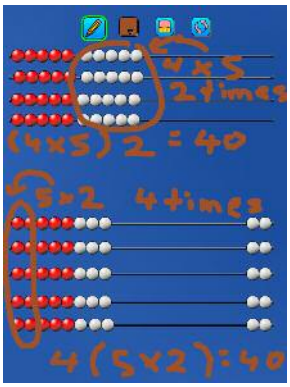


Shades can be used to highlight or conceal area to help build flexible thinking.



Annotations can be used to communicate student thinking.





Check out the wiki support page for the CLIPS Rekenrek tool at **http://mathclips.wikispaces.com/RekenrekTool** for further help and links to how-to videos.

Stay tuned for more tools and games available on **MathCLIPS.ca** and **mathies.ca**, and available for your mobile device in the App Store or Google Play. ▲

▲ Really Big Ideas: Adventures in Data

IAIN BRODIE
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Iain Brodie has been teaching in Scarborough, Ontario for 24 years. During that time, he has taught every possible combined grade from 1 to 6, as well as instrumental music. He teaches at St. Andrew's Public School, whose entire staff is currently engaged in math education research. Passionate about teaching math, Iain is a regular at the Fields Institute's Mathematics Education Forum, and is currently taking his M.Ed. in curriculum (math) online at Western University.

What if...?

"Look, Ma! No hands!"

Parents have heard this phrase and never failed to shudder. Their child, having mastered the basics of locomotion on two wheels, is pushing the boundaries of the possible. And what is it that the parent inevitably says to the child?

"Put both hands back on those handlebars before you get hurt!"



What if the parent did not accept the inevitability of a fall, but recognized the achievement and encouraged more?

"You know, if you lean a tiny bit to the right or left, you can turn your bicycle with no hands."

Parents try to limit their children's risk taking. The limits parents place upon them are frequently based on their own perceived limits, but the truth of the matter is that each generation of children is ever so slightly more capable than their parents' generation.



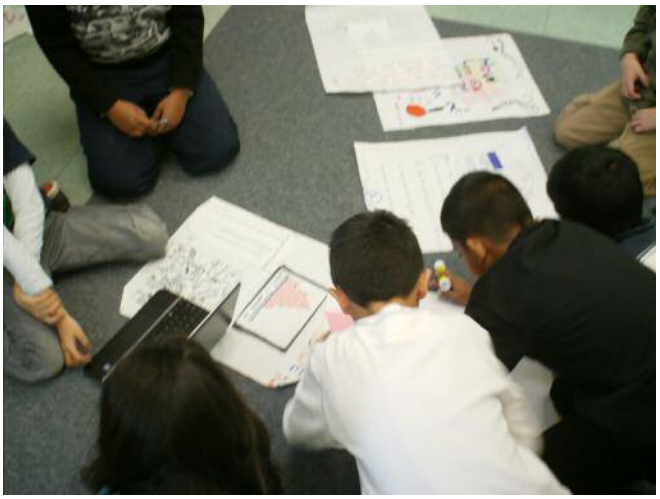
This is the quandary that faces us in mathematics education, especially at the elementary level. We have designed a curriculum that is a "both hands on the handlebars" type of curriculum that is supposed to safely take our children from one concept to another. All the different strands are carefully and neatly separated so that each can be learned without the intrusion of related concepts. The only problem with this is that while it is safe, it is also particularly well designed to beat the natural curiosity and risk taking out of the students by the time they have reached their teenage years.

There is a better way—a way that mixes concepts; a way that encourages risk taking; a way that allows for children to construct ideas, concepts and knowledge, and to make connections between big, important concepts. It is a way that involves really big ideas.

Big Ideas

For Wiggins and McTighe (1999, 2004), big ideas grew out of the idea that there were certain essential understandings that students needed to be able to grasp in any set of curricula. Big ideas "serve as a key concept for making important facts, skills, and actions more connected, coherent, meaningful, [and] useful." Big ideas should be about the core of a concept, not a trivial or basic coverage of a subject.

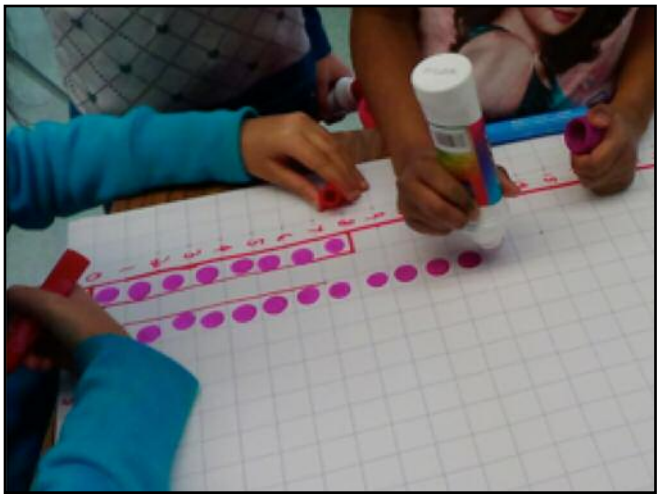
Charles (2005) defines a mathematical big idea as "…a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole." The commonality between both of these definitions is that of making connections between mathematical concepts that make for a greater whole than the sum of the parts.

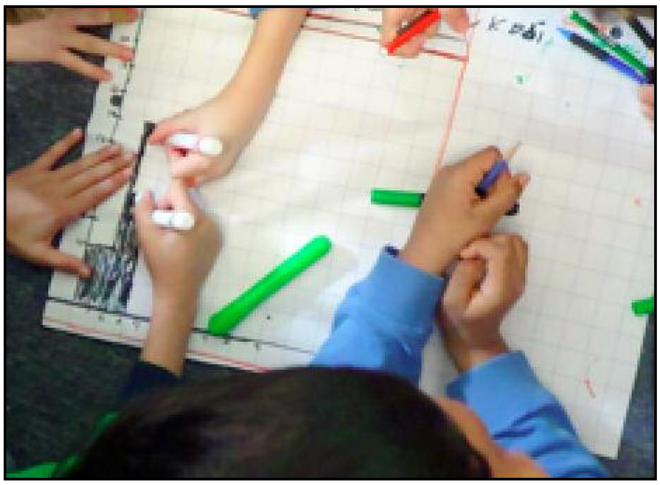


The Ontario Mathematics Curriculum (2005) is loosely organized around big ideas with specific expectations grouped together "under subheadings that reflect particular aspects of the required knowledge and skills" (p. 7). These subheadings and groupings "often reflect the 'big ideas' of mathematics that are addressed in the strand" (p. 7). More often, they do not. The subheadings in the Data Management and Probability strand for Grade 3 are Collection and Organization of Data, Data Relationships, and Probability. Verbs such as *demonstrate, conduct, collect, sort, classify,* and *read* all preface specific expectations that outline the basic skills and knowledge required of a Grade 3 student. These skills all build on the skills from previous years and carefully and precisely set out a path to the desired knowledge. Both hands on the handlebars, please!

Only one expectation uses the verb *interpret*, a higher-order thinking skill. One other uses *predict*, which could have some depth to it, but is given the example of predicting the frequency of rolling even numbers, using a single die. Both hands back on the handlebars, please!

How then, do we get at really big ideas that encompass and connect more than one strand of math? How do we make meaningful a set of curricula that is disconnected to encourage safe, "both hands on the handlebars" learning? How one class of Grade 3 students achieved this is a really good story involving a need for data, lots of measuring, and a basket full of bingo dabbers.





In the Beginning

In Context

In our classroom, we engage in inquiry. Science is an obvious place to conduct inquiries, but this process fares well in any subject. One reason we do inquiry is that it lets us bring in and integrate other subjects.

While learning all about our class's worms, a large group of students engaged in a quest that involved measurement and data management. During this process, which they eagerly shared with the other groups around them, they developed excellent measuring skills and discovered the idea of mode.

In #2 on next page, note that the students defined not only the basics of what mode is, but also started toward an understanding of how mode is related to the normal curve... before being taught about it.



1. How long is a worm?

A group of students wanted to know the length of a worm from our class vermicomposter. Measuring a moving target was challenging.

Disclaimer: *No worms were harmed during the course of this activity.*

2. Look at what we found!



"We think that the best answer to our question is 7 cm."

"Why?"

"More worms were 7 cm and it's sort of in the middle."

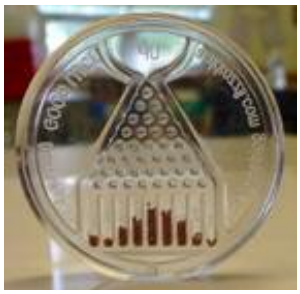
"Let me tell you about this idea called mode."

Flexibility and Patience

Flexibility is a virtue. Being able to see an opportunity and grab it and go with it lets you and your students delve into really big ideas when they come up naturally. Exploring normal curves is not something normally done in Grade 3, but the idea of how the curve is generated by large amounts of data and its basic shape gives us a low-floor entry into the topic.

Patience is essential. By choosing interesting, deep, and puzzling math activities, students will be able to construct new knowledge on top of their mathematical schema. Do not jump in too soon to teach. Let them enjoy the puzzles as much as finding the answers.

3. How is this the same?



Recognizing the basic shape of the students' graph of worm lengths as being like a normal curve, I brought out a toy called an *aakkozzll*. It drops balls at random, and as they fill the slots below, they form a normal curve.

"How are the shape the balls make and your graph similar?"

"They are both high in the middle, low at both ends, are kinda like a curve. Oh! The mode is right in the middle!"

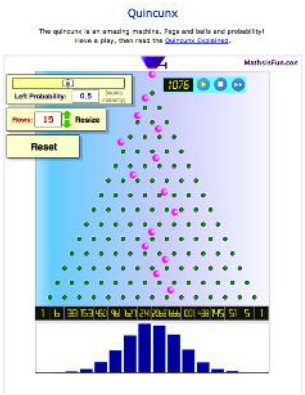
"One was random, the other was measured. How can they have the same shape?"

"I don't know." Looks puzzled.

"Want to find out?"

Grins. "Sure!"

4. Explore first



After the group of worm measurers had shared their scientific and mathematical discoveries, we dove into an activity that would let the students experience the forming of a normal curve and the large amount of data that is needed. The Quincunx at **www.mathsisfun.com/data/quincunx.html** lets students play and manipulate the dropping balls and see the results immediately.

"We don't get a normal curve right away. We need at least 200 balls. Can I leave it running during recess? I want to see what happens when there's 10 000 balls."

Grins. "Sure!"

Hunting a Normal Curve

Faith

When teaching through big ideas, it is important to have faith—faith that the skills and curriculum expectations will come out in the pursuit of big ideas, and that you will be able to assess, evaluate, and report on them.

Teaching from a big idea, the goal is to learn an important concept and along the way pick up the skills and knowledge needed to be able to understand and use the big idea. Most often, these come out as the students interact with and construct new knowledge. These skills, and the learning of curriculum expectations, become self-apparent as *part* of the new understanding, not separate from it.

Teaching the skills, knowledge, or specific expectations in isolation will not necessarily build into understanding the big idea.

1. Measuring ourselves

																X	
					X					X			X			X	
	X	X	X		X	X		X		X	X	X		X	X	X	X	X
	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42

1m ___cm

Would we have enough data if we measured ourselves? That was the first question we set about answering. We helped each other to measure our heights and quickly added our data to a class line plot.

"What shape are we looking for in our graph?"

"A curve."

"What does it look like?"

"High in the middle, low at the beginning and end."

"Did we find it?"

"No."

2. Scale changes the look of data

		x		
		x		
		x		
		x		
	x	x	x	
	x	x	x	
	x	x	x	
	x	x	x	
	x	x	x	
	20s	30s	40s	

Looking at the graph of our data, we wondered together why we might not have found a normal curve. Various ideas came out, including the fact that we had only measured 19 students. Maybe we needed more.

"There's a big difference between our smallest and our tallest. Our graph is too long."

"Can anyone remind us of what scale is?"

"When you count by twos or fives on your graph."

"If we counted by tens, would it change the way our data looks?"

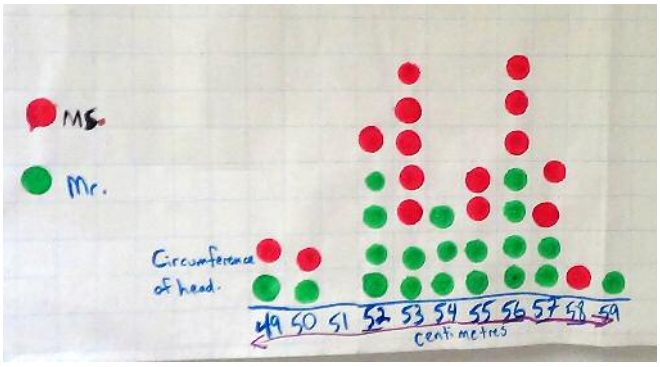
We quickly reorganized our data into a new graph.

"Oh, it's sort of like a curve."

"If we change the scale, we get something closer to what we're looking for. What do you think we need to really find it?"

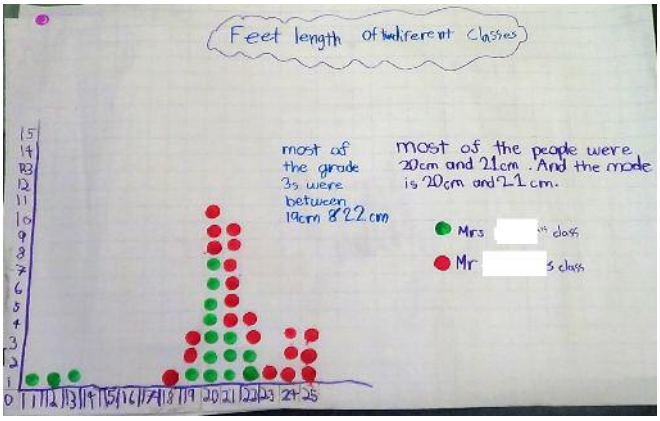
"More measurements."

3. Measuring ourselves and our partner class



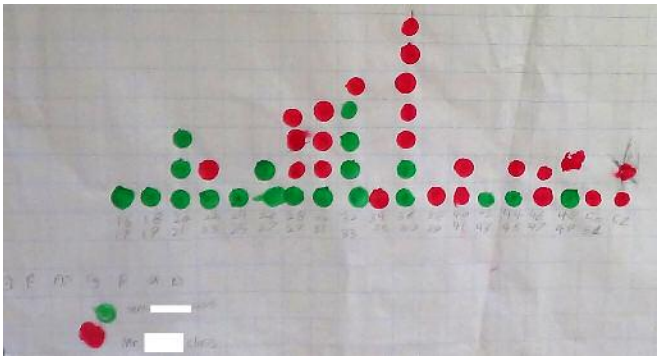
The circumference of our heads. Scale of one. Bi-modal.

We invited the other Grade 3's to help us measure ourselves. We measured the circumference of our heads, our heights, arm spans, and the length of our feet. Groups were in charge of measuring, recording data, and creating graphs to represent their findings.



The length of our feet. "Most of the Grade 3's are between 19 cm and 22 cm. Most of the people were 20 cm and 21 cm. The mode is 20 cm and 21 cm." Students are noticing that a range of data is a better way of describing results. Notice that these students identify two modes, even though one length had one more data point. The beginnings of realizing the limitations of mode.

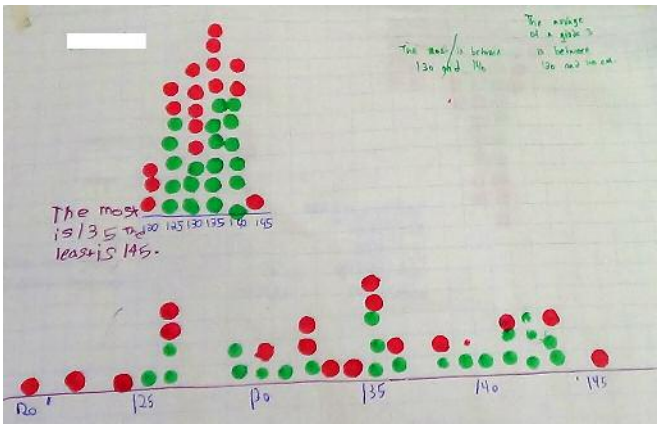
There was a purpose in the measuring beyond just measuring, so each group of students measured as quickly and as accurately as possible. Our partner class rotated through the four measuring stations, then we rotated ourselves through to add our data to that of the other Grade 3's.



Our arm spans. These students used a scale of two right away. Note the nice clear mode in the middle of the data.

While the students were organizing and displaying their data, I circulated through the groups, taking note of the skills of all of the students by observing them doing it. Not surprisingly to me, I found that just about every student was able to demonstrate an ability to conduct, collect, sort, classify, and read the data they had just generated. During the debriefing, we were able to talk and consolidate the concept of mode, and how there can be more than one mode. The students also showed that they were able to manipulate the appearance of their data through the use of different scales. During the discussion, we talked about the idea that data tells a

story about whatever has been measured or counted.



Our heights. Graphed first with a scale of one then a scale of five.

"The most is 135 and the least is 145."

"The average of a Grade 3 is between 130 and 140 cm."

By changing the scale, these students managed to create a graph of their data that was high in the middle and lower on either end. Close to what we were looking to find.

While we had found data that was "high in the middle and low at both ends," we still did not have enough data to produce a nice normal curve. We would have to look, and measure, somewhere else.

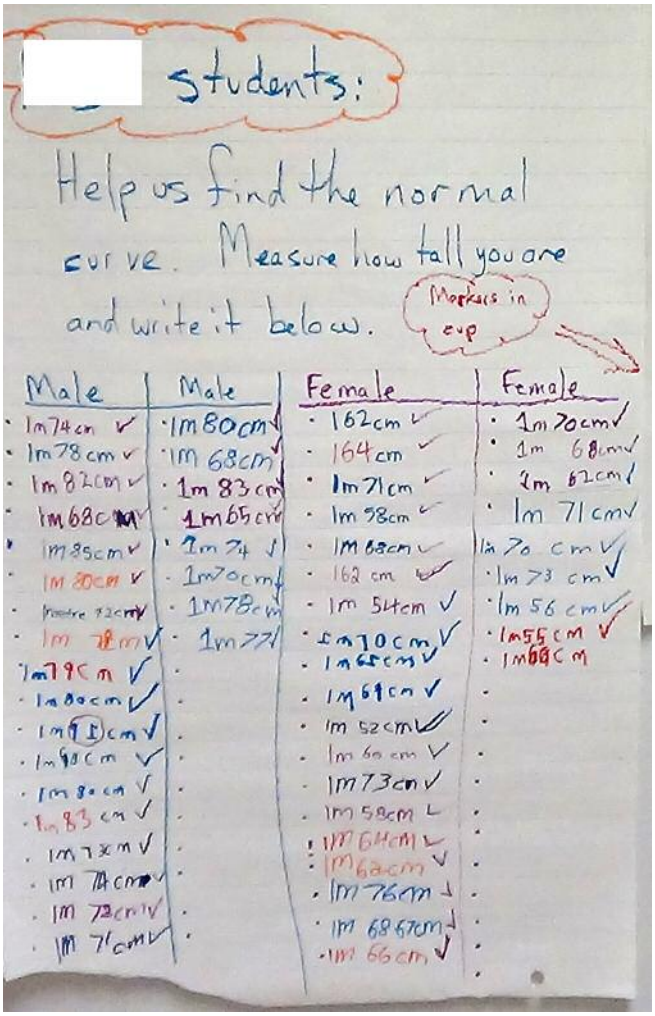
Side Effects

There were some unintended side effects caused by this hunt for the normal curve. Because of the large amount of measuring the students were doing to get data, they got very good at linear measurement. When I pre-tested their measurement skill before teaching that strand, I found that all the students could measure consistently and accurately.

We moved on.

4. Crowd sourcing our need for data

If you need to get something big done, ask a lot of people to each do a small bit . This is the idea behind crowd sourcing. We are very fortunate to share our school building with an alternative high school. The students are all in either Grade 11 or 12, so most have reached their full height. There are also quite a few of them, so we would be able to get a lot of measurements.



At first, we left the appeal up in the hall with a measurement strip taped to the wall. The teachers from the high school encouraged their students to stop and measure themselves to help us out. The Grade 3's needed to be much more hands-on than this. Small groups took turns standing out in the hall we share with the high school, asking students if they would let themselves be measured. "Excuse me. Could we measure how tall you are, please? We are trying to find a normal curve," asked a Grade 3 girl, who is all of 129 cm tall.

"Sure. Why are you doing this?" replied a mature young man who is somewhat taller than 180 cm.

"We need a lot of data to get a normal curve," she explains.

"What's a normal curve?," he asks.

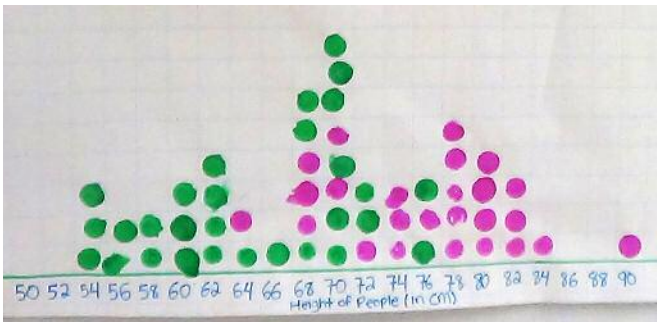
"It's a curve that is high in the middle and low on both ends. Most people are in the middle."

Low Floor, High Ceiling

Teaching curriculum around big ideas and designing activities with a low floor and a high ceiling can "potentially reduce learning boundaries and offer students the space to be mathematically curious and imaginative in their learning" (Gadanidis & Hughes, 2011).

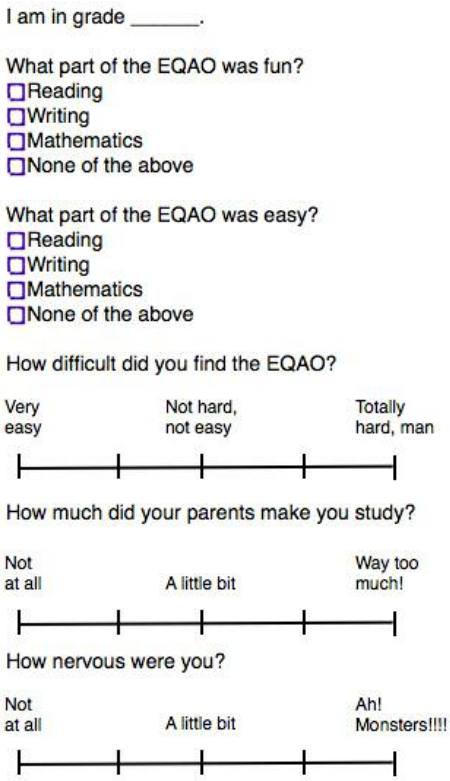
As everyone has a starting point and is able to engage in the learning to the ultimate extent of their potential, grade-level curriculum targets are frequently surpassed. These targets are not passed in a way that just proceeds to the next grade level, but in a way that lets students make connections, have a sense of wonder and see the beauty in mathematics.

The groups of students measured, recorded, and plotted the data from over 50 high school students. Frequently, the high school students participated in the graphing, adding their personal data point to our graph. In the end, we ended up with a set of data that pretty much formed itself into a nice normal-like curve. We celebrated finding the curve and noted that the mode was almost right in the middle of the range of our data.



Finding the curve was just one of the goals of this adventure. As the students were immersed in the greater idea that data can tell us something about groups of people such as how tall a Grade 11 or 12 student might be, they became acquainted with the idea that data can form itself into patterns like a curve is another. These ideas do not come from the Grade 3 curriculum; they come from a point much farther on. We took these advanced concepts and used them as big ideas to drive our investigations and satisfy our curiosity.

The Aftermath



Student-designed survey. Note the playful tone the students used in the rating scales.

The next activity we engaged in was a task designed to be assessed and evaluated at the Grade 3 level, but was still built around the big idea that data can tell us a story about a group of people. The students, being in Grade 3, were curious, and a little bit apprehensive about the EQAO test. In discussing what kind of question they would like to investigate, the students asked the big question: "Do we need to be nervous about doing the EQAO?" They made up a questionnaire, distributed it to all the students in the school from Grades 4 to 8. They thought nothing of dealing with this large amount of data. In fact, they knew that their question couldn't be answered without it.

As a product to be assessed, the students chose sections of the questionnaire and made a "beautiful graph" and told a "great story" about the data we had collected. In short, they found out that, no, we do not really need to be nervous about the EQAO. Yes, there are fun parts to the test, especially the mathematics part. And yes, most parents made their children study at least a little bit before the test. If only we could use this data to

help us get out of the studying!



Organizing large amounts of data is easier when groups each do a little bit. Students had no difficulty discussing the mode of each grade's data set and how it might change as you begin to combine all the grades' data.

Why Use Big Ideas?

Side Effects

Finding a curve by measuring worms and people predisposed the students to seeing curves as the presence of a pattern.

During an activity where we were finding the minimum perimeter for a given area, I asked the students to graph the length of each rectangle we found versus its perimeter. This results in a hyperbola. Neat.

When we were finding the maximum area for a given perimeter, I asked the students to graph the length of each rectangle we found versus its area. This results in a parabola. Cool.

"Can we graph everything?"

"Sure, almost everything."

"Are there more curves?"

"Oh, yes!"

"Can you show me?"

Using really big ideas in a purposeful and playful way allowed us to design data activities with a low floor and a high ceiling. Young students are able to understand much more than we generally give them credit for, and we want them to learn, make connections, and understand that mathematics is bigger than just adding and subtracting numbers.

Everyone can make a simple measurement and make a mark with a bingo dabber on a group graph—low floor. By letting the students interact with these big ideas and create graphs of data that told stories about us, they were able to make connections, draw conclusions, and construct knowledge around the concept of averages—high ceiling. Perhaps more importantly, the students were able and encouraged to be curious, to ask questions, and to discover ways that really big mathematical concepts are interconnected. They are ready and prepared to learn more.



"Look, ma. No hands!"

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▲ Improving Student Achievement in Mathematics by Using Manipulatives with Classroom Instruction

The National Council of Supervisors of Mathematics (NCSM) Improving Student Achievement Series No. 11/Spring 2013

Research-Informed Answers for Mathematics Education Leaders

NCSM is a mathematics leadership organization for educational leaders that provides professional learning opportunities necessary to support and sustain improved student achievement.

NCSM envisions a professional and diverse learning community of educational leaders that ensures every student in every classroom has access to effective mathematics teachers, relevant curricula, culturally responsive pedagogy, and current technology.

NCSM strives for excellence and equity in mathematics education and recognizes the important role leadership plays in achieving these outcomes.

OMCA is an affiliate of NCSM. NCSM's current president is Valerie Mills from Michigan and its current Canadian Regional director is Carol Matsumoto from Winnipeg, Manitoba. The current NCSM Ontario Leader is Peter Saarimaki.

NCSM Network Communicate Support Motivate

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Manipulative materials help students make sense of abstract ideas, provide students ways to test and verify ideas, are useful tools for solving problems, and make mathematics learning more engaging and interesting by lifting mathematics off textbook and workbook pages. ~ Burns, 2007

Our Position

It is the position of the National Council of Supervisors of Mathematics (NCSM) that in order to develop every student's mathematical proficiency, leaders and teachers must systematically integrate the use of concrete and virtual manipulatives into classroom instruction at all grade levels.

This position can be accomplished when leaders and teachers:

- Understand that manipulatives are not toys but are powerful learning tools which build conceptual understanding of mathematics;
- Use research to guide instructional use of manipulatives;
- Provide sustained professional learning opportunities in the use of manipulatives; and
- Recognize that learners—both adults and students—progress through varying levels of proficiency as they use manipulatives before they can realize their full impact.

The *Common Core State Standards* emphasize that concrete models are an essential tool for learning mathematics across all grade levels, K–12. This assertion is articulated most clearly in the Standard for Mathematical Practice 5, "Use Appropriate Tools Strategically," where students choose from concrete models (including manipulatives) and technology. Beyond this, the standards regularly suggest using models in initial steps of learning mathematics before students move to other representations. Therefore, students should have a variety of manipulatives and tools available to them at all times.

Manipulatives used in classroom instruction are physical objects handled by individual students and small groups. Virtual manipulatives are important tools for teacher modeling and demonstration and, additionally, provide students access to manipulatives both inside and outside of the school day via computers. However, virtual manipulatives do not replace the power of physical objects in the hands of learners.

Research That Supports Our Position

John van de Walle and his colleagues (2013) define a mathematical tool as, "any object, picture, or drawing that represents a concept or onto which the relationship for that concept can be imposed. Manipulatives are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (e.g., connecting cubes) or for other purposes (e.g., buttons)" (p. 24). Virtual manipulatives are "digital objects that resemble physical objects and can be manipulated with a mouse of a computer" (Moyer, 2002, p. 372). For example,

The NCSM Improving Student Achievement Series is a set of position papers designed to provide research-based practices for school and district mathematics education leaders.

virtual versions of Cuisenaire Rods and Tangrams are readily available online for instructional purposes. Users should be careful that the virtual versions are accurate matches for the physical tools.

In the opening quote, Marilyn Burns (2007) provides four reasons manipulative materials are fundamental to mathematics instruction. These ideas appear repeatedly in research and thoughtful commentary on the teaching of mathematics. The National Research Council's *Adding It Up* (2001) concludes its review of research on the role of manipulatives with the following statement:

> "The evidence indicates, in short, that manipulatives can provide valuable support for student learning when teachers interact over time with the students to help them build links between the object, the symbol, and the mathematical idea both represent" (p. 354).

Numerous studies have examined the effectiveness of specific manipulatives to teach specific topics. For example, the Milken Family Foundation analysis of NAEP data suggests that the use of hands-on materials is highly effective. The findings note that "when students are exposed to hands-on learning on a weekly rather than a monthly basis, they prove to be 72% of a grade level ahead in mathematics" (Wenglinsky, 2000, p. 27). Additionally, Sowell (1989) conducted a meta-analysis of studies focused on teaching with manipulatives and found them to have a positive impact on mathematics learning. Cramer et al. (2002) compared the performance of 1,600 fourth and fifth grade students studying fractions using both manipulative-based curricula and non- manipulative based curricula. They found that students in the manipulative-based program had higher mean scores at the end of the unit as well as higher retention scores.

Manipulatives are also considered an important element of teacher preparation. For example, the Conference Board of Mathematical Sciences' 2012 report, the *Mathematical Education of Teachers II*, includes numerous references to the use of manipulatives in classroom instruction and the importance of teacher preparation for this use. The authors continue by pointing out that teachers must work to help students see the connections between the manipulatives or other tools and the mathematical concept being taught. A number of studies cited in Van de Walle et al. (2012) suggest that manipulative instruction which follows a pattern of "do as I do" is one of the most widespread misuses of manipulatives. Stein and Bovalino (2001), for example, suggest three key features of successful manipulative lessons that avoid this pitfall. They conclude that 1) teachers have extensive training in the use of manipulatives; 2) teachers prepare by using manipulatives to complete the same instructional activities they would ask of their students; and 3) teachers prepare the classroom for activities by organizing students in groups, preparing materials, and thinking through the logistics of the lesson.

Similar findings on the importance of effective instructional strategies when teaching with manipulatives appear in the 2009 Institute for Education Sciences report on response to intervention in mathematics (Gersten et al., 2009). The report states that "research shows that the systematic use of visual representations and manipulatives may lead to statistically significant or substantively important positive gains in math achievement" (pp. 30–31). The report goes on to discuss the importance of transitioning from concrete objects to visual representations and then to abstract notation. It provides a comprehensive summary of the evidence supporting the use of manipulatives, including evidence supporting the Concrete—Representational—Abstract (CRA) method of instruction. This method, grounded in Bruner's (1966) constructivist discussion of enactive/iconic/ symbolic progression in learning, provides a basis for an effective framework for teaching with manipulatives. Under this framework, teachers begin with concrete manipulative experiences, transition students to using visual representations (drawings), and finally transition to using abstract mathematical notation.

Hattie (2012) states "when teachers see learning occurring or not occurring, they intervene in calculated and meaningful ways. In particular, they provide students with multiple opportunities and alternatives for developing learning strategies based on the surface and deep levels of learning some content or domain matter, leading to students building conceptual understanding of this learning, which the students and teachers then use in future learning" (p. 15). Hattie later cites research on the power of balance in the classroom: "There is a balance between teachers talking, listening, and doing; there is a similar balance between students talking, listening, and doing" (p. 76). Manipulatives provide a foundation around which teachers and students can talk, listen, and do. Other research from Hattie (2009) concludes that, more often than not, when students do not learn, they do not need "more;" rather, they need "different" (p. 83). Again, to ensure that every student

learns mathematics, a wide range of different strategies are needed for teaching and both physical and virtual manipulatives are a critical part of this toolkit.

Witzel et al. (2003) describe an example of successful implementation of the CRA approach in teaching algebra to middle grades students. The Association of Middle Level Education's research summary, Manipulatives in Middle Grades Mathematics (Goldsby, 2009), provides further information about this and other studies.

How NCSM Members Can Implement Our Position

As leaders, NCSM members must work to ensure that research-based recommendations are implemented in their schools, districts, states, and provinces. NCSM members must act to create and sustain the conditions and structures that will enable every mathematics teacher to use manipulatives successfully. Moreover, NCSM members must act to alert teachers, coaches, and administrators that it is time to move away from incidental to systematic approaches to manipulative-based instruction. NCSM members must act to move communities away from the "yes, but … I learned math without manipulatives" or, "they're playing with toys instead of learning mathematics" towards the power of multi-modal, conceptually based, hands-on instruction.

More specifically, NCSM members must:

- Ensure that curriculum documents K–12 support the use of manipulatives by their inclusion as an instructional tool on par with textbooks, technological tools, or other resources;
- Ensure access to manipulatives for every teacher and every student;
- Ensure ongoing professional development around the use of manipulatives;
- Ensure that teachers work collaboratively on grade level or subject area teams to provide equity among all student opportunities in using manipulatives;
- Ensure that the use of manipulatives is not viewed as optional by teachers, while recognizing that the nature and frequency of use will vary from course to course;
- Ensure the support of manipulatives to scaffold learning and in the problem-solving process;
- Ensure that teachers use manipulatives within the Concrete—Representational—Abstract Learning Cycle;
- Ensure that parents are educated about the place of manipulatives in the mathematics classroom;
- Ensure that manipulative-based activities are used for formative assessment in classrooms;
- Ensure that student background knowledge is considered in the variety of student choices; and
- Ensure that students have manipulatives available to help provide evidence in visualizing their thinking.

One of a series of position papers of the National Council of Supervisors of Mathematics mathedleadership.org © 2013

National Council of Supervisors of Mathematics

Mission Statement

The National Council of Supervisors of Mathematics (NCSM) is a mathematics leadership organization for educational leaders that provides professional learning opportunities necessary to support and sustain improved student achievement.

Vision Statement

NCSM envisions a professional and diverse learning community of educational leaders that ensures every student in every classroom has access to effective mathematics teachers, relevant curricula, culturally responsive pedagogy, and current technology.

To achieve our NCSM vision, we will:

N: Network and collaborate with stakeholders in education, business, and government communities to ensure the growth and development of mathematics education leaders

C: Communicate to mathematics leaders current and relevant research; and provide up-to-date information on issues, trends, programs, policies, best practices and technology in mathematics education

S: Support and sustain improved student achievement through the development of leadership skills and relationships among current and future mathematics leaders

M: Motivate mathematics leaders to maintain a life-long commitment to provide equity and access for all learners

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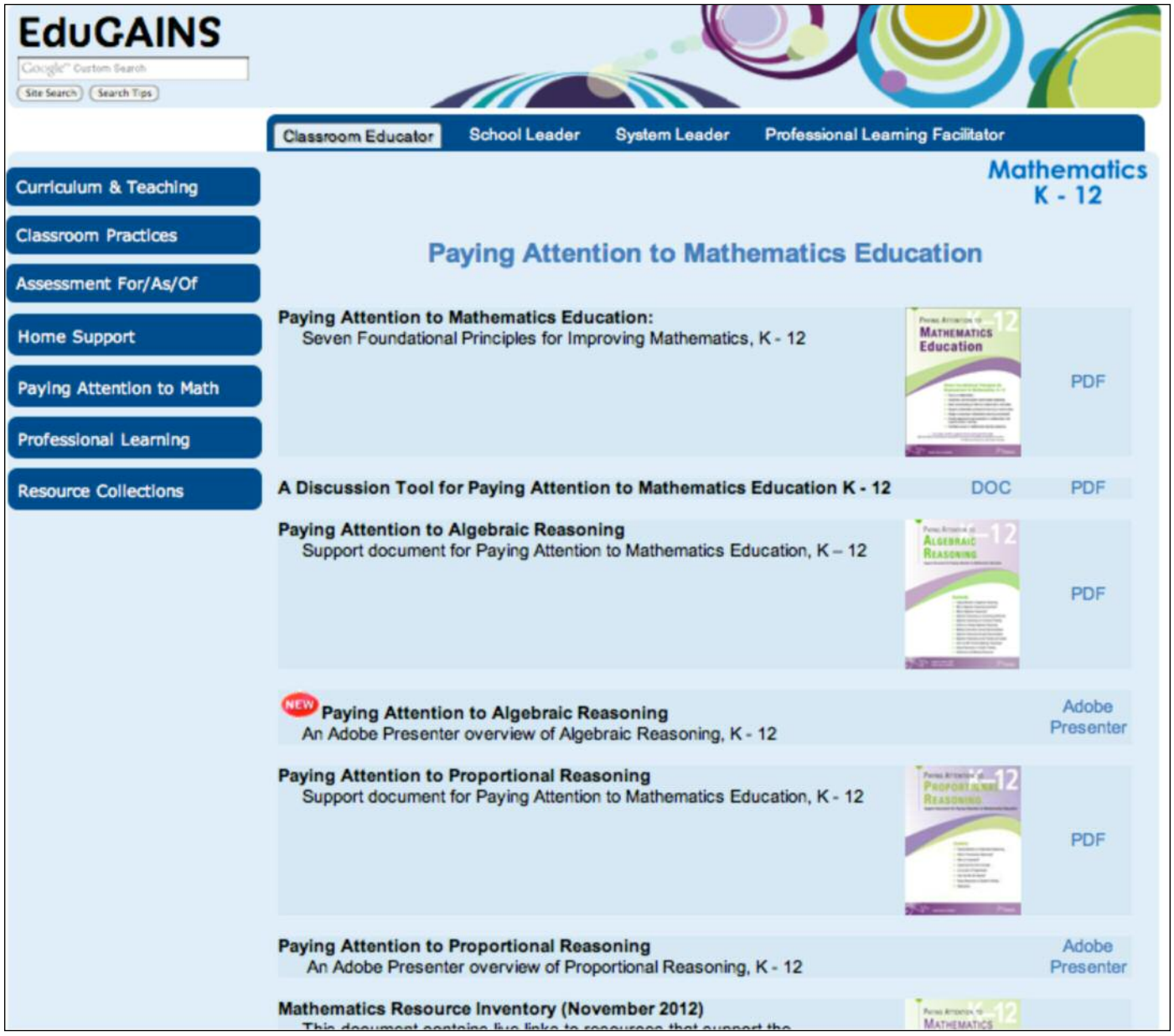
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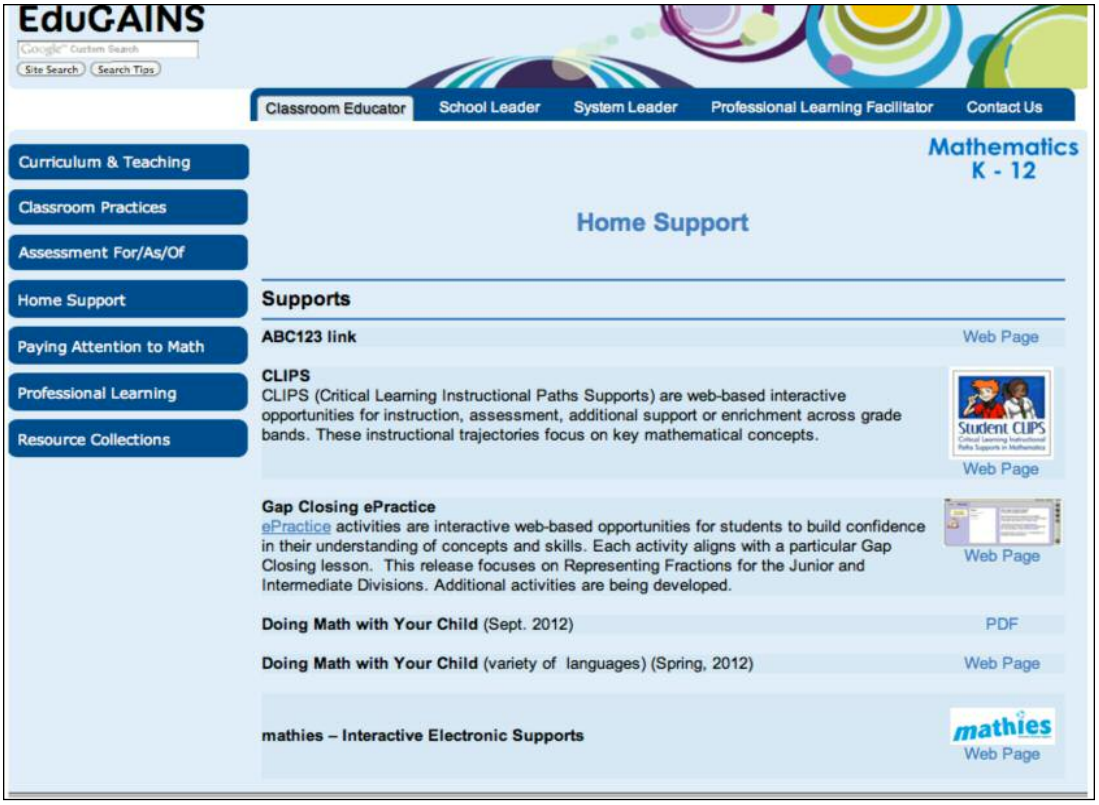
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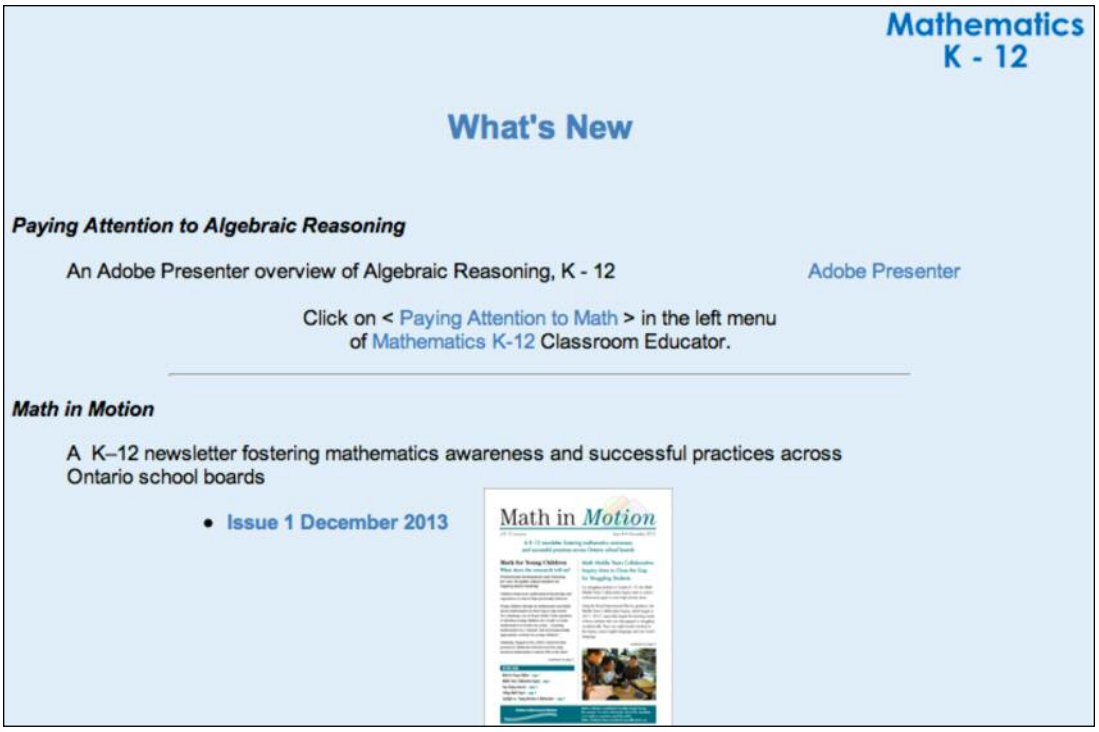
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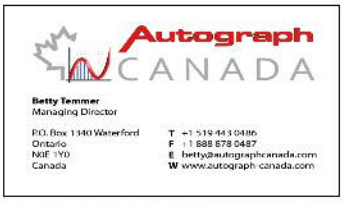
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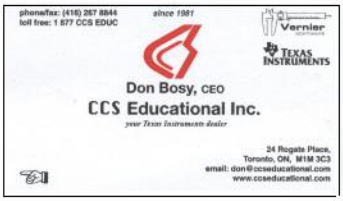






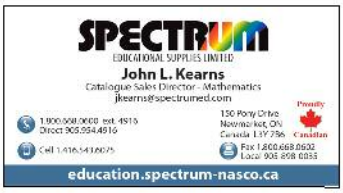










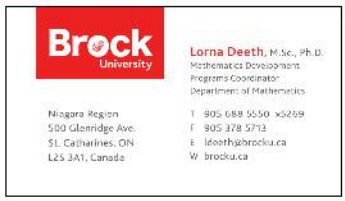








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PROPORTIONAL REASONING IN GEOMETRY AND MEASUREMENT

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Geometry and measurement provide many applications for proportionality. Proportional reasoning is used to determine the lengths, area and volume of 2-D and 3-D figures. Think about the use of proportionality in this context. "A rectangle has a length and width that are whole numbers of centimetres and the ratio of its length to its width is 4:5. Its area is 500 square centimetres." Could the problem be, "What are the rectangle's length and width?" or "If another rectangle has the same ratio of its length to its width, 4:5 and is double the area, what is the length and width of this other rectangle? In geometric contexts, proportional reasoning is related to the composition of similar 2D figures. For example, "What are least and greatest number of same-shape pattern blocks that compose 4 hexagons? 8 hexagons? 100 hexagons?"

In this issue, the Research Summary and problems (with multiple solutions) focus on the use of proportional reasoning within measurement and geometry contexts. The problems we provide are designed for use within a three-part problem-solving lesson structure.

LINKS TO LITERATURE AND MANIPULATIVES



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According to Douglas Clements almost 50% of every minute of children's play involves mathematics. This year I had the opportunity to observe students learning in play based kindergartens. Some of the students told me that long necked dinosaurs were plant eaters. Then they sorted the toy dinosaurs into two groups – long necked and not long necked. The dinosaurs were not all the same sized so students were applying the criteria of "long necks" by comparing the length of the necks to the height of the dinosaurs, proportionally. This experience reinforced the connection between measurement and proportionality. In this issue, I will look at using interlocking blocks such as LegoTM to continue exploring proportional relationships through play and a picture book that can be used to continue to investigate proportional relationships in bodies.

RESEARCH SUMMARY – PROPORTIONAL REASONING

According to Lesh, Post & Northern(1988), “proportional reasoning is a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information. Proportional reasoning is very much concerned with inference and prediction and involves both qualitative and quantitative methods of thought” (p. 93). A proportion is a relationship of equality between two ratios (Lobato & Ellis, 2010). In order to reason proportionally, students must not only reason with ratios, but they must also understand that the ratio of two quantities remains constant even as the corresponding values of the quantities change.

Proportional reasoning involves understanding that:

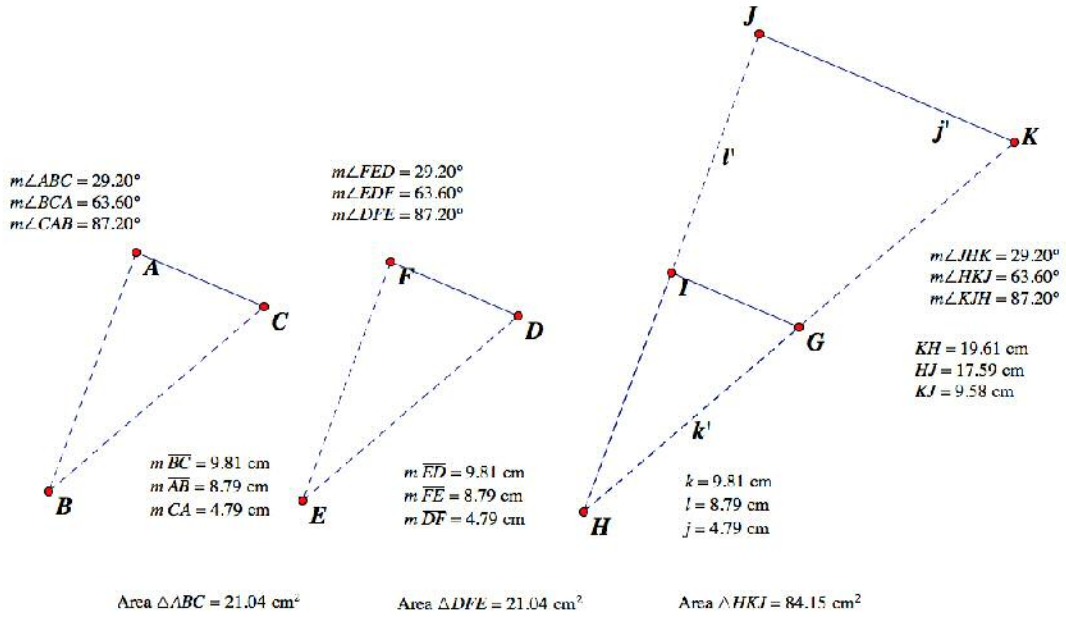
(a) equivalent ratios can be created by iterating and/or partitioning a composed unit and

(b) if one quantity in a ratio is multiplied or divided by a factor, then the other quantity must be multiplied or divided by the same factor to maintain proportional relationships.

So, proportional thinkers have a sense of **co-variation**. There is an understanding of proportional relationships as a change in one quantity, which causes a corresponding change in another quantity. In measurement and geometry contexts, Lobato, Ellis, Charles, & Zbiek (2010) interpret co-variation as a ratio-as-measure (ratio that measures a real-world attribute). A ratio-as-measure involves “isolating attributes and understanding the effect of changing a quantity on the attribute to be measured by the ratio” (p. 23). The attributes measured often include side lengths and angle measures.

Using Ratios to Compare the Attributes of Polygons

Polygons are often compared according to their corresponding side lengths, interior angles and interior area. In these ratios, we are comparing the measure of a side of the larger polygon to the corresponding measure of a side of the smaller polygon.



Similarity Ratio or Scale Factor describes a ratio between corresponding sides. Therefore, similar triangles have the same ‘shape’ but are just scaled differently.

- If two polygons are similar, then the ratio of the measures of any two corresponding sides equals the ratio of their corresponding perimeters. (perimeter of ΔABC is 23.39cm -> perimeter of ΔJKH is 46.78cm, which has a ratio of 1:2).
- Why is the area of the dilated triangle, ΔJKH 4 times the area of ΔABC?

LINKS TO MANIPULATIVES:

The German educator and founder of the Kindergarten movement Friedrich Froebel, developed a series of play materials that included geometric building blocks. In his book *Inventing Kindergarten*, Norman Brosterman argued that several major artists and designers such as Buckminster Fuller, Piet Mondrian, Paul Klee, Wassily Kandinsky, and Frank Lloyd Wright were greatly influenced by their experiences with Froebel's blocks in kindergarten.

While students play with blocks that use an arrangement of studs to connect such as LEGO™ blocks, they are also building their spatial and proportional awareness. Students can use a range of mathematics to compare two blocks.

Comparing the number of studs in the arrays numerically:



Red



Blue

There are 8 studs on the red block and 4 studs on the blue block.

The red block has more studs.

The red block has 4 more studs than the blue block.

The red block has 2 x as many studs as the blue block.

Comparing the blocks using measurement:

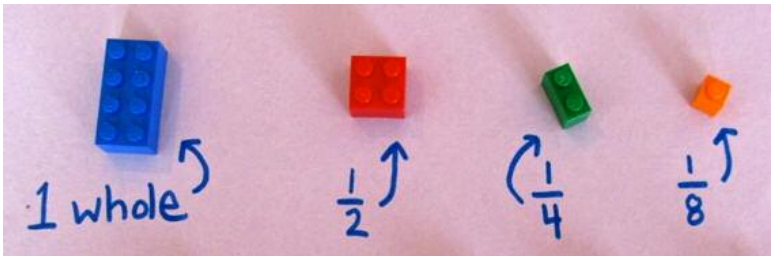
The blue block is half as long as the red block.

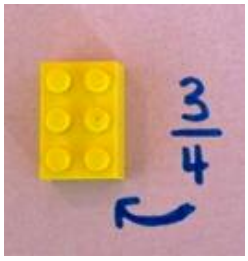
It would take two blue blocks to cover the red block

Comparing the blocks using geometry:

It would take two blue squares to make one red rectangle.

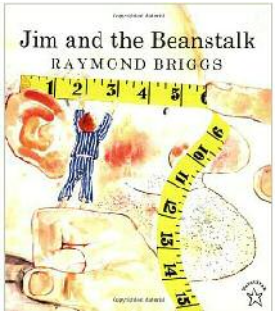
Students can use fractions of a whole, fractions of a set, and ratios to compare the blocks





http://www.scholastic.com/teachers/top-teaching/2013/12/using-lego-build-math-concepts

LINKS TO LITERATURE: *Jim and the Beanstalk*



In this sequel to *Jack and the Beanstalk*, Jim climbs a beanstalk and finds a bald, toothless, near –sighted giant. Jim measures the giant so that he can bring him a wig, dentures and glasses.

This story can be used to set the context for using mathematics to explore:

1. Measurement: Jim spends a lot of time measuring different parts of the giant's body. Find something in the room that is 3 times longer than your thumb. Find an object that has twice the circumference of your arm.
2. Proportional relationships in the body: How many heads tall are you?
3. Critical literacy skills: Measure some of the items in the illustrations from the book. If they were real, do you think they would actually be that size? Why or why not? Explain your thinking.

LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Tangram Runner Problem:

Explain how to make the image of the runner using 7 Tangram pieces.



Solution 1

The head is a square, which has the same area as 2 small triangles (feet). The 2 largest triangles are the shoulders (90^o angle points down) and bottom of the body (90^o angle points up). The left leg is the parallelogram and the right is the middle-sized triangle.

Solution 2

Shape	*# Pieces*	*Location on runner*	*Comparisons*
Small triangle (S)	*2*	*feet*	*2 S = 1 SQ*
"			*2 S = 1 M*
"			*4 S = 1 L = 2 M*
Medium triangle (M)	*1*	*right leg*	*1 M = 1 P*
Large triangle (L)	*2*	*body*	*1 L = 2 M = 4 S*
Square (SQ)	*1*	*head*	*1 SQ = 1 M*
Parallelogram (P)	*1*	*left leg*	*1 P = 2 S*
Total	*7*		

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to make a plan to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Tangram Problem 1:

Examine the relationships of size and shape in a set of Tangrams. If the medium triangle has an area of 4 square units, what is the area of the total Tangram?



What information are we going to use to solve the problem?

- *7 pieces in a Tangram set; Area of all 7 tangram pieces?*
- *Medium tangram -> area of 4 sqare units*

AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Proportional Quantities:

1 medium triangle = 1 square = 1 parallelogram
Ratio i-> 1 : 1 : 1 or M : Sq : P

1/2 medium triangle = 1 small triangle
1 medium triangle = 2 small triangle
Ratio -> 1/2 : 1 = (2 x $\frac{1}{2}$) : (2 x 2)
or $\frac{1}{2}$ M : 1 S = 2x1/2 M: 2 x 1 S

1 medium triangle = $\frac{1}{2}$ large triangle
2 medium triangles = 1 large triangle
Ratio -> 1 : $\frac{1}{2}$ = 2x1 : 2 x 1/2
or 1M : $\frac{1}{2}$ L = 2 x 1M : 2 x 1/2 L

Solution 1

1 Medium triangle (M)
= 4 sq units

2 Medium triangles (M)
= 1 L (large triangle)
=2 x 4 sq units =8 sq units

4 Small triangles (S) = 1 L
=4 x 2 sq units =8 sq units

2 S = 1 P (parallelogram)

2 S = 1 SQ (square)

Identifying Area Proportional Relationships Relative to 1 Shape

Solution 2

M + P + SQ
= (4 + 4 + 4) sq units
= 12 sq units
(or M=P=SQ=4 sq units)

2 S = 2 x 2 sq units
= 4 sq units
2 L = 2x 2 x 4 sq units
= 16 sq units

Total area of Tangram
M+P+SQ+2S+2L =
(12+4+16) sq units
= 32 sq units

Area Proportional Relationships Relative to Several Shapes

Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of students' mathematical discussion.

Grade 4	Grade 5	Grade 6
- describe relationships that involve simple whole number multiplication	- describe multiplicative relationships between quantities by using simple fractions	- represent ratios found in real-life contexts, using concrete materials, drawings and standard fractional notation

Proportional Relationships

As a change in one quantity, which causes a corresponding change in another quantity (co-variation)

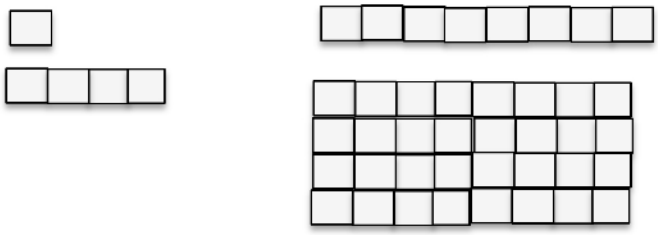
If 1 triangle is 4 square units,
then 8 triangles is 32 square units } *related by a multiple of 8 (scale factor)*

comparing ratio within and between objects

➔ *equivalent ratio* $1 : 4 = 8 : 32$
$1 : 4 = 8x1 : 8x4$

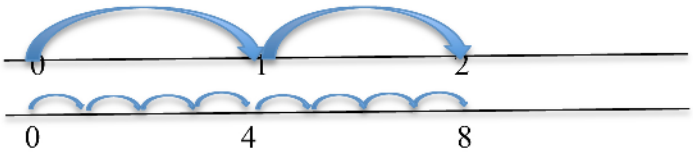
➔ *equivalent fraction* $\frac{1}{4} = 8/32$
$\frac{1}{4} \times 8/8 = 8/32$

If *Then*



1 : 4 = 8 : 32

equivalent ratios by a scale factor of 8



Solution 3

Whole Tangram	*L*	*M*	*S*	*SQ*	*P*
part	*1/4*	*1/8*	*1/16*	*1/8*	*1/8*
sq units	*8*	*4*	*2*	*4*	*4*

2 L + 1 M + 2 S + SQ + P
= (2 x ¼) + 1/8 + (2 x 1/16) + 1/8 + 1/8
= ½ + 1/8 + 1/8 + 1/8 + 1/8 = 1
➔ *8 + 8 + 4 + 2 + 2 + 4 + 4*
➔ *(2 x 8) + (3 x 4) + (2 x 2)*
➔ *16 + 12 + 4 = 32 sq units*

Solution 4

M (Medium Triangle) = 4 sq units and it is 1/8 of the whole trangram

Whole Tangram has units, so 8 x 4 sq units or 32 square units

Fractions of a Whole Proportional to 1 Shape

Unit Fraction Proportionality

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed

Highlights/Summary:

A proportional relationship shows a multiplicative relationship; that is, a change in one quantity which causes a corresponding change in another quantity using a scale factor (co-variation).
(e.g., If 1 triangle is 4 square units, then 8 triangles is 32 square units → scale factor of 8)

Compare quantities using multiplicative relationships by:

- *equivalent ratios (1:4 = 8x1 : 8x4 = 8 : 32)*
- *equivalent fractions: 1/4 = 8x1/8x4 = 8/32*

AFTER (Practice) – 5 to 10 minutes

- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Cat Tangram Problem:

If the tail of the cat is worth 100 points, what is the value of the whole cat?



Solution 1

Medium triangle (M), parallelogram (P), square (SQ) is 100 points; small triangle (S) is ½ of M; large triangle is 2xM

P	*SQ*	*M*	*2xS*	*2xL*
1	*1*	*1*	*2x1/2*	*2x2*
100	*100*	*100*	*100*	*400*

Whole cat tangram is worth 800 points

Solution 2

The parallelogram is 1/8 of the whole Tangram

1/8 Tangram is worth 100 points

Whole tangram has 8 1/8; so 8 x 100 points = 800 points

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.

LET'S DO MATH –



BEFORE (GETTING STARTED)

3 Generations Problem:

In the picture, you can see Josh, his father Jon, and Jon's father Jim.

When you compare their heights they are in the ratio 5 : 6 : 7.

If Jon is 150 cm tall, how tall are the others?

Solution 1:

5 : 6 : 7 = josh height : jon height : jim height
5 : 6 : 7 = josh height: 150 cm : jim height

josh height = (150 ÷ 6) x 5 = 125 cm
jim height = (150 ÷ 6) x 7 = 175 cm

Josh is 125 cm tall. Jon is 150 cm tall.
Jim is 175 cm tall.

Solution 2:

Josh Jon Jim

1 2 3 4 5 6 7

20 40 60 80 100 120 140 ←jumps of 20 too small
25 50 75 100 125 150 175 ←jumps of 25 just right

Grandfather Jim is 175 cm tall, father Jon is 150 cm and son Josh is 125 cm tall.



DURING (WORKING ON IT)

Enlarging Picture Problem:

The Photo Shop prints photos from negatives, smart phones or computers. They offer many sizes of prints. The chart was destroyed by a coffee spill. If a magnification of the small horse photo, 12 x 18 units, takes the length to 30 units, what will be the width and the magnification factor?

Width	8	12	16	20
Length	12	18	24	30

AFTER (CONSOLIDATION)

Solution 1:

I'm going to work in a ratio table to get the bottom number to be 30.

12 ÷ 2 | 6 ÷ 3 | 2 x 10

12	6	2	20
18	9	3	30

18 ÷ 2 | 9 ÷ 3 | 3 x 10

So the dimensions could be shrunk by a factor of 6 (2x3) and then enlarged by a factor of 10. The 12 x 18 photo will now be 20 x 30 units.

Solution 2:

The length changes from 18 units to 30 units.
18 units X factor of change = 30 units
18 units X _____ = 30 units
30 units ÷ 18 = 1.666 This is the magnification factor.

The width will be multiplied by 1.666 too.
l = 1.666 x 12 = 20

The 12 x 18 photo will be enlarged to 20 units x 30 units

Solution 3:

The pictures are similar shapes. Angles are equal and the sides are proportional.

$$\frac{\text{small length}}{\text{large length}} = \frac{\text{small width}}{\text{large width}}$$

$$\frac{18}{30} = \frac{12}{W}$$

OR

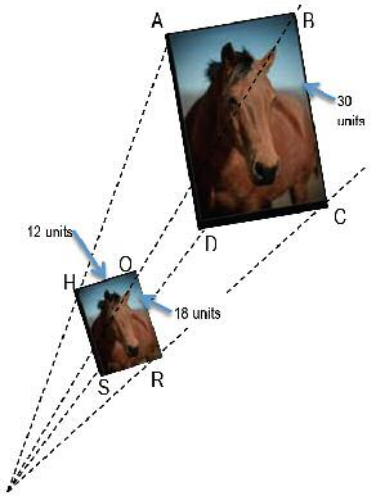
$$\frac{\text{large length}}{\text{small length}} = \frac{\text{large width}}{\text{small width}}$$

$$\frac{30}{18} = \frac{w}{12}$$

$$w = \frac{30x12}{18}$$

$$w = 20$$

The dimension of the blown up photo will be 20 x 30 units.



Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solutions 1 and 3?

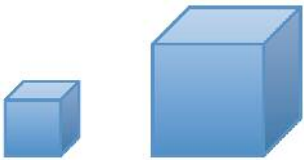
- *Solution 2* – This method honours the similarity of the figures and determines the scale factor necessary to take a length of 18 to a length of 30. Then the factor is used to determine the width using the constant scale factor.
- *Solution 1* – The ratio table sets up equivalent ratios with both width and length represented. Dividing and then multiplying the top and bottom units by the same number lets you determine the width when you know the length.
- Solution 3 – The solution shows equivalent ratios of side lengths because the polygons are similar. Once the ratios are created, the proportion can be solved using equation-solving strategies – whatever you do to one side, you do the same to the other.

LET'S DO MATH –

BEFORE (GETTING STARTED)

Stretching Volumes Problem:

Box A, a cube, has a volume of 1 cm^3. If the lengths of the sides are increased, to make Box B, what is the relationship between the volume of Box A and Box B, which is a box made by increasing the lengths of the sides?



Box A Box B

Solution 1:

Box A Let side length be 4 cm.

Volume (A) = l x w x h

= 4 x 4 x 4 = 64

Box A = 64 cm^3.

Box B is a cube with side lengths of 12 cm (4 x 3)

Volume (B) = 12 x 12 x 12 = 1728

Relationship $= \frac{V(A)}{V(B)} = \frac{64}{1728} = \frac{1}{27}$

So, Box B has 27 times greater volume that Box A.

Solution 2:

Box A = 1 x 1 x 1 = 1 cm^3

Box B = 2 x 2 x 2 = 8 cm^3

When you double the length of sides the area is 2 x 2 x 2 or 8 times bigger.

Solution 3:

Box A = s x s x s = s^3 cm^3

Box B = ns x ns x ns = n^3s^3 cm^3

When you multiply the length of the sides of a cube by any number, n, the volume increases by n^3 cm^3.

DURING (WORKING ON IT)

Suitcase Capacity and Mass Problem:

The ratio of dimensions between the smallest suitcase and the largest suitcase is about 1.4. The mass of an empty suitcase increases proportionally as the capacity of the suitcase increases.

Approximately what is the mass of the largest suitcase if the smallest suitcase has a mass of 3 kg?



Dimensions	L	W	D
small	50	33	20
medium	60	40	24
large	70	46	28

AFTER (CONSOLIDATION)

Anticipating Student Responses:

Solution 1:

V (small) = 50 x 33 x 20 = 33 000 cm^3

V (large) = 70 x 46 x 28 = 90 160 cm^3

Comparison of capacity (ratio) $= \frac{90\,160 \text{ cm}^3}{33\,000 \text{ cm}^3} = 2.7$

Comparing mass $= \frac{large}{small} = \frac{2.7}{1} = \frac{M}{3}$

M = 8.1 kg

Solution 2:

I know that if the side lengths on a polyhedron increase by a factor of n, then the volume / capacity increase by that number multiplied by itself 3 times.

So the capacity increases proportionally in terms of the ratio of each linear dimension:

1.4 x 1.4 x 1.4 = 2.74

So the mass should be 2.74 x 3 = 8 kg

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solutions 2?

Solution 1 – This solution shows the calculation of volume / capacity of both bags and confirms the increase will be factor of 1.4 for each length, then multiplies that factor by the mass.

Solution 2 – This student simply uses the comparison of 1.4 x 1.4 x 1.4 = 2.74 (an increase for length, increase for width, and increase for height) to multiply the mass of the smaller case by 2.74. (3 x 2.74 = about 8 kg).

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Try these proportional reasoning problems yourself first and then offer them to your students.
- To suit your students' learning needs, vary the types of measurements (length, area, volume) in relation to different polygons (e.g., triangle, rectangle).
- Consider also having students build similar polygons using single shapes and combinations of shapes to see proportional relationships.
- Practise noticing the different strategies and the breadth of the mathematics your students use to describe, show and make sense of proportional relationships, in terms of ratios of measurement attributes of 2D and 3D figures.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit key mathematical ideas and strategies for determining proportional ratios of linear, area and volume measurements as the students discuss and analyze each other's solutions to co-construct success criteria for the lesson learning goal.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below:

Cai, J. & Sun, W. (2002). Developing students' proportional reasoning: A Chinese perspective. In B. Litwiller and G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 195-204). Reston, VA: NCTM.

Kent, L., Arnosky, J., & McMonague, J. (2002). Using representational contexts to support multiplicative reasoning. In B. Litwiller and G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 145-152). Reston, VA: NCTM.

Lesh, R., Post, T., & Norther, M. (1988). Proportional Reasoning. In J. Hiebert & M. Behr (Eds.) *Number Concepts and Operations in the Middle Grades* (pp. 93-118). Reston, VA: Lawrence Erlbaum & National Council of Teachers of Mathematics.

Lobato, J, Ellis, A.B. (2010). *Developing essential understanding of frations, proportiosn and proportional reasoning for teaching mathematics, Grades 6-8*. Reston, VA: National Council of Teachers of Mathematics.

Lobato, J., Ellis, A. Charles, R., & Zbiek, R.M. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics in grades 6-8*. Reston, VA: National Council of Teachers of Mathematics.

Small, M. (2010). *Big ideas from Dr. Small, Grades K-3.* Toronto: Nelson Education.

Ontario Ministry of Ontario. (2012). *Paying attention to proportional reasoning, K-12.* Toronto, Ontario: Queen's Printer for Ontario.

Small, M. (2010). *Big ideas from Dr. Small, Grades 5-8.* Toronto: Nelson Education.

CLOSING

Our next issue will focus on proportional reasoning within the patterning and algebra and data management and probability strands. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a problem solving-based teaching and learning lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or mkestell@sympatico.ca