Volume 25
Number 2

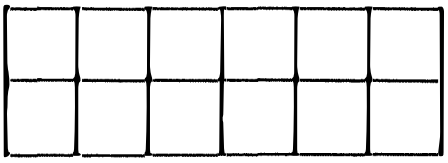
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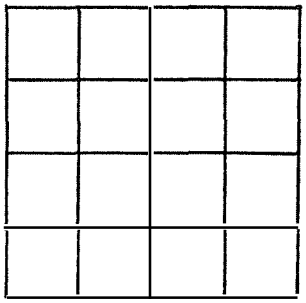
November 1986

PATTERNS AND PUZZLES

Here there are 10 straight lines and 17 squares.



Here there are 9 straight lines and 20 squares.



Find the fewest number of lines needed to make exactly 100 squares.

Editor: TREVOR BROWN
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TORONTO BOARD OF EDUCATION

Layout: LEARNXS PRESS

SPOT THE PATTERN

1. In this grid the counting numbers are arranged in a spiral pattern.

?												?
	?										?	
		?								?		
			37	36	35	34	33	32	31			
			38	17	16	15	14	13	30			
			39	18	5	4	3	12	29			
			40	19	6	1	2	11	28			
			41	20	7	8	9	10	27			
			42	21	22	23	24	25	26			
			43	44					?			
		?								?		
	?										?	
?												?

Where are the square numbers in this grid?

Without continuing the spiral, work out the numbers for the "?" squares.

2. This is another way to write the counting numbers in a grid.

Continue this pattern.

What are some of the patterns that you can identify?

What kind of numbers are in the top row?

1	3	6	10	15	21	28		
2	5	9	14	20	27			
4	8	13	19	26				
7	12	18	25					
11	17	24						
16	23							
22								
29								

3. Here is still another way to write the counting numbers.

1	4	9	16	25				
2	5	8	15	24				
5	6	7	14	23				
10	11	12	13	22				
17	18	19	20	21				
26					?			
?						?		
?							?	
?								?
?								

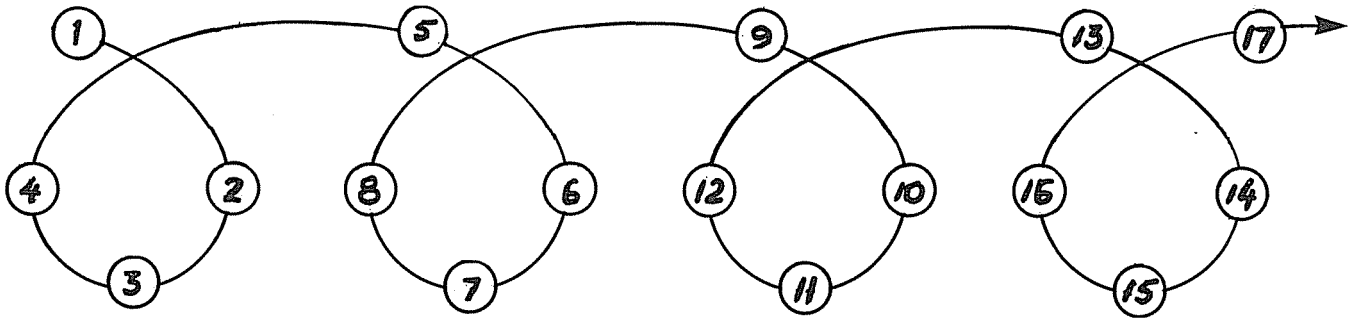
- Look at the numbers in the top row.

  What kind of numbers are they?

- What will the 9th number in the first column be?

- Without continuing the pattern, work out the numbers for the "?" squares.

4. Loops and a Spiral



- Which number will be at the bottom of the 10th loop?
  (Hint: Look at the last number in each loop.)

- Which of these numbers are at the bottom of a loop?

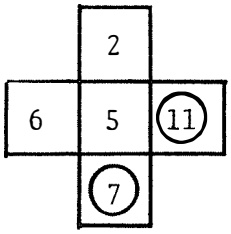
  21, 23, 42, 43, 47, 55, 60.

CROSS PUZZLES

This is an "addition cross".

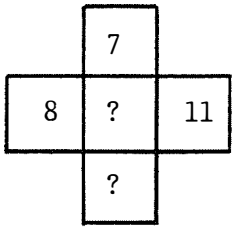
The numbers across make a sum: 6 + 5 = 11.

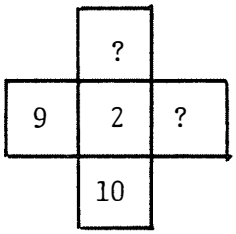
The numbers down make a sum: 2 + 5 = 7.

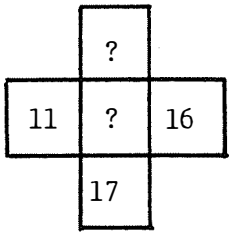


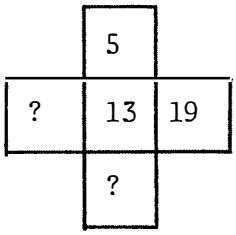
1. Copy and complete these crosses.

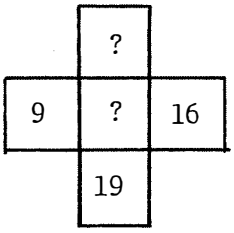
   The answers to the sums are circled.

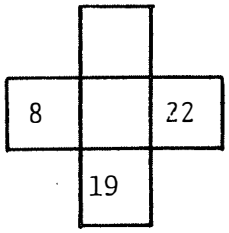






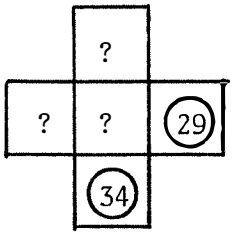


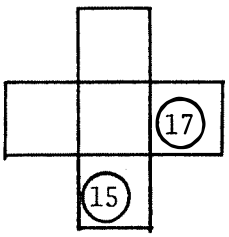


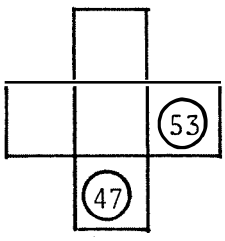


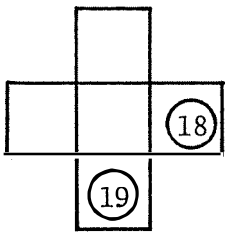
2. The following puzzles are far more difficult to solve.

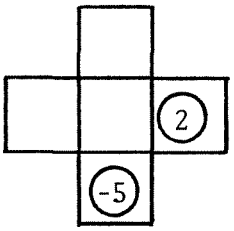
   Try to find an effective strategy for solving them.

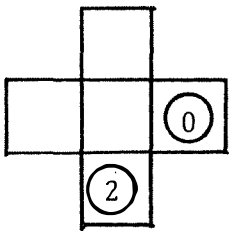








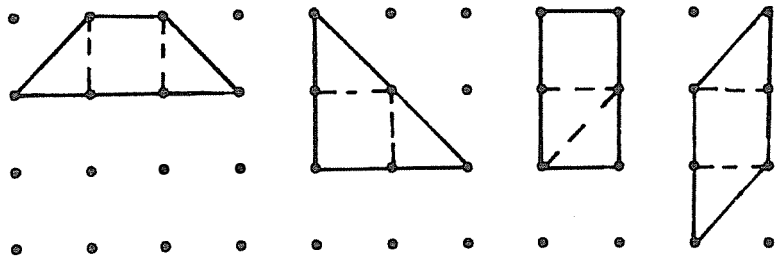




DISSECTION

You need geopaper for this exercise.

Look at these shapes.



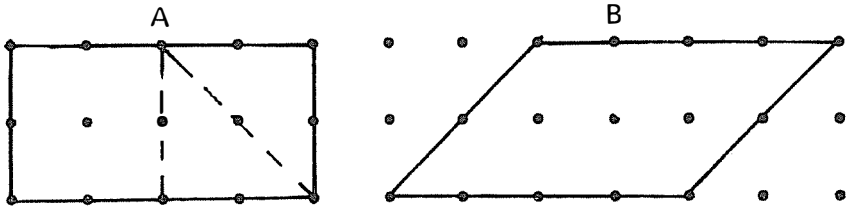
They can all be made from the same pieces.

We say the shapes have the same area.

1. Draw shapes A and B on geopaper.

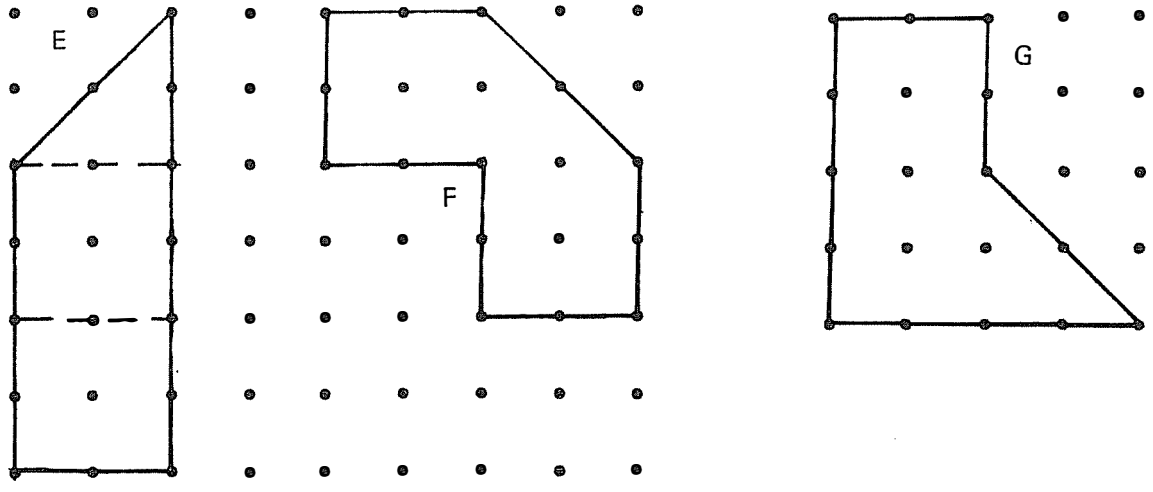
   Shape A is split into pieces.

   Split shape B into the same number of pieces with dotted lines to show that A and B have the same area.



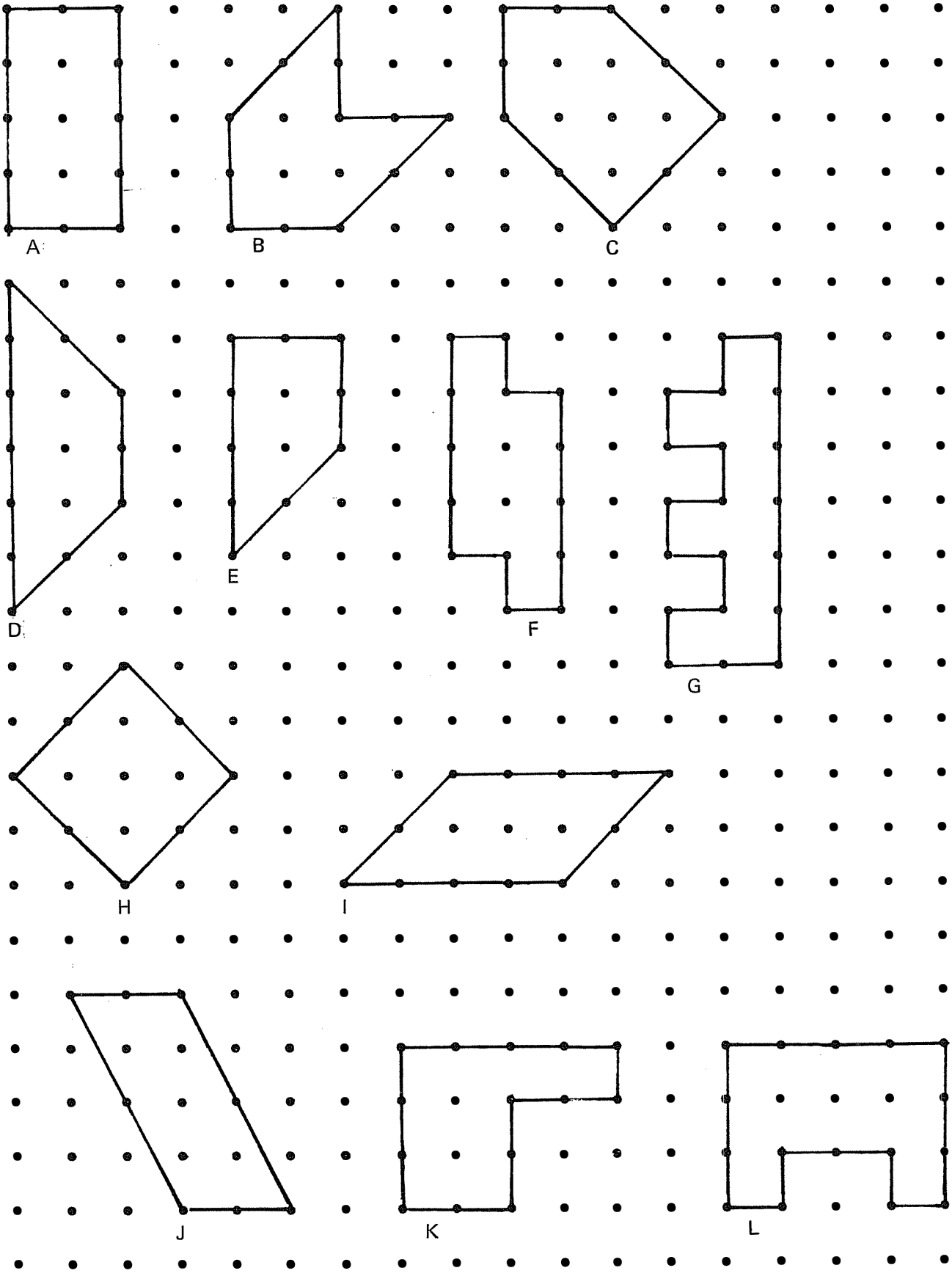
2. Draw shapes E, F and G on geopaper.

   Split up F and G into the same pieces as E.



3. Make a list of the shapes which have the same area as A.

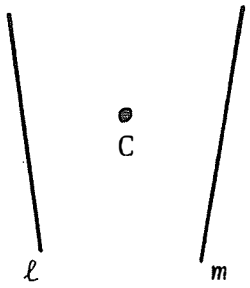
(a) Draw the shapes on geopaper. (b) Which of the following shapes have the same area as A?



GEOMETRICAL INVESTIGATIONS

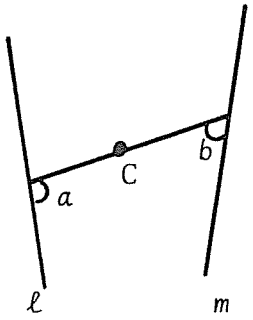
Draw two lines ℓ and m.

Mark a point C between them.



Draw any line through C which starts on ℓ and finishes on m.

Measure the two angles a and b.

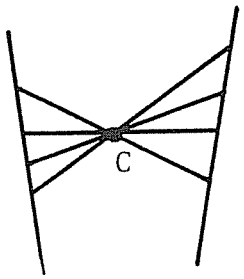


Make a table and write your results.

a	b

Draw additional lines through C.

Each time measure the angles a and b, and write your results in the table.



a	b

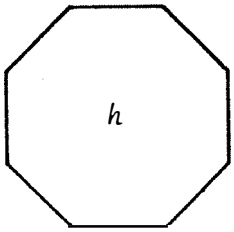
- Find a rule for the numbers recorded in the table.

- Write the rule and explain why it works.

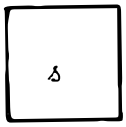
BRACKETS

Here is a way to explain why $h - (s-t) = h - s + t$.

1. Think of h as the area of this hexagon in cm^2.



Think of s as the area of this square.

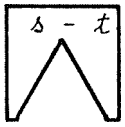


Think of t as the area of this triangle.



2. First subtract t from s.

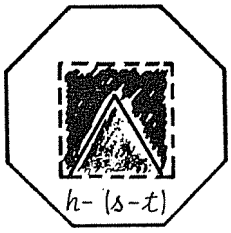
   The area of the shape left over is $s - t$.



3. Now subtract $(s-t)$ from h.

   The area left over is:

   $h - (s-t)$



Can you see from this diagram why $h - (s-t)$ must be equal to $h - s + t$?

The dotted line splits the area into two parts.

The unshaded part is $h - s$.

The lighter shaded part is t.

So the total is $h - s + t$.

From: Brackets 2 SMP11-16, Cambridge University Press

MATHEMATICAL DISCOVERY

(a) Think of an integer which contains more than one digit.

(b) Arrange the digits in decreasing numerical order so as to make the greatest number possible out of them.

(c) Arrange the digits a second time; this time in increasing numerical order so as to make the least possible number out of them.

(d) Now subtract the smaller number (c) from the larger one (b) to give a new number with the same number of digits; that is including initial zeroes if necessary.

(e) Repeat the operations (b), (c), and (d) on the new number and continue.

The object of the exercise is to see what happens as you go on and on.

Let us try an example to see what happens.

For this particular case the above procedure takes the form:

80 - 08 = 72	63 - 36 = 27
72 - 27 = 45	72 - 27 = 45
54 - 45 = 09	54 - 45 = 09
90 - 09 - 81	90 - 09 = 81
81 - 18 = 63	

and we notice that the last line is exactly the same as the fourth, so that we have formed a loop which regenerates itself over and over again.

- Does this sort of thing happen if we try another 2-digit number?

Now suppose we progress to 3-digit numbers; does anything different in principle happen for these?

Let us try an example to see what happens.

For this particular case the following would happen.

700 - 007 = 693
963 - 369 - 594
954 - 459 = 495
954 - 459 = 495

In this example we notice that our special number, 495, is not part of a loop; it stands on its own as the end product of the entire procedure - that is, it is self-reproducing forever.

But what if we selected another 3-digit number? Try it for yourself and see.

Was it 495 again? Any extensions?

FOR YOUR CONSIDERATION

METROPOLITAN REGIONAL EXAMINATIONS

BOARD

for the Certificate of Secondary Education

Paper 11(b)

(No time limit)

Answer One question only.

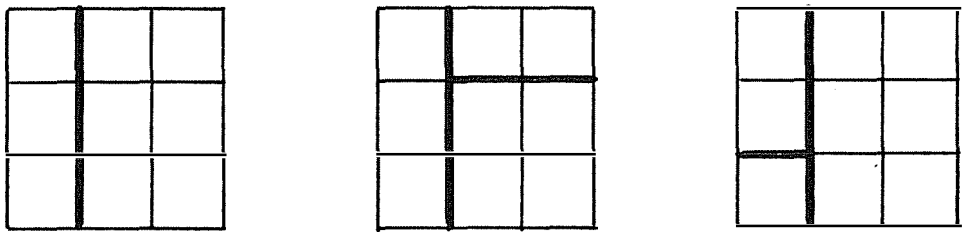
1. Write about the mathematics of a chessboard.

2. Investigate the set of numbers

   1, 1, 2, 3, 5, 8, 13, 21, 34, ...

3. A polygon is to be made from a piece of string 24 cm long. Investigate the possibilities.

4. When there were 5 houses in the school, 3 houses were in the hall for assembly each Wednesday, in rotation. Now that there are only 4 houses, 2 houses are in the hall each Wednesday. Compiling a rota for either situation can be complicated. Discuss the mathematics involved.

5. 

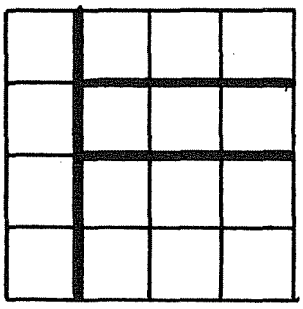
   The diagrams above show dissections of a 3-unit square into rectangles.

   (N.B., $\{\text{squares}\} \subset \{\text{rectangles}\}$) The rules are:

   (i) in each dissection the rectangles are all different,

   (ii) all the edges are a whole number of units long.

   The diagram below shows a similar dissection of a 4-unit square.

   Investigate other dissections of a 4-unit square, using the same rules. (Credit will be given for a systematic approach.)

SWITCHING OPERATIONS

1. Sometimes when you add two numbers and multiply the same numbers you get the same answer. e.g. Evaluate the following, and express your answer as a mixed number.

   (a) $1\frac{1}{2} + 3 =$ $\qquad$ $1\frac{1}{2} \times 3 =$

   (b) $1\frac{1}{3} + 4 =$ $\qquad$ $1\frac{1}{3} \times 4 =$

   (c) $1\frac{1}{4} + 5 =$ $\qquad$ $1\frac{1}{4} \times 5 =$

2. Find at least two cases when this does not occur.

3. Find some more cases when it does occur.

4. Write a statement describing when this will happen.

5. (a) Form algebraic expressions to illustrate your answer to #4.

   (b) Simplify each algebraic expression and show that they represent the same quantity.

Extension

Look at all the ways of representing the statements in #5(a) then simplify each one.

SWITCHING OPERATIONS 1

Objectives:

To prove numerical discoveries using algebra.
Making generalizations and forming algebraic expressions.
Simplifying rational variable expressions.

Possible problem-solving skills:

Look for patterns.
Make predictions based on data
Keep a systematic list.

Comments:

This can be used as an application of simplifying rational variable exprssions.

Answers:

1. (a) $4\frac{1}{2}$ (b) $5\frac{1}{3}$ (c) $6\frac{1}{4}$

2. Answers will vary.
3. Extend the pattern in #1.
4. The addend or multiplier is always one more than the denominator.
5. (a) One possibility is:

$$(1+\tfrac{1}{n}) + (n+1) = (1+\tfrac{1}{n})(n+1) = (n+2) + \tfrac{1}{n}$$

(b)

$(1+\frac{1}{n}) + (n+1)$	$(1+\frac{1}{n})(n+1)$	$(n+2) + \frac{1}{n}$
$= \frac{n+1+n(n+1)}{n}$	$= \frac{(n+1)(n+1)}{n}$	$= \frac{n^2+2n+1}{n}$
$= \frac{n^2+2n+1}{n}$	$= \frac{n^2+2n+1}{n}$	

PROBLEM SOLVING CHALLENGES

1. What integer multiplied by 52 631 578 947 368 421 will give all 9's?

2. Determine the sum of this series of numbers.

   101 + 301 + 501 + 701 + ... + 9 901

3. A goat is tethered by a rope 40 metres long. The rope is attached to a hook which is 10 metres from the corner of the longer side of a barn measuring 40 m by 20 m. Over what area of grass can the goat graze?

4. If a 20 cm pizza serves 2 people , then how many people eating the same portions would a 40 cm pizza serve?

5.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
.	.	.	.	.

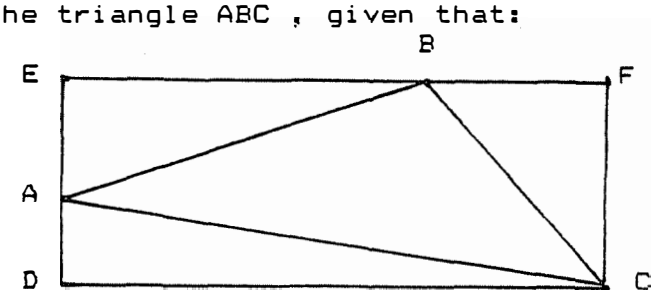
   For the table above , find the five numbers in the row which has a sum nearest to a value of 1 000.

6. If a , b , c are the smallest possible positive integers for which 2a = 5b = 7c , find the sum of a , b and c.

PROBLEM SOLVING CHALLENGES

1. Find the area of the triangle ABC , given that:

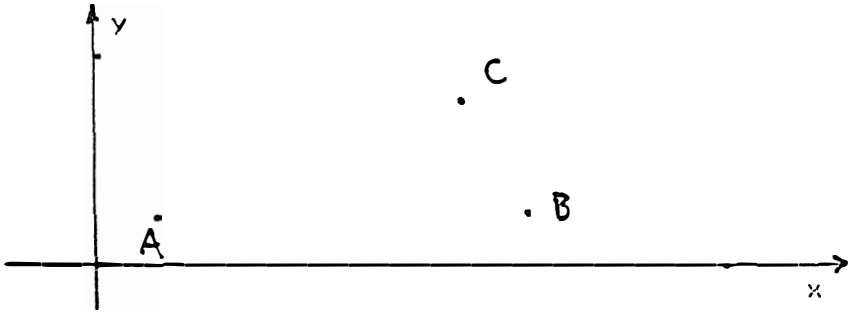
DA = 2
AE = 6
EB = 8
BF = 4



2. What numbers added to themselves give the same result as their squares.

3. The side , front and bottom faces of a rectangular solid are twelve , twenty , and fifteen square centimetres respectively. What is the volume of this solid?

4. Triangle ABC has vertices A(1,1) , B(7,1) , C(6,3). A triangle DEF has a base of 6 and has the same area as triangle ABC. Find the length of the altitude to the base for triangle DEF.



5. Show that no pair of integers satisfies the equation:

$3x + 6y = 38$

PROBLEM SOLVING CHALLENGES

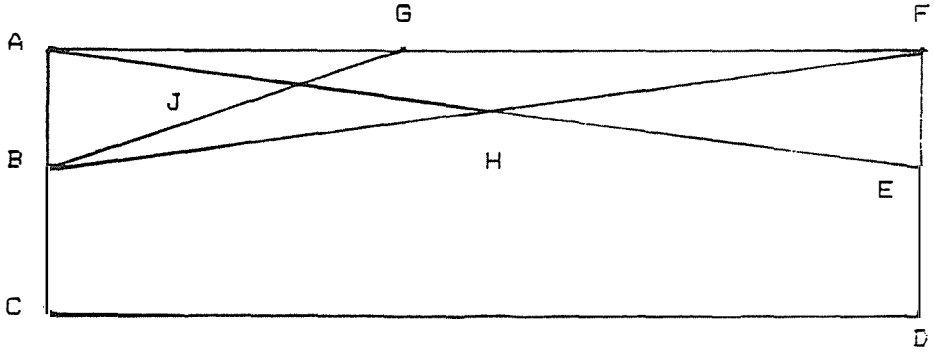
1. A special die was made. The faces were numbered 1,2,5,6,8,9. If two such die were rolled , then how many different sums were possible?
   How many different prime numbers appear as sums?

2. A rectangle has an area of 72 cm^2 and the perimeter of the same rectangle is greater than 20 cm . If the dimensions of this rectangle are integers , then what are the possible length and width?



3. A soup company wishes to change the dimensions of its cylindrical can. The old can has a height of 20 cm and a radius of 10 cm. Assuming that the new can will have integral dimensions , how many different dimensions are possible all of which result in a can of the same volume as the original.

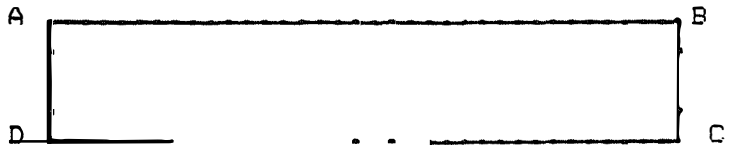
4. Every triangle in this shape is worth 3 points , every quadrilateral , 4 points. Five points will be given for a pentagon and six points for a hexagon. What would be the maximum score for this shape?



POSSIBLE SOLUTIONS FOR CHALLENGES

1. How many different sums were possible? 16
   Examples: 1,1 , 1,2 , 1,5 , 1,6 , 1,8 , 1,9 , 2,2 , 2,6 , 2,9 , 5,8 , 5,9 , 6,6 , 6,9 , 8,8 , 8,9 , 9,9.
   How many different prime numbers appear as sums? 6
   Example: 1,1 , 1,2 , 1,6 , 2,9 , 5,8 , 8,9.

2. A rectangle has an area of 72 square centimetres and the perimeter of the same rectangle is greater than 20 cm .



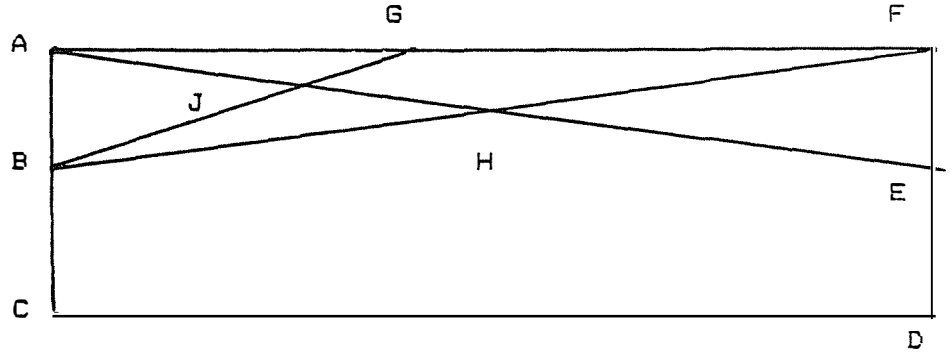
   Solutions: dimensions are 1 x 72 or 2 x 36 or 3 x 24 or 4 x 18.
   All units are centimetres.

3. A soup company wishes to change the dimensions of its cylindrical can. The old can has a height of 20 cm and a radius of 10 cm. Assuming that the new can will have integral dimensions , how many different dimensions are possible all of which result in a can of the same volume as the original.

   Solutions:

Radius	2	4	5	20
Height	500	125	80	5

4. Scoring this shape.



There are 9 triangles	27
Example: AJB , AJG , BJH , FHE , ABG , BGF , ABF , AFE , ABH.	
There are 3 quadrilaterals	12
Examples: AFDC , BFDC , AEDC ,	
There are four pentagons	20
Examples: BJEDC , BHEDC , AHFDC , BGFDC	
There are 2 hexagons	12
Examples: AJGFDC , BJHFDC	
SCORE	71