8-PAGE INSERT ELEMENTARY MATH

Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

Vol. 52 #4
June 2014

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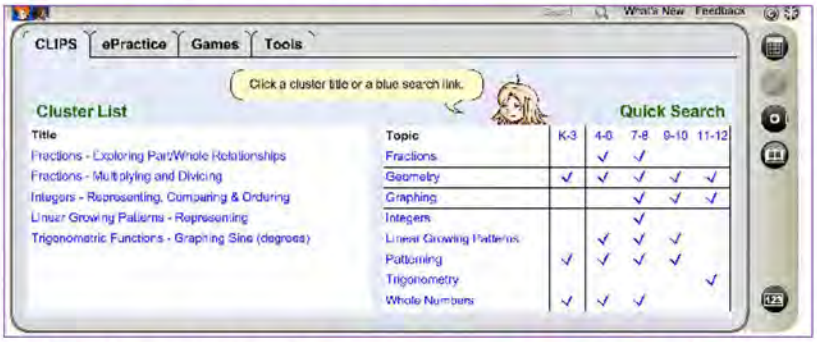
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See What's New with CLIPS? A Site Refresh and a New Search Engine!



See What's the Problem? Calculating Combinations of Children

The eleven partitions of 6 are shown in figure 5.

6,

1 + 5, 2 + 4, 3 + 3,

1 + 1 + 4, 1 + 2 + 3, 2 + 2 + 2,

1 + 1 + 1 + 3, 1 + 1 + 2 + 2,

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See Puzzle Brings... Olympic Rings



arc of radius

$$\sqrt{(r+a)(r+2a)} - \sqrt{(r-a)(r-2a)}$$

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Volume 46 – This resource for teachers includes the concepts of addition, subtraction, multiplication, and division of whole numbers and decimals, alternative algorithms, mental math strategies, manipulatives, and links to literature. It will be useful for exploring these aspects of teaching and learning personally or with a team of teachers. Includes one two-coloured bead string.





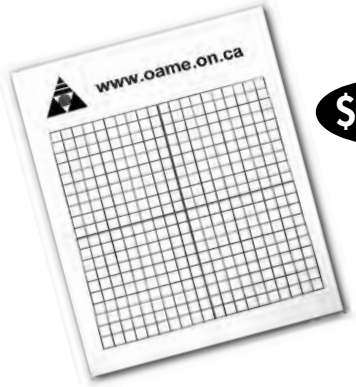
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Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Immaculate K. Namukasa, along with **separate files for each of the text, graphics, and/or photographs**. Please e-mail all files to Immaculate K. Namukasa at **immaculate.namukasa @oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and e-mail address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME.

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The views expressed or implied in this publication, unless otherwise noted, should not be interpreted as official positions of OAME.

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Submission of Advertisements
Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** at the above address. Courier is recommended to avoid possible delays. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

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▲ Editor's Message

IMMACULATE KIZITO NAMUKASA
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Immaculate Kizito Namukasa is an associate professor at the University of Western Ontario. Her research projects include "Mathematical thinking and activity of elementary and middle school learners," "New teachers learning to teach mathematics," "Expert teachers implementing innovative mathematics methods and environments in their classrooms," "Parents doing mathematics with their children in community settings," and "Complexity research inquiry and critical mathematics education in Canadian and international contexts."

In just a few weeks, the 2013–2014 school year will be coming to an end. Every school year, teachers and educators bring forth, with learners in their classrooms, a lived curriculum! This I speak of from two vantage points—as an educator and as a parent sharing in the school experiences of two elementary school children. Do they ever grow in their knowledge of school subjects, month after month, during the school year! As teachers work diligently hard to enact this curriculum that is experienced by learners, they find themselves faced with several key questions. Answers to these questions have far-reaching implications for each of the learners in a teacher's classroom. This issue responds to a set of key questions that arise for mathematics teachers, including: *In what ways can I demonstrate for students that this mathematics topic connects to the real world and to real people? How can I teach this difficult procedure in ways that make sense for learners? How do I make these innovative ways of teaching mathematics work for my students? In what ways is technology changing the mathematics I teach and how I teach it? How can I cultivate positive attitudes toward math in my classroom?*

The article entitled *An Examination of Mathematics Anxiety Research* offers tips on how math anxiety among students might be reduced and prevented. The *Technology Corner* column shares how an open- source technology, Desmos, may change one's teaching of patterning and algebra. *What's New with CLIPS?* responds to the question asked of any technology—

What new affordances are within CLIPS? This column demonstrates a new search engine now available within CLIPS. In the article *Collaborative Learning*, the author shares strategies that facilitate collaboration among high school and college students.

Several columns and articles in this issue respond to questions that go beyond grade bands. *Hey, It's Elementary* elaborates on how to connect "learners to the stories of the lives of authentic people, who played and solved mathematics mysteries, games, and puzzles" (this issue). The *In the Middle* column, explores the question of "making time for mathematicians—their lives, the aesthetic, history, and philosophy of their work"—*in the classroom*. The *Mathematics for and by Teachers* column responds to questions on various ways of teaching division of fractions and how to teach it in ways that make sense to the learners.

What's the Problem? shares a solution to a mathematics question and poses a math question for you and your students for the summer break. The report from NCTM shares new publications by NCTM and possible changes for the NCTM annual meeting. The *Fields Institute* reports on activities of the Mathematics Education Forum. Special features in this issue include a book review, a reprint of a book chapter, a geometry puzzle, and an announcement of 2014 Descarte Medalists. The book entitled *Mathematics Models For Teaching* is reviewed, and a chapter from the book *Why you Hate Math and How to Feel the Love!* is reprinted in this issue. Both books are authored by Ontario researchers who spend time working in schools.

It is in the June issue that an outgoing President records her thoughts as she finishes her term. For the *Gazette* team, the editors, and associate editors, it has been a great year of working under the leadership of its president, Sonia Ellison. In her message in this issue, she focuses the attention of the Ontario reader of the *Gazette* to a place of mathematics in the province, the OAME vision and mission. The president's message and the editorial are a "must read." Both articles address issues that concern mathematics educators and teachers as a collective. The editorial statement entitled *Mathematics Education and the Media* is approved by the OAME executive. The statement is a timely response to issues on mathematics education identified in the media in Ontario and elsewhere in Canada during this school year. ▲

▲ Fields Institute MathEd Forum Notes

STEWART CRAVEN
E-MAIL: numeratecitizen@mac.com



Stewart Craven presently teaches courses for the Continuing Education Department and the Transition Years Program at York University. He is an active participant in the Fields Mathematics Education Forum.

This summer, the Fields Institute for Research in Mathematical Sciences is hosting a workshop to celebrate Walter Whiteley's contributions to research in mathematics and math education. Walter has been, and continues to be, a champion of rich curriculum that will enhance the learning and teaching of geometry in our schools. The following is a very small cross-section of Walter's roles and initiatives:

- Researcher in Applied Geometry (in many disciplines), Geometry Education, and Spatial Reasoning
- Director and developer of the Honours undergraduate Mathematics for Education major for pre-service teachers of mathematics at York University
- York University representative and co-chair for the Council of Ontario Universities (COU) Mathematics Curriculum Review Task Force
- Past member of the Board of Directors of the Ontario Association of Mathematics Educators (OAME) and of the executive of the Canadian Mathematics Education Study Group
- Contributor to a project designed to develop computer programs that can predict rigidity and flexibility of protein structures (resulting in an online tool that allows researchers to make predictions relating to how proteins function and how drugs can change this)

The workshop, entitled "Workshop on Making Models: Stimulating Research in Rigidity Theory and Spatial–Visual Reasoning," takes place from August 5 to August 9, 2014 at the Fields Institute, University of

Continued on page 28

President's Message

SONIA ELLISON
E-MAIL: sonia.ellison@oame.on.ca



As I finish my term as President of our organization, I reflect on this school year. Mathematics education is definitely a provincial focus. This is not a surprise to any of us who care about mathematics, education, and mathematics education. We learned of the PISA results in the Fall, and news reporters wondered what was happening in classrooms in Canada (**www.theglobeandmail.com/news/national/education/canadas-fall-in-math-education-ranking-sets-off-red-flags/article15730663/**). But we also celebrated the PISA results because our Canadian students responded positively to the survey questions about their attitudes toward school. Our students generally like going to school. President-Elect Paul Alves spoke on our behalf on the CBC Radio show, The Current (**www.cbc.ca/thecurrent/episode/2013/12/04/is-canada-failing-the-grade-at-math/**), where he tried to convey not only his passion for mathematics education, but the vision of our organization. The Minister of Education released a statement in January regarding additional supports and funding that would provide more opportunities for educators to improve their mathematics instruction practices (**http://news.ontario.ca/edu/en/ 2014/01/new-math-supports-and-resources-for-the-classroom.html**). Many faculties of education and professional organizations are offering Mathematics Additional Qualifications courses at reduced rates due to the grants made available by the Ministry of Education. Math education still continues to be on the minds of teachers as we wait for our curriculum refresh.

This year has also been filled with positive opportunities to learn with and from each other. Chapter mini-conferences, the Leadership Conference, and the Annual Conference provided lots of occasions for sharing our learning and gaining new knowledge. Did you get a chance to learn something new this year? Have you considered ways to share your new learning—maybe writing an article for the *Gazette*, or presenting a session at OAME 2015? Being reflective on what we have learned, why we have learned it, and what we are going to do next will help ensure that OAME continues to have a voice in making a difference for students and educators across the province.

We find ourselves, as math educators, in very political times. As an organization, all members are asked to look to our mission statement to remind us that **we promote** *excellence in mathematics education throughout the Province of Ontario.* **We work** *for excellence for students by striving for a curriculum that will prepare them to meet the mathematical challenges in their lives, by fostering familiarity with the methods of mathematics and an admiration for its principles, and by giving recognition and respect to their own mathematical achievements.* **We work** *for excellence for educators by promoting their role as facilitators and resource persons, by supporting a variety of teaching methods to meet the varied learning styles and needs of students, by encouraging their students to gain facility in learning both co-operation and independence, and by celebrating the mathematical excellence of both individuals and groups.*

As I think back to the previous President's Message columns, I wonder:

- How did you get involved in OAME this year?
- How will you be a teacher-leader next year?
- Will you become a mentor to a beginning teacher or teacher candidate soon?
- Will you integrate blended learning and twenty-first-century fluencies into your classroom?, and
- How will you continue *promoting excellence in mathematics education*?

I would like to take this opportunity to convey a special thank you to those of you who have been there for me and for OAME this year:

- Executive Directors – Fred and Lynda Ferneyhough – Thank you for your commitment to our organization and your guidance throughout my term. I could not have done it without you.
- Past President – Cathy Hall – Thank you for your advice, your sense of humour, and your dedication to our organization. I would like to think of you as a co-president, not just a past president.
- Vice-Presidents – Bill Otto, Tim Sibbald – Thank you for helping me make sense of the often competing interests and to stay true to the vision of OAME.
- President-Elect, Paul Alves – Thank you for your devotion to mathematics education in the twenty-first century. I look forward to learning with you next year and supporting your vision for OAME.
- OAME Directors and Chapter Reps – Thank you for giving of your time and talents. Your work throughout

the past year is a testament to your commitment to OAME and to all of our members.

- OAME Webmaster – Greg Clarke – Thank you for your many hours in making sure we have an online presence, and to you and Kathy Pilon, for your MCIS support.
- The Executive and/or Council of our 15 regional chapters – CHAMP, COMA, GOLDEN, GVMA, ISOMA, MAC^2, NOMA, NWOAME, PRMA, QSLMA, SAME SWOAME, TEAMS, WOMA, Y4MA – Thank you for all the time you spend working on behalf of your chapter members and for providing valuable learning opportunities for them.
- Editor of the *Gazette* – Immaculate Kizito Namukasa – Thank you to you and your team, including *Abacus* Editors, Kathy Kybota-Zarivnij and Mary Lou Kestell, for publishing this respected resource for all math educators.
- MAC^2 – Thank you for hosting OMO this year. This event is always an important one, and all of the Grades 7 and 8 students around the province benefit from participating in it.
- CHAMP – Thank you for "OAME 2014: CHAMPions 4 Change." This conference was one of the best attended ever. I would like to thank Co-Chairs Dwight Stead and Paul Alves and their organizing committee, student volunteers, sponsors, and presenters for making this event accessible to so many people, both face-to-face and virtually.
- All OAME members – Thank you for your commitment to OAME. YOU are the reason why OAME is recognized as one of the most dynamic and relevant professional organizations.

It has been an honour and a privilege to be the President of OAME. I look forward to continuing this journey as Past-President.

All the best for a restful and rejuvenating summer,

Sonia Ellison
OAME President
2013–2014 ▲

▲ Mathematics Education and the Media

(An Editorial statement approved by the OAME Executive Committee)

Following the release of the Organization for Economic Co-operation and Development's (OECD's) Programme for International Student Assessment (PISA) mathematics results last winter, there were inquiries of the OAME executive to provide responses to the media. Often these requests were for hurried interviews looking for a response by the end of the day. As such, they were not always conducive to properly analyzing the results that had been released. In addition, requests for immediate responses do not allow for OAME to ensure that responses represent the voice of the entire organization. That said, the media interest is demonstrable evidence of an active issue in mathematics education and one that members of the OAME cannot ignore. At the very least, the *Gazette* provides the OAME with an opportunity to provide information about the issue (our stated objectives say no less).

Discussion around this matter took place at a meeting of the executive, with the editor of the *Gazette*, and with the Board of Director's Communications Committee. This editorial has been prepared by the Communications Committee, and reviewed by the executive, as a response to the mathematics issues confronting Ontario. We do not propose that this editorial is a final answer; rather, we invite responses, particularly through letters to the editor, that can engage the OAME membership in discussion of the issues the media has identified.

Several sources in the media cited declining scores in mathematics as measured by the PISA international testing. While the OAME recognizes information from all organizations involved in mathematics education, it does not align itself with any single point of view. In this respect, while PISA is viewed as a significant, and important, contribution to the understanding of mathematics education, it is conducted with a focus on economic success (Brochu, Deussing, Houme, & Chuy, 2013, p. 10). Without a doubt, this is an important consideration in education, but the OAME is in a position that requires accommodation of other points of view. The Council of Ministers of Education, Canada, facilitate PISA testing within Canada as part of its mandate to

examine education policies. It is notable that it uses other sources of information as well, such as the Pan-Canadian Assessment Program (PCAP). Details from PISA cannot, and should not, be interpreted in isolation. It should also be considered that they are primarily intended as a component for policy analysis, and do not warrant a reactionary response.

The stated purpose of PISA testing in mathematics gives an emphasis to mathematical literacy (Brochu, Deussing, Houme, & Chuy, 2013, p. 12). While PISA testing has been done five times, only two of the tests, 2003 and 2012, are deemed to have focused on mathematics (Brochu, Deussing, Houme, & Chuy, 2013, p. 12). This is an important detail that does not appear in media reports because, while the PISA report acknowledges policy changes, it does not mention that Ontario released a new curriculum in 2005 and that full implementation takes considerable time. These details highlight the fact that time must be taken to properly understand the full context of what is being reported.

Consider that the PISA testing is specifically for students aged between 15 years 3 months and 16 years 2 months (Brochu, Deussing, Houme, & Chuy, 2013, p. 20). In Ontario, these students are in both Grades 9 and 10, and the reporting acknowledges that there is a significant difference in performance between the grades. While the proportions of students in the two grades are provided in the 2012 reporting, it is not for the 2003 report. In addition, the 2003 report indicates that there were students from Grade 11 included in the testing. This is one example of the care that is needed in interpreting a significant study of this nature.

In Ontario, teaching is a profession. Teachers are professionals, and it is clear from their participation in OAME activities and conferences that they are diligent about their professional responsibilities. In this respect, it is not the role of the OAME to react to particular details extracted from particular reports according to the needs of reporters. Rather, the OAME exercises its power through collaboration, and promotion of professional growth and excellence. We acknowledge that the media periodically takes time to discuss mathematics, but not as frequently as the members of the OAME. Furthermore, the vision that the OAME works toward must be much deeper than any single conversation or report.

Those of us who are engaged in the many opportunities of an organization as large as the OAME recognize that there is a need for our organization to support the professionalism of mathematics educators. This exceeds any one issue and includes a multitude of positive stories that do not make headlines. Mathematics education in Ontario is very lively, active, and demonstrates that professional growth is ongoing. In effect, the mission, objectives, and vision of the OAME are the stated reaction that the organization has to any single issue or inquiry:

Mission Statement

The mission of OAME/AOEM is to promote excellence in mathematics education throughout the Province of Ontario (oame.on.ca/main/index1.php?code=principles):

- Excellence for students by striving for a curriculum that will prepare them to meet the mathematical challenges in their lives, by fostering familiarity with the methods of mathematics and an admiration for its principles, and by giving recognition and respect to their own mathematical achievements
- Excellence for educators by promoting their role as facilitators and resource persons, by supporting a variety of teaching methods to meet the varied learning styles and needs of students, by encouraging their students to gain facility in learning both co-operation and independence, and by celebrating the mathematical excellence of both individuals and groups

Objectives

OAME/AOEM is committed to the following objectives:

- To aid the professional growth of mathematics educators
- To encourage and to engage in research, development, and evaluation of curriculum design and curriculum teaching materials related to mathematics education
- To promote coordination in mathematics education at all levels and effective communication within the mathematics community of Ontario
- To provide liaison with other educational organizations in Ontario and with mathematics education organizations in other provinces and countries
- To maintain affiliation with the National Council of Teachers of Mathematics (NCTM), and to promote projects of mutual concern
- To maintain an active and representative mathematics educator's voice with the ministry that is

responsible for education in the Province of Ontario, the various other educational bodies in Ontario, and the public

- To encourage the development of local, regional, and special-interest mathematics organizations in Ontario, to grant such organizations chapter-affiliate status as defined in Article VIII, and to maintain a mutually supportive relationship with them

A Vision for Learning Mathematics

The Ontario Association for Mathematics Education envisions a learning environment where all students do, see, hear, and touch mathematics in a profound and meaningful way. Our association sees the classroom as a community where teachers and students work collaboratively to learn and value mathematics.

In such a classroom, students engage in meaningful mathematical experiences through the use of concrete materials and manipulatives, visuals, technology, and other resources. Students build on their prior knowledge of key mathematical concepts and connect these concepts to their world. They understand the purpose of their learning through clearly defined expectations, goals, and assessment criteria. Students engage in inquiry, pose questions, and actively discuss their understandings with one another. Students demonstrate their understanding on a regular basis and strive to become independent and collaborative mathematics problem solvers.

Teachers use a variety of strategies that accommodate different learning styles and use resources to develop the investigative process in mathematics. They provide multiple opportunities for students to learn and apply mathematics in creative, purposeful ways while assisting and supporting students to help them achieve success in both independent and collaborative tasks. To facilitate student learning, teachers provide feedback to students throughout the learning process. Teachers engage in opportunities for professional growth and continuously reflect on their teaching practices and classroom experience.

The learning of mathematics is supported through the active and ongoing involvement of students, parents/guardians, educators, and the community at large.

The learning of mathematics should be based upon the following beliefs and principles:

1. Learning is structured around the key concepts of mathematics.
2. Learners build on prior knowledge and understanding, and work at a developmentally appropriate level to construct new personal meaning.
3. Learners are better able to transfer and to use knowledge meaningfully when new ideas are presented within rich and meaningful learning contexts, and when learners make connections both within and outside of school.
4. The learning process is enhanced when new ideas are presented in a variety of ways and when all learners have access to “hands-on” and “minds-on” activities and opportunities. This includes the use of manipulatives, computers, calculators, and hand-held data-collection devices.
5. Learning is enhanced when students experience a variety of instructional approaches, ranging from direct instruction to inquiry-based learning.
6. Learners require opportunities to interact with each other during the learning process, to teach others, and to explain and justify the processes of mathematics. Learning is enhanced when the learners are encouraged to communicate their understanding.
7. Effective mathematics education requires strategies that engage and motivate learners, that help students develop positive attitudes about themselves and about mathematics, and that encourage learners to make positive contributions to the learning environment.
8. Learning is enhanced when students and teachers understand learning expectations and assessment standards and participate actively in assessing and evaluating their personal growth.
9. Learning is enhanced when teaching, coaching, assisting, supporting, and feedback is planned and designed to improve specific learning and mental processing skills and to accommodate different learning styles and needs.
10. Learning is enhanced when teachers plan and reflect together.
11. Learning is enhanced when all participants in the process are active partners within a community of learners (e.g., parents/guardians, educators, students, and the community).

Reference

Brochu, P., Deussing, M.A., Houme, K., & Chuy, M. (2013). Measure up: Canadian results of the OECD PISA study. Council of Ministers of Education, Canada. Retrieved from **http://cmec.ca/Publications/Lists/Publications/Attachments/318/PISA2012_CanadianReport_EN_Web.pdf** ▲

▲ Announcement of the Descartes Medalist Award, University of Waterloo

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The Society of Descartes Medalists was founded in 1968 by the René Descartes Foundation for the Advancement of Mathematics at the University of Waterloo. This Society has met annually since 1968 for the express purpose of paying tribute to mathematics teachers of Ontario who are identified as being worthy of special recognition. Each year, the Foundation admits new members, chosen for their contribution to the noble cause of mathematics education, to life membership in the Society of Descartes Medalists. A list of members may be accessed at **http://cemclinux1.math.uwaterloo.ca/descartes/index.php?option=com_content&view=category&id=35&Itemid=53**.

The Faculty of Mathematics at the University of Waterloo enjoys a close relationship with the teachers of mathematics in the schools of Ontario, and indeed across our great country. It is a relationship cherished by the Faculty, for it has led to a spirit of friendship experienced only rarely in other disciplines, and it has ensured that students in our schools have richer educational experiences.

Many of these teachers have laboured with no thought of personal reward, devoting themselves not only to the cause of mathematics education, but also to the general well-being of their students and their profession. It would be unfortunate indeed were their efforts not recognized.

This year, three teachers were presented the award on Saturday, June 7 at the University of Waterloo.

We are pleased to announce that

Elvis Husic, Cameron Heights Collegiate Institute, Kitchener

Stephen Lau, John Diefenbaker Secondary School, Hanover

Laurissa Werhun, Parkdale Collegiate Institute, Toronto

are the medalists for 2014. ▲

▲ Technology Corner – Desmos Activities

MARY BOURASSA
E-MAIL: mary.bourassa@oame.on.ca



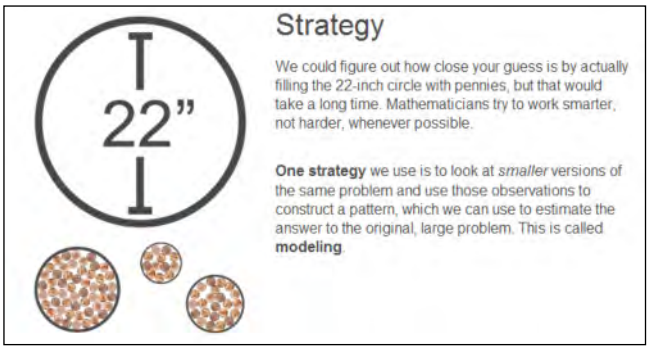
Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

The folks who work at Desmos (**www.desmos.com**) have created a great product that they back up with fantastic support. Anytime I have had a question about how to do something or whether Desmos has particular functionality, I tweet them (**@desmos**) and the reply is quick and on point. They love to feature graphs that you and your students create, and often will improve a file that you send them and will then return it to you. And with Desmos, you can always see what they did to your file and learn from it.

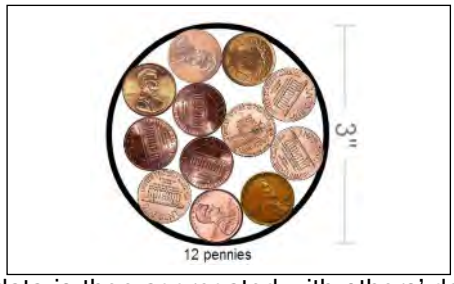
At Twitter Math Camp last summer, Desmos previewed their first activity: Penny Circle (**https://teacher.desmos.com/pennycircle**). This was in collaboration with Dan Meyer (who joined us remotely). Dan had done a three-act activity, in which he filled a small circle with pennies, then started to fill a large tile circle. The question that arose was, "How many pennies would fill the large circle?" Students were able to explore the number of pennies needed to fill circles of various diameters and come up with a model for the relationship between the number of pennies and the diameter of the circle. A video reveal concluded the activity.



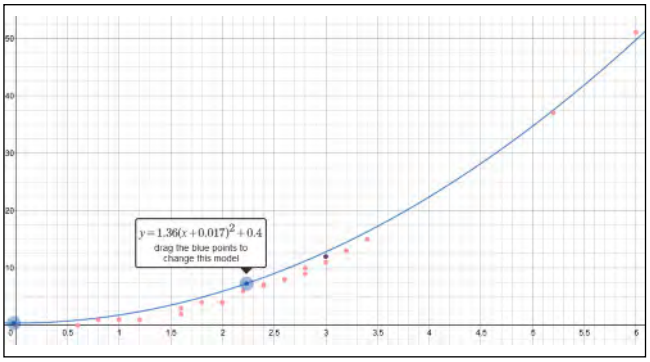
Desmos has digitized this activity. It starts with the same video, then prompts the user to guess the number of pennies. Then this screen appears:



This is followed by data collection. You then virtually fill circles with pennies (which is extra good for Canadians, since we no longer have pennies). This is really slick – your students will love it!

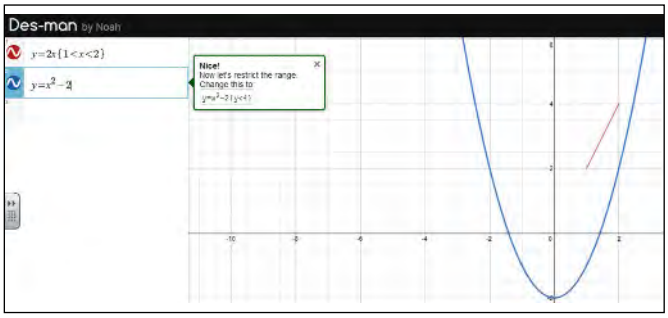


Your data is then aggregated with others' data – how cool is that? A scatterplot is shown, and you need to select a model: linear, quadratic, or exponential. The curve appears with two points that you can drag to adjust the equation. If you don't like the model you chose, you can change that too.

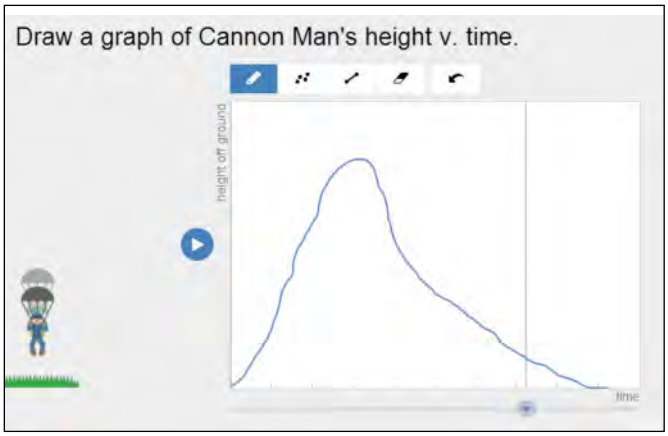


Once you are satisfied with your model, it is extrapolated out to beyond a 22” diameter circle, and your model's prediction for the number of pennies in the large tile circle is shown. If you are not satisfied with this, you can go back and adjust your model. The Act 3 video is subsequently shown to determine how closely you arrived to the correct answer, followed by some pedagogically important closing questions.

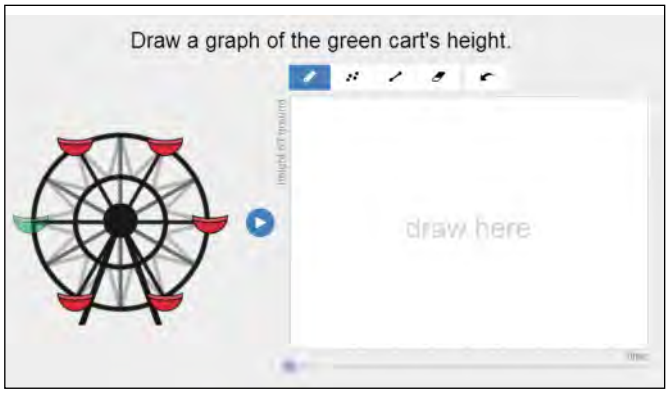
Des-man (**https://teacher.desmos.com/desman**) was inspired by Fawn Nguyen (**@fawnpnguyen**) and is a great way of introducing your students to Desmos. The activity leads them through the beginning of creating a person's face by prompting them to enter equations and restricting the domain or range and changing the colour. Then students are set free to make their best des-man!



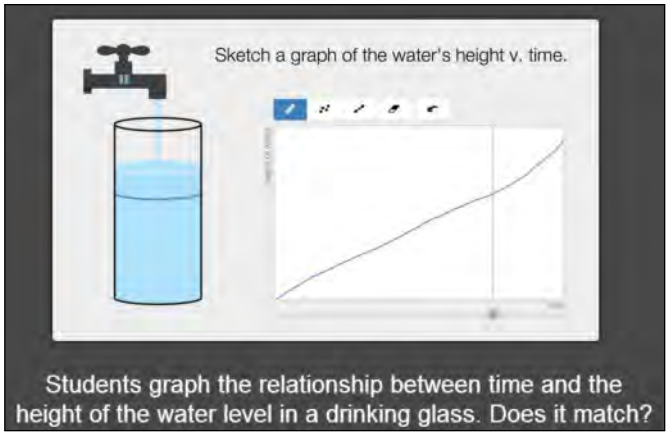
Function Carnival (**https://teacher.desmos.com/carnival/**) allows students to gain a better understanding of how to graphically represent a situation. Students start by watching a video of a man being shot from a cannon. Next, students draw the graph representing the cannonball man's height versus time. They can then see the animation of their graph along with the original animation and can adjust their graph as needed. If their graph isn't a function, multiple people will appear!



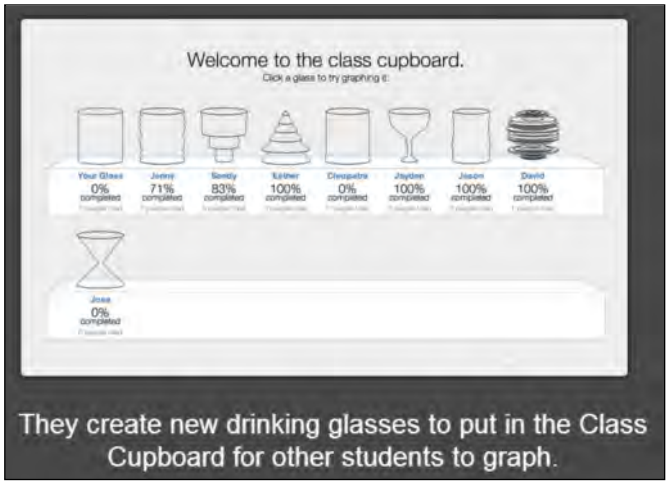
This is followed by other great scenarios, allowing students to analyze errors and create more graphs. Here is another part of Function Carnival:



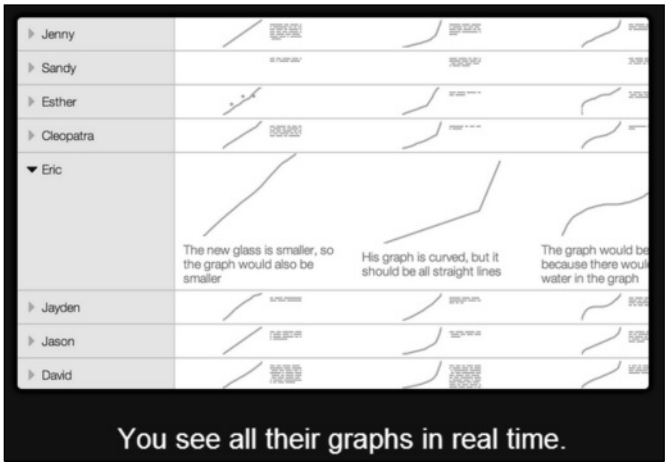
Water Line (**https://teacher.desmos.com/waterline**) is the latest Desmos activity. This is perfect for exploring relationships in Grade 9. After watching a video of a cylinder being filled with water, students draw a height-versus-time graph. Students then play the animation to see how closely their graph matches graph. They can scrub out parts of it or start over until they get a good match.



This is repeated with different containers. Students analyze errors and create graphs for many different-shaped containers being filled with water. They even get to create their own container to add to the class cupboard.



As with each of these activities, students join your class (via desmos), and you can see their graphs in real time. You know if someone is struggling, without the student having to raise his or her hand. This is formative assessment at its best – students know that if their graph doesn't match, they can fix it and check their new graph. If they need help, you will see it.



Desmos is doing a fantastic job of working with teachers to create activities that fill a void in student understanding. Students work at their own pace through dynamic activities that provide immediate feedback and are very engaging. Head over to **https://teacher.desmos.com/** to get started now! ▲

Graphic Organizing Opportunity!

Math teachers needed to create graphic organizers for Grades 7-10.

Preparation, revision, and rights for subsequent use of the graphic organizers are compensated with $150.

Please contact Dr. Timothy Sibbald at timothys@nipissingu.ca for additional details.

▲ OAME/NCTM Report

TODD ROMIENS
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Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a past president of OAME and a life member, as well as a previous editor of the Gazette*.*

NCTM has been looking carefully at its array of professional learning offerings. The review was prompted by a decade-long decline in attendance at the annual meetings and increased participation in Fall regional meetings. The annual meeting has always been held in the Spring, usually April.

There has been a marked decline in attendance at the annual meeting by classroom teachers. A few years ago, classroom teachers accounted for about 75 percent of attendees; in 2013, it was 53 percent.

NCTM has always considered the annual meeting to be its greatest and largest learning and networking experience for teachers, and NCTM wants this to continue to be the case. As a result, a motion to move the annual meeting to the Fall is now under consideration. Because of early commitments and the need to give ample warning of a change, this would not occur until 2020.

The lead time will allow for input from all of the NCTM affiliates, of which OAME is one. So if you have an opinion or ideas regarding this issue, feel free to contact me.

Twenty-five years ago, NCTM released *Curriculum and Evaluation Standards for Teaching Mathematics*, which presented a comprehensive vision for mathematics teaching, learning and assessment for Grades K–12 and helped mold the approach to teaching mathematics in Ontario. NCTM continues to provide publications to identify what it believes students should know and be able to do throughout their school mathematical experience.

In keeping with this tradition, a new publication called *Principles to Actions: Ensuring Mathematical Success for All* is now available through the NCTM website. This publication attempts to define and describe the principles and actions, including specific teaching practices that are essential for a high-quality mathematics education for all students. It offers guidance to teachers, mathematics coaches, administrators, parents/guardians, and policy-makers. It builds on the 1989 document mentioned above and supports the implementation of the Common Core Standards.

An additional two new publications are also available—*Success from the Start: Your First Years of Teaching Secondary Mathematics* and *High-Yield Routines for Grades K–8*—both of which may be purchased through NCTM. A complete list of resources and prices may be accessed through the NCTM website, **www.nctm.org/catalog**.

NCTM president Linda M. Gojak left the presidency in April of 2014. In her final message, she mentioned a number of things that the mathematics community has accomplished since 1989 and the challenges that must continue to be addressed. Here is what she said about teacher preparation:

> There continues to be a shortage of qualified math teachers. Too often elementary teacher preparation does not include ample coursework to provide pre-service teachers with the deep knowledge of mathematics content and pedagogy necessary to provide their students with the mathematical foundation and productive disposition needed to ensure success in middle and high school mathematics courses… A lack of focused work in building pedagogical content knowledge is likely to encourage teaching the way they were taught, rather than expressing research-based effective teaching and learning practices that contribute to student success.

While this may be somewhat true for Ontario teachers, a move to make teacher preparation a two-year program may help to alleviate this problem.

As we begin a new year of teaching, continue to provide your students with exciting learning experiences, and challenge them every day to become creative thinkers. ▲

▲ Call for Manuscripts

The *Ontario Mathematics Gazette* is inviting manuscripts for all grade levels.

Instructions for submission of manuscripts are found on page 1 of any *OMG*.

Contact the Editor for further details.

▲ What's the Problem? Calculating Combinations of Children

SHAWN GODIN
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Shawn is the head of Mathematics, Business Studies, and Library at Cairine Wilson Secondary School in Orleans, Ontario. He is a past member of the OAME Board of Directors and is currently Editor-in-Chief of the Canadian Mathematical Society's problem-solving journal, Crux Mathematicorum.

Welcome back, problem solvers. Hard to believe it is June already! This is especially true, since I am writing this on a cool, rainy Saturday afternoon in April. From my office window I can see the deep, dirty, melting snow in my front yard. Hopefully it will be gone by the time this appears in print! Last time, I left you with the following task:

A man was visiting his friend and noticed that there were a number of children playing in the backyard. When he asked his friend if they were his, he stated that his children were playing with children from three other families. The friend explained that the numbers of children satisfied the following conditions:

- There were fewer children from family A than from his family.
- There were fewer children from family B than from family A.
- There were fewer children from family C than from family B.
- There were fewer than 18 children in total.
- The product of the number of children from each family is equal to the friend's house number.

The man took to calculating, but he didn't have enough information. He asked his friend if family C had more than one child. As soon as his friend answered the question, he easily determined how many children from each family were present.

Using only the information given above, it is possible for you to also determine the number of children in each family.

This problem was inspired by Chapter 12, "Nine More Problems," from *Hexaflexagons, Probablility, Paradoxes, and the Tower of Hanoi*, by Martin Gardner. I mentioned Mr. Gardner's work in the December 2013 column "Gardner's Grid."

Often students want some sort of magic bullet, like a precise formula, to solve a problem. Interesting problems require insight. Frequently, when problem solving, you must make connections between various concepts, or use familiar tools in unfamiliar settings. In other cases, you have to jump right in and do some calculating and hope inspiration strikes, or some pattern begins to unfold.

The person from the problem has an advantage over us. We have to assume that he or she knows the house number of the friend and can count the number of children playing. Unfortunately, we do not have this information, but we know that the problem can be solved without it (unless the problem proposer is a nasty person!). Thus the best strategy is to jump in with both feet.

The problem only involves addition and multiplication, so it can be handled at many different grade levels. The key to solving the problem will be organization of a large number of possibilities and some subtle logic.

The problem can be solved with pencil and paper, but since there are so many cases involved, I will get some support from a spreadsheet. The set-up of the spreadsheet is shown in figure 1. The families are listed from the smallest number of children to largest, to help systematically search for all potential solutions. Note that values will be typed into columns A, B, C, and D for the number of children in each family. Also, the formula = A2+B2+C2+D2 has be put into cell E2, and the formula = A2*B2*C2*D2 appears in cell F2, which can be seen in the top right corner of figure 1.

F2 f_x =A2*B2*C2*D2

	A	B	C	D	E	F
1	Family C	Family B	Family A	Friend	Sum	Product
2	1	2	3	4	10	24

Figure 1

Now, if cells E2 and F2 through E50 and F50, say, are selected and the "fill down" option is used, the formula will be copied into rows 3 through 50, so that when the numbers for the families are filled in, the calculations will be done automatically.

I will do my search systematically, starting with the smallest family sizes that satisfy the conditions, that is, 1, 2, 3, and 4 children, respectively. Then the largest family number will be increased until the sum gets to 17, then I will increase one of the other numbers and continue the process. It is very important in a problem like this to tackle things methodically, so that we don't leave out any cases. Figure 2 gives a complete listing of all possible cases, where D represents the family of the friend.

C	B	A	D	Sum	Prod.	C	B	A	D	Sum	Prod.
1	2	3	4	10	24	1	2	6	8	17	96
1	2	3	5	11	30	1	3	4	5	13	60
1	2	3	6	12	36	1	3	4	6	14	72
1	2	3	7	13	42	1	3	4	7	15	84
1	2	3	8	14	48	1	3	4	8	16	96
1	2	3	9	15	54	1	3	4	9	17	108
1	2	3	10	16	60	1	3	5	6	15	90
1	2	3	11	17	66	1	3	5	7	16	105
1	2	4	5	12	40	1	3	5	8	17	120
1	2	4	6	13	48	1	3	6	7	17	126
1	2	4	7	14	56	1	4	5	6	16	120
1	2	4	8	15	64	1	4	5	7	17	140
1	2	4	9	16	72	2	3	4	5	14	120
1	2	4	10	17	80	2	3	4	6	15	144
1	2	5	6	14	60	2	3	4	7	16	168
1	2	5	7	15	70	2	3	4	8	17	192
1	2	5	8	16	80	2	3	5	6	16	180
1	2	5	9	17	90	2	3	5	7	17	210
1	2	6	7	16	84	2	4	5	6	17	240

Figure 2

How do we sift through these 38 cases to find the solution? Looking at the first case, with 1, 2, 3, and 4 children, respectively, we can tell that this cannot be the solution, since these numbers add to 10, and that is the only way 10 children satisfy the conditions of the problem. So if he saw 10 children, there would only be one solution, and the person would know it. Similarly, if the house number was 40, there is only one set of children, satisfying the conditions of the problem, which give a product of 40, so this couldn't be the case either. We are searching for situations where there are several cases that add to a specific number, and similarly where several cases have a product of a given number. If we focus on the products that repeat, we can shorten our list to what is given in figure 3.

C	B	A	D	Sum	Prod.
1	2	3	8	14	48
1	2	4	6	13	48
1	2	3	10	16	60
1	2	5	6	14	60
1	3	4	5	13	60
1	2	4	9	16	72
1	3	4	6	14	72
1	2	4	10	17	80
1	2	5	8	16	80
1	2	6	7	16	84
1	3	4	7	15	84
1	2	5	9	17	90
1	3	5	6	15	90
1	2	6	8	17	96
1	3	4	8	16	96
1	3	5	8	17	120
1	4	5	6	16	120
2	3	4	5	14	120

Figure 3

Now we run into a bit of a problem. In each of the cases, if we knew the number of children, we would be able to solve the problem. For example, if the house number was 96, and if there were 16 children, the numbers in each family would have to be 1, 3, 4, and 8, but if there were 17 children, the numbers would be 1, 2, 6, and 8. Since the person could not solve the problem, it seems that our assumption that the number of children was known is invalid. Anyone familiar with children knows this is reasonable, since in a large group of children, some could be hiding, while others are running hither and yon. As a result, the census taker runs the risk of counting a child multiple times, or missing one or more altogether. In retrospect, the person was told that there were less than 18 children, implying that the exact number of children was unknown.

The key to the problem is the question about how many children were in family C. That question would only be asked if the product was 120, since in all other cases, family C has one child. Since we know that the problem could be solved after the question was answered, there must be 2 children in family C. Hence, the other families have 3, 4, and 5 children, respectively, for a total of 14 children, and the house number would thus be 120.

It is sometimes necessary, when we solve a problem, to make assumptions. We should be cautious when we assume something (there is the old saying…), but

sometimes the problem is unsolvable otherwise. In these situations, our assumption should be stated, so that anyone reading the solution knows a condition has been inserted into the problem. Having made an assumption, we may have to, upon reflection, discard the assumption when other information comes to light, or the assumption contradicts an implication from the given information.

As a side note, before we get on to the homework, let's take a quick look at another tool that could have been used to solve the problem. To visualize the problem, we could use coloured counters, linking cubes, or some other manipulative. For example, all the possibilities that satisfy the conditions of the problem when there are 14 children are shown in figure 4. We would then just need to find the product of each set of four numbers. Different models and representations can sometimes give us insight into the problem that would have been missed otherwise.

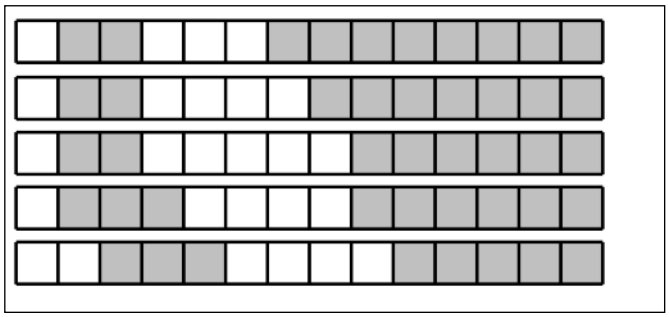


Figure 4

Our problem involved writing a number as the sum of four positive integers, where no number was repeated. In number theory, a partition of a positive integer n is a way of writing the number as a sum of positive integers. For example, the eleven partitions of 6 are shown in figure 5.

6,

1 + 5, 2 + 4, 3 + 3,

1 + 1 + 4, 1 + 2 + 3, 2 + 2 + 2,

1 + 1 + 1 + 3, 1 + 1 + 2 + 2,

1 + 1 + 1 + 1 + 2,

1 + 1 + 1 + 1 + 1 + 1.

Figure 5

Notice that the order of the numbers is not important in a partition. There are a vast number of things that students could play with involving partitions. Problems like "How many partitions of a number exist that only has two terms?" should be accessible to a broad range of ages and abilities. More inquisitive students may have fun exploring general questions involving all partitions of a number. These are examples of extensions, or "what if… ?" questions that can be asked upon solving the given problem.

Now for your homework:

At the beginning of the winter, there were at least 66 students registered in a ski class. After the class started, 11 boys transferred into this class and 13 girls transferred out. The ratio of boys to girls in the class was then 1 : 1. Which of the following is not a possible ratio of boys to girls before the transfers?

(A) 4 : 7 (B) 1 : 2 (C) 9 : 13 (D) 5 : 11 (E) 3 : 5

Until next time, have a great summer and happy problem solving! ▲

▲ Puzzle Brings… Olympic Rings

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After moving to Saskatchewan in 2004 and completing his teacher certification, Dr. Akulov began teaching Calculus, Mathematics, and Physics classes at Luther College High School in Regina. Before coming to Canada, he was an associate professor at the National Pedagogical Dragomanov University in Ukraine. He has a PhD in probability theory and mathematical statistics from Taras Shevchenko National University in Kyiv. His research interests include theory of functions, foundations of geometry, and mathematics curriculum development. In his spare time, Gregory enjoys travelling to scenic locations, playing music, and writing poetry.



Let the **centres** of the red, blue, yellow, and green rings of the Olympic Symbol (shown above) lie on the **arc** of radius *r*. If the distance between centres of green and yellow rings is 2*a*, show that the distance between the centres of red and yellow rings is

$$\sqrt{(r+a)(r+2a)} - \sqrt{(r-a)(r-2a)}$$

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▲ Hey, It's Elementary – Improving Mathematics Achievement with (Im)possible Thoughts

LYNDA COLGAN
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Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.

> "Can't you?" the Queen said in a pitying tone. "Try again: draw a long breath, and shut your eyes."
>
> Alice laughed: "There's no use trying. One can't believe impossible things."
>
> "I daresay you haven't had much practice," said the Queen. "When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."
>
> ~ Lewis Carroll, Alice in Wonderland

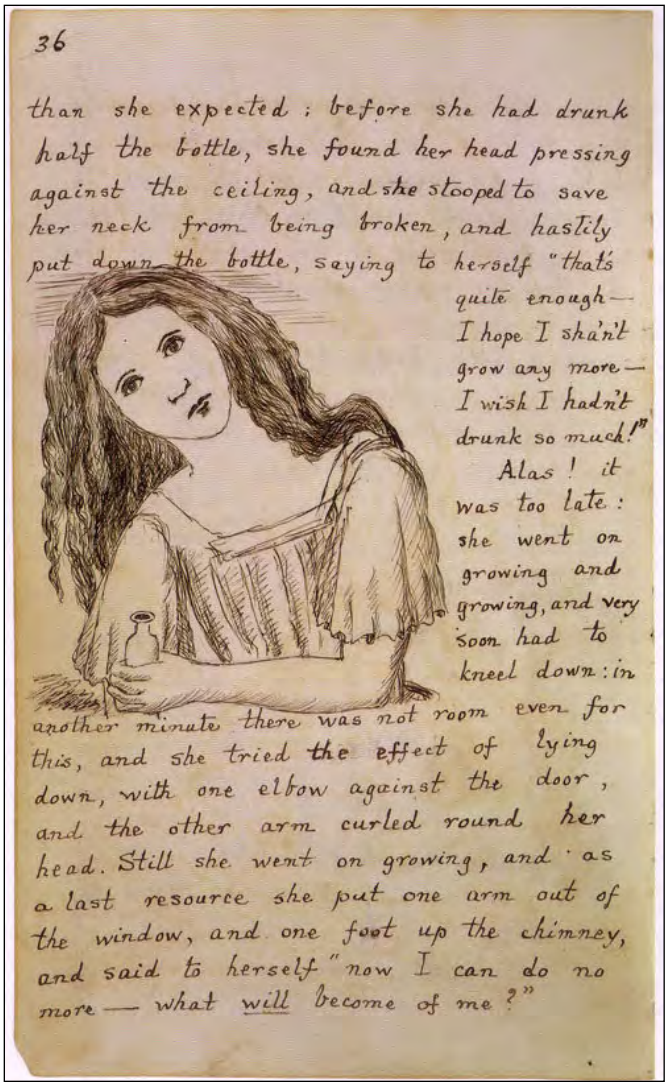


A LESSON IN ARITHMETIC: "Can you do division?" asked the White Queen. "Divide a loaf by a knife." "Try another subtraction sum," said the Red. "Take a bone from a dog."

A look at the negative trend reported in recent EQAO data shows a worrisome downturn in the number of students who report that they like mathematics—a 5 percent drop in girls who like math (55 percent in 2009 to 50 percent, in 2012), and a more drastic drop for boys (from 61 percent liking math in 2009 to 52 percent in 2012) (EQAO, 2012). It is not a stretch to imagine that at least one impossible thought is widely shared by children in Ontario: MATH IS FUN.

If we are to take the advice of cognitive scientists, then this impossible thought must become possible before we can begin to address the decline in mathematics achievement scores. Why? Research suggests that children with mathematical difficulties are most influenced by affective factors (Lebens, 2011), and empirical findings posit that it is the affective domain that governs a student's willingness to learn, his or her ability to accept new ideas, and the student's motivation to use new knowledge (Krathwohl, Bloom, & Masia, 1964). Quite simply, a learner's motivation, emotion, interest, and attention, as well as the relationship that the learner has with the content and process of learning, matters; i.e., when individuals engage in tasks in which they are motivated, they tend to exhibit a number of pedagogically desirable behaviours, including increased time on task, persistence in the face of failure, more elaborative processing and monitoring of comprehension, selection of more difficult tasks, greater creativity and risk taking, and selection of deeper and more efficient performance and learning strategies (Lepper, 1988).

Where does one begin to make the impossible, possible? Who among us cannot admit to the fact that our own early years were largely spent in play and exploration? It is likely that you, like me, experienced the world of childhood through direct engagement, and that we learned by being told bedtime stories, fairy tales, and listening to family stories around the dinner table. We became story-listeners when we were young, a fact that is not lost on neuroscientists, who have discovered that our brain reacts to personal stories and positive interactions in a particular way: "New advances in understanding the human brain [point out that it is] not a purely cognitive information processing system, but a system in which affective functions and cognitive ones are inextricably integrated with one another" (Picard, Papert, Bender, Blumberg, Breazeal, Cavallo, Machover, Resnick, Roy, & Strohecker, 2004). So what better strategy than to connect young learners to the stories of the lives of authentic people, who played and solved mathematics mysteries, games, and puzzles daily, voluntarily, and happily—just for fun.



So here begins my "Once upon a time" story… the aim of which is to dispel the myth that mathematics is impossible and to tell the tale of one famous citizen mathematician who made a difference to the lives of many people in many ways.

Charles Dodgson is better known to the world as Lewis Carroll, a man of mathematics and science, a famous photographer and illustrator, and an individual who also clearly delighted in writing delightful fantasy, first to entertain his younger siblings, and later the world.



Charles Lutwidge Dodgson was born on January 22, 1832 in the village of Daresbury in Cheshire, England. He was the third of 11 children born to Charles, an Anglican clergyman, and Frances.

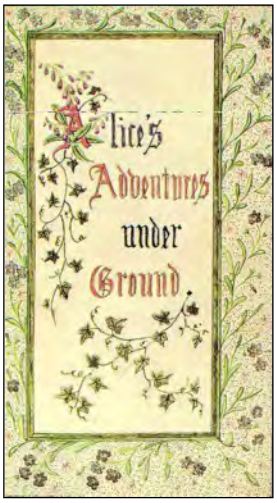


By age 11, his teacher, James Tate, declared that the young Charles had enormous potential in mathematics, and by age 14, he was enrolled at Rugby School in Warwickshire. In 1850, he began his education at Oxford, where he was befriended by the Liddell family. Henry Liddell, who was the Dean at Christ Church College, had four children: Harry, Lorina Charlotte, Alice, and Edith.

By 1855, he had graduated from Oxford with first-class honours, and began his career as a mathematics lecturer (a position he held until 1881). He published two mathematical textbooks in 1860 and by February 1861, had completed a third and was working on four more. He continued to be a prolific writer of mathematics books throughout his life, including The Formulae of Plane Trigonometry (1861), Elementary Treatise on Determinants (1867), Euclid and His Modern Rivals (1879), and, for a known insomniac, the aptly titled, Curiosa Mathematica, Part II: Pillow Problems Thought Out During Sleepless Nights (1893).

To keep his mathematical works and fictional works distinct, Dodgson decided he needed a pen name, which is said to have come from translating his middle and last names into Latin, i.e., Lutwidge = Ludovicus = Lewis and Charles = Carolus = Carroll.

In 1862, while Charles was travelling with the Liddell family, the Alice stories were born. He wrote the stories down in 1863, and gave them the title Alice's Adventures Underground (which was later titled Alice in Wonderland). By 1868, he had completed Through the Looking Glass.



Although Lewis Carroll is probably most well known for his imaginary masterpieces, The Mad Hatter, The Cheshire Cat, Tweedle Dum and Tweddle Dee, the March Hare and The Queen of Hearts, in mathematical circles he is highly respected for his enormous efforts to increase students' understanding of mathematics by viewing it as a game.

One of the algorithms (or rules) attributed to Lewis Carroll is that for determining the day of the week on which any date falls.



You can try Carroll's mathemagical trick yourself. Think of a date (year, month, and day). To test Lewis Carroll's algorithm, I used my parents' wedding day, July 19, 1952.

Begin with the century:

- Divide by four.
- Subtract the remainder from three.
- Multiply the remainder by two.

So for "19," I ended up with 0.

Next use the year:

- Add together the number of dozens, the remainder, and the number of fours in the remainder.

For "52," I ended up with 9.

Use the following table for the month:

January	0	July	6
February	3	August	2
March	3	September	5
April	6	October	0
May	8	November	3
June	4	December	12

For July, I got 6.

Use the day of the month.

- Add the results of each part together.
- Divide the answer by seven.
- Use the remainder to determine your day.
- The remainder gives the day of the week (refer to the table below).

Sunday	0
Monday	1
Tuesday	2
Wednesday	3
Thursday	4
Friday	5
Saturday	6

By adding 0 + 9 + 6 + 19, I ended up with 6, which means my parents were married on Saturday—a fact I know to be true from the pressed roses and carnations my mother carefully pressed between small squares of dated waxed paper and carried with her always between the pages of her bible.

Lewis Carroll also popularized a set of thousand-year-old logic brainteasers known as the *river-crossing puzzles*. He took great joy in posing these problems to children and watching them act out their attempted solutions.

Here is the problem: *A farmer has to transport a fox, a goose, and some grain across a river. He has a boat which he can row back and forth from shore to shore. He can take himself and one other thing in the boat. However, unless he is present, the goose will eat the grain and the fox will eat the goose. How can he transport all three things across the river?*

So, what does the farmer do? Here's one answer. The farmer crosses the river with the goose and leaves the goose on the far shore, then returns alone to the near shore. He brings the fox to the far shore and brings the goose back to the near shore. He then brings the grain to the far shore and returns alone to the near shore. Finally, the farmer brings the goose to the far shore. It takes seven trips.

Here is another river-crossing puzzle for you to try. This time, a circus owner wants to transport his elephant, his dog, his cat, and his mouse across a river. He can take one at a time across the river. But, unless he is there, the cat will fight with the mouse, the dog will fight with the cat, and the elephant will fight with the dog. However, as everyone knows, the elephant is frightened of the mouse, and if the mouse is present, the elephant will not fight with the dog. By taking the animals back and forth, how does the circus owner get all across without any fighting breaking out?

Lewis Carroll was also known for his creation of doublets puzzles. Every step in the word ladder differs by one letter and must formulate a new word.

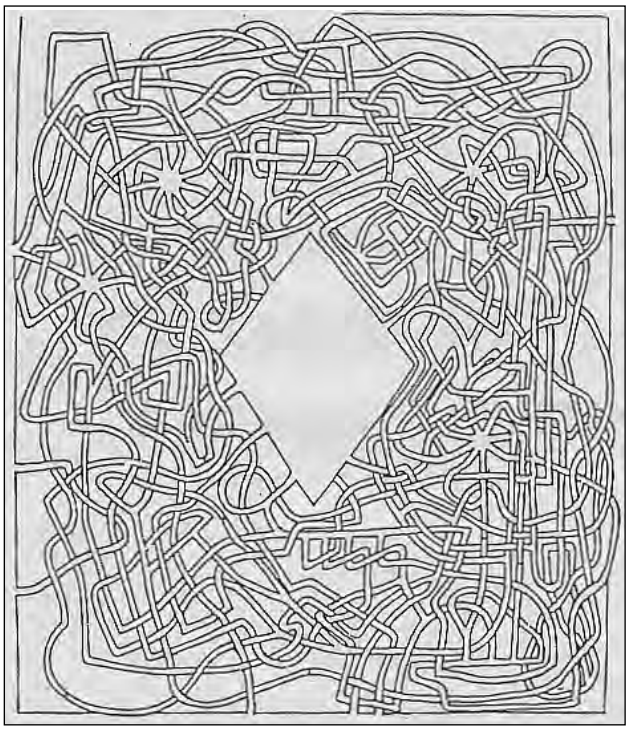
For example, how do you drive the PIG into the STY? By changing one letter at a time, the following word sequence is produced.

- PIG
- WIG
- WAG
- WAY
- SAY
- STY

Here are a few more doublets to challenge you: Raise FOUR to FIVE; Make WHEAT into BREAD; Dip PEN into INK; Touch CHIN with NOSE; Change TEARS into SMILE; Change WET to DRY; Make HARE into SOUP; and PITCH TENTS.

As we work together to improve children's learning and achievement in mathematics, it seems fitting to close with Lewis Carroll's famous maze problem. Although it is said that the maze represents Alice lost in the woods, starting from the little park in the centre and trying to

reach home, it may be appropriate to see the maze symbolically, as a teaching challenge filled with twists and turns, backs and forths, steps forward, and steps retraced. May you have fun as you find your way through this maze, and in turn, may you give your students as much fun along their mathematics education journey.



References

Bevan, B., Dillon, J., Hein, G.E., Macdonald, M., Michalchik, V., Miller, D., … Yoon, S. (2010). Making science matter: *Collaborations between informal science education organizations and schools.* Washington, DC: Center for Advancement of Informal Science Education.

Center for the Advancement of Informal Science Education. (2013). Welcome to Informal Science. Retrieved from **informalscience.org/**

Education Quality and Accountability Office. (2012). EQAO 2012 primary and junior divisions assessment results: Retrieved from **www.eqao.com/ProvincialReport/ProvincalReport.aspx?Lang=E&yr=12&cat=e**

Krathwohl, D.R., Bloom, B.S., & Masia, B.B. (Eds.). (1964). *Taxonomy of educational objectives: Handbook II: Affective domain.* New York: David McKay.

Lamon, S.J. (2007). *Rational numbers and proportional reasoning: Toward a theoretical framework for research.* In F. Lester (Ed.), Second handbook of research on mathematics teaching and learning, (pp. 629–666). Reston, VA: National Council of Teachers of Mathematics.

Lebens, M., Graff, M., & Mayer, P. (2011). The affective dimensions of mathematical difficulties in schoolchildren. *Educational Research International, 20*, 1–13.

Picard, R.W., Papert, S., Bender, W., Blumberg, B., Breazeal, C., Cavallo, … Strohecker, C. (2004). Affective learning – A manifesto. *BT Technology Journal, 22*(4), 253–269. ▲

▲ In the Middle – Making Time for Mathematicians in the Classroom

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> "What if at school you had to take an art class in which you were only taught to paint a fence? What if you were never shown the paintings of Leonardo da Vinci and Picasso? Would that make you appreciate art? Would you want to learn more about it? I doubt it. (…) While the paintings of the great masters are readily available, the math of the great masters is locked away."
>
> ~ Edward Frenkel (2013),
> Love and Math: The Heart of Hidden Reality.

How often do we use the word "mathematician" during a math lesson in relationship to a concept or idea discussed? How often do mathematics teachers have a chance to work closely with mathematicians or present mathematics as a subject that continues to discover and produce new ideas?

In his book *A Mathematician's Lament,* research mathematician and teacher Paul Lockhart (2009) wonders, "What other subject is routinely taught without any mention of its history, philosophy, thematic development, aesthetic criteria and current status? What other subject shuns its primary sources—beautiful works of art by the most creative minds in history (…)?"

Picker and Berry (2001) point out the fact that "in no other subject in school are students placed at such distance from the discipline as they are in mathematics."

Mathematician and University of California – Berkeley

professor Edward Frenkel (2013) is getting a lot of attention with his recent mathematical memoir, *Math and Love: The Heart of Hidden Reality*. He mentions that initially, he hated mathematics, until a family friend, a mathematics professor introduced him to the concept of symmetry group. The books received from his friend were unlike the textbooks used at school in that they allowed him to see "this mysterious beast—Mathematics—from different sides and get excited about it." On numerous occasions, when asked where teachers should begin when trying to bring real mathematical beauty into the classroom, he answered right away that the concept of symmetry is a good start. "Transformation of an object which preserves its shape and location is called a symmetry. Anyone can understand this. (…) The way physicists learned about elementary particles and were able to theorize their existence was precisely through a theory of symmetries. The Higgs Boson is this elusive particle recently discovered by the Large Hadron Collider in Geneva, and for this prediction, two physicists were awarded the Nobel Prize." (In a comment posted on the Prospero Blog of *The Economist* on December 5, 2013)

He goes on to blame mathematicians for not advertising well for this "magic world," unlike scientists do. In various interviews (e.g., a comment on the Podcast "Inquiring Minds" with Chris Mooney and Indre Viskontas on February 28, 2014), he points out that students around the world are stuck with a curriculum that is 1000 years old and that focuses mostly on the work of the great classics, like Euclid or Archimedes. Thus, it contributes to the general perspective that mathematics is equivalent to formulas and numbers and a subject that ceased to progress and innovate, so ultimately, why pursue it? He challenges us to wonder, if the same were true for science, would we have learned anything about atoms (let alone quarks), the solar system, or DNA?

On a recent trip to the Johnson Space Center in Houston, Texas, my eyes fell on a poster that read "MathAlive," which advertised a very special program offered by NASA at the centre, where students learn about robotics, do math simulations, and realize basically one thing: Math is alive! What a great idea. Nevertheless, it only proves again that students do not see math's applicability to the real world, and most of them do not know what a mathematician really does, nor have they met one. I don't think a science program would need to be called "Science Alive," as all students already know this to be true.

Mathematics needs this type of advertising. Mathematics needs to be seen as "alive" and studied by real people who have dedicated a lifetime to the quest of a proof, an idea, who left problems that have yet to be solved, whose ideas served the latest scientific discoveries, and who regard mathematics as an art.

Therefore, we teachers of mathematics must remember to seize its beauty and present it to our students.

Research done by Picker and Berry (2001) and by Wong (1995) indicates that when students are asked to represent a "mathematician at work," most of them display severe misconceptions and lack of knowledge about this area of study, with huge implications on their attitudes toward learning it. The two research articles above were consistent in their findings, even though one was done on students from Singapore (Wong). The authors pointed out that most of the students draw male characters, usually white, with glasses, messy hair, and a ragged appearance resembling the "mad scientist" stereotype. Their explanations mostly depict mathematicians in the proximity of chalkboards, teaching or writing simple arithmetic sentences or nonsense formulas. This is another reason why making time for mathematicians in the classroom is necessary.

I was curious about how my students would represent mathematicians, and I surveyed my class too, using the survey proposed by Picker and Berry (2001) and inspired by the "Draw-a-Scientist Test" (DAST), originally developed by David Chambers in 1983. The survey was given to my Grade 6 class after the students had completed a project called "The Great Mathematician Project" (Goldberg, 2013). I was surprised to find that when it comes to the gender of the mathematician drawn, 76 percent of the girls I surveyed drew female mathematicians, which is in contrast to the 41.34 percent that Picker and Berry mention in their study. This high percentage of female representation is due, arguably, to the success of the project.

Even though some of the drawings were somewhat related to the mathematician my students had researched, I realized that it made an impact on them, as many components of their drawing were alluding to their researched mathematician. One of the students drew himself as a mathematician in the picture, mentioning, "I would like to become a mathematician and inspire students (about) what real math looks like and how beautiful it is" (Figure 1), and one female student explains, "Here I drew a mathematician working on the chalkboard. I did not include a face or any facial features

because a mathematician can look like anything. They do not have specific features like, for example, glasses. It can also be any gender, but I drew a girl just 'cause it's easier. Also the math she is doing up on the board does not have to be that or even complicated" (see Figure 2).

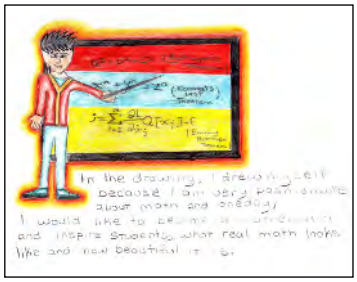


Figure 1: *A Grade 6 male student's explanation*



Figure 2: *A female student's explanation of his drawing of "Mathematician at Work." Explanation of her drawing.*

Making Time for Mathematics in the Classroom

Earlier this school year, I read a wonderful article in one of NCTM's *Mathematics Teaching in the Middle School* magazines, entitled "The Great Mathematician Project," by Sabrina R. Goldberg (2013). The project endeavours to expand students' knowledge about various mathematical ideas and mathematicians, while developing their digital literacy. Students' inquiry has four components: selecting biological data of one mathematician of their choice, the educational background, professional life, influential ideas and real-life applications. Intended initially for gifted students, the project became very successful among mixed-ability students. Just like the author, I quickly realized one thing: our students crave these kinds of projects, and they display such a strong curiosity about and eagerness toward them.

When I assigned this project, I made a few modifications. The choice of the mathematician to study was an individual one made by every student, but I encouraged the female students to research a female mathematician. Education researchers have pointed out on numerous occasions that middle school girls fall victim to stereotypical views regarding mathematics, which impacts their decisions to pursue a STEM career later on in life.

Additionally, TED talk speaker and author of the forthcoming book *The New Science of Smart*, Annie Murphy Paul (2013), points out that girls tend to be demotivated by "impossible role models"; that is, presenting them with the story of success and achievement, without mentioning the many failed attempts that it took to realize such achievements, does more damage than good. Therefore, I wanted to make sure that studying someone who reflected the same gender or heritage would give female students in my class a chance to reshape their perspectives toward mathematics. In other words, reading about the mathematicians' endeavours would give them a chance to learn about the importance of failure and persistence in one's life.

Ultimately, the project has become one that emphasized the power of equitable learning experiences and culturally relevant teaching practices in the mathematics classroom.

In his article "Gliding Through Time Toward Equitable Mathematics" (2014), author Kenneth A. Whaley presents a similar project that he had assigned his students. They were required to use a free online tool, Timeglider.com (see Figure 3), to learn about remarkable mathematicians and create interactive timelines of the life of one of those mathematicians. The author based his endeavour on the hypothesis presented by Fink (2003) as a "significant learning experience"—that this type of project will develop students' knowledge about mathematics. Moreover, Whaley emphasizes the fact that these types of teaching practices promote equity and we have a moral duty to employ them.

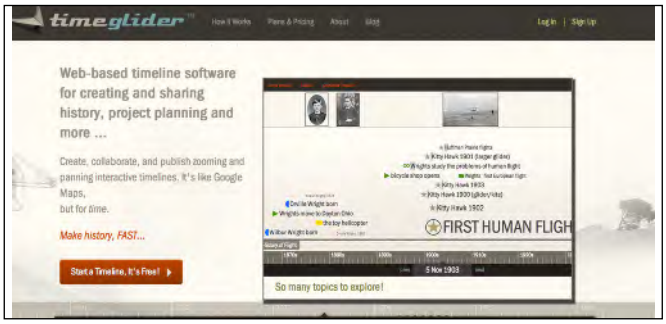


Figure 3: *Timeglider.com, a free online tool that allows students to create interactive timelines*

The Outcomes

Students felt empowered by the uniqueness of the project, which gave them access to a more complex pool of ideas, and they felt proud to discover someone who shares parts of their identity makeup. That is not to say that all students chose mathematicians who reflected their cultural heritage, but some did, and they felt a sense of renewed pride for sharing something in common with that mathematician. Among the

mathematicians researched were Tabbit Ibn Qurra, who gave the most elegant proof of the Pythagorean theory; the father of Algebra, Al-Khowârizmi; August Mobius, famous for his strip; Rubik; Buckminster Muller; Emmy Noether; Pythagora; Thalles of Miletus; Hypathia of Alexandria; Julia Robinson; and Dr. Fan Chung.

The project culminated with the Great Mathematician Expo (see Figure 4) when, during carousel presentations, students had a chance to use a presentation tool of their choice (e.g., PowerPoint or Prezi) and to explain one idea or discovery for which that mathematician was known.



Figure 4: *The Great Mathematician Expo*

Additionally, each presenter was required to prepare a hands-on experiment, a task, or a mathematical challenge that he or she wanted to share with the audience (see Figures 5 and 6).

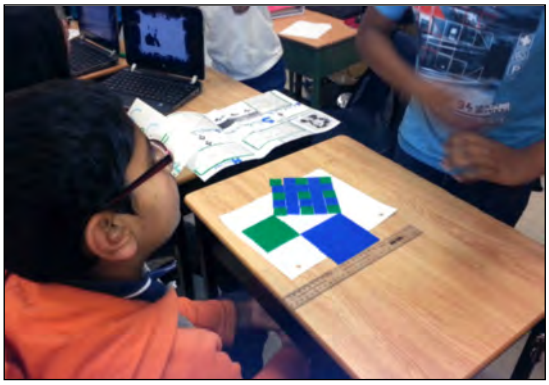


Figure 5: *A student explaining Pythagoras' Theorem to a peer.*

While not all students developed a deep understanding of the mathematical ideas they presented, as some were quite complex, this project deepened their appreciation for mathematics and mathematicians and strengthened our math community.

The popularity and the success of this project among the students proves the following:

a. As students developed models or tasks for their presentations to their peers, they become more familiar with an area of study they had regarded as abstract, challenging, and intangible for their age or grade level. Irrespective of their ability, students became the teachers and proved that, given the opportunity and challenge, they can rise to the occasion. A struggling Grade 6 student could successfully explain Thales' theorem, demonstrating to his peers that an inscribed angle in a semi-circle is a right angle (see Figure 7).



Figure 6: *Student doing an experiment with Mobius' strip assigned by a peer presenter.*



Figure 7: *A Grade 6 student explains Thales' theorem about an angle inscribed in a semi-circle to another student.*

b. By reading about the history of an idea, students get closer to mathematical beauty, to its meaning. Additionally, they realize that mathematics belongs to all cultures, not just to mathematicians of European descent, often portrayed in math textbooks, who contributed and continue to develop this field of study.

c. As students learn about the birth of a mathematical idea, they also begin to see the power of intuition in posing questions and developing such ideas.

d. While preparing their presentations, students do an exercise in mathematical communication that is concise, clear, and uses logic and specialized vocabulary. Mathematics is, in the words of Paul Lockhart, "the art of explanation."

e. Stereotypes surrounding the gender of mathematicians are gradually fading as the female students read about the works of various female mathematicians. Math textbooks tend to present mathematicians of European descent, all males, mostly the classics, such as Descartes, Pascal, or Pythagora.

f. Students get to see that math is indeed "alive" as they discover the direct applicability of mathematics in real life. Doing experiments with a Mobius strip or constructing Buckminster Fuller's models of geodesic

domes, students emphasized the connection between math and architecture (see Figure 8).

g. Additionally, biographies like that of Emmy Noether, for example, provide an opportunity for students to learn about how sexism, racism, and anti-Semitism were substantial challenges that interrupted career and dreams.



Figure 8: A student's model of one of Buckminster Fuller's Geodesic structures.

Other Implications

In order to make room for mathematicians in the math class, teachers must familiarize themselves with knowledge about mathematicians.

Ideally, just like Paul Lockhart and Edward Frenkel suggested, mathematical ideas should not be separated from the individuals who worked on them. Think of a piano teacher trying to teach Beethoven's "Moonlight Sonata" without mentioning the composer's name or anything about him. What about studying a novel without mentioning its author, but trying to get to the heart of that book without knowing anything about the author's literary intent?

I am not advocating for a math class transformed into a math history class, but for classes that, as Paul Lockhart describes, focus more on the why and less on the what. The why here is the birth of an idea, the habits we want our students to develop as "young mathematicians." So, when we call students "mathematicians," do we ensure that they have encountered them, if only through research or virtually, or that they know how they work, in order to begin to display their habits?

Resources and Ideas

1. Meet The Mathematicians (MTM), United Kingdom

A good starting point is "Meet The Mathematicians" (MTM). This is an annual conference held at University College in London, England. Its focus is on having middle school students familiarize themselves with classic as well as recent mathematical discoveries. Students are able to meet real mathematicians and to familiarize themselves with the latest applications or discoveries of this area of study. The event booklets can be downloaded from the MTM website, **www.meetmaths.org.uk**. This is a unique resource that provides classroom teachers with great background information and contexts that can be shared with middle school students. Topics range from applications of mathematics to modern economy (e.g., games of chance) or ecology, to biology and medicine (e.g., predicting the foot and mouth epidemics or, through modelling, how certain treatments can be improved). Famous math problems are also included.

2. Notable Women in Mathematics

Another great resource that focuses on modern female mathematicians, scientists, and engineers of all cultural backgrounds is the one created by The College of the Redwoods. It can be found at **http://online.redwoods.edu/womenmse/NotableWomen/NotableWomen.htm**.

3. Being a Professional Mathematician

"Being a Professional Mathematician" is an excellent website, created by University of Greenwich and University of Birmingham. It contains great teaching resources about what mathematicians do or study. It is primarily geared to intermediate students who are interested in pursuing STEM careers. Materials such as case studies about historical and contemporary mathematicians, worksheets, and interviews can be downloaded from **www.BeingAMathematician.org/resources.html**.

References

Being a Professional Mathematician. (2012). Retrieved from **www.BeingAMathematician.org/**

British Applied Mathematics Colloquium. (n.d.). *Meet the mathematicians (MTM).* Retrieved from **www.meetmaths.org.uk/**

Fink, D.L. (2003). Creating significant learning experiences: An integrated approach to designing college courses. San Francisco: Jossey-Bass.

Frenkel, E. (2013). *Love and math: The heart of hidden reality.* New York: Basic Books.

Goldberg, S.R. (2013). "The Great Mathematician Project." *Mathematics Teaching in the Middle School, 19*, 273–279.

Lockhart, P. (2009). *A mathematician's lament: How school cheats us out of our most fascinating and imaginative art form.* New York: Bellevue Literary Press.

Picker, S.H., & Berry, J.S. (2001). Your students' images of mathematicians and mathematics. *Mathematics Teaching in the Middle School, 4*, 202–208.

Whaley, K.A. (2014). Gliding through time toward equitable mathematics. *Mathematics Teaching in the Middle School, 19,* 478–485.

Wong, K.Y. (1995). "Images of Mathematicians." Retrieved from **http://math.nie.edu.sg/kywong/Images%20of%20Mathematicians%20Wong.pdf** ▲

▲ MB4T... Fraction Division Procedures – Not Just Invert and Multiply!

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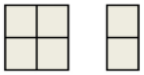


Ann Kajander is an experienced classroom teacher currently teaching mathematics and mathematics education at Lakehead University. Her research interests relate to teachers' enhanced learning of mathematical concepts. She and her classroom-teacher colleague, Tom Boland, are currently writing a new book for teachers called Math Without Memorization: Big Ideas in Modelling and Reasoning for Elementary Teachers and Other Learners.

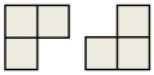
Sometimes we jokingly refer to the "…it's not ours to wonder why, just invert and multiply" ditty as exemplifying the extreme case of the learning of algorithms without understanding. There is indeed a place for procedures and algorithms, but only when based on their proper development.

As discussed in many of these columns, models and reasoning can form the developmental basis for deep understanding, but in turn, they *also* can actually support the *development* of generalizations such as procedures. While numeric methods that "always" work provide quick computational methods to determine answers, when such methods are developed via models and reasoning, students have a greater chance of understanding why and how they work, as well as when they are appropriate to use. This also provides a fallback in case a method is forgotten. A flexible conceptual understanding also means we have the capacity to pick the most efficient method for a given circumstance.

For example, imagine we are looking to purchase lengths of $\frac{3}{4}$ metre from a spool of $1\frac{1}{2}$ metres of chain. If we can imagine or visualize that two groups of $\frac{3}{4}$ can be found in $1\frac{1}{2}$, then the question, which we might also recognize as $1\frac{1}{2} \div \frac{3}{4}$, has the answer 2. Here is one possible model of $1\frac{1}{2}$ (using a 2 x 2 grid as one whole):



Next we can look for groups of $\frac{3}{4}$:



The *two* groups of $\frac{3}{4}$ yield the answer of 2.

In this case, our mental image, based on understanding that division by this fraction ($\frac{3}{4}$) can be thought of as "how many $\frac{3}{4}$'s are in $1\frac{1}{2}$", is much more efficient and reliable than a numeric procedure. For a different question, however, a procedure may be the best bet; the key is to be able to make a good judgment based on the methods available.

As a pedagogical note, it would not be appropriate to ask a student who solved $1\frac{1}{2} \div \frac{3}{4}$, using a mental image to "show his or her work" with a (now redundant) procedure. In case this, a sentence such as "I know there are two groups of $\frac{3}{4}$ in $1\frac{1}{2}$" might be a more appropriate explanation.

Same-Size Parts or Common Denominator Method

Contrary to popular understanding, there are actually several fraction-division procedures (in addition to the use of concrete representations or mental models such as in the example just given). The simplest case involves dividing two fractions with the same denominator, or same- size parts, such as $\frac{3}{4} \div \frac{1}{4}$.

In the previous column, the idea of the *measurement* model of division as a critical conceptual building block for understanding division by a fraction was developed. The measurement models helps us to interpret $\frac{3}{4} \div \frac{1}{4}$ as *how many* $\frac{1}{4}$*'s are there in* $\frac{3}{4}$*?* The reasoning behind $\frac{3}{4} \div \frac{1}{4}$ is now analogous to that for $3 \div 1$ because the units (one-fourths) are the same. We can generalize to the fact that fractions with like denominators can be divided by simply dividing the numerators. For example, you might try $\frac{6}{8} \div \frac{2}{8}$. We can also use this method when dividing fractions that can be easily converted to have like denominators. For example, converting thirds to twelfths is helpful when completing the following calculation:

$$\frac{\mathbf{2}}{\mathbf{3}} \div \frac{2}{12} = \frac{\mathbf{8}}{\mathbf{12}} \div \frac{2}{12} = 8 \div 2 = 4$$

The method of dividing numerators also works even if the division doesn't work out to a whole number. For example, we might draw a model (or visualize) and use reasoning to verify that $\frac{3}{4} \div \frac{1}{2} = 1\frac{1}{2}$. Compare this to the *dividing numerators* method shown below:

$$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \div \frac{2}{4} = 3 \div 2 = \frac{3}{2} \text{ or } 1\frac{1}{2}$$

It seems that if we are dividing *pieces of the same size* (fractions with like denominators), we can simply *divide the numerators* to find the answer, which makes sense because the size of the pieces is the same. The method is, however, limited to fractions with the same, or easily convertible, denominators; otherwise, it can be cumbersome.

Dividing Both Numerators and Denominators

Another strategy that is used less often involves dividing *both* numerators and denominators. Since this method is not useful for all fractions, it may be slightly less useful, but in the cases where it does apply, it works very simply.

Beginning with an example equivalent to the previous $1\frac{1}{2} \div \frac{3}{4}$, namely $\frac{12}{8} \div \frac{3}{4}$, we can predict that the answer will still be 2 (because $\frac{12}{8}$ is equivalent to $1\frac{1}{2}$).

We have already developed (in the December column) the idea that when *multiplying fractions*, a numeric method involves multiplying the numerators and multiplying the denominators. In fact, a parallel method exists for division! We can also divide the numerators and divide the denominators. Hence,

$$\frac{12}{8} \div \frac{3}{4} = \frac{12 \div 3}{8 \div 4} = \frac{4}{2} = 2, \text{ as expected.}$$

Other examples (in which numerators and denominators respectively divide evenly) will verify this method. While a full development of this procedure is not included here, it makes sense that if we think of dividing by 3 (from the numerator of $\frac{3}{4}$), we might instead multiply by $\frac{1}{3}$, and we know that a parallel procedure exisits for multiplying fractions.

The Traditional Procedure – Multiply by the Reciprocal

Lastly, we turn to developing a procedural method that will work when dividing all fractions. It is important to note, however, that in cases in which a simpler method can be used to successfully solve a problem, the simpler method should indeed be used. The following method should be seen as a last resort, used only if the methods described above—visualizing parts, dividing numerators, or dividing both parts of the fraction—do not work out simply.

Developing this procedure draws heavily on our flexible understanding of operations, as well as order of operations. In particular, we are reminded that when dividing by a whole number such as 20, we can first divide by 10 and then by 2, or vice versa (because 2 x 10 = 20). Conceptually, dividing by 20 might be thought of as making 20 equal groups. We can do this by first making 10 equal groups, and then dividing each of these 10 groups into 2 equal parts. This gives 20 equal groups in all—which is division by 20. Similarly, if we were dividing by $\frac{3}{4}$, we can *first* divide by 3 and *then* divide by $\frac{1}{4}$.

For the derivation, we will use the example of $\frac{1}{6} \div \frac{2}{3}$. You may first wish to explore this example with a model in order to be clear on both the meaning of the problem and the answer. A model could be used to show that the answer to the question of *how many (or how much) of* $\frac{2}{3}$ *fits into* $\frac{1}{6}$ is that only $\frac{1}{4}$ (of the $\frac{2}{3}$) fits into the $\frac{1}{6}$ (because $\frac{2}{3}$ is 4 times as large as the $\frac{1}{6}$). The answer is $\frac{1}{4}$.

To follow are the four main conceptual ideas needed to develop the multiply-by-the-reciprocal (sometimes nicknamed "invert and multiply") numeric method, which then follows. You may wish to refer back to each of the following paragraphs as you go through the subsequent formal derivation.

The first conceptual idea is the flexible use of division strategies, as mentioned. We can divide by $\frac{2}{3}$ by first dividing by $\frac{1}{3}$, and *then* dividing by 2 (observing that $\frac{2}{3} = \frac{1}{3} \times 2$).

The second idea we need to be familiar with is the result of dividing by a unit fraction, which was fully explored in the previous column. We must be familiar with the idea that dividing by, for example, $\frac{1}{3}$ is like asking *how many pieces of size* $\frac{1}{3}$ *are there*, and realizing that since there are 3 thirds in each whole, the answer will be the initial amount times 3. We see that dividing by one-third, "$\div \frac{1}{3}$," and multiplying by three, "$\times$ 3," are numerically equivalent.

The third idea is also related to unit fractions. It is the converse of the previous idea, namely that *division* by a whole number, for example 2, is equivalent to *multiplying* by the related unit fraction, for example, $\frac{1}{2}$.

Lastly, we need to apply the conceptually simple idea that a whole number multiplied by a unit fraction is just the product of the two—for example, $3 \times \frac{1}{2} = \frac{3}{2}$.

Understanding each of these concepts allows us to complete all of the intermediate steps that lead to the final formal fraction division "invert and multiply" method. In practice, we would like to eventually be able to skip the reasoning and development steps and just use the final result as an accepted procedure. Before we can do so, we need to be sure that the procedure makes sense.

We will work through the previously mentioned example, $\frac{1}{6} \div \frac{2}{3}$. Each step in the method relates to a concept described in the previous paragraphs, which could be reviewed as needed.

$\frac{1}{6} \div \frac{2}{3}$	We know that $\frac{2}{3} = \frac{1}{3} \times 2$ from fraction multiplication.
$= \frac{1}{6} \div \left(\frac{1}{3} \times 2\right)$	We can divide in two stages, first by $\frac{1}{3}$ and then by 2.
$= \frac{1}{6} \div \frac{1}{3} \div 2$	Counting thirds by *dividing by* $\frac{1}{3}$ is like *multiplying* by 3.
$= \frac{1}{6} \times 3 \div 2$	*Dividing by* 2 is the same as *multiplying by* $\frac{1}{2}$.
$= \frac{1}{6} \times 3 \times \frac{1}{2}$	We use fraction multiplication for $3 \times \frac{1}{2}$.
$= \frac{1}{6} \times \frac{3}{2}$	To conclude, we see that $\frac{1}{6} \div \frac{2}{3} = \frac{1}{6} \times \frac{3}{2}$

Using fraction multiplication, we can finish the division calculation in the example as follows:

$\frac{1}{6} \div \frac{2}{3} = \frac{1}{6} \times \frac{3}{2} = \frac{3}{12} = \frac{1}{4}$, as predicted earlier

Another Example

Returning to the example used earlier in today's column, $\frac{12}{8} \div \frac{3}{4}$, similar reasoning is provided:

$\frac{12}{8} \div \frac{3}{4}$	(dividing by $\frac{3}{4}$ in two stages)
$= \frac{12}{8} \div 3 \div \frac{1}{4}$	
$= \frac{12}{8} \div 3 \times 4$	(division by $\frac{1}{4}$ is like multiplying by 4)
$= \frac{12}{8} \times \frac{1}{3} \times 4$	
$= \frac{12}{8} \times \left(\frac{1}{3} \times 4\right)$	(dividing by 3 is equivalent to multiplying by $\frac{1}{3}$)
$= \frac{12}{8} \times \frac{4}{3}$	(flexible order of multiplication)

Again, the procedure is the result!

If desired, the multiply-by-the-reciprocal procedure can be derived more algebraically (this may not be necessary for students, but is provided here for teachers!). The expression $A \div \frac{n}{d}$ is used to explore the division of any number *A* (which could be a fraction) by the fraction $\frac{n}{d}$, where *n* and *d* are non-zero whole numbers . The reasoning is analogous to that used in the two previous examples.

$A \div \frac{n}{d}$	dividing in stages
$= A \div \underbrace{\frac{1}{d} \div n}$	since there are d pieces of size $\frac{1}{d}$ in *A*
$= A \times \underbrace{d \div n}$	
$= A \times \underbrace{d \times \frac{1}{n}}$	dividing by n is like multiplying by $\frac{1}{n}$
$= A \times \frac{d}{n}$	multiplying the last two terms

It seems that the method of multiplying by the reciprocal fraction works for dividing by any fraction $\frac{n}{d}$. If the previous value *A* was indeed a fraction of the form $\frac{a}{b}$, where *b* is not zero, and remembering the previous restriction that *d* and *n* are also non-zero, it is always true that $\frac{a}{b} \div \frac{n}{d} = \frac{a}{b} \times \frac{d}{n}$.

Thus, the multiply-by-the-reciprocal method is more formally established.

In conclusion, there are a number of ways to divide fractions. A key idea is to help students develop the understanding required to choose the best method for a given circumstance—and to encourage them to do so. ▲

What's New with CLIPS? A Site Refresh and a New Search Engine!

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The CLIPS team has been working steadily on a number of different fronts over the past few months. The Rekenrek app submitted to the iTunes store, mentioned in the last issue, took a longer time through the approval process than hoped, but is ideally available by the time you are reading this. The Pouring Containers and Whole Number Comparison Tools have recently been approved and others are being field-tested and revised. However, the biggest changes to the CLIPS site include unifying its look to support the mathies.ca website, totally redesigning the search engine, and making the initial "matrix" table of topics on the opening page much more functional.

Site Refresh

When you first arrive at MathCLIPS.ca and click the Start button, you will be greeted by a new and improved CLIPS tab. It looks very similar to the previous incarnation, so return visitors will feel at home.



However, the topics table is now an interactive Quick Search chart. Clicking a topic, grade band or check mark will take you directly to a list of all activities, tools, games, or resources that match that criterion. For example, clicking on the check mark opposite Fractions and beneath "7-8" will display results related to Grades 7 and 8 Fractions, whether they are from CLIPS, ePractice, or are standalone resources. The results are displayed in the new and improved search results dialog (more on that shortly!).

You can still explore whole clusters of activities by clicking on the appropriate title on the left. A rollover effect has also been added to the links in the cluster list so that you can see how many activities and other resources are included in each collection.

The What's New link at the top has a new look to it, and has been updated to include the latest releases of games, tools, and other improvements to the site. The team has also improved its method of keeping this information current, so remember to check this out upon future visits.

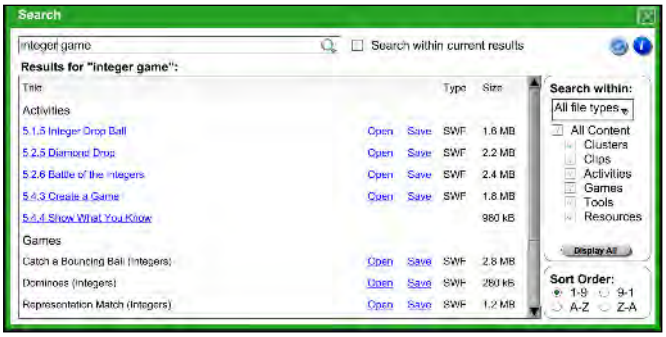
Search



The static search link has now been replaced by a dynamic Search Box—enter any search term and click the magnifying glass icon, or just hit ENTER, and the new improved search dialog will appear.

Search Keywords

If you search for more than one word, like "integer game," then all records tagged with "integer" ***and*** with "game" are displayed.



If you use "or" or "not" in a search term, they will be treated as Boolean operators.

e.g.,

- "integer or game" will show a lot more records than "integer game"
- "integer not game" will show any records tagged with "integer," but not with "game"

"• not game" will show all records that do not have a tag of "game"

Using the Search Dialog



A search text field is at the top left of the Search Dialog. Enter search terms into this field. Click the *Magnifying Glass Icon* to search all records. Select Search *within current results* to only search through the records currently displayed. For example, when you search for "integer or game," you will get some entries with titles like "Who am I?" To exclude these from the list, type "not who" in the box, select *Search within current results*, and click the *Magnifying Glass Icon*, or press Enter.

The *Reset* button allows you to clear out the results and start over.

Search Filters

At the right of the Search Dialog, you will find a drop-down menu and check boxes that are used to narrow your search. Select one of the file types from the drop-down list and check up to six content categories, and the search results will automatically be filtered based on those selections.



All activities are accessed either from the CLIPS or ePractice tab on the main screen. Each is assigned to a topic or cluster. CLIPS activities are also assigned to a clip. When activities are opened, this information is displayed at the bottom of the screen. See below for what is displayed for the Fractions Dropball activity:

If the search term is "Fraction," and the boxes for Clips and Clusters are checked, the results will include "1 Fractions – Exploring Part/Whole Relationships" and "1.5 Representing Improper Fractions as Mixed Numbers."

In addition to searching for clusters, clips, and activities, you can search for the following:

- Games – The main CLIPS page has a Games tab that lists all the games. Each cluster has a games icon (a rook) that, when clicked, opens a list of games in that cluster.
- Tools – The tool container is accessed using the gears icon at the right of the screen.
- Resources – These are listed on each cluster's Downloads tab and accessed from the link icon at the right of the screen. The resources are given a number that indicates to which cluster they belong.

You can also opt to search for a specific file type. Common file types include the following:

- SWF – Flash activities playable in a browser or embeddable in some applications, including interactive whiteboard software
- ZIP – Compressed folders consisting of several files. Often there is a SWF file, together with support documents or videos.
- PDF – Adobe's Portable Document Format, used for print documents and viewable with Adobe Acrobat Reader
- GSP – The Geometer's Sketchpad® document

The *Display All* button first clears the search text field, then shows all the records in the selected content categories of the selected file type.

Search Examples



1. Select GSP files and click Display All to find Sketchpad sketches.
2. Check only the Games box and click Display All to show all games.
3. Select PDF files, check only the Games box, and click Display All to show all printable games.

4. All activities are tagged by their type, so "MindsOn," "Action," "Consolidation," and "Show" are all possible search terms.
5. All activities are tagged with "CLIPS" or "ePractice," so "ePractice Add" will show ePractice activities related to addition.
6. Type "K-3," "4-6," "7-8," "9-10," or "11-12" to find content appropriate for a specific grade band.
7. To find all games associated with Cluster 1, search for "CL001 game."

Sorting

Each record is assigned a number, as appropriate, and a title. The records are displayed in numeric, ascending order by default. Click on any one of the buttons to change the sort order.

Navigating from the Search Results

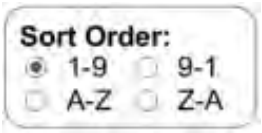
The search results are displayed in a scrolling window, subdivided by content type. Notice that some titles are links. Clicking on a title will take you to that location.

For example, clicking on *"101.1.13 Adding Fractions*" will close the Search Dialog and open the Adding Fractions activity. Click the search icon to reopen your current results. Click the *Open* link to open the item in a new browser window (in standalone mode). Click the Save link to download a local copy. The file Type and Size are provided as information in the last two columns.



Frequently Asked Questions

1. **Why is the record I want not displayed when I perform a search?**
   The search is based on search tags. It is possible that we are not using the tags that you expected. Please submit *Feedback*, using the link at the top right of the screen and indicating the search term(s) you used and what you expected to find.
2. **Why do I get a lot of duplicates when I search for "cluster movie"?**
   These movies were produced in a variety of formats and sizes. Check the file type and file size columns to find a download suitable for your needs.
3. **When I use the Quick Search table to search for "Graphing," why does the search text field show "Graph"?**
   Using a root word for a search term will find a broader set of results. In this example, searching for "Graph" will also find references to "Graphing," "Graphs," "Graphical," and even "telegraph." This is a useful strategy for your own searches. In general, "fraction" will return more results than "fractions," and "navigate" will return more results than "navigating."



We hope that you find these changes helpful. We always enjoy getting feedback from the link at the top right. If you haven't already done so, why not search for *K-3 Whole Numbers* and try out the two games there. ▲

Continued from Page 3

▲ Fields Institute MathEd Forum Notes

Toronto. The following is a short quotation describing the workshop (the full description can be found at **www.fields.utoronto.ca/programs/scientific/14-15/making_models/**):

> The first part of the workshop will bring together leading experts in the field of rigidity theory to share new research and work on open problems. The second part of the workshop will focus on key issues in education research and practice that connect to the mathematical content and thinking processes inherent in rigidity theory, with specific attention to visual and spatial reasoning in and for school and post-secondary mathematics. All attendees are warmly invited to participate in both parts of the workshop.

The Fields Education Forum will resume next September. Dates and topics will be posted at **www.fields.utoronto.ca/programs/mathed/**. We are looking forward to seeing you there! ▲

▲ An Examination of Math Anxiety Research

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Math anxiety can be defined as a feeling of nervousness, unease, or tension that "interferes with math performance" (Ashcraft, 2002, p. 181). Taking liberties with this definition, math anxiety can be considered within the category of transmissible or communicable "diseases," which may lie dormant within individuals for many years. From this new perspective, this article will provide a brief look at the characteristics of this illness, will outline some of the most common symptoms exhibited by hosts, and detail some of the recent advances in science that aim to manage or control and alleviate some of these damaging symptoms. Alternatively stated, this article is an "examination" of math anxiety research. Further, we provide a brief outline of possible measures that can be taken to prevent the further spread of this infectious disease.

Classification

Arguably, most everyone in society today is a student of math to some extent. Math is all around us throughout our daily lives, from knowing how far one can travel on the gas remaining in a tank; understanding and using plans for three-dimensional structures; interpreting and questioning charts on the evening news or polls on a Twitter feed; measuring ingredients to create a meal; estimating the volume of luggage versus trunk space; to figuring out a monthly budget or doing annual taxes. The need for at least a basic level of numeracy in modern society is inescapable. However, in the hope that there might be some practical applications, this paper will focus on math anxiety from two distinct perspectives in the educational institution—that of the math student and from the position of the math teacher.

While anxiety runs rampant in Western society today, we focus on a particular strain that is found in educational institutions around the world. Even with general anxiety and anxiety in school not being new subjects, the fields of math anxiety and math teacher anxiety are considered relatively new areas of study. Math teacher anxiety, itself, likely existed as far back as the first time, for example, a mathematician tried to explain the lunar phase counter tally marks on a "Lebombo Bone" to her husband, or for Thales of Miletus the first day as he walked into the Academy of Athens to teach. Much more recently, Dutton (1951) wrote that teacher candidates, who were studying math methods, had a negative attitude toward mathematics and the teaching of the subject. Since that time, there has been increasing interest and broadening study of math anxiety.

The multiple causes of math anxiety can be classified in two interrelated groups: internal and external causes. We are interested, here, primarily in those external factors or causes that can be linked to the classroom and home environment. We will turn our focus to cause and transmission of math anxiety in the next section. It should be noted here, however, that at least some research indicates that student math anxiety is a result of how the student learns math—the style or type of authority used in the classroom (Stodolsky, 1985), the imposition of timed responses and exposure of the student (among other factors) (Ashcraft, 2002), emphasis on the correct answer or fear of mistakes (Tobias, 1978)—can all contribute to the level of anxiety that a student might develop.

Math anxiety is troublesome for both sets of teachers and students, albeit for different reasons. For the student of math, symptoms of math anxiety might include a rise in heart rate, sweating, nausea, stomach discomfort or disorders, headaches, confusion, and even panic. A recent study by Lyons and Beilock (2012) even linked higher anxiousness related to math work with increases in activity in "regions associated with bodily threat detection and the experience of visceral pain itself" (para. 24). The varying symptoms can lead to many negative behaviours,

including lack of confidence, panic, poor performance, and avoidance. Students that suffer these symptoms may tend to avoid studying math, which then potentially limits future education and career paths as well as impairs their ability to survive and excel in modern society. Ashcraft (2002) calls this a "global avoidance tendency" (p. 181), which can limit students in two ways. First, they rush through work in an attempt to limit the amount of time that is spent working on math. Secondly, they try to reduce, via course selection, the amount of time spent studying math. Some work done is this area suggests the impact of anxiety on course selection and math success is not significant (e.g., Meece, Wigfield, & Eccles, 1990). There are also a number of studies (e.g., Beilock, Holt, Kulp, & Carr, 2004; Beilock, Gunderson, Ramirez, & Levine, 2010; Furner & Berman, 2003, 2005; and others) that have examined the link between math anxiety and math performance, finding a significant relationship between the two. The field of work in anxiety research continues to grow today (e.g., Clark, 2013; Rancer, Durbin, & Lin, 2013; Ganley & Vasilyeva, 2014; Jameson, 2013; Harari, Vukovic, & Bailey, 2013; and others). Leaving aside the debate over anxiety and performance results, we can clearly see that there are a number of negative outcomes or limitations (at the least), for anyone suffering from math anxiety. For this reason alone, we contend there is a benefit to trying to alleviate this anxiety, if possible.

For the teacher of math, anxiety can result from similar roots, but play out in a far different fashion than for that of the student. Bush (1989) determined that teachers who were math anxious retreated from current best practice guidelines and instead used more traditional teaching styles involving individual seat work and focusing on skills instead of concepts. The relationship between teacher and student anxiety will be the subject of the next section on transmission.

Transmission

The research indicates that a student is made more or less susceptible to math anxiety by certain conditions in the classroom. "Common practices leading to increased anxiety were emphasis on drill and practice, getting the right answer and using the right method, taking timed tests, memorizing formulas, and applying rules" (Harper & Daane, 1998, p. 30). Said conditions are some of the very same that the math-anxious teacher may retreat to when instructing mathematics. While both boys and girls are impacted in the teacher-to-student transmission of this anxiety, in early elementary education, where the teachers are predominantly female, girls may be more severely impacted than boys. Beilock, Gunderson, Ramirez, and Levine (2010) recently confirmed that "where teachers are almost all female, teachers' math anxiety carries consequences for girls' math achievement" (p. 1861). The transmission by teachers of math avoidance, and general unease with math, appears to be consistently supported in the literature (e.g., Furner & Bermann, 2005; Vinson, 2001; Hembree, 1990; and others). While transmission of math anxiety is possible, so is prevention, which we now address.

Prevention

Math teachers play a critical role both via internally held beliefs as to their own mathematical adequacy, skills, and talents, as well as externally in their daily practice or approach to teaching, through which the mathematics instruction is delivered. As such, this section will provide a brief review of several suggested approaches as to how math anxiety can be reduced or prevented in students. Owing to the related issue of teachers' math anxiety, this section will also touch on approaches to reducing or preventing math anxiety in students.

Most of the suggestions for reducing math anxiety in students align well with current thought in prospective teachers' programs at various universities across Canada. In their review of related research, Furner and Berman (2003) compiled the following list from the National Council of Teachers of Mathematics Standards: teaching how to think for oneself; working in groups at all levels of math; efficiently using technology; teaching estimation; including more statistics and probability in the early grades; incorporating fewer computational processes and drills; using manipulatives; and focusing on more realistic problem-solving experiences. Other best practices gleaned from current writing practices in the math class include: incorporating more writing to explore mathematical thinking and problem solving (Pugalee, 2001; Schoenfeld, 1992); importance of questioning and conjecture (da Ponte, 2001; Ball, 1993); revising assessment to allow for multiple ways to demonstrate understanding (Pirie & Kieren, 1994; Simon, 1995); and alternative forms of assessment including performance tasks, self-assessment (Kulm, 1994; Artzt, Armour-Thomas, Gurl, & Curcio, 2008), portfolios, and writing samples (Midkiff & Thomasson, 1993).

In order to reduce or prevent math anxiety in the student then, the math teacher must engage in careful and reflective practice that ensures a more humanistic approach suited to each learner in the classroom. He or she must engage in helping the student to actively, excitedly, and

independently inquire into problems, rather than memorize procedures, and also allow for multiple ways to solve a problem and demonstrate that understanding.

A critical first step in ensuring the success of the learner in the math classroom is to be sure that the teacher will not be transmitting personal biases or negative practices to students. Harper and Daane (1998) suggest that the first step is for teachers to be aware of their own personal level of math anxiety. With this knowledge, they are then better able to make adjustments in their own practice to prevent passing on negative dispositions to students. Gresham (2007) points to teacher education programs as the critical point to make teachers aware of their bias, and work to change it to a more positive one.

Conclusions

Math anxiety is recognized as a concern for many students today. If it is not identified and addressed, math anxiety could potentially lead to lower performance in math, which can further impact students via limitations in education and career options. Relatedly, math anxiety in the math teacher has similar root causes as it does in students; however, with instructors, math anxiety, unfortunately, can be transmitted to students. Fortunately, math anxiety can be overcome by reflective, diligent, and confident instruction.

References

Artzt, A.F., Armour-Thomas, E., Gurl, T., & Curcio, F.R. (2008). Becoming a reflective mathematics teacher: A guide for observations and self-assessment. (2nd ed.). Oxford, UK: Routledge.

Ashcraft, M.H. (2002). Math anxiety: Personal, educational, and cognitive consequences. *Current Directions in Psychological Science, 11*(5), 181–185.

Beilock, S.L., Gunderson, E.L., Ramirez, G., & Levine, S. L. (2010). Female teachers' math anxiety affects girls' math achievement. *Proceedings of the National Academy of Sciences of the United States of America, 107*(5), 1860–1863.

Beilock, S.L., Holt, L.E., Kulp, C.A., & Carr T.H. (2004). More on the fragility of performance: Choking under pressure in mathematical problem solving. *The Journal of Experimental Psychology, 133*(4), 584–600.

Bush, W.S. (1989). Mathematics anxiety in upper elementary school teachers. *School Science and Mathematics, 89*(6), 499–509.

Clark, M. (2013). *Teaching the math anxious female student: Teacher beliefs about math anxiety and strategies to help female students in all-girls schools*. Toronto: University of Toronto Research Repository.

da Ponte, J.P. (2001). Investigating mathematics and learning to teach mathematics. In F.L. Lin & T.J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 53–72). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Dutton, W.H. (1951). Attitudes of prospective teachers toward arithmetic. Elementary School Journal, 52(2), 84–90.

Furner, J.M., & Berman, B.T. (2003). Review of research: Math anxiety: Overcoming a major obstacle to the improvement of student math performance. *Childhood Education, 79*(3), 170–174.

Furner, J., & Berman, B. (2005). Confidence in their ability to do mathematics: The need to eradicate math anxiety so our future students can successfully compete in a high-tech globally competitive world. *Dimensions in Mathematics, 18*(1), 28–31.

Ganley, C.M., & Vasilyeva, M. (2014). The role of anxiety and working memory in gender differences in mathematics. *Journal of Educational Psychology, 106*(1), 105.

Gresham, G. (2007). A study of mathematics anxiety in pre-service teachers. *Early Childhood Education Journal, 35*(2), 181–188.

Harari, R.R., Vukovic, R.K., & Bailey, S.P. (2013). Mathematics anxiety in young children: An exploratory study. *The Journal of Experimental Education, 81*(4), 538–555.

Harper, N.W., Daane, C.J. (1998). Causes and reduction of math anxiety in preservice elementary teachers. Action in *Teacher Education, 19*(4), 29–38.

Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. *Journal of Research in Mathematics Education, 21*, 33–46.

Jameson, M.M. (2013). Contextual factors related to math anxiety in second-grade children. *The Journal of Experimental Education,* (ahead-of-print), 1–19.

Kulm, G. (1994). *Mathematics assessment. What works in the classroom*. San Francisco: Jossey-Bass Inc.

Lyons, I.M., & Beilock, S.L. (2012). When math hurts: Math anxiety predicts pain network activation in anticipation of doing math. *PLoS ONE*, 7(10): e48076.

Meece, J.L., Wigfield, A., & Eccles, J. (1990). Predictors of math anxiety and its influence on young adolescents' course enrollment intentions and performance in mathematics. *Journal of Educational Psychology, 82*, 60–70.

Midkiff, R.B., & Thomasson, R.D. (1993). *A practical approach to using learning styles in math instruction.* Springfield, IL: Charles C. Thomas.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*(2), 165–190.

Pugalee, D.K. (2001). Writing, mathematics, and metacognition: Looking for connections through students' work in mathematical problem solving. *School Science and Mathematics, 101*(5), 236–245.

Rancer, A.S., Durbin, J.M., & Lin, Y. (2013). Teaching communication research methods: Student perceptions of topic difficulty, topic understanding, and their relationship with math anxiety. *Communication Research Reports, 30*(3), 242–251.

Schoenfeld, A.H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145.

Stodolsky, S. (1985). Telling math: Origins of math aversion and anxiety. *Educational Psychologist, 20,* 125–133.

Tobias, S. (1978). *Overcoming math anxiety*. New York: Norton.

Vinson, B. (2001). A comparison of preservice teachers' mathematics anxiety before and after a methods class emphasizing manipulatives. *Early Childhood Education Journal, 29*(2), 89–94. ▲

▲ Collaborative Learning – It's More Than "Divide and Conquer"

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As soon as I finished handing out a worksheet on solving linear equations, I asked students to get into groups of two or three. "Please work through these problems together. After about 10 minutes, I will take some of them up and ask you to share some of them too." I often make use of collaborative activities as a teaching strategy because the *Growing Success* policy document identifies collaboration as one of the key learning skills that students need in order to become effective learners (Ontario Ministry of Education [OME], 2010). In addition, allowing students to work in groups gives me the opportunity to make informal observations and have individual conversations to obtain assessment information (OME, 2010). Specifically in mathematics learning, Francisco (2013) reported that high school students used ideas of their peers to re-examine their mathematical reasoning. Building ideas as a group, students were able to construct more complex arguments and had shown greater mathematical understanding (Francisco, 2013). Thus, collaborative learning could promote development of learning skills, formative assessment opportunities, and growth of mathematical understanding. Theoretically.

As I circulated from group to group, I heard my first-year college students say, "Okay, so I'll do questions 1 to 5, you do 6 to 10." After that, not a word was heard. There was no exchange of ideas. There was no, "Hey, how did you tackle this question?" There was no, "Let's figure this out together." The collaborative activity I assigned turned out to be students sitting in groups and working independently. At that moment, this class was ideal in terms of classroom management. What more can a teacher ask for? But I asked for more. Where is the collaborative learning?

To combat this "divide and conquer" approach that my students took, I explored a variety of strategies and types of collaborative activity throughout the year with both college students and secondary school students. Some approaches worked better for some topics than others. Here is the list I would like to share:

1) "Pass the marker to your right!"

   This approach allowed me to ensure everybody in the group gets a turn in participating. Basically, I would begin with assigning a set of questions, projected on screen, written on the board, or provided as handouts. Approximately every 2 minutes, I make an announcement for the students to pass on the marker. The student who holds the marker in the group at any given time is responsible for writing down the solution, while at the same time, explaining to the group members about the steps he or she is taking. When it is time to pass on the marker, the new marker holder will have to pick up from the point left off by the previous marker holder, which is not necessarily the point at which the solution is completed. The beauty of this is that it requires everyone in the group to pay attention. This approach works really well with lengthy and procedural types of problems, such as completing the square of a quadratic function in Grades 10 and 11, long division of polynomials in Grade 12, factoring in Grades 10 and 11, and curve sketching in Grade 12.

   In both the high school and college classrooms, I found it necessary to have a "bank" of questions because of the likelihood that some groups would complete the set of assigned questions more quickly than others.

   In the college classroom, I recently used this activity for a test review with my Linear Algebra class. This method appeared to work better for high-order thinking questions than procedural problems such as performing matrix algebra, solving linear systems, and calculating determinants. When working on an assigned problem that required students to complete a mathematical proof, it appeared that students were more engaged in their discussions. They dove deeply into their mathematical reasoning and contributed equally in their discussions. From my observations, the challenging problems appeared to drive collaboration more successfully than procedural problems.

2) Placemat

I prepare pieces of chart paper, with each divided up into however many sections are needed, and you have an activity that is suitable for identifying and classifying items. This group activity worked well for the topic of factoring different types of polynomials because students need to identify the type of special product prior to deciding on a factoring strategy. On the "placemat," students had to identify trinomials, perfect square trinomials, difference of squares, perfect cubes, sum and difference of cubes from a given list of special products.

Originally, the placemat activity was designed for students to brainstorm individually in their own section of the chart paper. Ideas are then shared among group members, and students record ideas about which they have reached a consensus in the centre of the sheet. In a case study reported by Phillips (2010), such activity facilitated mathematics learning. It was observed that using mathematical language, students communicated with one another mathematics concepts they used to solve problems (Phillips, 2010).

I adopted the placemat layout simply for the need for students to sort items, which worked well in my class of first-year college students because it helped them understand that the type of polynomial expression needs to be identified prior to deciding on a factoring method. After observing the rich discussions my Linear Algebra students had when asked to complete mathematical proofs, I will try to use Placemats in the near future to truly capture the advantage of this collaborative activity suggested by current research.

3) Carousel

This activity is great for letting students move around and interact with one another. Set up between five and eight stations, depending on your classroom space, class size, and the number of students you would like in each group. I found that the ideal size was three students per group. Every 5 minutes, students rotate from station to station as a group. This worked well for me when I wanted my students to collaboratively discover a concept by investigating different cases. For example, on the Grade 10 topic of solving linear systems, each station was set up to contain a linear system that would either have one solution, no solution, or infinite solutions, which have not been taught prior to this activity. At each station, students were asked to solve the linear system algebraically and graphically.

After visiting all stations, I asked students to explore patterns between the linear equations and the number of solutions in their groups. The goal was to have students recognize the relationship between slopes, y-intercepts, and number of solutions. To consolidate the concept, students discussed as a class their conclusions and summarized how they would predict when a linear system would have one, none, or infinite solutions.

Because solving linear systems was also a topic in a first-year college mathematics course that I taught, I also tried the Carousel activity in my class of 15 students. Fortunately, the class size and classroom set-up allowed this activity to run smoothly. On the other hand, in a typical college mathematics class of 30 students, the Carousel could be a challenge to organize and implement.

Other suggestions for the Carousel activity include:

- Facilitating Grade 9 students' discovery of the formula of volume of pyramids in relation to volume of prisms with the same base; and providing students with manipulatives for them to calculate and recognize that the volume of pyramids is a third of the volume of prisms, given that their areas of the base and the heights are the same. Each station could be set up with a pyramid and a prism of the same height and base. The shape of the base and the height of the solids could vary from station to station. After the Carousel activity, create a scatterplot of Volume of Prism versus Volume of Pyramid as a class. Calculate the slope of the line of the best fit to recognize that the volume of a prism is three times as much as the volume of a pyramid with the same base and height.
- Facilitating Grade 11 students' understanding of the role of the discriminant, and of connecting the value of the discriminant to the number of real roots in quadratic equations. Quadratic equations could vary from station to station. Students would rotate along the Carousel to solve a quadratic equation at each station. When rotations are completed, allow students to discuss within their groups their observations about the discriminant and the number of solutions.

4) Jigsaw

The Jigsaw works well when I would like students to revisit a concept or review for an upcoming test. I place students in groups of four or five. Each group is responsible for reviewing a topic by solving a specific

problem. After 20 minutes or so, the groups are shuffled into new ones. Each new group consists of a member from each of the previous groups. Each student is now the "expert" of his or her topic and is responsible for sharing with the new group members. When the new groups are formed, allow 5 minutes for each "expert" to make a presentation to his or her new group members. At the end of each "expert" presentation, encourage group members to ask questions.

This activity needs to be highly organized. Groups will need to be pre-determined by the teacher. I organized students, using the following chart:

	Group 1	Group 2	Group 3	Group 4
A				
B				
C				
D				
E				

In my class of 20 students, students were placed in Groups A, B, C, D, and E first. New groups were then formed into Groups 1, 2, 3, and 4. Each of the new groups had an "expert" from the previous groups.

In this activity, students require time to solve problems, become the "expert," reorganize into new groups, present their ideas, and discuss ideas as a group. Because time was needed for this collaborative activity to be meaningful, time could be a constraint in a high school classroom. I made use of this activity in a Grade 12 summer school environment and in a college math classroom, both of which allowed ample time for the activity to be carried out successfully.

Through these activities, meaningful conversations surfaced. I heard one student say to his group members, "Okay, so do you want to try it first and then I show you?" as he took on the leadership role. One of the most memorable moments I could recall was during the Jigsaw activity. One student struggled to understand factoring and said, "It's 'cause I'm stupid." His group member responded, "No, no, don't say that. Here, I'll show you."

Despite the benefits, there were also challenges that I needed to overcome. One major difficulty I had in the college classroom that I did not encounter in a secondary school classroom was the need to prompt students to collaborate. As a college teacher, it was essential to consider the diversity of age, culture, professional background, and educational background of the students, which is more likely found within a college classroom. Collaborative learning opportunities could be valuable in meeting the needs of a class with high diversity (Jao, 2013). To eliminate the common view of collaboration as merely "working in groups," teachers need to recognize that there is a need to facilitate students' mathematical activity (Francisco, 2013). Students need to be shown explicitly ways to interact with their peers, to contribute constructively, and to build mathematical ideas using others' ideas. To ensure that students participate in groups in a way that is expected of them, Jao (2012) suggested that teachers need to communicate the goals of the task clearly.

References

Francisco, J.M. (2013). Learning in collaborative setting: Students building on each other's ideas to promote their mathematical understanding. *Educational Studies in Mathematics, 82*(3), 417–438. doi:10.1007/s10649-012-9437-3

Jao, L. (2012). The multicultural mathematics classroom: Culturally aware teaching through cooperative learning and multiple representations. *Multicultural Education, 19*(3), 2–10.

Ontario Ministry of Education. (2010). *Growing Success: Assessment, evaluation, and reporting in Ontario* schools. Toronto: Queen's Printer for Ontario. Retrieved from **www.edu.gov.on.ca/eng/policyfunding/growSuccess.pdf**

Phillips, S.R. (2010). *Student discussions in cooperative learning groups in a high school mathematics classroom: A descriptive multiple case study* (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses. ▲

▲ Book Review of *Mathematical Models for Teaching: Reasoning Without Memorization*

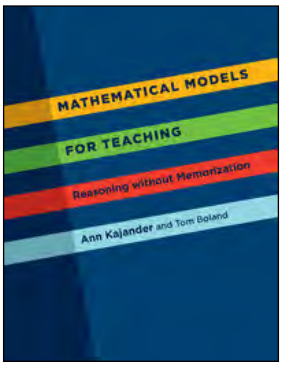
(Boland, T, & Kajander, A. (2014). *Mathematical models for teaching: Reasoning without memorization.* Toronto: Canadian Scholars' Press.)

REVIEWED BY: PATRICIA KEHOE
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What manipulatives best support a student's understanding of addition of fractions? What kind of exploration would help students create a model that justifies why two negative numbers, when multiplied, give a positive result? What can a teacher do to support students who have difficulty explaining a pattern, using an algebraic model? The answers to these and many more questions posed by teachers of elementary mathematics are answered in this text, *Mathematical Models for Reasoning*, written by Tom Boland and Ann Kajander. These Ontario authors have worked with more than 700 teachers over a number of years, conducting research around best practices in supporting teacher mathematics development. Drawing on this research, and that of many others in the field of mathematics education, they have concluded, "There is a body of specialized mathematical knowledge that teachers require to facilitate effective classroom teaching" (p. 3). This book can be used to help teachers acquire this specialized knowledge. It can be used as a personal reference text, as a text for pre-service teachers, or a professional learning resource.



In the first chapter of the book, the authors set the stage by presenting a convincing argument supporting problem-based learning in classrooms. They acknowledge how challenging this can be for teachers, especially if they themselves had the experience of learning mathematics by following set procedures that led to one correct answer. Throughout the chapter, however, the point is made that research over the past several decades indicates that in order for students to learn best and retain their knowledge, they need to go through the process of constructing understanding of mathematical concepts for themselves. Making sense of mathematics, using inquiry-based methods, supports learning that is "rewarding and well retained, rather than frustrating and easily forgotten" (p. 6).

Implementing this process in a classroom requires teachers to have the content knowledge to connect ideas to the students' level of development and facilitate a process that helps them model and reason through problems. The authors acknowledge that this requires teachers, on a daily basis, to:

- design useful contexts/examples
- provide students with models and strategies to unpack that context
- support alternate student methods and solutions
- pose good questions
- help students identify misconceptions

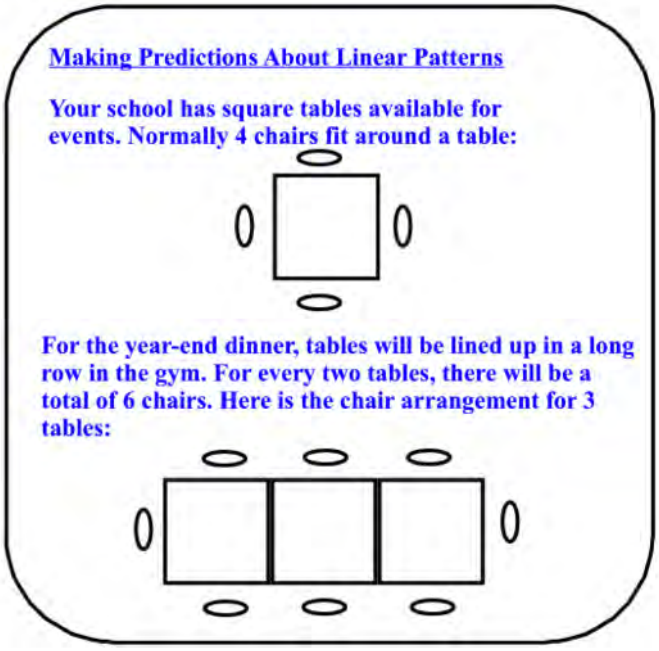
The creation of such classroom experiences can be a daunting, but exciting, responsibility for teachers. As readers journey through the remaining 14 chapters of this book, they are presented with a mountain of concrete examples and support to explain how these experiences can be orchestrated. Each chapter addresses specific content areas in the elementary mathematics curriculum. The table of contents is very detailed, allowing the reader to flip to a specific topic easily. Eight of the chapters deal with Number Sense, two are on Patterning and Algebra, and one each on Geometry, Measurement, and Data Management.

The chapters are divided into a number of sections, each of which deals with a very specific topic. Each section begins with an explanation of the topic and a discussion as to why it can be problematic for teachers. Research findings on the topic are also presented, and appropriate models/manipulatives for problem solving are suggested. This portion also provides teachers with the language or terminology of the mathematics being explored. Mathematical terms are defined and clarified in the margins. Connection boxes also pop up frequently in the margins, linking the current topic to others in the text.

Following the initial introduction to a topic, a sample task is usually presented, followed by a number of different student solutions. The reader is then provided with a "Follow-up and Discussion" section and a "Practice and Further Exploration" section, where next steps for digging deeper into the topic are suggested. This format for unpacking a topic is an important feature of this resource and sets it apart from other texts of its kind. It lends itself to the possibility of individual study. If the reader does not have colleagues with whom to share, then the authors point toward student solutions provided. The development of each topic also provides facilitators of professional learning with a guided tour for exploring student thinking and misconceptions.

Each chapter finishes with a series of Chapter Problems that incorporate the key ideas that have been presented. Although there is a complete bibliography at the end of the book, further reading suggestions are provided at the end of every chapter.

Let's look at a sample task from the Patterning and Algebra strand in section 11.7 (see Figure 1). In the opening comments of this chapter, the authors argue that, "Introducing algebraic descriptions of concepts and relationships that children have not visualized or understood concretely contributes to algebra's reputation of being mysterious and confusing and thus incomprehensible" (p. 153). The scenario presented in the task lends itself to a number of different representations, each of which contributes to the student visualization/understanding that the authors suggest is so critical.



Students are asked to make predictions based on the model. If the largest number of tables that fit in a row in the gym is 15, then how many chairs could be arranged around the 15 tables? If 28 people are coming to dinner, then how many tables are needed?

Four different solutions are presented, each providing a window into unique student thinking. Each solution is discussed in the "Follow-Up and Discussion" section. The authors make the important point here that "the art of being an effective teacher is developing the skill of understanding others' solutions methods, rather than just the one you prefer" (p. 160).

The problem is further explored in the next section, where a graphical representation of the problem is developed. Readers are taken through a step-by-step explanation of the strength of a graphical representation and the conventions associated with it. The terminology is explained clearly and is consistent with language used in the CLIPS series on Patterning, available to all teachers and students on the EduGAINS website Mathematics K-12 website. Later in the same chapter, the problem is revisited, introducing the concept of a variable. With its many representations of the same scenario, this is the perfect type of problem for cross-grade study groups, as it reaches across expectations covered in many grade levels.

The release of this text is well timed, as last Fall's PISA test scores have led to a flurry of articles in newspapers across the country, questioning the pedagogy used in delivering the mathematics curriculum. Some parent groups have petitioned the provincial government for a return to basic skills and memorized algorithms. Classroom teachers need to be able to defend a curriculum that supports learning by understanding. Boland and Kajander have provided us with all the research necessary to be confident to do so. If we as an educational community believe that Mathematics is learned best when students are actively engaged in constructing their own understanding, then we face the challenge of choosing appropriate tasks for our students. This resource will help teachers find those tasks and use them in a manner that will create classrooms that value students mathematical reasoning and produce confident problem solvers.

▲

▲ Why You Hate Math and How to Feel the Love! In search of cool answers to "What did you do in math today?"

BY GEORGE GADANIDIS, PHD

iBook from iTunes and Amazon ($3.99) Also available in print from **www.BrainyDay.ca** ($6 + shipping) All proceeds from this book are used to donate children's math literature to schools in need.





George Gadanidis is an education professor at Western University. George spends 50–60 days each year in elementary school classrooms, collaborating with teachers to develop and research math experiences for students and their parents that offer the pleasure of mathematical surprise, insight and beauty. Using documentaries from this work, George is authoring a freely accessible online Grade 3 mathematics textbook, available at ***www.researchideas.ca/wmt3****.*

Why do so many people hate math?

Why do so many people fear, dislike or even hate math?

A clue lies in your answers to the following quiz.

Check ( ) the statements that are true.

Parallel lines *never* meet.
You can *never* hold infinity in your hand.
2 plus 2 is *always* 4. ✓

Most people are surprised by the quiz. The typical response is, "All the statements are true!"

The reason so many people hate math is that all of these statements are in fact false. How is it possible we don't know this? What strange math did we study in school?

Chapter 2. Can you hold infinity in your hand?

Here is what William Blake said about infinity, in the first stanza of his poem *Augeries of Innocence:*



To see a world in a grain of sand,
And a heaven in a wild flower,
Hold infinity in the palm of your hand,
And eternity in an hour.

Can you walk out the door?

Do you know that if you can't hold infinity in the palm of your hand, then you can't walk out of the room you're in? At least not if you think about it using this riddle:

Start 1 m away from an open doorway.
Walk half the distance to the doorway.
Then half the remaining distance.
Then half the remaining distance.
Then half the remaining distance.
How far will you travel if you do this forever?



Will you ever be able to reach the doorway and walk past it?

The interesting thing about this riddle is that if you don't think about it, then you have no problem walking to the doorway and beyond.



But, when you think about the infinite number of fractional distances you have to travel on your way to the doorway, it's difficult to imagine how it's possible.

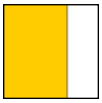
If you can solve this riddle, then you're well on your way to understanding infinity and limit, which is the foundation of Calculus.

2.A. An infinite number of fractions

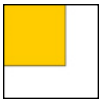
Here's how we've explored the "walk to the door" infinity riddle with children in third grade.

Fractions as area diagrams

Let's take the first 5 fractions that you would walk on your way to the door (1/2, 1/4, 1/8, 1/16 and 1/32) and represent them as shaded areas of a square.



Here is how we could represent the fractions 1/2 and 1/4.

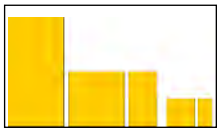


Imagine how squares could be shaded to represent the next three fractions: 1/8, 1/16 and 1/32.

Draw five squares and shade in the five fractions.

Putting it all together

Use a pair of scissors to cut out the shaded parts, as shown here.



Now imagine doing this for all of the fractions you would walk on your way to the door.

Then, imagine joining all of the shaded parts to form a new shape. How big would that shape be?

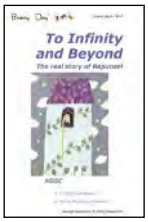
2.B. Infinity can be small

In a third grade classroom, we read *To Infinity and Beyond*, a retelling of the story of Rapunzel. See **www.BrainyDay.ca**.

The real story of Rapunzel

Rapunzel is trapped in her cell, even though her cell door is open.

Rapunzel doesn't think she can get past the open door because she's imagining that she has to first walk half way to the door, then half of the distance left, then half of the distance left, and so on forever. No matter how many times she does this, there is always some distance between her and the door.



Infinity in a square!

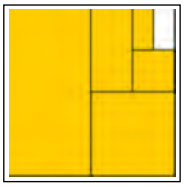
After students shaded five squares to represent the five fractions 1/2, 1/4, 1/8, 1/16 and 1/32, and then cut out the shaded parts, I asked:



Imagine doing this for all of the fractions that you would walk on your way to the doorway. Then imagine joining all of the shaded parts to form a new shape. How big would that shape be?

I've asked this question in several third grade classrooms and there's always quick consensus that "All the fractions fit in a single square."

Here is how the first five fraction areas fit inside the square. If you keep going, adding shaded fractions to represent all of the other fractional distances you would walk on your way to the door, you would never leave the square.



There is a limit

Curious what third-grade students might say about the sum of the fractions, I wrote the following on the board and asked, "What is the sum of these fractions?"

1/2 + 1/4 + 1/8 + 1/16 + 1/32 + ... = ?

It's cool that students quickly convince one another that "The sum is a whole or 1, because all of the fractions of a square fit inside a single square."

2.C. Math art

Using colour and your imagination, you can create cool representations of this sum:

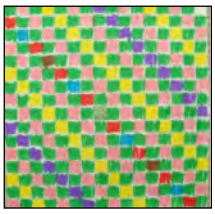
1/2 + 1/4 + 1/8 + 1/16 + 1/32 + ... = 1

Area and volume

You can spiral the pattern towards the centre of the square. You can shade the fractions as triangles instead of rectangles. You can create a three-dimensional representation.



The image on the left shows how you can use a 16 by 16 square grid. Shade every other square in the grid with one colour to represent 1/2. Then shade every other un-shaded square with another colour to represent 1/4. And so on. There will eventually be a single square that is un-shaded. Divide this last square into a 16 by 16 grid and keep going. The process never ends.



Stained glass

Here is a stained glass version of "infinity in the palm of your hand".

2.D. Infinity in a song



In one third-grade classroom, students worked in groups to write dialogues of how they might share their learning with family and friends.

Infinity in my hand

Below is one of their ideas, written as song lyrics. Serena uses the walk-to-the-door puzzle to avoid having to take out the garbage.

Infinity in my hand

Hey Serena, do your chore
Take the garbage out the door
I can't Daddy,
I learned in math
That I can't do it any more

It can't be true Serena
Your teacher is crazy
You can walk out the door
You've done it
Many times before

Halfway there Dad
And half of what's left
And half of what's left
It never ends
Infinity gets in the way
I heard my teacher say

Math concert

See a music video of the song *Infinity in my hand* at **youtu.be/y638CsJJ8vA**, performed by Bob Hallett of Great Big Sea and



Aboriginal recording artist Tracy Bone, at the Regent Theatre in Oshawa, in 2012.

I like infinity!

Below is a short poem written by two second-grade students about infinity, while participating at a summer math and science camp.

I Like Infinity

I like infinity
I can hold it in my hands
It goes on forever
I like infinity
I can hold it in my hands
I can fold it up small
1/2, 1/4, 1/8, 1/16
1/32, 1/64, 1/128
We love infinity!



Celebration of student learning



Songs like *Infinity in my hand* celebrate student learning.

We sing these songs in math concerts for K–8 schools, funded by the Fields Institute for Research in Mathematical Sciences.

See more about our band at **www.researchideas.ca/jx**.

2.E. Infinity as a painting



The painting on the right shows images of students as they explore infinity and discover they can hold it in their hands.

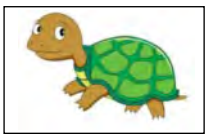
The painting was created by artist Ann Langeman, based on a design by George Gadanidis. It is freely available as a poster from **www.researchideas.ca/resources.html**.

The painting can be seen hanging at St. Matthew the Evangelist Catholic School, in Whitby, Ontario.

2.H. Inventing Calculus

Historically, mathematicians had a difficult time getting their minds around infinity.

Zeno's paradoxes



The infinity paradoxes posed by Zeno of Elea (490–430 BC) puzzled and amused mathematicians for centuries. In one of his paradoxes, Zeno proposed that if Achilles gave a tortoise a head start, he would never be able to catch up.

To catch up to the tortoise, Achilles must first reach the point where the tortoise started.

No matter how fast Achilles runs, each time he gets to where the tortoise was, the tortoise will have travelled some distance farther.

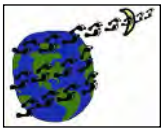
This process of Achilles reaching the point where the tortoise was never ends.

Zeno proposed that this infinite process prevents Achilles from catching up to the tortoise.

Of course, we know that in real life, Achilles would not have a problem in catching up to and passing the tortoise. However, when we think of the infinite process described by Zeno, we get a paradox.

Infinity in every step

A student in one of my math-for-teachers courses described her thinking about such paradoxes in this way:



Thinking about all the distances I have to cross vs "not" thinking about it, I realized that we are ALWAYS crossing these distances, and we arrive at all kinds of destinations. So whether I think about it or not, I'm crossing fractions of a distance all the time. I can think about it in terms of decreasing fractions of space, or not think about it at all.

In search of good math stories



We have developed a tradition in math education where being able to share a good math story is not valued. Math has become something we only do in school, something most of us have learned to dislike, something we try to avoid. We need to change this!

Math can be something we are attracted to, something we can lust for, says poet Molly Peacock. See a music video of her poem, *Math Lust*, at **youtu.be/qpkGxKdMHpc**.



Good storytelling involves solving the artistic puzzle of creating surprise and offering insight (Boyd, 2001).

The math stories in this book are examples of solving such artistic puzzles, to add excitement to children's math learning.

Good math stories need a low floor and a high ceiling

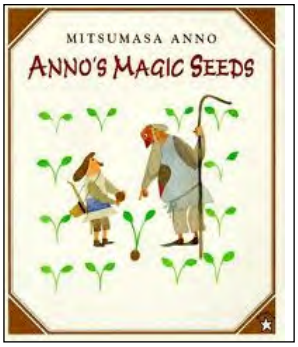
A *low mathematical floor* allows engagement with

Editor's Note: Sections 2F, 2G and 2I are not included in this reprint due to space limitations.

minimal math knowledge. A *high mathematical ceiling* extends concepts and relationships to more complex connections and more varied representations.

For example, the "Infinity in my hand" activity starts with the low floor puzzle of walking to a door, by repeatedly travelling half of the distance that is left. It also connects to the familiar concept of representing fractions by shading areas. At the same time, the activity uses the high ceiling ideas of infinity (the fractions never stop) and limit (their sum is 1).

Using a low math floor and a high math ceiling solves the artistic puzzle by teaching area representations of fractions in the context of "I can hold infinity in my hands!"



We also use children's math literature to introduce, reinforce or extend math ideas.

Good children's math literature offers math surprises and insights. Good stories also serve as lesson plans, with natural places to pause and investigate, as well as opportunities to synthesize and extend ideas.

Content and context

Teachers plan lessons with the goal of covering mandated curriculum expectations. Curriculum expectations determine the content they teach. A challenge with individual curriculum expectations, however, is that they can be quite *small* mathematically.

Content based on *small* math ideas does not make for a good story. Small ideas do not offer math surprise and insight. To solve this problem, we teach math *content* in a *context* of big math ideas, often from higher grades, like "area represenations of fractions" in the context of "infinity and limit".

Math is big – children's minds are bigger

Thanks to some of Jean Piaget's work, there is a strong belief in math education that children go through defined stages of development, and that young children are not capable of abstract thinking (Egan, 2002). However, Piaget (1972/2008) cautioned about how generally his stages of development might apply, noting that:

> we used rather specific types of experimental situations... it is possible to question whether these situations are, fundamentally, very general and therefore applicable. (p. 46)

Papert (1980), who studied with Piaget, says that Piaget's stages are not in children's minds but in the "poverty of the culture" of schooling. Egan (2002) says that if young children cannot think in abstract ways, they would not be able to develop language.

"Generations of school children, deprived of challenging tasks because Piaget said they were incapable of them, bear the evidence of his impact" (Fernandez-Armesto, 1997, p. 18).

Ginsburg (2002) suggests "children possess greater competence and interest in mathematics than we ordinarily recognize" and we should aim to develop a curriculum that challenges them to understand big mathematical ideas and offers opportunities to "achieve the fulfilment and enjoyment of their intellectual interest" (p. 7).

The math shared in this book challenges that common idea that students learn best when learning is made easy by designing experiences where students *think hard*.

Children need an audience



Children love to share their ideas. As one teacher said, "They love presenting. It gives them a chance to express themselves, and that's exciting."

Good stories and an audience work together. Young children are eager to share math stories that will surprise their family and friends.

Audiences demand stories that offer new, wonderful and surprising ways of looking at math.

References

Boyd, B. (2001). The origin of stories: Horton hears a who. *Philosophy and Literature, 25*(2), 197–214.

Egan, K. (2002). *Getting it wrong from the beginning. Our progressive inheritance from Herbert Spencer, John Dewey, and Jean Piaget.* New Haven: Yale University.

Fernandez-Armesto, F. (1997). *Truth: A history and a guide for the perplexed.* London: Bartam.

Ginsburg, H.G. (2002). Little children, big mathematics: Learning and teaching in the pre-school. *Proceedings of the 26th Conference of PME* (Vol.1, pp. 3–14), University of East Anglia: PME.

Papert, S. (1980). *Mindstorms: Children, computers and powerful ideas.* New York: Basic Books.

Piaget, J. (2008). Intellectual evolution from adolescence to adulthood. *Human Development, 51*(1), 40–47. (Original work published 1972). ▲

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Abacus

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PATTERNING AND ALGEBRA, DATA MANAGEMENT AND PROBABILITY

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Proportional reasoning is the underpinning of pattern relationships. Think about the use of proportionality in this context. What is the 5th figure? How does the pattern grow? What changes as the pattern grows? What remains the same as the pattern grows? These kinds of questions provoke students to think about multiplicative relationship between the figure (term) number and the number and type of shapes in each figure (term).



In this issue, the Research Summary and problems (with multiple solutions) focus on the use of proportional reasoning within data management and patterning and algebra contexts. The problems we provide are designed for use within a three-part problem-solving lesson structure.

LINKS TO LITERATURE AND MANIPULATIVES



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As we come to the end of another school year, I thought that I would share some criteria for choosing books so I began by searching the Internet for some of Marilyn Burns' articles about using picture books to teach math. I came across this article on Scholastic's website, "Teaching Math With Picture Books, Part 1" by Alycia Zimmerman (2012). In this article, the author classifies books and sorts them into three categories: (a) fundamental math picture books in which the math content is the primary purpose of the book (b) embedded math picture books in which the plot has deliberate connections to math, but the story stands on its own and (c) connected math picture books in which the connections to math are not explicit or obvious, but can be made by problem posing and inquiry. In this issue, I will use graphs to represent the math in some "fundamental" and "embedded" math picture books. See if you can tell which titles belong to which category. I will also share a math manipulative that appears in the English translation of a Japanese textbook.

RESEARCH SUMMARY – PROPORTIONAL REASONING WITH PATTERNING AND ALGEBRA

Proportional reasoning involves the deliberate use of multiplicative relationships to compare quantities and to predict the value of one quantity based on the value of another. The development of proportional reasoning is a gradual process, underpinned by the multiplicative thinking and the capacity to compare two quantities in relative (multiplicative) rather than absolute (additive) terms (Small, 2010a, 2010b). More specifically, "proportional reasoning [is] a form of mathematical reasoning that involves co-variation sense and multiple comparisons, as well as the aptitude to mentally gather and process several groups of information … it is related with inference and prediction and it involves qualitative and quantitative thought (Lesh, Post, & Northern, 1988, p. 93).

Types of Proportional Relationships

There are three main types of proportional relationships:

(a) *Simple Direct Proportion (or Direct Variation)*

A direct proportion occurs when two quantities change in the same direction, that is, as one quantity increases (or decreases), the other quantity also increases (or decreases), but the relative size of the change stays the same (Cai & Sun, 2002). For example, as the number of cans increases and the price of cans increases with a relative size of change, or the c*onstant of proportionality,* is $0.52

Number of cans	1	2	5	10	100
Price of cans	$0.52	$1.04	$2.60	$5.20	$52.00

(b) *Inverse Proportion (or Inverse Variation)*

Inverse proportion is when one quantity increases and the other quantity decreases and the relative size of the change stays the same (Cai & Sun, 2002). For example, as speed increases, the travel time decreases and as speed decreases, the travel time increases or speed is inversely proportional to travel time. In another example, suppose it takes 6 students, 4 days to complete the stuffing of school flyers in envelopes. Suppose it takes 4 days to complete the stuffing of 1000 envelopes. If we double the number of students, the stuffing should be completed in half the time. Another way of looking at this situation is to look at the product: as one factor increases (or decreases), the other factor decreases (or increases), but the product remains the same (e.g., number of students/days is # students x # days = 24)

Number of days	2	3	4	6
Number of students	12	8	6	4
Number of students/day	24	24	24	24

(c) *Complex Proportion (or Joint Variation)*

A complex proportion situation occurs when proportions involve more than two variables, and one of the variables may change simultaneously, in the same direction as one and in the opposite direction from the other.

Types of Representations for Proportional Relationships

Proportional reasoning includes explaining, interpreting and recognizing proportionality between two or more quantities that are being compared. Such proportional relationships are often represented as:

(a) *Equations* - represent a proportion by expressing two ratios as equal. An example is comparing 2 quantities such as 4 boys : 8 children = 1 boy : 2 children or 4/8=1/2.

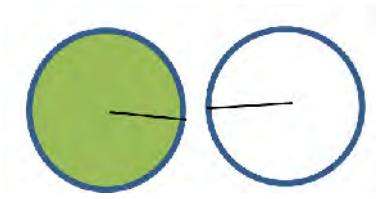
(b) *Ratio Tables* - are a convenient way of keeping track of the quantities within proportional situations, and for supporting proportional thinking strategies (e.g., the use of halving, doubling, multiplying by 10) and number strategies. For example, if there is 1 sports team per 14 children, how many children can are there on 14 sports teams?

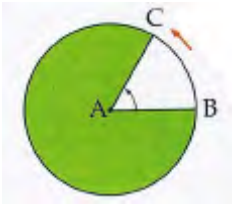
Number of teams	2	10	5	15	14
Number of children	28	140	70	210	196

(c) *Table of Values and Graphs* - enable students to see both direct and inverse proportional relationships. To see proportional relationships between 2 quantities, students record data in a table and graph the results on a Cartesian coordinate plane. When coordinate pairs of a simple direct proportion are plotted, the points all lie on the same line, either with an increasing trend (simple direct proportion) or decreasing trend (inverse proportion). In fact, all direct proportional relationships are linear relationships. More specifically, Lobato, Ellis, Charles and Zbiek (2010) suggest that students consider ordered pairs to be ratios, that is, multiplicative comparisons, and think about how many times greater each term value (*y*-value) is than the corresponding term number (*x*-value). This constant of proportionality (or constant of variation) is referred to as the slope of the line.

LINKS TO MANIPULATIVES: CONGRUENT PAPER CIRCLES

This manipulative is an easy to make, inexpensive yet powerful tool. Cut two congruent circles from two different coloured paper (the size doesn't matter). Then, cut a slit in each circle from the circumference of the circle to the centre. Join the two circles together using the slits.







Provide each student with congruent paper circles and challenge them to describe the mathematics they see:

- I see fractions – one fourth, one half
- I see angles – right angle, obtuse angle, acute angle, straight angle
- I see how I could make a circle target to use as a spinner for probability

The manipulative is useful in the "before" part of a lesson as each student can respond to challenges (e.g., show what could a spinner could look like if it had two colours (green and white) and represents a situation that it is less likely to land on green than on white.) Students can share and compare responses and analyse the solutions (e.g., each solution is more than half white) Use the manipulative to make connections between proportional reasoning and spinners and circle graphs.

LESSON LEARNING GOALS

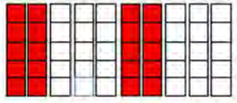
Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

Grade 5	Grade 6	Grade 7
-describe multiplicative relationships between quantities using fractions/decimals -simple multiplicative relationships involving whole number rates	-represent relationships using unit rates -identify quantities in an equation that vary and those that remain constant	-demonstrate understanding of rate as a comparison or ratio of two measurements with different units; solve problems involving calculations of unit rates

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Quilting Planning Problem A:

A quilt has 60 pieces. Every 5 pieces in a quilt consists of 2 red pieces and 3 white pieces. How many pieces are there of each colour?



Solution 1

2 red : 3 white → 5 total
4 red : 6 white → 10 total
6 red : 9 white → 15 total
8 red : 12 white → 20 total
20 red : 30 white → 50 total

24 red
36 white
60 total

same

Solution 2

Sets	1	2	3	4	5	6	7	8
Red	2	4	6	8	10	12	14	16
White	3	6	9	12	15	18	21	24
Total	5	10	15	20	25	30	35	40

Total of 20, 12 white, 8 red
Total of 30, 18 white, 12 red
Total of 60, 18x2 white, 12x2 red (36+24=60)

x2

5 groups of (2+3) -> 25
10 groups of (2+3) -> 50
12 groups of (2+3) -> 60

1:6
2:12
3:18
5:30
x6

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to make a plan to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Tangram Problem 1:



Travis does 6 good deeds every 4 days. How many good deeds will he do in 32 days?

What information are we going to use to solve the problem?
- *4 days with 6 good deeds*
- *32 days - How many total good deeds to do?*

AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Multiplicative Relationship – *How many times larger is the y value (deeds) than the x-value (days)*
-> *x-value X 1.5 = y-value (between 2 quantities)*

Equivalent (Part-Part) Ratios *are related by ratio constant or constant of proportionality … X 1.5*

for 4 days -> 6 deeds
4x 1.5 = 4x1+4x.5 = 4+2 = 6
for 8 days-> 12 deeds
8x1.5 = 8.1+8x.5 = 8+4 =12
for 12 days -> 18 deeds
…
for 32 days -> 48 deeds
32x1.5 =32x1+32x.5 =32+16= 48

 # days x 1.5 good deeds

4:6
8:12
12:18
16:24
20:30
24:36
28:42

part-part equivalent ratios x:y

Solution 1

days
4 8 12 16 20 24 28 32
6 12 18 24 30 36 42 48
deeds

Solution 2

x-value y-value

days	deeds
4	6
8	12
12	18
16	24
20	30
24	36
28	42
32	48

In-Between 2 Quantities
Using Double Number Line
4 X 1.5 = 6

In-Between 2 Quantities
Using Table of Values
x-value X 1.5 = y-value

Proportional Relationship
As a change in one quantity, which causes a corresponding change in another quantity with size of change staying same (co-variation - across sets of quantities)
→ 4 days/6 deeds -> X 8/8 -> (4x8)/(6x8) = 32/48

Simple Direct Proportion (or Direct Variation)
- two quantities change in the same direction
- as one quantity (# of days) increase, the 2nd quantity increases (# of good deeds)
→ 4:6 (X4/4) … 16:24 (X2/2) … 32:48
X 8/8

1 = 2/2 = 4/4
= 8/8

Does the order of the ratio matter to predict?
6:4 (x4/4) … 24:16 (X2/2) … 48:32
→ # good deeds : # days … y-value : x-value

Equivalent Ratios
-multiplicative relationship between x:y is X 1.5
4:6, is 4x1.5= 6
8:12 is 8x1.5=12
32:48 is 32x1.5=48

Predict Quantity Across Sets of Quantities
4 : 6
4(÷4) : 6 (÷4)
1 : 1.5 — *unit rate*
1(X32) : 1.5 (X32)
32 : 48
x : y

Rate *is a special ratio - comparing 2 quantities of different units (days/deeds)*

Equivalent Rates
4 days / 6 deeds (÷4/4)
1 day / 1.5 deeds (X32/32)
32 days / 48 deeds

Solution 3

	÷4		X32	
Days	4	1	16	32
Deeds	6	1.5	24	48
	÷4		X32	

Divide both by 4/4 and multiplied up by 32/32

Solution 4

equivalent rates

4 days / 6 deeds = 32 days / ?? deeds

X 8

4 days / 6 deeds = 32 days / 48 deeds

X 8

Across Sets of Quantities
Ratio Table
Using Unit Rates

Across Sets of Quantities
Predicting Unknown Rate
Using Equivalent Rates

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed

Highlights/Summary:

A simple direct proportion shows a change of 2 quantities in the same direction with size of change staying the same.

Relate and predict rates using proportional relationships by comparing:
- *in-between quantities (part-part equivalent ratios related by ratio constant/constant of proportionality)*
  - *double number line (4 days x 1.5 = 6 deeds)*
  - *table of values (x-value x 1.5 = y-value)*
- *across sets of quantities (multiplicative relationships between equivalent rates)*
  - *ratio table - all rates are equivalent - multiply and divide by 1 (1=4/4=32/32)*
  - *equivalent rates - 4 days/6 deeds x8/8 =32 days/48 deeds*

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Good Deeds Problem B:

Clifford does 6 good deeds every 4 days (same as Travis). How long will it take Clifford, working at the same rate, to complete 21 good deeds?

Solution 1

+2 x6

Deeds	6	3	18	21
Days	4	2	12	14

3 + 18 = 21 (+2, x6)
7 + 12 = 14

Solution 2

6 deeds / 4 days — 18 deeds / 12 days — 3 deeds / 2 days
3/2 x 7/7
14 days he could do 21 good deeds.

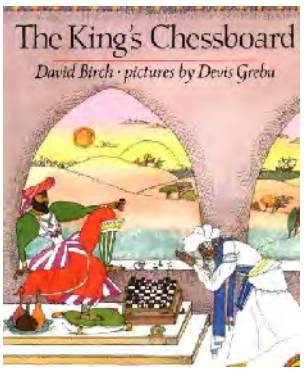
Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.



Michael Dahl has a collection of counting books, by ones, twos, fives and tens in his "Know Your Numbers Series" (e.g., titles for counting by twos; Pie for Piglets, Footprints in the Snow and Eggs and Legs.) The book Eggs and Legs introduces counting by twos by counting the number of legs coming out of the eggs and then create a bar graph of counting by twos to visualize the +2 growing pattern.



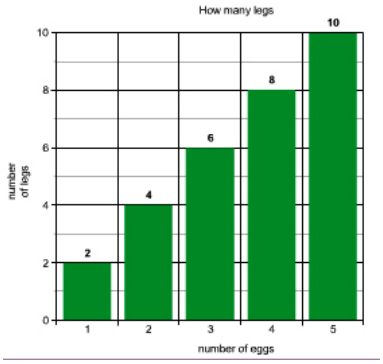


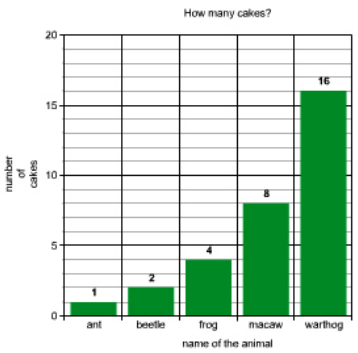
LINKS TO LITERATURE:

Representing the changes in the numbers, in bar graphs, in picture books is a strategy that can be used to compare growing patterns.

According to the expectations in the patterning and algebra strand, students in grade 2 are expected to "represent a given growing or shrinking pattern in a variety of ways (e.g., using pictures, actions, colours, sounds, numbers, letters, number lines, **bar graphs**)."

Compare bar graphs for counting by twos (from the book, *Eggs and Legs*) to a bar graph that shows doubling (from the book, *The Lion's Share*). *The King's Chessboard* also has a growing pattern that is based on doubling.





Students in junior grades can use equivalent ratios to describe the counting by twos pattern (e.g., 1 egg: 2 legs; 2 eggs: 4 legs; 3 eggs: 6 legs) and fractional language to describe the proportional relationships in the doubling pattern (e.g., the frog is making one fourth as many cakes as the warthog.)

LET'S DO MATH – DICEY PROBLEMS

Before (Getting Started)



Dicey Problem 1:

What do you notice about the number cube? If you roll it 10 times, how many times do you think a 5 will land face up?

Solution 1

The number cube has 6 sides numbered 1, 2, 3, 4, 5, and 6.

If you roll it 6 times each number should come up once. If you roll it 10 times a 5 might come up twice.

Solution 2

Probability (5) = 1/6
If you roll it 12 times, you should get 5 two times.
If you roll it 18 times, three 5s should come up.

So, rolling the cube 10 times suggests you will get a 5 one or two times.

During (Working On It)

Dicey Problem 2:

If you win when you roll an even number less than 5 on a die (same as number cube), what is the probability that you will win in a game involving 100 rolls of the die?

Scaffolding Questions

- *How many times should you win when the die is rolled 6 times?*
- *How many times should you win when the die is rolled 60 times?*



After (Consolidation)

Anticipating Student Responses:

Solution 1

I win when 2, or 4 are rolled. So, in 6 rolls I win 2 times or 2/6 or 1/3 of the time.

In 60 rolls, I should win 20 times.
(60 x 2/6 = 20)

In 100 rolls, I should win about 33 times.
(100 x 0.33 about 33 times)

Solution 2

# rolls	12	24	36	72	96	4	100
# times get even # < 5	4	8	12	25	32	4/3	33.3

In 100 rolls, I should win 33-34 times times.

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solution 2?

- *solution 1* – based on proportional reasoning as a winning roll comes up 1 in 3 times or in the ratio of 1 : 2. So for 100 rolls the ratio would be ((1 x 33.3) : (2 x 33.3) = 33.3 : 66.7)
- *solution 2* – ratio table shows the relationships between the numbers growing or shrinking at the same rate. So when we know 96 rolls + 4 rolls makes 100 rolls, we know 32 + 4/3 = 33.3 successful rolls.

LET'S DO MATH – SURVEY PROBLEMS

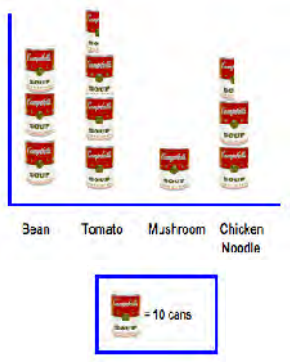
Before (Getting Started)

Survey Data Problem 1:

Examine the graph. What do you notice?

Compare the numbers of different kinds of soup. Give evidence.

Favourite Kinds of Soup



Solution 1

- I see there are more cans of Tomato soup than any other kind (35). (1 picture = 10 cans; 3 pictures = 30 cans and Half a picture = 5 cans)
- The fewest number of cans are mushroom (10). (1 picture = 10 cans)
- 25% of the cans are chicken noodle. (10 whole pictures = 100 cans The ratio of (chicken noodle) : (all kinds) = 25 : 100 or 25/100 or 25% so 25% of the choices are chicken noodle.

Solution 2

- I see a legend that says a picture of a can of soup represents 10 cans. So, the total can be considered 100% (per cent)
- So, I think the image of the half-cans must represent 5 cans
- In order, the least popular to most is: mushroom 10; chicken noodle 25; bean 30; and tomato 35.
- The ratio of bean : chicken noodle = 30 : 25. If this ratio is maintained, and there were 100 chicken noodle cans then there would be 120 cans of bean.

During (Working On It)

Survey the Survey Data Problem 2:

Students at DeGrassi Middle School collected cans and packages of food for the food drive for Second Harvest. The chart to the right reports the results of their month-long effort.

Food	Number of items
Can fish	5
Can beans	45
Can soup	25
Rice bags	30
Can fruits	10
Peanut butter jars	35
Can vegetables	30
Macaroni & cheese	20

Two other schools joined the food drive. Northern school matched DeGrassi's total and East Meadows school had twice as many of each item than DeGrassi. How many items, in total did the 3 schools collect?

If Second Harvest estimates they need 100 bags of rice for the weekend, would they have enough from the 3 schools?

After (Consolidation)

Anticipating Student Responses:

Solution 1

The total number of items that DeGrassi collected:

In pairs: 30 + 40 + 75 + 55 = 200 items

Northern total = 200 items
East Meadows = 2 x 200 = 400 items
Total 200+200+400 = 800 items

Rice : 30 + 30 + (30 + 30) = 120
So yes, Second Harvest has enough rice for the weekend.

Solution 2

DeGrassi : Northern : East Meadows = 1 : 2 : 1 So, altogether there would be 4 times what DeGrassi collected.

Total items = 200 x 4 = 800 items
Or 1 : 2 : 1 = 200 : 400 : 200 with a total of 800 items

Rice: Well it says that Northern's total was the same as DeGrassi's but not that each item was matched so for sure 3 x 30 is 90 bags of rice but we can assume that there would be at least 10 compared to 30 bags of rice in the collection at Northern so Second Harvest would have enough rice.

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion, followed by solution 2?

- *solution 1* – shows the amounts taken for each school and added together.
- *solution 2* – shows the ratios comparing the 3 schools' totals as a proportion 1 : 2 : 1 = 200 : 400 : 200.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Try these proportional reasoning problems yourself first and then offer them to your students.
- To suit your students' learning needs, choose growing and shrinking (linear relations) problems that are relative to the grade specific curriculum expectations in the Patterning and Algebra or Data Management and Probability strands.
- Provide problems that have students explain, extend, predict and create growing patterns in order to use multiplicative reasoning in order to see proportional relationship between the term number and term value in terms of its expanded arithmetic structure.
- Practise noticing the different strategies and the breadth of the mathematics your students use to describe, show and make sense of proportional relationships, from ratio tables, table of values (term number, term value, arithmetic structure of term value), graphs (relationship between x and y values).
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit the multiplicative relationships between term number and term value of a growing or shrinking linear pattern as the students discuss and analyze each other's solutions to co-construct success criteria for the lesson learning goal.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below:

- Cai, J. & Sun, W. (2002). Developing students' proportional reasoning: A Chinese perspective. In B. Litwiller and G. Bright (Eds.), Making sense of fractions, ratios, and proportions (pp. 195-204). Reston, VA: NCTM.
- Kent, L., Arnosky, J., & McMonague, J. (2002). Using representational contexts to support multiplicative reasoning. In B. Litwiller and G. Bright (Eds.), Making sense of fractions, ratios, and proportions (pp. 145-152). Reston, VA: NCTM.
- Lesh, R., Post, T., & Northern, M. (1988). Proportional Reasoning. In J. Hiebert & M. Behr (Eds.) Number Concepts and Operations in the Middle Grades (pp. 93-118). Reston, VA: Lawrence Erlbaum & National Council of Teachers of Mathematics.
- Lobato, J., Ellis, A. Charles, R., & Zbiek, R.M. (2010). Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics in grades 6-8. Reston, VA: National Council of Teachers of Mathematics.
- Small, M. (2010a). Big ideas from Dr. Small, Grades K-3. Toronto: Nelson Education.
- Small, M. (2010b). Big ideas from Dr. Small, Grades 5-8. Toronto: Nelson Education.

CLOSING

Our next issue will focus on spatial reasoning within geometric and measurement contexts. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within learning and teaching mathematics within a three-part problem solving lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or mkestell@sympatico.ca

The Ontario Association for Mathematics Education
L'Association ontarienne pour l'enseignement des mathématiques

OAME 2014 Leadership Conference

For ALL Mathematics Educators

Promoting Mathematical Mindsets



Jo Boaler



Amy Lin



Cathy Bruce



Shelley Yearley

November 6 – 8, 2014
Holiday Inn, Toronto International Airport
970 Dixon Rd, Toronto ON

All Inclusive Registration Fee: $500/$545 (OAME/non-OAME)
See Page 4 for Group Incentives



Dr. Jo Boaler

Dr. Jo Boaler is Professor of Mathematics Education at the Stanford Graduate School of Education. Her research focuses on the how different teaching approaches impact students' learning, how equity is promoted in mathematics classrooms and the opportunities students receive to develop active relationships with mathematics. Her work also focuses on gender and learning, the role of group work and mathematical discussions in the development of understanding and the ways teachers may be supported in moving towards equitable and effective teaching environments. She is also the driving force behind YouCubed, a non-profit organization that provides K-12 resources for parents and teachers.



Amy Lin

Amy Lin's background in the sciences and the arts helps explain why her passion in mathematics education is to develop both the logical and creative aspects of the subject. She is currently teaching at Craig Kielburger SS, is a PhD candidate in math education and is co-director of her global education social enterprise, Blue Skies Teaching and Learning. Amy is often asked to speak on mathematics teaching and learning around the province and beyond and she is an author on a number of mathematics resources for students and teachers at both the elementary and secondary levels. Amy recently presented a TEDx talk entitled "Teaching Math-ishly". Amy has won awards for teaching and leadership, and has worked as a teacher, department head, math coach, program consultant, ministry researcher and provincial lead. Amy has significant expertise with differentiated instruction for student success, transitions, and the use of technology supports for students. (www.amylin.me)



Dr. Cathy Bruce

Dr. Cathy Bruce is Associate Professor in the School of Education and Professional Learning at Trent University. She is passionate in her conviction that all learners are capable of success in mathematics, from the young child to the experienced educator, and that our ultimate goal is to support learners to become capable and creative solution finders for the day-to-day and world-class problems we face. Her research areas include teacher and student efficacy, assessing the effects of professional learning models for educators, and technology use in the mathematics classroom. Dr. Bruce initiated the Math for Young Children (M4YC) project in Ontario with Dr. Joan Moss to feature, and learn more about, young children's mathematics thinking and capabilities by collaborating with JK to Grade 2 educators and students in the form of lesson study. She is also conducting a three-year study in collaboration with the Ministry of Education on fractions teaching and learning using collaborative action research with teachers in Ontario schools. You can see her work featured at www.tmerc.ca.



Shelley Yearley

Shelley Yearley is currently on assignment with the Ontario Ministry of Education (from Trillium Lakelands DSB) as a Provincial Mathematics Lead. She is engaged in a multi-year research project with Dr. Cathy Bruce and the Trent University Mathematics Education Research Team. This project, focused on the teaching and learning of fractions, allows her to work collaboratively with educators from seven school boards on the design and field testing of high-quality tasks. Additionally, Shelley supports school board leaders in their planning and implementation of professional learning plans, including connecting such boards with Provincial Mathematics Facilitators to lead the professional learning. During her time as a consultant with TLDSB, Shelley supported the development of the TLDSB Mathematics Profile, explored teaching and learning through lesson study in grade 7-10 classes, and co-organized a number of GAINS Math CAMPPPs. You can contact her at s.yearley@tldsb.on.ca

The Ontario Association for Mathematics Education
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2014 Leadership Conference Overview

DAY	TIME	ACTIVITY
Thu. Nov. 6	6:00pm – 7:30pm	Registration
	7:30pm – 9:00pm	Plenary Session with Amy Lin ***Tales of Passion*** **-** As educational leaders, we are committed to taking action and to bring about challenge and change. In this presentation, Amy will share and reflect on stories that will bring to life the importance of speaking success mindsets, building relationships, increasing confidence by developing skills and deep understanding, and the power of voice.
Fri. Nov. 7	7:30am – 9:00am	Breakfast and Registration
	9:00am – 12:00pm	Plenary Session with Jo Boaler ***Promoting Mathematical Mindsets*** – A revolution in the way we see and teach mathematics.
	12:00pm - 1:00pm	PUBLISHERS'/SUPPLIERS' LUNCHEON: preview of new products and an opportunity to view displays
	1:00pm – 4:00pm	Plenary Session with Jo Boaler
	5:30pm – 7:00pm	DINNER
Sat. Nov. 8	7:30am – 8:45am	BUFFET BREAKFAST
	8:45am – 11:30am	Plenary Session with Cathy Bruce and Shelley Yearley ***The Visible and Invisible Work of Math Leaders*** - Cathy and Shelley will describe the activity and results of a collaborative action research program with math facilitators. In this project, facilitators of mathematics professional learning identified some key features of their work that are unique to math. They also mined particular tensions and challenges as facilitators, with the goal of articulating what it means to facilitate mathematics learning with educators. Additionally, our work with educators has centred squarely on the topic of fractions - a difficult to teach and difficult to learn area of mathematics. (mid-morning nutrition break included)

The Ontario Association for Mathematics Education
L'Association ontarienne pour l'enseignement des mathématiques

Registration Fees:

WITH ACCOMMODATIONS: OAME MEMBER **$500;** NON-MEMBER **$545**

Fee includes:

- Full session
- Two nights at Holiday Inn Toronto Airport (2 people/room)
- parking
- 2 breakfasts, workshop refreshments
- 1 lunch
- 1 banquet
- taxes & gratuities

Note: Surcharge for Single Room - $70.

WITHOUT ACCOMMODATIONS: OAME MEMBER **$400;** NON-MEMBER **$445**

Fee includes:

- Full session
- 2 breakfasts, workshop refreshments
- 1 lunch
- 1 banquet
- parking
- taxes & gratuities

Early Bird and Group Incentives:

- **10% discount when groups of 4 or more register and pay before July 31, 2014.**
- **5% discount for individuals who register and pay before July 31, 2014.**
- **5% discount for groups of 4 or more who register and pay between July 31st and Oct. 1, 2014.**

REGISTRATION DEADLINE: October 1, 2014
SPACE IS LIMITED TO 200.

2014 Leadership Registration

To complete registration, please
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