8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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FROM CONCEPTS TO ALTERNATIVE ALGORITHMS AND MENTAL MATH STRATEGIES - DIVISION OF WHOLE NUMBERS AND DECIMALS

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ABACUS CO-EDITORS



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Greetings! What aspects of division have you taught this year? Which model(s) of representation for division have you used: set model (groups of, array), area/region model, and/or a measurement/linear model? These models are connected with conceptual notions of addition, subtraction, and multiplication.

Your choice of lesson problems are important, as they provide students with learning experiences about different aspects of division, such as partitive and quotative division models, the relationship between division and fractions, and division as a ratio and proportion.

Think about this division expression, 38 ÷ 4. Write problems or stories in contexts that lead to different responses to the question. The quotient, (i.e., 9.5 or 9 ½) for this division expression, may be represented as 9, 9 ½ , or 10 depending on the question you pose in your problem. You may even ask a question that prompts the answer 2 which would be a question asking about the remainder, 2. Which aspects of division do your problems leverage?

Through such mathematical work, Deborah Ball explained that we are developing specialized content knowledge (SCK), a subset of mathematics knowledge for teaching (MKT).

LINKS TO LITERATURE AND MANIPULATIVES



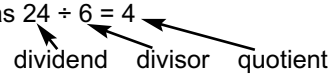
PAT MARGERM is an independent literacy and mathematics consultant with more that 30+ years of K to 8 teaching experience with students, in-service and pre-service teachers. She is a long-time OAME board member and works in various aspects of mathematics education.

Do you sometimes feel that there is so much to teach and so little time? In this issue, I will look at a way to be more efficient in using time and resources. I will look at using two math picture books for both setting contexts for division, as well as for character education lessons. How do you predict we will use bead strings and the 10 x 10 rekenrek to model division?

RESEARCH SUMMARY – ABOUT DIVISION

Krysten has 4 bags of marbles. There are 6 marbles in each bag. All together she has 24 marbles.

A division problem can be symbolically written as 24 ÷ 6 = 4



The 3 quantities in the problem are the **number of bags**, the **number of marbles in each bag (per bag)**, and the **total number of marbles**. Carpenter and Fennema (1999) explain that any one of the three quantities can be unknown in a problem.

- When the **total number of marbles** Krysten has altogether is unknown, the problem is a Multiplication problem. For example, Krysten has 4 bags of marbles. There are 6 marbles in each bag (6 per bag). How many marbles does Krysten have altogether?
- When the **number of marbles in each bag or group** is unknown, the problem is called a Partitive Division problem. Partitive Division problems give the total number of objects and the number of groups. The number of objects in each group (marbles per bag in this case) is unknown. Children partition the total number of objects into a given number of groups. For example, Krysten has 24 marbles. She put the marbles into 4 bags with the same number of marbles in each bag. How many marbles are in each bag?
- When the **number of bags or groups** is unknown the problem is called a Measurement or Quotative Division problem. Measurement/Quotative Division problems give the total number of objects and the number of objects in each group. The number of groups (the number of bags of marbles) is the unknown. Children use the number of objects in each group to deconstruct the whole set of objects and measure the number of groups. For example, Krysten has 24 marbles. She wants to put 6 marbles in each bag. How many bags does she need?

Children initially solve multiplication and division problems by directly modeling the action and relationships described in the problems. Kamii (2000) explained that providing children with counters are helpful, because they are movable and conducive to the physical action of dividing. Baroody (1998) described four direct modeling strategies

For **Partitive Division** situations, Baroody noted two informal, direct modeling strategies that children use:

- Fair Sharing – This strategy entails dealing out items into groups. To have 24 objects shared fairly among 6 people, a child might count out 24 blocks, deal out one block to each of the 6 piles until all the blocks are distributed, then count the number of blocks in one of the piles.
- Trial and Error Grouping – This strategy involves estimating how many items each person gets and then making the appropriate adjustments. To have 24 shared fairly among 6, a child might estimate 2 each. After making 6 piles of 2 each, the child would realize there are more counters left, and would continue distributing the remain 12 counters among the 6 piles.

For **Measurement/Quotative Division** situations, Baroody identified another two informal, direct modeling strategies that children use:

- Measuring Out – This strategy entails making groups of a specified size. To have 24 objects divided into shares of 6 in a group, a child might count out 24 blocks, form piles of 6 blocks until all the objects are grouped, then count the number of piles or groups (4).
- Repeated Subtraction – This strategy involves subtracting one group of 6 blocks at a time, so that the child is doing the following: 24 – 6 – 6 – 6 – 6 = 0. This might be recorded as 24, 18, 12, 6, 0. The answer is determined by counting the number of times 6 was subtracted from 24 (4).

Yet, over time, Carpenter and Fennema (1999) noted that these direct modeling strategies are replaced by more efficient strategies based on counting, adding/subtraction, and/or the use of derived number facts. In summary, Baroody (1999) outlines two types of division problems:

- **Partitive Division** – amount ÷ number of groups = number in each group
- **Measurement/Quotative Division** – amount ÷ number in each group = number of groups

LET'S DO MATH:
CONCEPTUAL REPRESENTATIONS OF DIVISION

About Problem #1

For this problem, conceptual models of representation for division will be shown as student sample solutions. According to the Ontario Curriculum (2005), by grade 6, students must be able to demonstrate the operation, division, using concrete materials, diagrams, student-generated algorithms, and standard algorithms. It is important for students to understand the concept of division, before they learn to use alternative and standard algorithms.

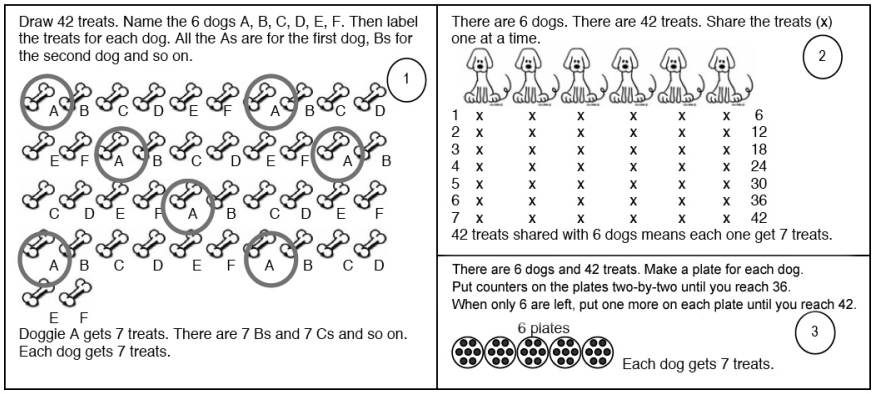
Problem #1:

a) A dog walker brought 42 treats to share with 6 dogs. How many treats does each dog get?

b) Siri was caring for her neighbour's horses. She took 28 apples and gave each horse 3. How many horses was she caring for?

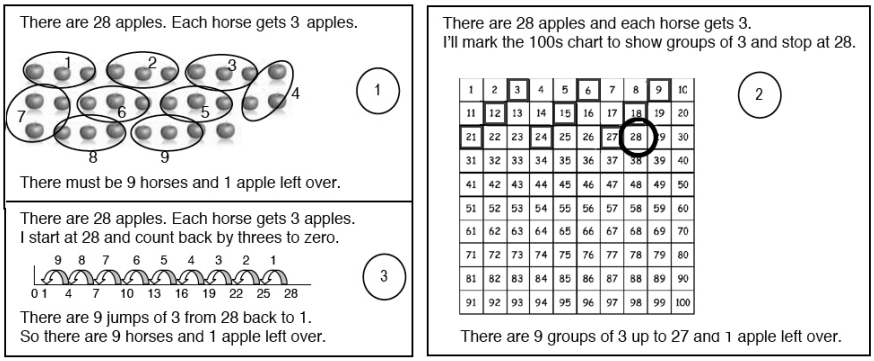
Anticipating a Range of Mathematical Solutions – Fair Sharing/Partitive Division

a) What mathematics do you see in each student sample work?



Anticipating a Range of Mathematical Solutions - Measurement/Quotative Division

b) What mathematics do you see in each student sample work?



Possible Responses - Fair Sharing/Partitive Division 1a)

- *solution 1 shows all 42 treats being shared, one at a time with each of the six dogs – then students counted the number of treats for dog A, then dog B, and so on – each dog got 7 treats*
- *solution 2 shows that students know that the treats are shared, one at a time, with each dog – when the total number of treats has reached 42, the sharing is done – the skip counting on the right shows students are keeping track of the running total as they move along – they kept track of how many treats each dog got on the left – this could easily be transformed into a t-chart*

- *solution 3, students shared the treats two-by-two and then shared by ones at the end (which we only know because they explained that) – they used counters to represent the treats and plates to represent the dogs and shared fairly*

In sharing/partitive problems, we know the number of groups and we have to determine the number in each group. Material actions (i.e., actions with your hands on manipulatives or numbers) show 1 for you, 1 for you, and so on in a "giving out" kind of way until all are shared. Solutions 1 and 2 show sharing by ones. Solution 3 shows sharing with treats grouped together (2 for you, 2 for you, …). All solutions show counting up and decomposing the whole (42) into groups of six (6) parts. The "answer" lies in counting the number of times the whole could be shared 6 ways which is 7. So, sharing 42 treats fairly with 6 dogs means that each dog gets 7 treats.

Possible Responses – Measurement/Quotative Division 1b):

- *solution 1 shows all 28 apples represented and grouped into 3s – the number of groups of 3 tells the number of horses she was caring for, 9 and the number of apples left over, 1. (3 x 9 = 27 and 28 – 27 =1)*
- *solution 2 shows students counting up to 28 by 3s to determine the number of horses that could be fed with the 28 apples; the number of counts shows 9 groups of 3 and 1 apple left over*
- *solution 3, shows starting from 28 and counting backwards by 3s until all groups of 3 are identified; that is, the number of groups of 3 in 28 is 9 (3 x 9 = 27), but you end at 1 not at 0, so that means 1 apple is left over*

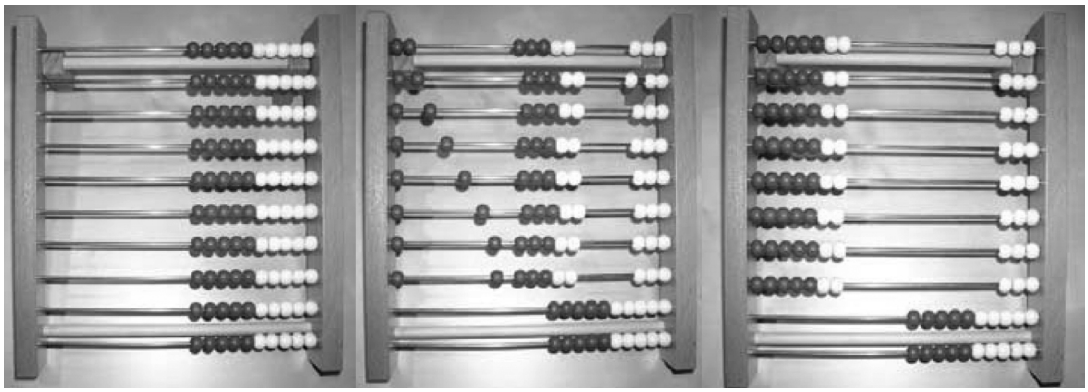
In measurement/quotative problems, we know the number in each group and we have to determine the number of groups. Material actions show making groups from the total (e.g., 3 for you, 3 for you, 3 for you, and so on) until all are organized into groups of 3. All three solutions show grouping, counting up or down by 3s. The "answer" is in the number of groups of 3 that can be formed with 28 which is 9. So, giving 28 apples to horses in groups of 3 means that 9 horses are fed 3 apples each and 1 apple is left over.

LINKS TO MANIPULATIVES: VISUALIZING DIVISION FACTS WITH A REKENREK

Using both the 100 bead string and the 10 x 10 rekenrek allows students to compare fair sharing (partitive) and measuring (quotative) models for any division fact. Challenge students to create a problem for any division fact, such as 56 ÷ 8 = 7. Have students share and compare the problems. Students could use the 10 x 10 rekenrek and the 100 bead string to compare the mathematical actions for each problem.

Trading Cards Problem A

I have 56 trading cards. I have 8 sets of trading cards with the same number in each set. How many cards are in each set?



My rekenrek has 10 beads per row, 5 red and 5 white; I will only use 8 rows to show 8 groups.

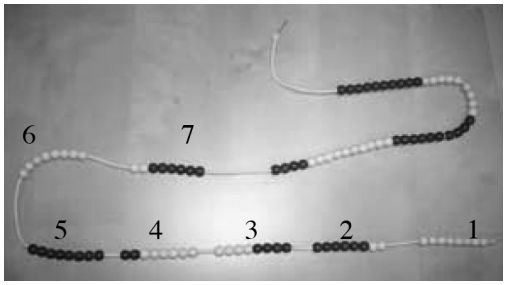
I move the beads across each of the 8 rows one at a time. One bead across each row means I count 1, 2, 3, 4, 5, 6, 7, 8. Then I go back to the first row and move a second bead into each group – 9, 10, 11, 12, 13, 14, 15, 16. I keep moving beads and counting until I have moved and counted 56 beads. Then I count how many beads I have in each row. That tells me the number in each set (7 beads with 5 red and 2 white in each of 8 rows).

Trading Cards Problem B

There are 56 cards and 8 cards in each set.

How many sets of cards are there?

I show 56 on the bead string. Then I count 8 beads back from 56 and mark that set. I continue to count (56, 48, 40, 32, 24, 16, 8, 0) and mark each group of 8. Then I count how many groups I made (7). So, I have 7 sets of cards.



LINKS TO LITERATURE

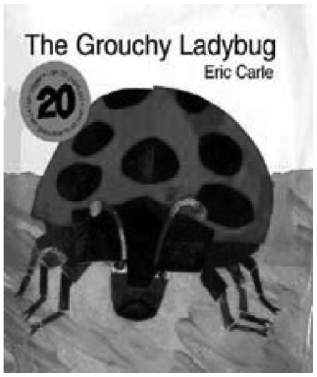
Sharing is both a model for division and a character trait. Picture books that have stories which involve sharing can be used as contexts for mathematics, as well as for character education.

In the picture book, *Bean Thirteen*, two ants demonstrate character traits, such as perseverance, fairness, and sharing as they try to solve a problem. Their problem is to figure out how to share 13 beans among their friends so that no one ends up with the 13th bean. In the end, their dinner guests share the beans by taking the number that each one wants rather than an equal number of beans.



Ask the students to compare how the story would be the same and different if the number of beans was 15, 17, 19, 21, and so on. What do they notice about the possibilities for sharing each of these numbers of beans? Students might notice that prime numbers cannot be shared equally. Also, students might notice that solving the problem of sharing 15 beans did not require perseverance, since inviting 1 more friend meant that there were 3 friends to share the beans equally.

In *The Grouchy Ladybug*, a friendly ladybug invites a grouchy ladybug to share aphids on a leaf. Ask the students if the aphids in the picture can be shared equally by the two ladybugs. Students might count the aphids in the picture, model the number of aphids, and show sharing of the aphids in a variety of ways (e.g., separating aphids and ladybugs). Identifying the number of aphids that can be shared equally by the 2 ladybugs will provide an experience that begins to develop the concept of halving, an important mental math strategy. Further, students might notice that all even numbers of aphids can be shared equally and all odd numbers of aphids have a remainder of one.



LET'S DO MATH:
ALTERNATIVE ALGORITHMS FOR DIVISION

About Problem #2

For this problem, different samples of student-generated algorithms will be provided.



Problem #2:

Early today, 70 cut flowers were delivered to the community centre for the spring dinner. We want to decorate tables with 4 flowers in each bouquet. How many tables can we decorate with 70 flowers?

Anticipating a Range of Mathematical Solutions:

What mathematics do you see in each student sample work?

1

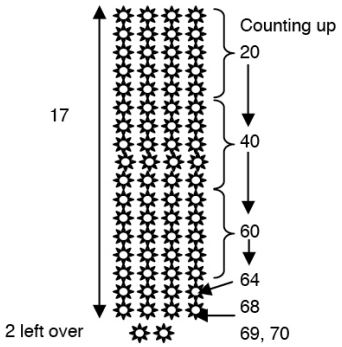
There are 70 flowers to be put into bouquets or groups of 4.

10 bouquets of 4 means 40 are used	70 – 40 = 30	30 left
6 bouquets of 4 means 24 are used	30 – 24 = 6	6 left
1 bouquet of 4 means 4 are used	6 – 4 = 2	2 left not enough to make another bouquet.

10 + 6 + 1 = 17. So, 17 bouquets of 4 flowers can be made with the 70 flowers. There will be 2 flowers left.

2

70 counters organized in groups of 4.



17 tables can be decorated.

3

Make groups of 4 with 70 altogether.

Number of groups	Number of flowers used so far
1	4
2	8
3	12
4	16
5	20
6	24
7	28
8	32
	double
16	64 not enough
17	68
18	72 too many

So we can decorate 17 tables and have 2 flowers left over.

4

I will take away as many 4s as I can.

4	70	
	40	10
	30	
	20	5
	10	
	8	2
	2	17 bouquets

We can make 17 bouquets.
But we'll have 2 flowers not used.

Think about an ***algorithm*** as a structured series of procedures that:

- can be used across problems regardless of the numbers
- promotes accuracy
- are efficient.

Possible Responses:

- *solutions 1 and 4 show similar thinking – groups of 4 are removed and the related number of bouquets is counted or added up; solution 4 seems a little more efficient than solution 1 because all the writing/narrative is not required*
- *solution 2 shows that students know that 4 flowers form 1 bouquet – they started drawing rows of 4 until they showed an arrangement of 4 x 17 which involved 68 flowers – their array showed 4 x 17 = 68 and 70 – 68 = 2 – so they make 17 bouquets and have 2 flowers left*
- *solution 3, students used a t-chart to build relationships between the number of bouquets and the number of flowers used, knowing they needed to get to 70 in the right column – notice how they used proportional reasoning after they had the number of flowers needed for 8 bouquets – they doubled both numbers to show 16 bouquets would use up 68 flowers – this use of ratio starts to appear in the grades 5 and 6 curriculum*

LET'S DO MATH:
MENTAL MATH STRATEGIES FOR DIVISION

About Problem #3

These division mental math strategies can be used when the numbers are friendly.

Problem #3:

Solve each question in these strings and describe your strategy.

a)	b)	c)
900 ÷ 9 =	80 ÷ 4 =	10 ÷ 2 =
81 ÷ 9 =	48 ÷ 4 =	60 ÷ 12 =
981 ÷ 9 =	128 ÷ 4 =	35 ÷ 0.5 =
306 ÷ 6 =	114 ÷ 6 =	10 ÷ 1/4 =
	152 ÷ 8 =	2.7 ÷ 0.5 =

Anticipating a Range of Mathematical Solutions

What mathematics do you notice in each student example?

a) 900 ÷ 9 = 100 (1)
81 ÷ 9 = 9
981 ÷ 9 = (900 + 81) ÷ 9 = 100 + 9 = 109

306 ÷ 6 = (300 + 6) ÷ 6 = 50 + 1 = 51

b) 80 ÷ 4 = 20 (2)
48 ÷ 4 = 12

128 ÷ 4 = (80 + 48) ÷ 4 = 30 + 12 = 32
114 ÷ 6 = (60 + 54) ÷ 6 = 10 + 9 = 19
152 ÷ 8 = (160 – 8) ÷ 8 = 20 – 1 = 19

c) I know 10 ÷ 2 = 5 If I multiply the divisor and dividend by the same number I get the same answer – like this, for example 10 ÷ 2 = 5. Now multiply each by 6, 10 x 6 = 60 and 2 x 6 = 12 and 60 ÷ 12 = 5 just like 10 ÷ 2 = 5 (3)

So, if I do 35 x 2 and 0.5 x 2, I do not have to divide by a decimal and I will get the same answer.
x2 35 ÷ 0.5 x2 = ?
70 ÷ 1 = 70 so, 35 ÷ 0.5 = 70. This makes sense because 35 split up into halves is 70.

x both by 4
10 ÷ 1/4 = 40 ÷ 1 = 40 so, 10 ÷ 1/4 = 40 This makes sense because 10 split up into fourths is 40.

x both by 10
2.7 ÷ 0.9 = 27 ÷ 9 = 3 so, 2.7 ÷ 0.9 = 3

Possible Responses - Mental Math Strategies for Division

- *solution 1 shows decomposing the dividend (the whole) using expanded notation into parts and then dividing each part so 981 becomes 900 + 81 and 306 becomes 300 + 6 – the divisor divides each part of the whole and the sum of the quotients is the answer*
- *solution 2 shows finding compatible numbers that sum to the dividend and are both divisible by the divisor, 4, so 128 is decomposed into 80 and 48, each of which can easily be divided by 4 so that 114 = 60 + 54 both multiples of 6; then 152 becomes 160 – 8 and each of the numbers is divisible by 8*
- *solution 3 shows dividing by balancing – when you multiply the dividend and divisor by the same number, the quotient remains the same – so 35 ÷ 0.5 gives the same answer as 70 ÷ 1 = 70 – and when you consider 10 ÷ 1/4 , you know you will get the same answer when you multiply the dividend and divisor by 4 to make 40 ÷ 1 = 40 – same reasoning for 2.7 ÷ 0.9 = 27 ÷ 9 = 3 (multiplying each by 10)*

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

Try these division problems with your students. To suit your students' learning needs, you might adjust the numbers (e.g., larger or small whole or decimal numbers, using fractions or integers) and/or the context of the problem.

Practise noticing the different strategies and the breadth of the mathematics your students use in their solutions. Take note of which solutions show groups of, a set model (groups of, arrays), a region model (area), or a linear model of representation. In preparation for your class discussion of the solutions, decide which solution you will highlight first, second, and third. Think ahead of the type of mathematical comments you will make to render explicit key mathematical ideas, strategies, and terms about division, as the students share their solutions. As well, choose 4 student work samples to analyse further, in terms of its mathematical details related to division.

What if your students need to learn decimal division? Using similar strategies when making connections between whole and decimal number addition and subtraction, how will you adjust the problems and anticipated solutions for decimal number division?

Sample Responses:

- *for conceptual representations using arrays, think of the number of groups to be a decimal number and the number of objects in one group to be a decimal number*
- *for conceptual representations using an area model (with linear dimensions), use decimal numbers to be the linear dimensions*
- *for mental math strategies, decompose the decimal numbers into products of whole numbers and decimal numbers of tenths, hundredths, and/or thousandths (e.g., 0.35 = 35 x (0.01) or 0.7 = 7 x (0.1)) – rearrange your expression to track the powers of 10 and simplify it to apply the multiplication to the integer number derived by the division (e.g., 0.35 ÷ 0 .7 = 35 x (0.01) ÷ 7 x (0.1) = 35 ÷ 7 x 0.01 ÷ 0.1 = 5 x (0.1 x 0.1 ÷ 0.1) = 5 x 0.1 x (x 0.1 ÷ 0.1) = 5 x 0.1 = 0.5)*

Suggested Readings

Some readings that we used in our Research Summary and in the development of the division problems and solutions are listed below:

- Baroody, A. (1998). *Fostering children's mathematical power: An investigate approach to K-8 mathematics instruction.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.
- Carpenter, T., Fennema, E., Franke, M., Levi, L., and Empson, S. (1999). *Children's mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann.
- Donovan, S. and Bransford, J. (2005). *How students learn: History, mathematics, and science in the classroom.* Washington, D.C. The National Academies Press.
- Fosnot, C. & Dolk, M. (2001). *Young mathematicians at work: Constructing multiplication and division.* Portsmouth, NH: Heinemann.
- Kamii,C. (2000). *Young children reinvent arithmetic: Implications of Piaget's theory (2nd edition)*. New York, NY: Teachers College Press.
- Kilpatrick, J., Swafford, and J., Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

CLOSING

Our next issue will focus on the relationship between arithmetic to algebra, in terms of key mathematical concepts, strategies, and models of representation. As well, we will explore possible ways that teachers need to know it for teaching, as well as, ways that teachers can use these mathematical ideas and pedagogical strategies through problem solving-based teaching and learning strategies.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email