8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Abacus

Volume 50 • Number 1 • September 2011

COMPOSING AND ANALYSING 2-D SHAPES

TABLE OF CONTENTS

Abacus Editor Greetings 1
RESEARCH SUMMARY
- Composing and Analysing 2-D Shapes 2
LET'S DO MATH: Polygons Problems . . . 4 and 5
LINKS TO MANIPULATIVES
- Pattern Blocks 6
LET'S DO MATH:
- Composing Shapes Problems 6
LINKS TO LITERATURE
The Warlord's Puzzle by Virginia Walton Pilegard 7
LET'S DO MATH:
- Classifying Triangles Problems 7
Next Steps for Your Professional Learning 8
- Application to Your Classroom
- Suggested Readings

A Publication of the OAME/AOEM

ABACUS CO-EDITORS



MARY LOU KESTELL works on K to 12 mathematics professional learning at the provincial level. She has spent more than 30+ years working in all aspects of Ontario mathematics education and is a past-president of both OAME and OMCA.



KATHY KUBOTA-ZARIVNIJ currently focuses on K to 6 mathematics professional learning at the provincial level. She is a long-time OAME board member and OAME past president. In relation to her work in mathematics education, she uses complexity thinking to interpret mathematics teaching and learning for students and teachers.

Geometry is an important branch of mathematics that involves shapes, spatial sense, symmetry and proportion. For elementary education, geometry is more than just naming lines and shapes, as it also includes moving, combining, and comparing lines and shapes. Students use their spatial sense in everyday life through their interaction with shapes and objects in their environment, such as classroom objects, artwork, and architectural designs within students' local community.

So, to develop students' understanding of the geometric attributes of 2-D shapes, what kinds of tasks and problems should teachers use and in what sequence from grade 1 to grade 6?

In this issue, the Research Summary and problems (with multiple solutions) focus on strategies for composing and decomposing shapes, identifying and describing geometric attributes of 2-D shapes, understanding geometric relationships between 2-D shapes and using geometric reasoning. The problems we provide are designed for use within a three-part, problem-solving lesson. Consider solving these 2-D shape problems yourself first, before examining the solutions we provide.

LINKS TO LITERATURE AND MANIPULATIVES



PAT MARGERM is an independent literacy and mathematics consultant with more that 30+ years of K to 8 teaching experience with students, in-service and pre-service teachers. She is a long-time OAME board member and works in various aspects of mathematics education.

Some manipulatives and picture books develop an understanding of shapes as line segments that connect points in order to enclose space (e.g., geoboards or anglegs; *When a Line Bends . . . A Shape Begins* by Rhonda Gowler Greene or *What Can Simon Be?* by Gene Yates and Thomas Frank (where a snake turns himself into various shapes). There are manipulatives that include non-traditional geometric shapes, such Power Polygons™ or BLMs from the *Guide to Effective Instruction: Geometry, K to 6*. Because I believe that it is better to do fewer things in more depth, I will focus on pattern blocks, as they have been investigated in previous *Abacus* issues. I hope there are some new ideas for you!

RESEARCH SUMMARY – COMPOSING AND ANALYSING 2-D SHAPES

Some Key Concepts in Elementary 2-D Geometry

According to Randall (2005), two-dimensional objects can be described, classified and analysed by their geometric attributes. Small (2007, 2008, 2009) explains that 2-D shapes can be analysed and sorted according to their geometric attributes (e.g., number of sides, vertices, inside angles, parallel lines, side lengths, lines of symmetry). From these attributes, 2-D shapes or polygons can be classified in particular ways:

- naming polygons by the number of sides (e.g., triangle – 3 sides, quadrilateral – 4 sides, pentagon – 5 sides)
- classifying triangles by length of sides (e.g., scalene, isosceles, equilateral) and inside or interior angles (e.g., right, acute, obtuse)
- classifying quadrilaterals by pairs of parallel sides (e.g., 2 pairs of parallel sides – parallelogram, rectangle, square, rhombus, kite; 1 pair of parallel sides – trapezoid), yet all quadrilaterals are polygons with 4 straight sides, 4 vertices, 4 angles, 2 diagonals that divide the quadrilateral into 2 triangles
- regular and irregular polygons where all regular polygons are convex shapes that have all sides equal in length and all angles equal in size.

As well, polygons can be constructed to form polygons or decomposed into other polygons. Clements and Sarama (2003) and Clements, Wilson and Sarama (2004) elaborate on the ways that students compose shapes through a shape composition learning trajectory: pre-composer, piece assembler, picture maker, shape composer, substitution composer, shape composite iterater, shape composer with superordinate units. Further, “how a shape can be cut up (dissected) and rearranged (combined) into other shapes helps us attend to the properties of the shape, for example where right angles are and where the equal sides are” (Small, 2009, p. 93).

Development of 2-D Geometric Thinking

Battista (2009) describes four theories that outline how students learn geometry and implicate ways for understanding teaching approaches to geometry in elementary classrooms.

1. Van Hiele Levels Theory

This theory describes how students progress through a sequence of hierarchically, discrete and qualitatively different levels of geometric thinking. These levels have been elaborated and refined by Clements and Battista (1992) and Battista (2007). For elementary geometry, three out of five levels are described below.

- *Level 1 (Visual-Holistic Reasoning)* - Students identify, describe and reason about shapes according to their appearance as visual wholes. (e.g., squares are like rectangles except rectangles have shorter and longer sides)
- *Level 2 (Descriptive-Analytic Reasoning)* - Students explicitly attend to, conceptualize and specify shapes by describing their parts and spatial relationships among the parts. As students develop formal geometric concepts like angle measurement and parallelism, they use a combination of informal and formal descriptions of shapes. During this level, students use and reason about formal definitions for classes of shapes (e.g., geometric properties of quadrilaterals or types of triangles)
- *Level 3 (Relational-Inferential Reasoning)* - Students make empirical inferences among geometric properties of shapes, based on what they have seen. When they analyse how shapes can be built one part at a time, they conclude that when one property occurs, another property must occur. By increasing the specificity and detail of analysis and using a broader set of 2-D shapes (e.g., regular, irregular, convex, concave), students can make simple logical inferences about properties of 2-D shapes. In fact, students use such logical inferences to reorganize their classification of shapes into a logical (more formal) hierarchy. Towards the final phase of level 3, students are increasingly able to understand and appreciate minimal definitions for classes of shapes (e.g., rectangle has 2 sets of equal length, parallel sides).

2. Abstraction Theory

Battista (2009) explains that abstraction as a cognitive process of incorporating objects, actions, and ideas in consciousness and memory. When objects, actions and ideas are abstracted at a sufficiently deep level, they can be mentally operated on (e.g., compared, decomposed, analysed). Two forms of abstraction are as follows.

- *Spatial structuring* is the mental act of organizing an object or set of objects by identifying components and establishing interrelationships among them.
- *Mental models* are nonverbal mental versions of situations that capture the structure of the situations they represent.

3. Concept Analysis and Objects of Geometric Analysis Theory

According to Battista (2009), this theory acknowledges that students' geometric reasoning requires analysis of objects and mental concepts. Learning geometry involves forming both natural and formal concepts. Natural concepts are formed in everyday activity and are rarely accompanied by concept definitions. In primary math classrooms, natural concepts are often derived from physical objects, such as a geoboard shape, pictures and drawings. Formal concepts, such as rectangles, have definitions that explicitly specify a sufficient set of properties to identify examples. While students form natural shape concepts outside school, subsequent school instruction requires students learn formal concepts for shapes. However, difficulties arise as students reconcile natural and formal concepts for the same category.

4. Diagrams and Representations Theory

Battista (2009) explains that physical objects are considered as input for geometric conceptualization. Geometric concepts are generally derived through an analysis of physical objects or represented objects, including diagrams that represent formal geometric concepts. When diagrams rather than objects are used to represent formal concepts, students often reason about things they see in the diagram and connect irrelevant characteristics to the geometric concept that they were intended to represent. Therefore, students could be confused about the represented and the representing role of the diagram. So, geometric concepts need to be formed from interactions with physical objects, as well as with diagrams and objects that represent the formal shape.

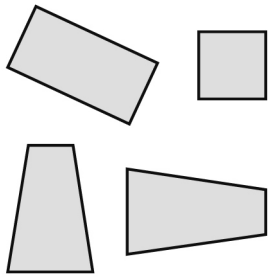
Geometry Instructional Approaches

All of these theories support that notion that students cannot directly be given knowledge or concepts; they must construct it from their own perceptions, experience and inquiry.

Geometry instructional tasks and problems need to engage students in exploring and discussing geometric concepts. Language facilitates communication among the students in a group, allowing them to describe, clarify and discuss the structures they observe and re-conceptualize identified misconceptions. Students discuss how to solve problems and learning occurs in a context of collaborative, social interactions that lead to understanding. Patsiomitou and Emvalotis (2010) emphasize the significance of language in moving students through the hierarchy of van Hiele levels. They explain that each van Hiele level defines its own language (symbols) within a network of relations. Only when students have realized the interrelations and connections between the structures can they progress up the levels.

Common Errors and Misconceptions

According to Small (2007, 2008, 2009), students often confuse the attributes of 2-D shapes with their properties. An attribute applies to only some of the shapes in a group, while a property applies to all of them. Further, students think that the orientation of a shape contributes to the definition of the shape. This error makes it difficult for students to recognize congruence among shapes with different orientations. Having students manipulate the concrete models of 2-D shapes, such as pattern blocks, helps them to see that shapes oriented in different positions maintain their properties.



LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal:
- to see the development of mathematical knowledge, skills, and strategies across grade levels
- to determine the mathematical focus of the problems chosen for Before (activation), During (lesson), and After (Practice) parts of the lesson

BEFORE (Getting Started) – 5 to 10 minutes

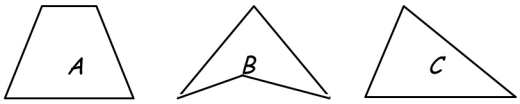
- Activating students' mathematical knowledge and experience using a prompt or problem that is directly relates to the mathematics in the lesson problem.
- Includes student responses to prompt or problem to highlight ideas, strategies, and tools that will provide access to lesson problem

AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates.
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies,

Polygons Problem A:

How are these polygons similar and different? Explain.



Solution 1
B and C look like triangles. If you pulled out the bottom edge you would get another straight line and a triangle.

Solution 2
A and B both have 4 sides and 4 angles and 2 equal sides.

Solution 3
A has 4 sides and 4 vertices, C has 3 sides and 3 vertices.

geometric attributes
- *number of sides*
- *number of angles*
- *number of vertices*

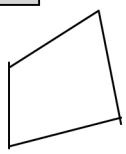
Name a Polygon by Number of Sides:
- *triangle – 3 sides*
- *quadrilateral – 4 sides*
- *pentagon – 5 sides*

DURING (Working on It) – 15 to 20 minutes

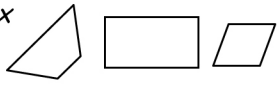
- Understanding the problem – Teacher asks, "What information from the problem are we using to make a plan to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for whole class discussion) in pairs or in small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Polygons Problem B:

What different polygons can be made, by cutting a convex quadrilateral with one straight line?



What information will we use to solve this problem?

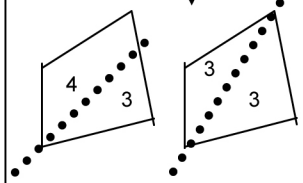
- *concave*  *convex* 

- *use a convex quadrilateral, 4-sided*
- *one straight line cuts the shape to make 2 polygons*
- *show all possible polygons with explanation*

cut from different vertices but dealt with only one cut from a side to an adjacent side

cut the shape twice and named polygons by its number of sides

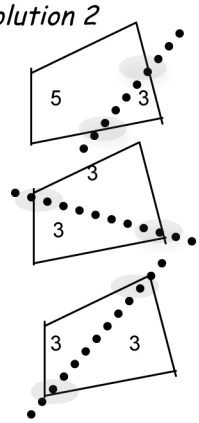
Solution 1



I made a quadrilateral and a triangle with 1 cut. I made two triangles with another cut.

1 Side/1 Vertex and 2 Opposite Vertices

Solution 2



I made triangles and a 5-sided polygon with straight line cuts:
- *vertex to vertex*
- *side to side*

Side to Side and 2 Sets of Opposite Vertices

Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of the students' mathematical discussion.

Grade 2	Grade 3	Grade 4
- compose and describe pictures, designs, and patterns by combining two-dimensional shapes - compose and decompose two-dimensional shapes	– solve problems requiring the greatest or least number of two-dimensional shapes (needed to compose a larger shape in a variety of shapes) - identify and compare various polygons	– identify and compare different types of quadrilaterals (rectangle, square, trapezoid, parallelogram, rhombus) and sort and classify them by their geometric properties

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal.
- Teacher lists key ideas, strategies, and models of representation separately, so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal.

Highlights/Summary:

Ways to cut a convex quadrilateral include:

- *vertex to vertex*
- *vertex to each of the non-adjacent sides (in a quadrilateral there will be two non-adjacent sides; in a pentagon, three)*
- *side to opposite side*
- *side to adjacent sides*

Polygons are named by the number of sides:

- *triangle – 3 sides; quadrilateral – 4 sides;*
- *pentagon – 5 sides; hexagon – 6 sides*

all cases: vertex to vertex; vertex to side; side to adjacent side; side to non-adjacent side (opposite)

begins to be systematic - cuts from 2 different sets of vertices and from 2 different sets of sides

AFTER (Practice) – 5 to 10 minutes

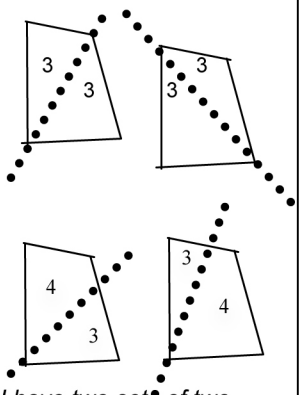
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Polygons Problem C:

What combinations of polygons can be made by cutting a convex pentagon with one straight line?



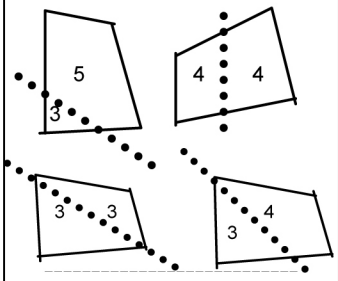
Solution 3



I have two sets of two triangles and two sets of a quadrilateral and a triangle. I always started with a vertex so I would not miss a shape.

2 Sets of Side to Side and 2 Sets of Opposite Vertices

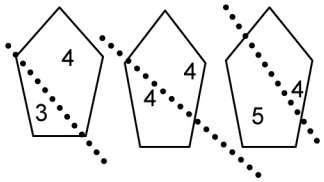
Solution 4



- Vertex to vertex makes two triangles
- Vertex to either opposite side will always make a triangle and a quadrilateral
- Side to opposite side will always make two quadrilaterals
- Side to adjacent side will make a triangle and a pentagon

Adjacent and Opposite Sides, Opposite Vertices, Vertex/Side

Solution 1



- Vertex to vertex – triangle, quadrilateral
- Vertex to side -> 2 quadrilaterals
- Side to side -> pentagon, triangle

Solution 2

- Vertex to vertex –> quadrilateral, triangle
- Vertex to 1st non-adjacent side – triangle, pentagon
- Vertex to 2nd non-adjacent side –> 2 quadrilaterals
- Side to non-adjacent side – quadrilateral, pentagon
- Side to adjacent side –> triangle, hexagon

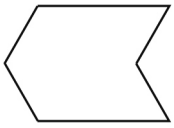
Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO MANIPULATIVES: PATTERN BLOCKS

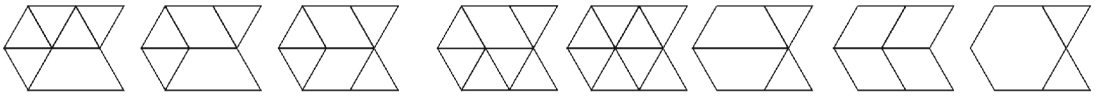


Composing and Decomposing 2-D Shapes

Pattern blocks can be used to create quick images. A geometric image is shown briefly, then students are asked to draw what they see on a piece of paper. Picturing, both on paper and in one's mind is an important aspect of geometric reasoning. Through quick images, students learn the mathematics of making accurate drawings, by paying closer attention to and reasoning about the geometric attributes and relationships they see. In fact, when students recognize parts of wholes or see shapes within shapes, they begin to see relationships between the geometric attributes of different polygons and geometric properties among classes of 2-D shapes.



Sample Quick Image Activity: Flash a pattern block image (outline) on the overhead or SmartBoard™ for 3 seconds. Ask students to draw what they see. Then have them partition the image into different combinations using pattern blocks. For each quick image, the teacher records the students' ideas and compares them to the image. Here are some examples.



To develop students' geometric reasoning pose questions like these:

- For each image, what did you notice the first time you saw the 2-D shape?
- Which geometric attributes did you use in your first drawing?
- What did you miss when you first saw each shape? How did you revise your drawing?
- What type of 2-D shape is the original image? How do you know?
- What geometric attributes (e.g., side length, inside or outside angles) are you using?

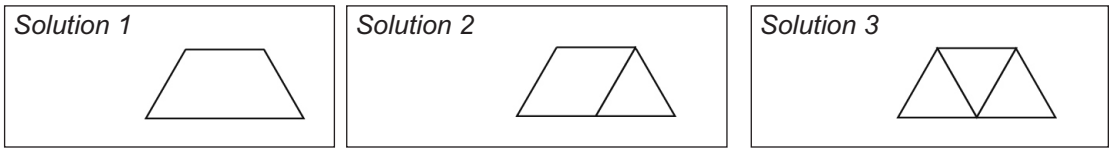
LET'S DO MATH – COMPOSING SHAPES



Before (Getting Started)

Composing Shapes Problem A:

What do you see? Make it using pattern blocks.
(Show a trapezoid for 3 seconds. Ask the students to make it with at least 2 pattern blocks.)





During (Working On It)

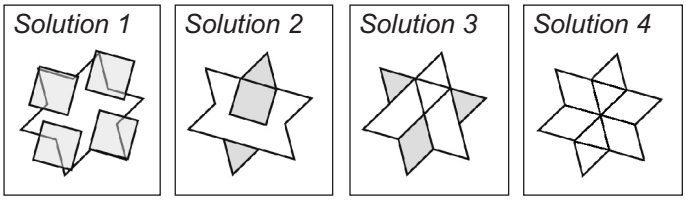
Composing Shapes Problem B:

Cover this outline puzzle with pattern blocks in two different ways.

After (Consolidation)

Anticipating Student Responses:

Why might solution 1 be chosen first for student discussion, followed by solutions 2, 3 and 4?



This rationale is based on the work of Clements and Sarama (2003):

- solution 1 – *pre-composer* – manipulates individual shapes, but is unable to combine shapes to make a larger shape; shapes placed do not match simple outlines
- solution 2 – *piece assembler* – fills in simple puzzles with limited use of turns and flips because cannot use motions to see shapes from different perspectives; has difficulty filling in areas with multiple shapes
- solution 3 – *shape composer* – combines different shapes, uses turns and flips with intentionality and anticipation, based on the shapes' attributes
- solution 4 – *substitution composer* – deliberately forms composite units of shapes (e.g., 2 equilateral triangles make a rhombus) and uses substitution relationships among these shapes to compose a shape.

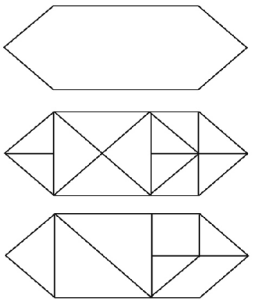
LINKS TO LITERATURE: *THE WARLORD'S PUZZLE*

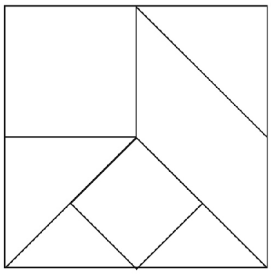


This story is by Virginia Walton Pilegard. It is about a fierce warlord in China, who receives a ceramic tile as a gift. The tile falls and smashes into seven pieces. A contest is held to find the person clever enough to put the tiles back together. As you read through the story, notice the ways in which the contestants compose the shapes to make new shapes. Identify the names of the polygons illustrated in the story.

Extension Activity:

- Ask students to use one set of tangrams to create new shapes.
- To develop students' understanding of polygons, have them sort the solutions by the number of polygons.
- Provide the following puzzle for students to cover with a collection of tangram pieces.
- Identify relationships between the pieces that are used in solutions to the problems. (e.g., Two small triangles can cover the same area as one medium triangle. The longest side of the small triangle is the same length as the two equal sides of the medium triangle. Two angles in the small triangle combine to form the same angle as the right angle in the medium triangle.)





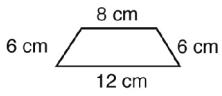
LET'S DO MATH – CLASSIFYING TRIANGLES

Before (Getting Started)

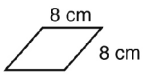
Classifying Triangles Problem A:

Build two different polygons using straws of lengths: 12 cm, 10 cm, 8 cm, and 6 cm, 4 cm. Label the lengths of the sides.

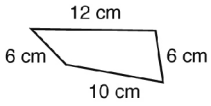
Solution 1



Solution 2



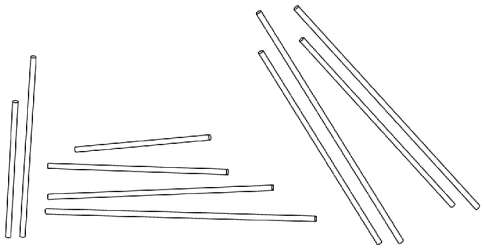
Solution 3



During (Working On It)

Classifying Triangles Problem B:

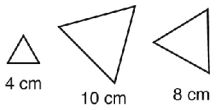
Build different triangles with a collection of straws of lengths: 12 cm, 10 cm, 8 cm, and 6 cm.



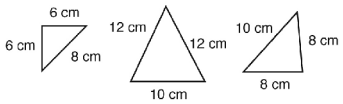
After (Consolidation)

Anticipating Student Responses:

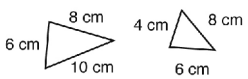
Solution 1 – 3 same side lengths



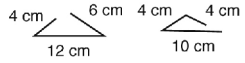
Solution 2 – 2 same side lengths



Solution 3 – different side lengths



Solution 4 – no triangles made



Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion, followed by solutions 2, 3, and 4?

- solution 1 – all three sides are the same length – equilateral triangles
- solution 2 – two adjacent sides are the same length – isosceles triangles
- solution 3 – three different side lengths – scalene triangles
- solution 4 – could not make triangles as lengths of the 2 shorter sides added together was not greater than the length of the longest side.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Try these 2-D geometry problems yourself first and then with your students.
- To suit your students' learning needs, you might change the type of concrete material (e.g., tangrams, power polygons, pattern blocks), the number or type of geometric attributes used.
- Think about how the context of the problem evokes visual images and serves as a tool for understanding and determining the geometric attributes of a set of 2-D shapes, the sorting rule for 2-D shapes (e.g., puzzles, sorting through building pieces, building shapes of toys and buildings).
- Provide students with non-examples of the geometric attributes of 2-D shapes (e.g., Is it true that all rectangles have 1 pair of parallel sides? Jess asks Matt why he thinks that a rectangle is not a square. Who is right, Matt or Jess?).
- Practise noticing the different strategies and the breadth of the mathematics your students use in their solutions. Take note of the number and kinds of solutions that show strategies for sorting and classifying 2-D shapes, composing and decomposing 2-D shape. Which strategy do students use more often? (e.g., counting the number of sides, comparing the side lengths, measuring the inside angles)
- In preparation for your class discussion of student solutions, decide which solution you will discuss first, second, and third by referring to the developmental learning sequence for composing and decomposing 2-D shapes and classifying 2-D shapes.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit key mathematical ideas, strategies for geometric attributes of 2-D shapes, methods for composing and decomposing 2-D shapes when the students discuss and analyse each other's solutions.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below:

- Battista, M. (2009). Highlights of Research on Learning School Geometry. In T. V. Craine and R. Rubenstein (eds.), *Understanding geometry for a changing world* (pp. 91-108). Reston, VA: National Council for Teachers of Mathematics.
- Battista, M. (2007). The development of geometric and spatial thinking. In F. L. Lerner (ed.), *Second Handbook of research on mathematics teaching and learning* (pp. 843-908). Greenwich, Conn: Information Age Publishing.
- Clements, D. & Battista, M. (1995). Geometry and proof. *Mathematics Teacher, 88*(1), 48-54.
- Clements, D. & Sarama, J. (2003). Building blocks of early childhood mathematics. *Teaching Children Mathematics, 9*, 480-484.
- Clements, D., Wilson, D. & Sarama, J. (2004) Young Children's Composition of Geometric Figures: A Learning Trajectory. *Mathematical Thinking and Learning. 6* (2), 163-184.
- Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education Leadership, 7*(3), 9-24.
- Patsiomitou, S. & Emvalotis, A. (2010). Students movement through Van Hiele levels in a dynamic geometry guided reinvention process. *Journal of Mathematics and Technology*, ISSN: 2078-0257.
- Small, M. (2009). *Big ideas from Dr. Small, Grades 4-8*. Toronto, ON: Nelson Education.
- Small, M. (2008). *Making math meaningful to Canadian students, K-8*. Toronto, ON: Nelson Education.
- Small, M. (2007). *Geometry: Background and strategies*. Toronto, ON: Nelson Education.

CLOSING

Our next issue will focus on key notions of 3-D geometry, how teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a problem solving-based teaching and learning lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email