8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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SPATIAL REASONING WITHIN MEASUREMENT CONTEXTS

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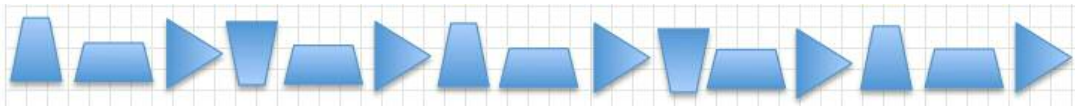
When we use mental images to see patterns and relationships among concepts, we are using spatial reasoning. Think about the aspects of spatial reasoning that you need to use in order to examine and compare the two patterns. Describe pattern A and B, by telling:

- how the pattern starts
- how the pattern changes
- the attributes and number of attributes.

Pattern A



Pattern B



In this issue, the Research Summary and problems (with multiple solutions) focus on the use of spatial reasoning within patterning, data management, and probability contexts. The implicit and explicit use of spatial reasoning is threaded throughout student solutions to problems. The problems we provide are designed for use with a three-part problem-solving lesson structure.

RESEARCH SUMMARY – SPATIAL REASONING WITH PATTERNING, DATA MANAGEMENT, AND PROBABILITY CONTEXTS

How is spatial reasoning implicated in learning about growing patterns, selecting an appropriate measure of central tendency and/or understanding probability? Newcombe (2010) explains that "spatial thinking involves noticing and remembering the locations of objects and their shape, and being able to mentally manipulate those shapes and track their paths as they move" (p. 33). Further, Whiteley, Sinclair and Davis (2005, p. 5) identified Tahta's (1989) work as linking spatiality to the visual; that is, three "powers involved in working with space:

- imagining, which involves seeing what is said
- construing, which involves seeing what is drawn or saying what is seen; and
- figuring, which involves drawing what is seen."

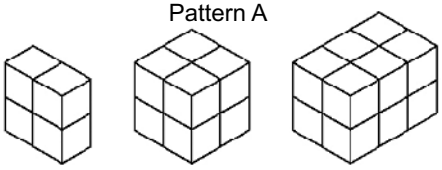
Owens (2015) refers to Battista (2007a, p. 843) who explains that "'spatial reasoning … includes generating images, inspecting images to answer questions about them, transforming and operating on images, and maintaining images in the service of other mental operations.' Owens adds "mental operations on mental visual imagery are about space. The term 'visuospatial reasoning' emphasizes the reasoning associated with and dependent on visual imagery, but also expressed and argued with spatial references. Spatial abilities, spatial skills and both spatial visual imagery are part of this reasoning… . Recreating the visible in terms of visualizing from alternative perspectives, re-seeing, and reconstructing shapes is supplemented." (p. 7).

Using Spatial Reasoning for Creating and Extending Growing (Linear) Patterns

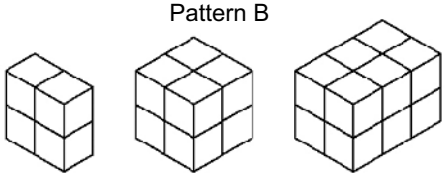
Think about the ways that spatial reasoning is used to describe and extend a growing pattern using different visual representations.

- Are you imagining the space that one cube takes up and determining the total number of cubes for each figure, even if you can't see every cube?
- How many cubes are at the start of the pattern and what is the constant change of number of cubes from figure to figure?
- How are Pattern A and Pattern B similar? How are they different?
- What would the graphic representation of Pattern A and Pattern B look like?

Pattern A



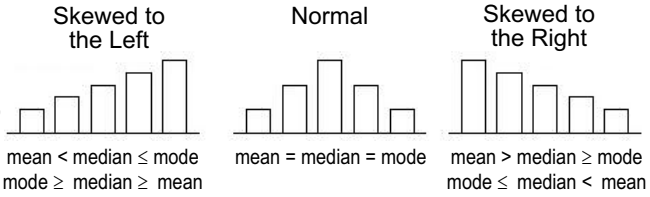
Pattern B



Using Spatial Reasoning for Selecting Appropriate Measures of Central Tendency

Think about how spatial reasoning is used to visualize concrete representations of measures of central tendency (i.e., mean, median, and mode)

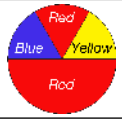
- Imagine that these graphs are created using square tiles.
- Use square tiles to model mean, median, and mode in these 3 different sets of data.



Using Spatial Reasoning for Determining the Probability of Events

Think about the ways that spatial reasoning is used to determine the probability of events. Theoretical probability is based on what should happen as ratio of the favourable outcomes in relation to the total number of possible outcomes.

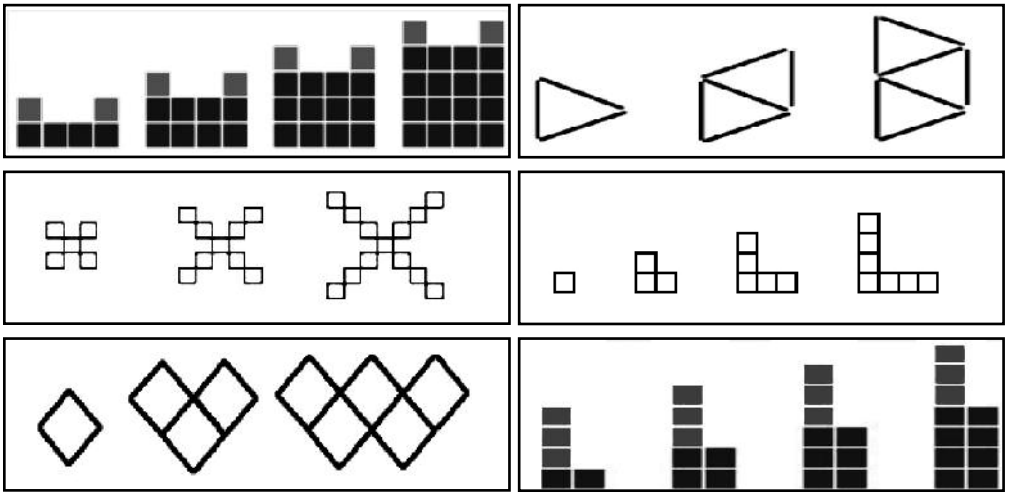
- If each section of the spinner is of equal size and each number shows up only once, then each digit will have an equal chance of being spun.
- Which probability representations show fairness? Which are unfair? How do you know?
- How could spatial reasoning be used to examine fair and unfair events?

Spinning red 	Pulling out a red cube from a bag with 20 red cubes and 5 blue cubes	Rolling doubles with 2 regular die (1 to 6)	Spinning yellow 	Getting a grape bar in a box of 13 orange, 12 grape and 15 strawberry bars

LINKS TO MANIPULATIVES: USING SQUARE TILES AND TOOTHPICKS

Use square tiles and toothpicks to build models of different sorts of growing patterns. Pose problems like:

- Model the next 3 figures using square tiles or toothpicks.
- How many square tiles or toothpicks would there be in the 5th figure, if the first diagram is the 1st figure? Explain what you are visualizing.
- What does the growing pattern look like as a graph? Visualize first, and then use a table of values to organize the number of square tiles or toothpicks for each figure to generate coordinate points for graphing.
- Describe each growing pattern. Imagine where the pattern starts and how it changes. Use words like “Start at <number> and change by <number>”
- Compare 2 growing patterns. Which growing pattern changes at a greater rate? (e.g., For every figure number or term number, the pattern grows by a multiple of <number>.)



LINKS TO LITERATURE: ABOUT MATH PICTURE BOOKS

Since the 1990s, picture books have commonly been used by teachers as rich contexts for activating students’ mathematical thinking. Van den Heuvel-Panhuizen, Elia and Robitzsch (2014) identify several ways that math picture books are used in the classroom. Math picture books provide:

- meaningful concrete and real life contexts for learning mathematics, as the story form is a basic tool for establishing meaning;
- a story context and narrative that improves the learners’ retention and recall of learning mathematics knowledge embedded in the story;
- a cognitive framework or cognitive hook to emphasize, explore, and/or construct mathematical concepts and skills;
- different ways to develop mathematical language, in terms of number and spatial reasoning;
- informal experiences with mathematical operations, objects and structures that can be developed to more formal, precise and in-depth level of understanding through instruction;
- a problematic situation that provokes learners to ask questions, develop solutions, examine different points of view, exchange ideas, and adapt their mathematical thinking;
- ways for learners to attach personal mathematical meaning to the mathematical objects included in the picture books.



Five friends: a donkey, a cow, a sheep, a pig, and a mouse, decide to go for a boat ride. Each of them enters the boat from biggest to smallest. As each animal steps on the boat, it tips and sits a bit lower in the water than before. Guess who gets in last. Guess who sinks the boat.

- Describe a growing pattern from the story.
- Describe a shrinking pattern from the story.

LESSON LEARNING GOALS

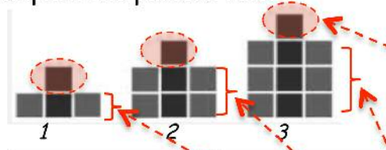
Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

Grade 5	Grade 6	Grade 7
-create, identify and extend numeric and geometric patterns using a variety of tools -make predictions related to growing and shrinking geometric and numeric patterns	- describe pattern rules (in words) that generate patterns by adding or subtracting a constant or multiplying or dividing by a constant to get the next term	-develop and represent the general term of a linear growing pattern using algebraic expressions involving one operation

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Growing Squares Problem 1:

Explain the pattern rule.



Stays the Same (Constant) *– each figure has 1 square tile*

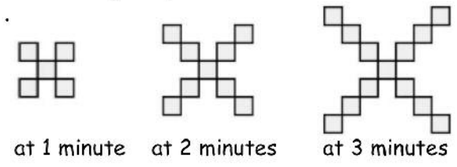
Changes (Variable) *increasing number of groups of 3 square tiles (1 group, 2 groups, 3 groups)*

Solution 1
If 1st diagram is figure 1, then start with 1 square tile and add 1 group of 3 square tiles for every figure.

Solution 2
If the 1st diagram is figure 0, then start with 4 square tiles and add 1 group of 3 square tiles for every figure.

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Growing Squares Problem 2:



at 1 minute at 2 minutes at 3 minutes

Assuming the pattern grows in the same way, how many squares are there at 5 minutes? Explain its pattern rule.

What information are we going to use to solve the problem?
- *1 minute has 5 squares*
- *2 minutes has 9 squares*
- *3 minutes has 13 squares*
- *How many squares at 5 minutes?*

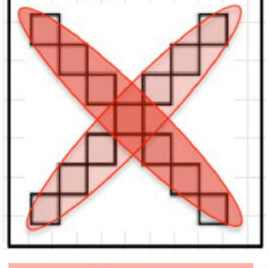
AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Visualize the starting number and the size of the grouping for the growing pattern.

Constant
- this growing pattern always starts at 1 (at term number, 0)

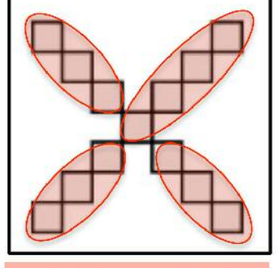
Variable
- number of groups of 4 changes from term number to term number (1 group of 4 for term number 1)

Solution 1
At 3 minutes, double the long arms (2 groups of 3) plus the start (3x2)x2+1=13
At 5 minutes, double the long arms (2 groups of 5) plus the start (5x2)x2+1=21



Crossed Arms *(double term number, double term number1 plus start)*

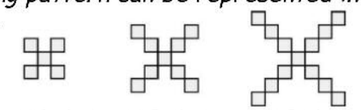
Solution 2
At 3 minutes, have 3 groups of 3 and 1 group of 3 plus the start (3x3)+(3+1)=13
At 5 minutes, have 3 groups of 5 and 1 group 5 plus the start (5x3)+(5+1)=21



Arms *(term number)* ***and Extended Arm*** *(term number and start)*

Growing Pattern has a constant increase (e.g., 4) Though a growing pattern can be represented in different ways:

(a) ***diagram***

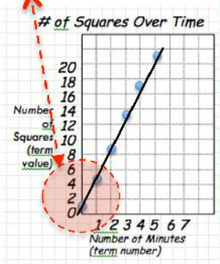


at 1 minute at 2 minutes at 3 minutes

(b) table of values *term number term value*

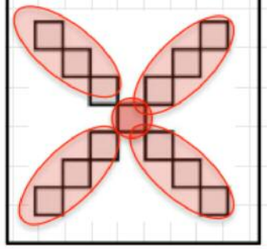
time	0	1	2	3	4	5
# squares	1	5	9	13	17	21
expression	1+4(0)	1+4(1)	1+4(2)	1+4(3)	1+4(4)	1+4(5)
coordinate pairs	(0,1)	(1,5)	(2,9)	(3,13)	(4,17)	(5,21)

(c) ***graph***
time and # squares for each term number and term value makes co-ordinate pairs (0,1), (1,5) to plot on the graph



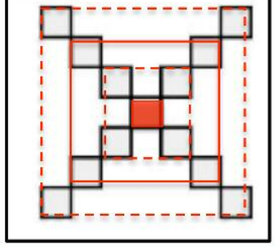
(d) ***pattern rule*** *explains what stays the same (constant) and what changes (variable) in the growing pattern; can be used to predict the term value for any term number.*
(e.g., Start at 1 and multiply size of one group by its term number)

Solution 3
At 3 minutes, start at 1 and add 4 groups of 3 or (1+3+3+3+3=13)
So, at 5 minutes, start at 1 and add 4 groups of 5 or (1+5+5+5+5 = 21)



Arms *(term number)* ***and Centre*** *(start)*

Solution 4
At 3 minutes, have 3 square groups of 4 with a start of 1. ((4x3)+1=13)
At 5 minutes, have 5 square groups of 4 with a start of 1 (1+(4x5)=21)



Size of One Group x Term Number Plus Start

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed

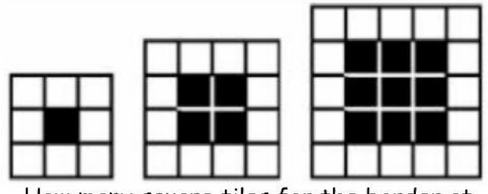
Highlights/Summary

A growing pattern *has a constant increase (e.g., multiple of 4; straight line graph)*
Term number *shows the figure number (independent variable).* ***Term value*** *shows the result of the pattern for each term number (dependent variable).*

Growing pattern rules explain the start of the pattern and how the pattern changes from term number to term number.
(e.g., Start at 1 and multiply size of one group by its term number);
-> for term number 1 -> 1+4(1) = 5
-> for term number 3 -> 1+4(3)= 13
-> for term number 5 -> 1+4(5)= 21

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Growing Squares Problem 3:



How many square tiles for the border at the 5th figure? Explain the pattern rule.

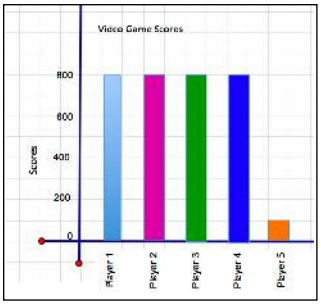
Solution 1
Start at 4 and add 1 group of 4 squares for every figure.

Solution 2
Add double the number of square tiles for the:
-> vertical side length
-> horizontal side less 2 squares.

Note: Mathematical annotations include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy, and questions to further thinking. All annotations are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical annotations are recorded above the solutions with arrows, rather than on and around the solutions.

Before (Getting Started)



Game Score Problem A:

Here are some video game scores. Which central measure best represents the data: mean, median, or mode?

Solution 1

Well, I can see the mode score (most frequent) is 800.

Also the score in the middle when organized highest to lowest is 800.

Mean is $\frac{3200+100}{5}+\frac{3300}{5}=660$

The mean is way lower than the median or mode. Because of the outlier of 100 – the average (mean) is pulled lower.

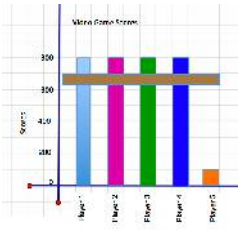
Solution 2

To find the mean, I take the horizontal ruler and move it down until all 4 pieces of bars above the dashed line go together to make the right bar equivalent in length to the other 4 bars.

It looks like I need to take off about 150 units from each of the first 4 bars. (150 x 4 = 600). New bar is 600 + 100 = 700).

Others are only 650 now.

So, each decomposed part will be a little less – like 140. The mean is 660.



Solution 3

The mode can be seen on the graph – occurring the most often. Mode is 800.

Median is score in the middle (median runs down the middle of the road). Median is 800.

The mean is going to be pulled lower because of the one score of 100.

So, median or mode best represents the data.

During (Working On It)

Game Score Problem B:

Five friends played 10 rounds of Rock, Paper, Scissors. In each round, A winner scored 2 points, ties scored 1 point and loses resulted in 0 points.

These are the scores:

Ria 16, Lambo 8, Travis 8, Edward 6, Arthur 2.

What is the mean score of this data? What do you visualize?

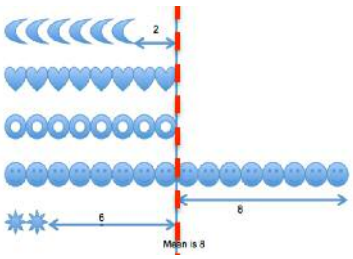
After (Consolidation)

Anticipating Student Responses:

Solution 1

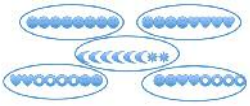
I lined up the scores and moved my dashed line. When I moved the point counters on the right to fill empty spots on the left, I knew I had the mean right.

Mean is 8.



Solution 2

I wanted to break the points into 5 groups of equal size. I distributed (shared) the points one-by-one into 5 groups. I found the mean was 8.



Solution 3

Mean = $\frac{\textit{sum of all the scores}}{\textit{number of players}}=\frac{40}{5}=8$ The mean score is 8.

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solution 1, and 3?

- Solution 2 – distributing the point tiles evenly among 5 groups to see how the points are equally distributed; mean shows as 8.
- *Solution 1* – point tiles lined up and a dashed line to estimate mean; when the number of point tiles above the mean (on the right) is equal to the number of empty spaces below the mean (on the left) then the dashed line shows the mean.
- *Solution 3* – total number of point tiles is determined and the sum divided by the number of groups of tiles there were in the beginning (5); represents other 2 processes using calculations.



Before (Getting Started)

Duck, Duck, Goose Problem 1:

In the game *Duck, Duck, Goose* each person has an equally likely chance of being picked as the Goose. If you are one of 13 people playing, what is the probability that you will be picked as the Goose?

Solution 1

I am seeing me, 1 person, and there are 12 others.



It is very unlikely that I will be picked.

Me
x x x x
x x x x
x x x x

Solution 2

$$P(me) = \frac{favourable\ outcome(s)}{all\ possible\ outcomes}$$

$$P(me) = \frac{1}{13} = 0.0769 \approx 8\%$$

8% is pretty low. It means if there were 100 people, 8 of us would likely NOT be chosen.

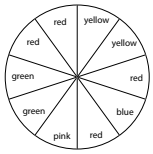
During (Working On It)

Colours Problem 2: (EQAO Grade 6. 2012-2013)

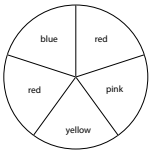
Consider the two spinners to the right. Spinner A is divided into 10 equal sections, and Spinner B is divided into 5 equal sections.

Which colours have the same probability of being spun on Spinner A as they do on Spinner B? Justify your answers.

Spinner A



Spinner B



After (Consolidation)

Anticipating Student Responses:

Solution 1

I want to rearrange the colours on Spinner A.
I put the same colours together so it would look more like Spinner B.

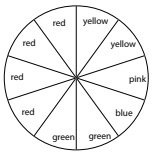
RED:

Now I can see there are four 1/10 ths on Spinner A and that this is the same area as two 1/5 ths on Spinner B.

YELLOW:

I can see there are two 1/10 ths on Spinner A and this is the same area as one 1/5 th on Spinner B.

Spinner A

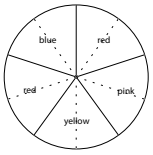


Solution 2

I split up Spinner A so there were 10 sections like Spinner B.

Now I can see there are 4 reds on both spinners. There are 2 yellows on both.

Spinner B



Solution 3

Colour	Spinner A		Spinner B	
Red	4	$\frac{4}{10} = 40\%$	2	$\frac{2}{5} = \frac{4}{10} = 40\%$
Yellow	2	$\frac{2}{10} = 20\%$	1	$\frac{1}{5} = \frac{2}{10} = 20\%$
Blue	1	10%	1	20%
Pink	1	10%	1	20%
Green	2	20%	0	0%

Red and yellow both have the same likelihood of being spun on Spinner A and Spinner B.

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion, followed by solution 1 and 3?

- *Solution 2* – equi-partitioning of sectors of a circle graph in order to make a spinner with 10 sectors rather than 5; comparing by counting the number of favourable sectors and comparing it to 10 sectors; it's the same on both spinners.
- *Solution 1* – repositioning the colours on Spinner A so the sectors could be merged to see equivalent fractions or sectors with equal areas on both spinners.
- *Solution 3* – equi-partitioning and merging are done in an invisible way by manipulating the fractions to get common denominators and percents for comparison.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how these patterning and algebraic reasoning problems include visual-spatial reasoning with an emphasis on multiplicative relationships (e.g., between the term number and the size of the group of objects within a growing (linear) pattern).
- To suit your students' learning needs, select and design reasoning tasks or problems that are relative to grade specific curriculum expectations in the Patterning and Algebra strand, being mindful of the transition from additive to multiplicative reasoning (e.g., adding repeatedly the number of objects in each group to multiplying the number of objects in a group by its term number).
- Incorporate daily practice of spatial reasoning through visual-spatial, growing (linear) pattern Quick Images. Quick images materials could include a growing (linear) pattern represented by a diagram (see examples in "Links to Manipulatives – Square Tiles and Toothpicks"), table of values or pattern rule described in words. Quick images are presented to the learner for a few seconds, then the learner is prompted to either explain the pattern rule in words (e.g., Starts at … changes by …), model the pattern using square tiles or toothpicks and/or create a table of values.
- Practise noticing the different strategies and the breadth of the mathematics your students use to solve growing (linear) pattern reasoning tasks or problems, such as, identifying the parts of the growing pattern that stay the same and the part(s) that changes from term number to term number.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit spatial reasoning strategies for creating, describing, or modeling a linear (growing) pattern.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

Beatty, R. (2014). Exploring the power of growing patterns. What works? *Research into practice, 55*.

Newcombe, N. (2010). Picture this: Increasing Math and Science learning by improving spatial thinking. *American Educator,* 29-35.

Ontario Ministry of Ontario. *Paying attention to spatial reasoning, K-12.* Toronto: Queen's Printer for Ontario.

Owens, K. (2015). *Visuospatial reasoning: An ecolcultural perspective for space, geometry and measurement education.* Switzerland, Springer International Publishing.

Van den Heuvel-Panhuizen, M., Elia, I., & Robitzsch, A. (2014). *Effects of reading picture books on kindergartners' mathematics performance.* **http://dx.doi.org/10.1080/01443410.2014.963029**.

Whiteley, W., Sinclair, N., & Davis, B. (2015). What is spatial reasoning? In B. Davis (Ed.) *Spatial reasoning in the early years: Principles, assertions and speculations,* pp. 3-14. New York: Routledge.

Yilmaz, H. B. (2009). On the development and measurement of spatial ability. *International Electronic Journal of Elementary Education, (1)*2, 83-96.

CLOSING

Our next issue will focus on Algebraic Reasoning and outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within learning and teaching mathematics within a three-part problem solving lesson framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email