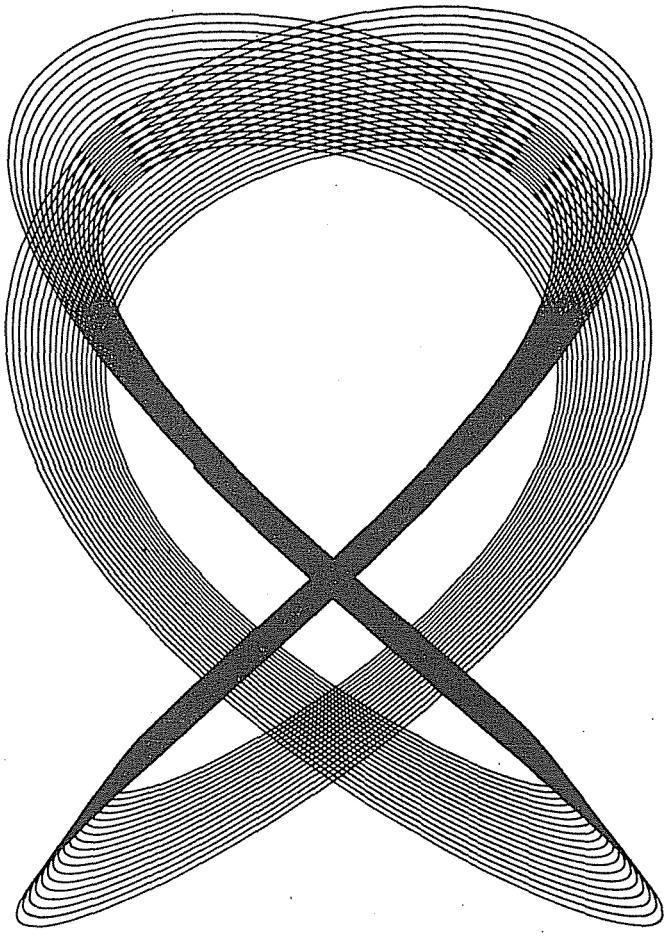
Volume II
Number 1

Abacus

A publication of the O.A.M.E.

September 1987



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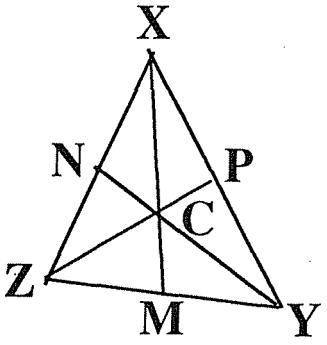
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A Generalization

Again this past weekend, somebody asked me why I called my yacht 'CEVA'. "There are two reasons," I usually respond, "One, it is an acronym and two, it's the name of a little known 17th century mathematician and since this vessel is owned by a little known 20th century mathematics teacher, it seemed appropriate".

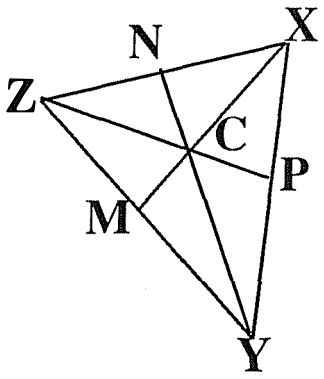
I used to re-introduce my grade 10 mathematics classes to geometry with accurately drawn diagrams of the various centres and circles associated with the triangle. Now, mathematicians are always generalizing particular results, so why not do the same for the median-centroid result.

By definition, a median is a line segment joining a vertex of triangle to the midpoint of the opposite side. The 3 medians are concurrent and this point is the centroid. Necessarily, too, the medians divide each side of the triangle in the ratio 1:1.



I get my students to draw a large triangle, XYZ and pick any point, C, within the triangle. They then join X to C and produce it to M, Y to C, etc. Unless by chance C is the centroid, M,N and P will not be the mid points of the sides. For convenience I will call XM, YN, ZP. Cevians.
I ask my students to find, by measuring, the 3 ratios

$$\frac{XP}{PY}, \frac{YM}{MZ}, \frac{ZN}{NX}.$$



I ask, "Are they equal? etc. Finally, I lead them to the product. Its approximately 1! Eventually we produce "If any 3 cevians of a triangle are concurrent, then the product of the ratios, taken in order, in which they divide the opposite sides, is 1".

i.e., If XM, YN, ZP are concurrent,

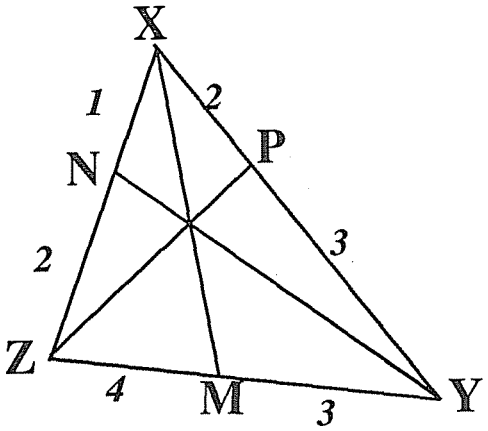
Then $\frac{XP}{PY} \cdot \frac{YM}{MZ} \cdot \frac{ZN}{NX} = 1$

I then set the class onto verifying the converse statement i.e. suppose $\frac{XP}{PY} \cdot \frac{YM}{MZ} \cdot \frac{ZN}{NX} = 1$

Does it follow that XM, YN, ZP are concurrent? I set the students to think of 3 easy ratios whose product is 1.

Suppose one picks $\frac{2}{3}, \frac{3}{4}, \frac{2}{1}$.

It's a simple matter, with a calculator to determine the points M, N and P. A carefully drawn figure suggests that indeed XM, YN and ZP are concurrent.



It's a good exercise in grades 9 and 10. Medians are but special cevians and the centroid is that point of concurrence in the degenerate case.

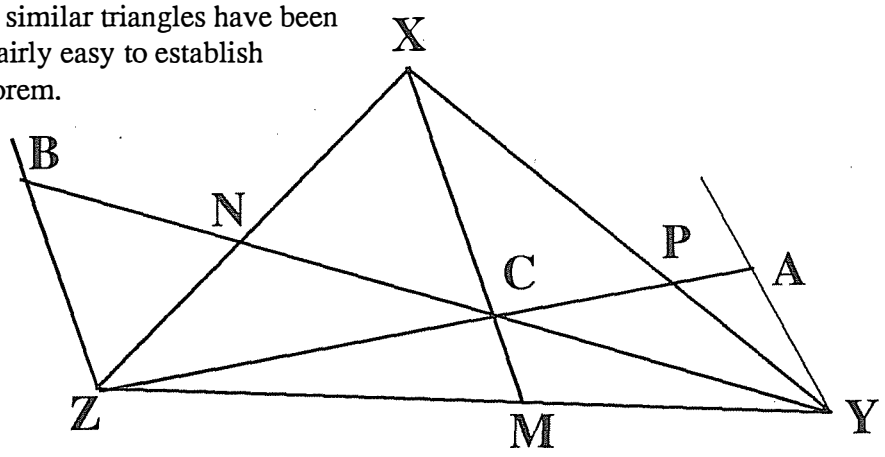
Def. A cevian of a triangle is a line segment formed by joining any point of a side of a triangle to the opposite vertex.

Theorem:

Three cevians XM, YN, ZP of ΔXYZ, are concurrent if and only if

$$\frac{XP}{PY} \cdot \frac{YM}{MZ} \cdot \frac{ZN}{NX} = 1$$

N.B. When similar triangles have been taken, it is fairly easy to establish Ceva's Theorem.



Draw YA // MCX // ZB where A and B are on the productions of ZP and YN respectively.

$\Delta XPC \sim \Delta YPA$, $\Delta ZYA \sim \Delta ZMC$, $\Delta YBZ \sim \Delta YCM$ and $\Delta BNZ \sim \Delta CNY$

So,

$$\left.\begin{array}{l} \dfrac{XP}{PY} = \dfrac{XC}{AY} \\ \\ \left\{\begin{array}{l} \dfrac{YM}{YZ} = \dfrac{CM}{BZ} \\ \\ \dfrac{ZY}{ZM} = \dfrac{AY}{CM} \end{array}\right. \\ \\ \dfrac{ZN}{NX} = \dfrac{ZB}{XC} \end{array}\right\} \quad \text{Product} = 1$$

So,

$$\frac{XP}{PY} \cdot \frac{YM}{MZ} \cdot \frac{ZN}{NX} = 1$$

The converse is easily proved too - use the indirect method.

David Wing
Frontenac Secondary School

GEM:

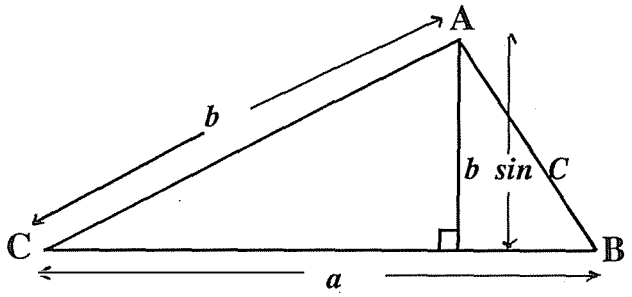
THE SINE ANGLE SUM FORMULA

One of the drawbacks of introducing the trigonometric ratios as functions is that it takes some time to reach the point where students can use trigonometry for problem solving. There is still a lot to be said for introducing trigonometry, after a discussion of similar triangles, as the angle dependent ratios of sides of right-angled triangles. My favourite mnemonic for remembering the names of the ratios is "Some Old Hen Caught Another Hen Taking Oats Away". By reciting this, you can write down the equations: $s = \frac{o}{h}$; $c = \frac{a}{h}$; $T = \frac{o}{a}$.

I have a large poster in my classroom depicting a bossy looking hen, wings tucked onto her hips, crossly scolding another hen that is scampering away with a satchel slung over her shoulder from which oat grains are spilling. The mnemonic is written on the poster and it attracts attention. Students are always curious about it, and by the time we come to the trigonometry unit, they have it memorized.

From the very first lesson, students are able to use trigonometry in practical applications. By the second lesson, the students are able to deduce that the height of ΔABC is b sin C and, therefore, that its area is

1/2 base x height = 1/2 ab sin C

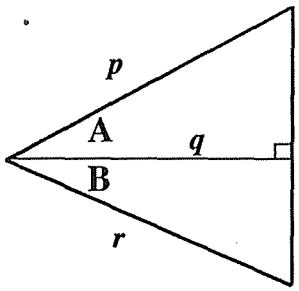


It is worth pointing out here that this formula uses two sides and the included angle and is the area formula that must necessarily exist because any two triangles having the same measures for two sides and the included angle are congruent.

Now for the gem. If your students have been prepared as above, you could now prove the "sine angle sum" formula in a way that they would find readily acceptable.

To begin with, in order to consider sin (A+B) you need a model of the angle A+B. Obviously, you make A and B adjacent angles. Now draw a line perpendicular to the common ray and extend it both ways to intersect the other two rays forming A and B. Let thelengths of the intercepts of these three rays be p, q and r as shown. Then the area of the whole triangle is 1/2 pr sin (A+B).

But this is simply the sum of the areas of the two right angled triangles of which it is composed.



Thus:

1/2 pr sin (A+B) = 1/2 pq sin A + 1/2 qr sin B multiply by 2 and divide by pr

$$\sin (A+B) = \frac{q}{r} \sin A + \frac{q}{p} \sin B$$

$$= \sin A \cos B + \cos A \sin B$$

Peter Battey-Pratt
Ernestown Secondary School

A Problem That Won't Quit!

George, who was 1/3 his father's age 4 years ago and whose present age now makes a sum of 64 with his dad's, is mixing a 40% salt solution with a 25% salt solution to get 600 kg of 28% solution. He plans to sell this mixture by driving from Montreal to Cornwall at 80 km/h and then from Cornwall to Belleville at 110 km/h. The total distance is 395 km, and the time is 4 hours. He will sell the mixture for $1.50/kg, and the money will be composed of a total of 54 $10 and $20 bills. George then will invest part of his money at 12% and part at 16% in order to earn $112 interest for one year.

1. How old is George?
2. How much 25% salt solution will he need?
3. How far is it from Cornwall to Belleville?
4. How many $20 bills will he have?
5. What amount will he invest at 16%?
6. Why would a teacher ever dream up a crazy problem like this?

Bruce Black
Frontenac Secondary School

Flashing Lights

I have designed an advertising sign with three flashing lights on it. The lights flash at different but regular intervals. The orange light flashes every 2 seconds, the green light flashes every 3 seconds, and the blue light flashes every 4 seconds.

How often do the orange and blue lights flash simultaneously (at the same time)?

How often do all three lights flash simultaneously?

What would be the shortest simultaneous flash-time if their separate times of flashing were 3, 4, and 5 seconds?

If there were four lights with different flash-times of 2, 3, 4, and 6 seconds, what would be the shortest simultaneous flash time?

Can you design a three-light board with different flashing time for each light and a shortest simultaneous flash-time of 8 seconds?

Can you design a four-light board with a shortest simultaneous flash-time of 30 seconds?

How many different three-light boards have a shortest simultaneous flash-time of 24 seconds if:

(a) all three lights have different flash-times?

(b) two of the lights may have the same flash-time?

Is it possible to design a three-light board with different flash-times for each light, and having a shortest simultaneous flash-time of 4 seconds? 5 seconds?
6 seconds? ... ?

Make up some problems of your own flashing light signs and have a class-mate solve them.

Hugh Allen
Faculty of Education
Queen's University

ANALYTIC GEOMETRY
Quiz 2

Answer all of the following questions for equation 1 or 2 or 3

	Equation 1 $x - 2y + 6 = 0$	**Equation 2** $2x - y - 5 = 0$	**Equation 3** $y = 2x - 1$
(a) Find the x-intercept			
(b) Find the y-intercept			
(c) State the coordinates of the point where the line crosses the Y-axis			
(d) State the coordinates of the point where the line crosses the x-axis			
(e) Find the slope of the line.			
(f) State the slope of a line parallel to the given line.			
(g) State the slope of a line perpendicular to the given line			
(h) Give an equation of a line that has the same Y-intercept as the given line			

Note: The quiz will be marked out of 10. There will be one mark for each of the eight parts. If you use equation 1 for your answers, you will get an additional 2 marks. If you use equation 2 for your answers, you will get an additional 1 mark. If you use equation 3 for your answers, there will be no additional marks.

(I use this type of quiz so that students who understand the concepts but cannot apply them to the difficult questions are able to tackle the questions and pass the test; and yet the best student is also challenged. I encourage students to answer all three strands if they have time and I give them their best mark.)

M. Elaine Harvey
Frontenac Secondary School

FUNCTION GRAPHICS

by Jack Sontrop
Electronics Department
Loyalist College of Applied Arts and Technology
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One future direction indicated by Ontario Ministry of Education Memorandum 91 is the use of graphics editors in a wide range of activities. I first wrote graphics programs on my home computer to dazzle myself, my family, and visitors to our rec room. This article aims to promote such activity and to foster awareness of mathematical concepts involved in certain types of applications.

New visitors to the computer labs at Loyalist College are usually spellbound by a plotter performing the magical ritual of precision drawing. Compared to a single graph of a particular function, a composite of multiple graphs of related functions can create a striking visual impact. Composites like those reproduced in this article are difficult to produce without the aid of a computer. They are also difficult to produce with commercial computer software such as MacPaint, AutoCad and AccuCad. Yet programs to produce such composites are relatively simple and can be easily modified to create interesting illusions of curved or jagged surfaces, depth and tension. The creation of unique, function-based, graphic objects can be a challenging and satisfying experience.

To illustrate, start with the set of cubic polynomials represented by $\mathbf{y = K\,x\left(x^2 - a^2\right)}$ with intercepts at $(-\mathbf{a},0)$ $(0,0)$ and $(\mathbf{a},0)$. Figure 1 below shows sixteen such polynomials plotted over the domain $-\mathbf{a} \leq \mathbf{x} \leq \mathbf{a}$. Each graph consisted of 128 segments. $\mathbf{K}_1$ through $\mathbf{K}_{16}$ formed an *arithmetic* sequence, terminating with $\mathbf{K}_{16} = 0$.

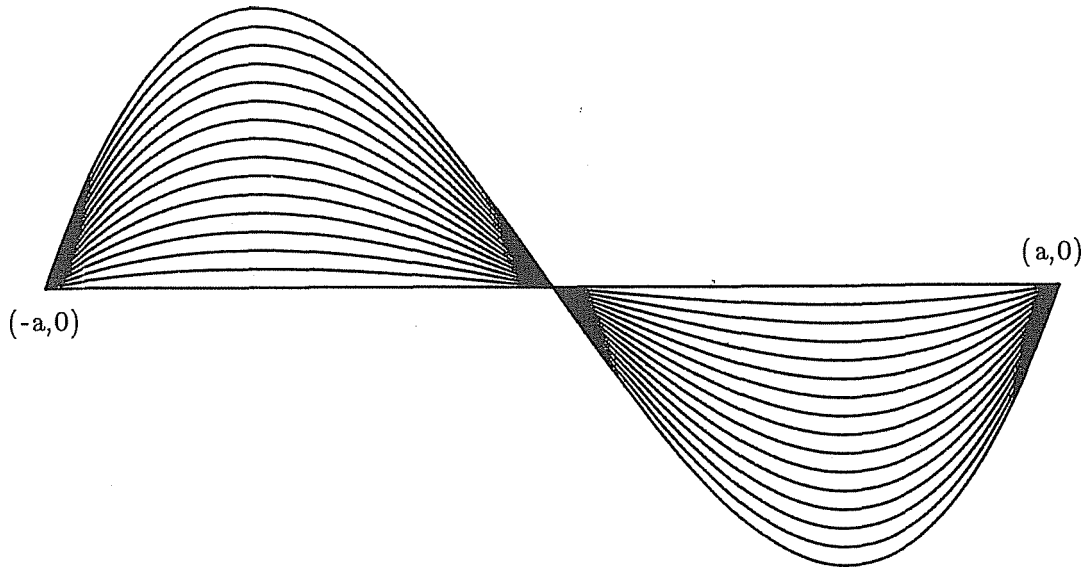


Figure 1: Family of Cubic Polynomials: $\mathbf{y = K\,x\,(x + a)\,(x - a)}$
The **K** coefficients formed an arithmetic sequence. $\mathbf{K \geq 0}$

On the next page is an algorithm for producing plots similar to Figure 1. You will have to translate it into your favourite programming language. If it's been a while since you graphed a function, it may help to get out the calculator and graph paper and do a few hand plots until you are thoroughly accustomed to the steps involved in producing graphs of functions. You may also need to review the topics of programming loops, sequences (arithmetic and geometric) and translation of coordinates.

ALGORITHM FOR FIGURE 1:

1. Initialize the output device. If it's a video display monitor, specify the screen resolution, colour, etc. If it's a plotter, specify graphic limits, grid size, scale, pen colour, pen size, etc.

2. Get (input, read, fetch, assign or scan) **a**, the absolute value of the x-intercepts.

3. Get x_i and x_f, the limits of the domain over which each plot will be made.

4. Get **n**, the number of segments used to plot each polynomial.

5. Compute x_{incr}, the common difference in the arithmetic progression of abscissae as **x** proceeds from x_i through x_f.

6. Get **p**, the number of similar polynomials to be plotted.

7. Get K_1 and K_p, the values of the numerical coefficient **K** to be used in the first and **p**'th plot respectively.

8. Compute K_{incr}, the common difference in the arithmetic progression of numerical coefficients as **K** proceeds from K_1 through K_p.

9. Compute X_{off} and Y_{off} which are offsets required to translate coordinates (Step 11) onto the grid established in Step 1. Note that if you establish a grid such that address (0,0) is near the centre of your monitor display or plotter, you can set both of these parameters to zero.

10. Initialize the numerical coefficient **K** to K_1.

11. Repeat the following:
    a. Initialize abscissa **x** to x_i.
    b. Compute its ordinate mate **y**, according to the functional relationship.
    c. MOVE the cursor or pen to $(x + X_{off} , y + Y_{off})$.
    d. For segment number **1** to **n** perform the following:
        Increment abscissa **x** by x_{incr};
        Compute its ordinate mate **y**, according to the functional relationship.
        DRAW the cursor or pen to $(x + X_{off} , y + Y_{off})$.
    e. Increment **K** by K_{incr}.

    until the **p**'th plot is finished.

Once you have implimented the basic algorithm, refine it to do some simple error checks. For example, what's to be done if the calculated coordinates are off the grid? (If your graphic output is to a monitor, this may not be a problem.)

Next you could modify Step 8 to create a graphic like Figure 2. Notice that the spacing between successive graphs (at $x = \pm a/2$) is no longer constant as it was in Figure 1. This may be important if you wish your composite to have a 3-D effect like Figure 6. Experiment with *geometric* rather than arithmetic increments to **K**. My increments were determined as follows: let the initial smallest difference between successive **K**'s be **d**; then let the sequence of subsequent *differences* be **2d, 3d, 4d ..... 4d, 3d, 2d, d**. It is left to you to determine the relationship between **d, p,** K_1 and K_p for odd and even values of **p**.

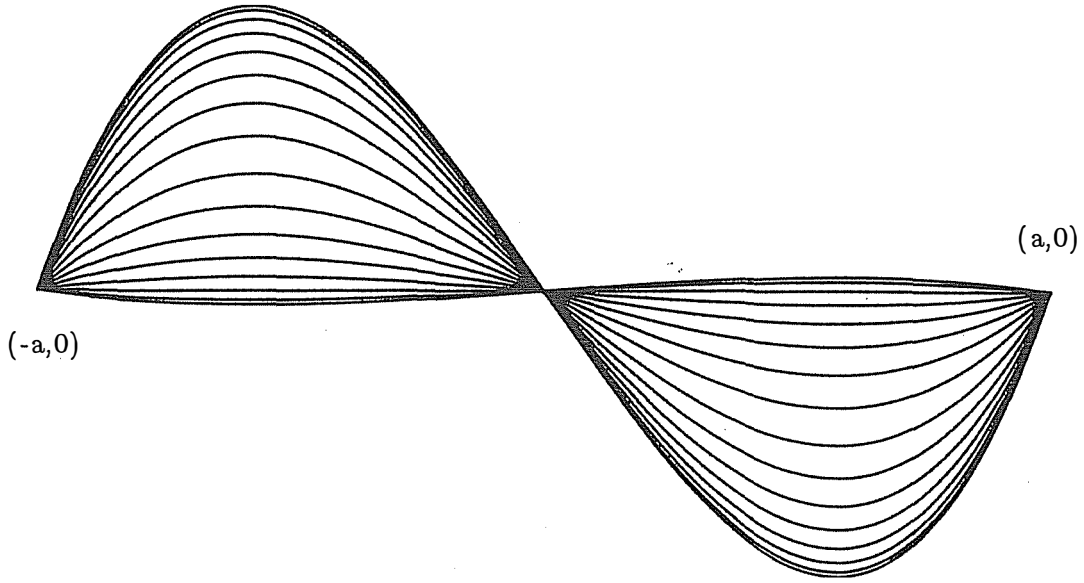


Figure 2: The **K** coefficients did *not* form an arithmetic sequence.

By keeping **n** small and extending the values of x_i and x_f beyond the intercepts, you can create a more jagged effect like Figure 3. Interesting patterns can be produced with a colour monitor. Experiment with different colour combinations. Don't hesitate to use *low* resolution to give a "painted with a wide brush effect".

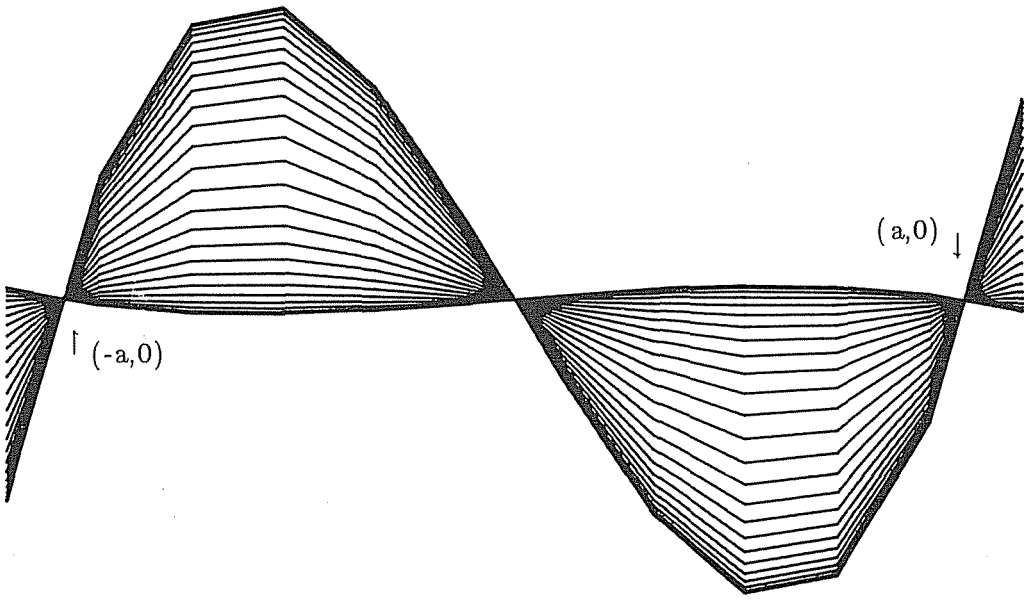


Figure 3: $\mathbf{n} = 11$; $\mathbf{x_i} < -\mathbf{a}$; $\mathbf{x_f} > \mathbf{a}$

The symmetry of the composite can be most easily upset by choosing unequal absolute x_i and x_f values. This is not illustrated here. A second way is to put in a selective y-scaling factor at Step 11. For example (Figure 4) at all abscissae greater than or equal to zero, scale down and invert the ordinates. In BASIC the code is simple:

```
IF X >= 0 THEN Y = -0.35 * Y
```

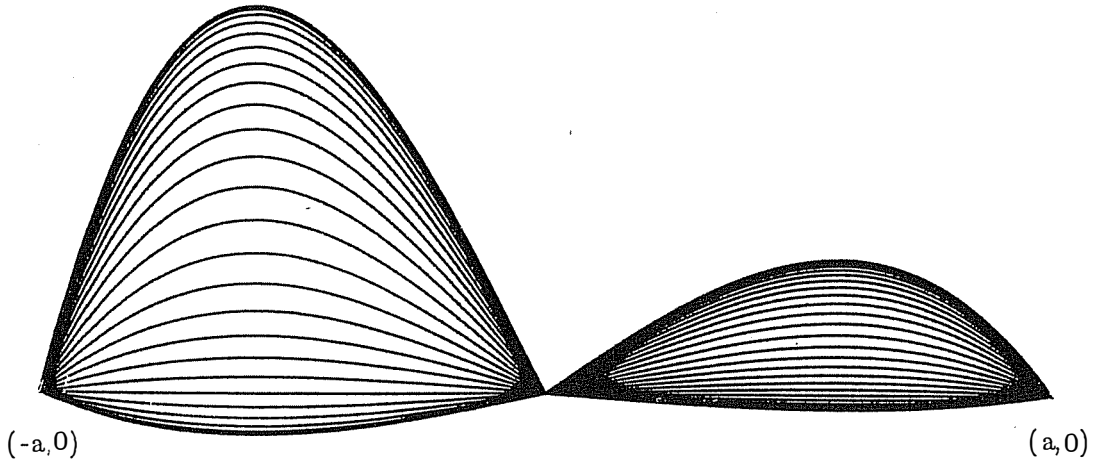


Figure 4: At $x \geq 0$ the ordinates were scaled down and inverted.

For those Artsies who dislike symmetry, there are lots of interesting non-symmetrical functions with which to experiment. Figure 5 was produced using the polynomial function $y = K x^3 (x + a) (x - b)$ with intercepts at at $(-a,0)$ $(0,0)$ and $(b,0)$. Make sure you modify Step 2 of the algorithm to get **a** and **b** from the user. Then if you want symmetry again (Figure 6) set K_1 equal to-K_p.

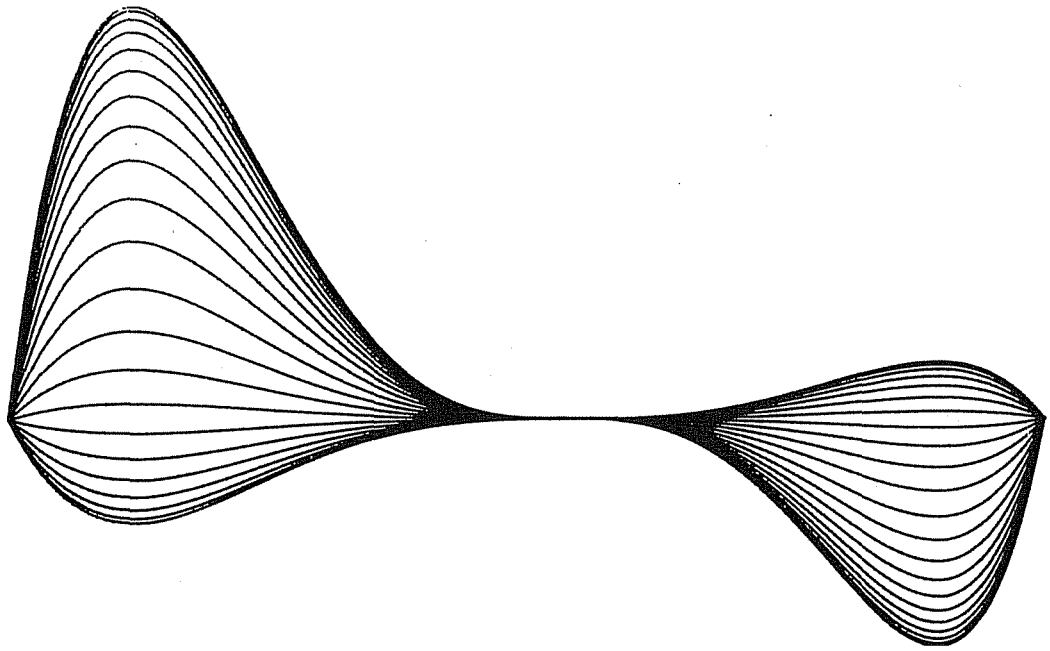


Figure 5: Polynomials: $y = K x^3 (x + a) (x - b)$ ($x_i = a$; $x_f = b$; $n = 128$)

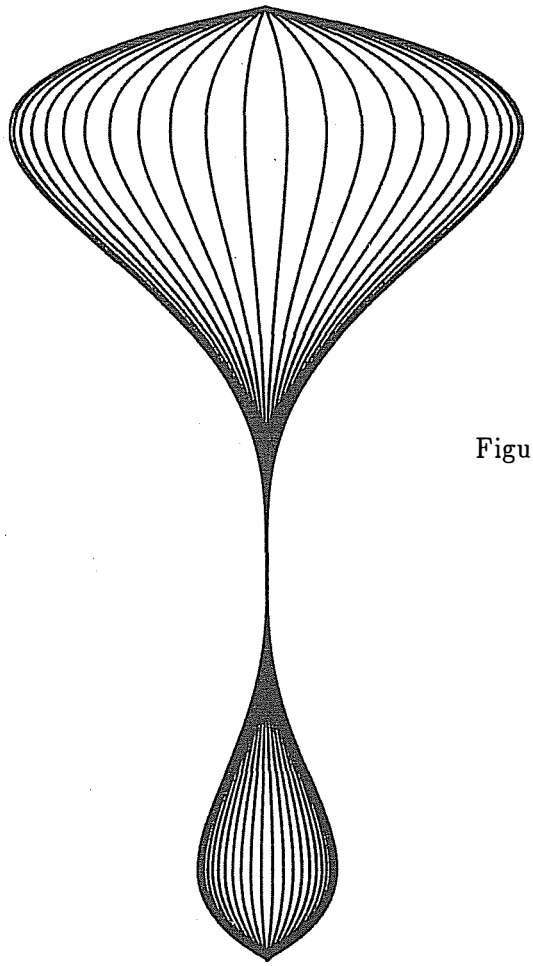


Figure 6: Polynomials: $y = K x^3 (x + a) (x - b)$ $-K_1 = K_p$. Graphs rotated 90 degrees.

Once you master polynomials, you may want to go on to more challenging functions. Another challenge is to make your programs run efficiently. There may be some heavy number crunching involved in programs that use trigonometry to produce composites like Figures 7 through 10. Modularize your programs. Use functions and procedures (subroutines). That way you won't be reinventing the program every time you have a new idea and you can cannibalize old programs to make new ones.

Finally, a warning - you could get so caried away with graphics, that you'll end up drawing something that everyone will want a copy of. See Figure 11.

All figures in this article were produced on an Hewlett Packard Model 7221A plotter using programs that I recently converted from BASIC to C. These programs now reside on a DEC VAX 11/750 running under UNIX 4.3 bsd.

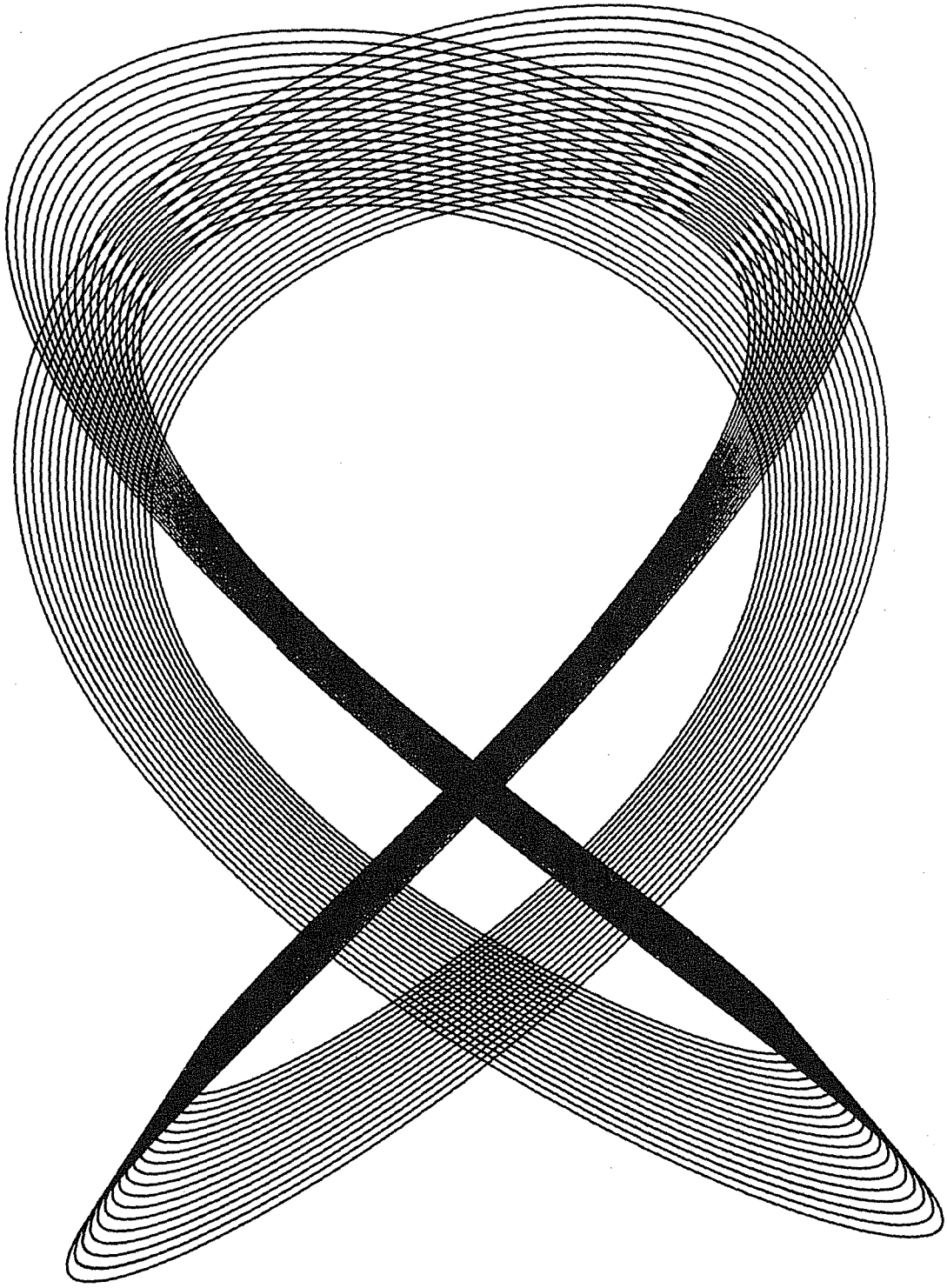


Figure 7: Lissajou Figure - Vibration in a Plane $x = A_x \cos\left(2\pi f_x t + \Theta_x\right)$ $y = A_y \cos\left(2\pi f_x t + \Theta_y\right)$

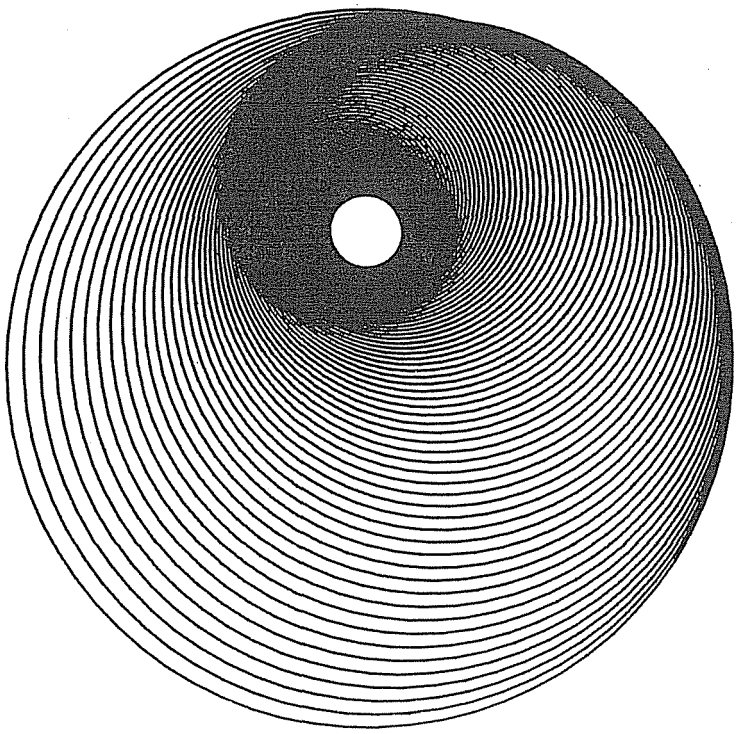


Figure 8: Circles of decreasing radii along a spiral path.

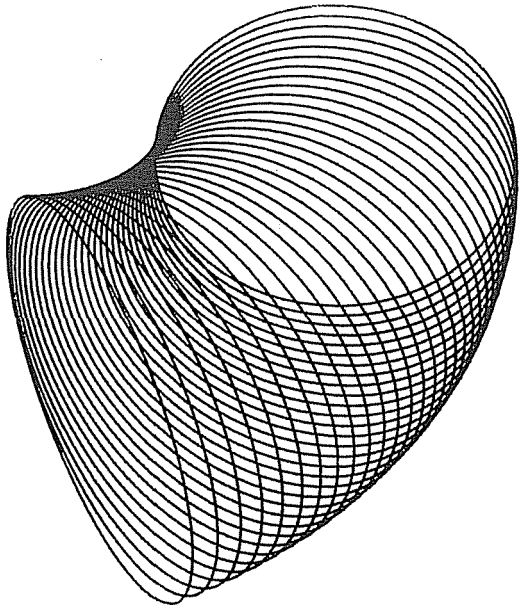


Figure 9: Ellipses of varying size, eccentricity and orientation along an elliptical arc path.

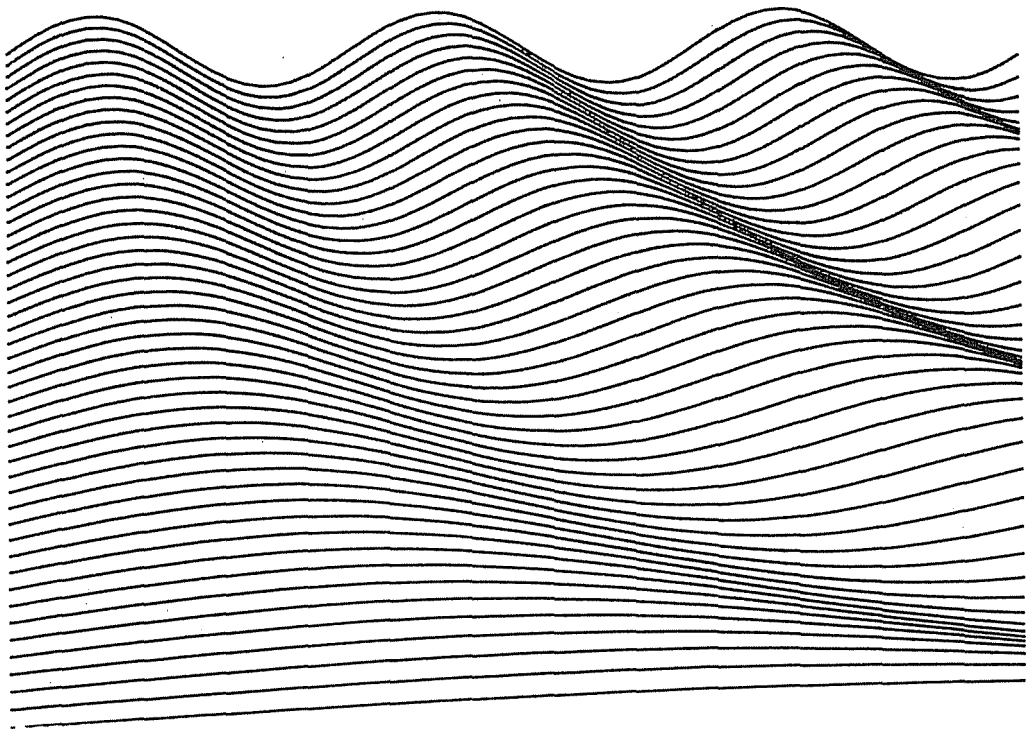


Figure 10: $\mathbf{y} = \mathbf{K} + \mathbf{A} \sin 2\pi \mathbf{f} \mathbf{x}$; $\mathbf{f}_1 \leq \mathbf{f} \leq 12\,\mathbf{f}_1$. Amplitude **A** was the same (constant) for each function. Vertical space between the graphs as determined by **K**, is a geometric progression. From bottom to top each *change* in **K** was 99 percent of the previous change.

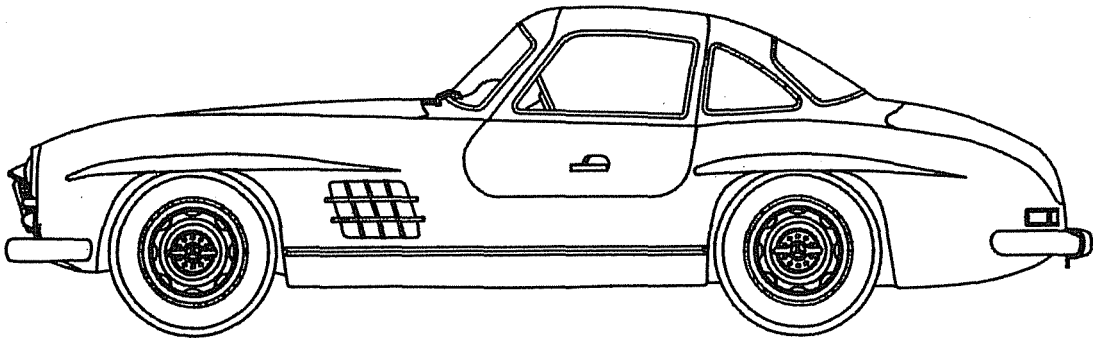


Figure 11: *1955 Mercedes-Benz 300-SL.*
The car was defined as a composite of 582 lines and arcs.