8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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EQUIVALENCE AND RELATIONAL THINKING - MULTIPLICATION

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A Publication of the OAME/AOEM

ABACUS CO-EDITORS



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The balance scale, number line model is an effective and logical visual model for determining the equivalence of addition and subtraction expressions, as described in the previous two Abacus issues. What model(s) and image(s) could be used to represent the concept, equivalence within the context whole number multiplication expressions.

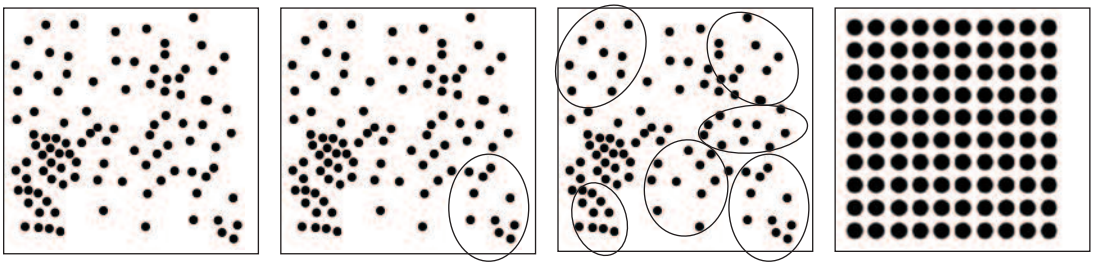
Which arithmetic properties are being developed through the equations below? Imagine what mathematical models could represent these multiplication equations? If you can imagine appropriate mathematical models, then what manipulatives would be useful to model these equations or model the comparison of two multiplication expressions?

- 6 x 4 = 3 x 8
- 6 x □ = 3 x 2 x □
- (3 x 4) x 25 = 3 x (4 x 25)
- 120 x 30 = 12 x □ x 3 x 10
- 30 x 120 = 100 x 3 x 12

In this issue, the Research Summary and reasoning tasks (with multiple solutions) focus on equivalence and relational thinking, within multiplication of 1, 2 and 3-digit whole number contexts. The reasoning tasks in this issue are designed for use with a three-part problem-solving lesson or a Mental Math mini lesson structure. Consider including in your mathematics program planning daily learning and practice for solving whole number multiplication true/false number sentences and open number sentences.

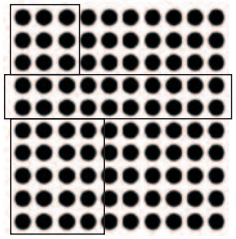
RESEARCH SUMMARY – EQUIVALENCE AND RELATIONAL THINKING FOR MULTIPLICATION

How many dots are below? Are you counting by 1s, 2s, 5s or 10s? Are you visualizing equal size groups of 2s, 5s or 10s? (Diagrams adapted from Davis, 2010)



(1) What's your count? (2) Grouping by 10s? (3) How many groups of 10? (4) Rows x Columns?

What different ways could you prove that there are 100 dots in the first diagram?



- Count the dots by 1s or by groups of 10.
- Organize the dots in an array of 10 rows x 10 columns, written as 100 = 10 x 10.
- What smaller dot arrays can you see in the 10 x 10 array?
  = (5 x 4) + (2 x 10) +(3 x 3) + (3 x 7) + (5 x 6) = 20 + 20 + 9 + 21 + 30
- Imagine what this 10 x 10 array looks like, when decomposed into these arrays.
  100 = (5 x 4) + (2 x 4) + (3 x 4) + (5 x 6) + (2 x 6) + (3 x 6)

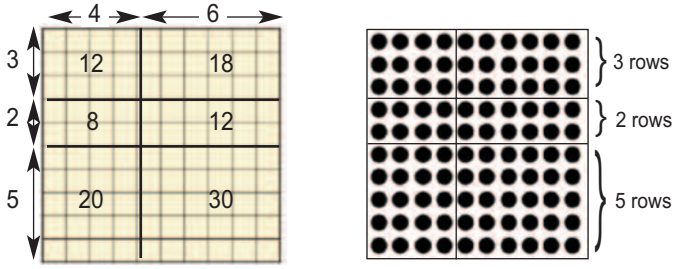
Fundamental Properties of Multiplication

Mason, Stephens and Watson (2009) argued that "appreciating [mathematical] structure is powerfully productive [and that] attention to structure should be an essential part of mathematical teaching and learning" (p. 10). The following arithmetic properties of multiplication comprise key mathematical structures that underpin students' multiplicative and relational reasoning.

(a) *Distributive Property for Multiplication*

The product of the sum of two numbers is the same as the sum of their products.

100 = 10 x 10 = (3 + 2 + 5) x (4 + 6) = (3 x 4) + (2 x 4) + (5 x 4) + (3 x 6) + (2 x 6) + (5 x 6)
= 12 + 8 + 20 + 18 + 12 + 30



(b) *Commutative Property for Multiplication*

Numbers can be multiplied in any order and get the same product.

a x b = b x a ⇒ 5 x 2 x 10 = 10 x 2 x 5

(c) *Associative Property for Multiplication*

The product of a set of numbers is the same, regardless of how the numbers are grouped.

(a x b) x c = a x (b x c) ⇒ 5 x (2 x 10) = (5 x 2) x 10

Equivalence and Relational Thinking Revisited

In order to calculate mentally, Heirdsfield (2011) argued that students need to (a) understand numeration (i.e., the size and the value of numbers), (b) recognize the effect of an operation on a number, (c) know number facts (or relationships), and (d) estimate the results of one or more operations to check the reasonableness of a solution.

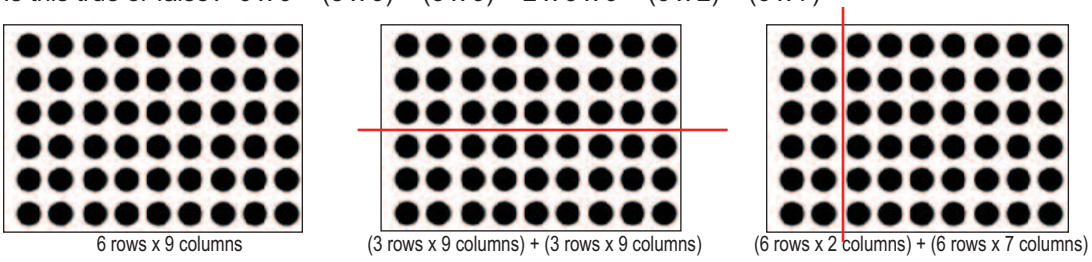
More specifically for solving true/false number sentences, Molina, Castro and Mason (2007) identified these student-generated strategies:

- determine numerical values of each side and compare them
- apply a known arithmetic principle after having noticed a particular relation or characteristic in the sentence which led them to recall a related fact
- notice sameness between numbers in the sentence and knowledge about the effect of operations
- notice number fact contained in the sentence and sameness between sentences
- notice numeric relations between numbers in the sentence and sameness between them
- identify differences in the magnitude between numbers and knowledge about the effect of operations
- identify numeric relations between numbers in the sentence and knowledge about the effect of operations.

LINKS TO MANIPULATIVES – DOT ARRAYS AND SQUARE GRID RECTANGLES

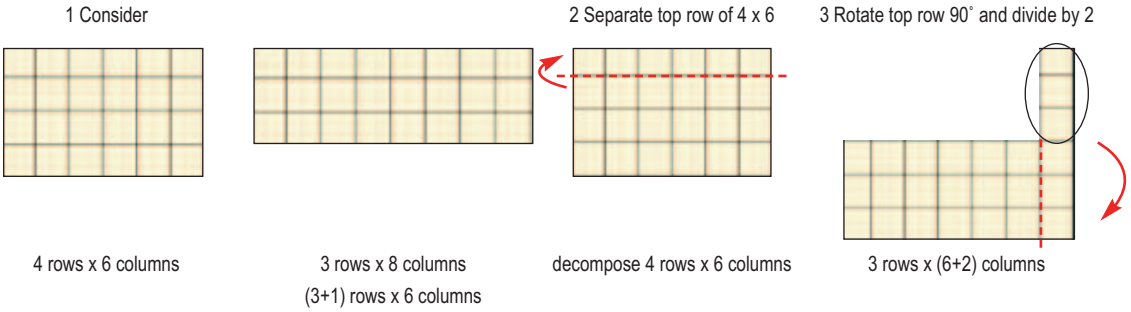
Counting large quantities of objects is inefficient and often leads to errors. Multiplication models, like arrays and area of rectangles offer greater precision and efficiency in determining quantities. Students' understanding of and flexibility with multiplication is developed using visual models, such as dot arrays and square grid rectangles.

Is this true or false? 6 x 9 = (3 x 9) + (3 x 9) = 2 x 3 x 9 = (6 x 2) + (6 x 7)



6 rows x 9 columns | (3 rows x 9 columns) + (3 rows x 9 columns) | (6 rows x 2 columns) + (6 rows x 7 columns)

Is this true or false? 4 x 6 = 3 x 8



- Decompose array of 4 rows x 6 columns into two arrays of 3 rows x 6 columns and 1 row x 6 columns.
- Rotate 1 row x 6 columns 90˚ clockwise and divide 6 rows x 1 column into two sets of 3 rows x 1 column.
- Translate 2 sets of 3 rows x 1 column to make 3 rows x 8 columns.

LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

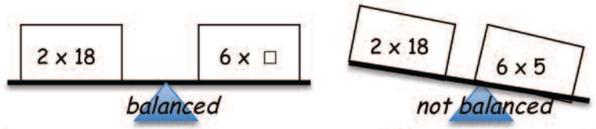
Grade 4	Grade 5	Grade 6
-determine the missing number in equations involving multiplication of one- and two-digit numbers –multiply one-digit by two-digit whole numbers	-determine the missing number in equations involving addition, subtraction, multiplication, or division and one- or two- digit numbers, using a variety of tools and strategies -multiply two-digit by two-digit whole numbers	-solve problems that use two or three symbols or letters are variables to represent different unknown quantities -solve problems involving multiplication (four-digit by two-digit whole numbers)

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

What is the Unknown Factor?
What is □? Show your work.
2 x 18 = 6 x □

factor - a number or expression that is multiplied by another to yield a product



Solution 1
2 x 18 = 36
What is the result of the product of 1st expression, 36 divided by 6 or
6 x □ = 36 □ = 6

Solution 2
2x18 = 2x3x6
2x3x6 = 6x□
6 = 6
2x3x6 = 6x2x3
2x3 = □

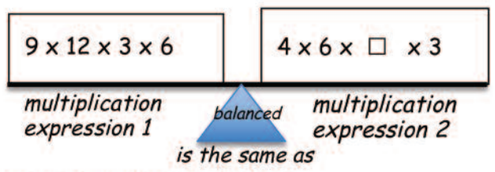
Solution 3
2x18 = 6 x □
2x(3x6) = (2x3)x6
6 x □ = (2x3)x6
□ = 2x3 = 6

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

What is the Unknown Factor?
What is □? Show your work.
9 x 12 x 3 x 6 = 4 x 6 x □ x 3

What information are we going to use to solve the problem?

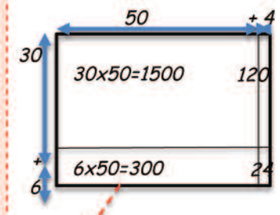


□ means the unknown number

AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Partial Products
(or Distributive Property for Multiplication
-Decompose the rectangle with linear dimensions of 36 and 54 into 4 rectangles.
-Decompose factors into addends of tens and ones. (34=30+4; 54=50+4)
-Determine area of each of the 4 rectangles.



Same Product - the result of multiplying numbers in the first expression is the same quantity as the result of multiplying numbers in the 2nd expression

Solution 1
9 x 12 x 3 x 6
= 9 x 6 x 12 x 3
= 54 x 36
= (50+4) x (30+6)
= (50x30) + (4x30) + (50x6) + (4x6)
= 1500 + 120 + 300 + 24
=1944
1944 | 3 | 4 | 6
= 648 | 4 | 6 = 162 | 6 = 27
□ = 27

Solution 2
9 x 12 x 3 x 6
= 9 x 6 x 12 x 3
= 54 x 36
= (50+4) x (30+6)
= (50x30) + (4x30) + (50x6) + (4x6)
= 1500 + 120 + 300 + 24
= 1944
4 x 6 x □ x 3 = 1944
72 x 20 = 1440
72 x 25 = 1440 + 360 = 1800
1944 – 1800 = 144 (72x2=144)
72 x 27 = 1800 + 144 = 1944
□=27

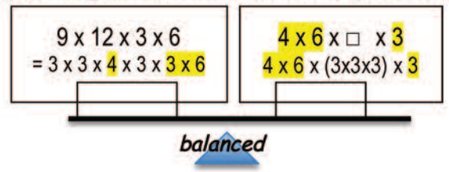
Operational
Product of 1st Multiplication Expression Used as Dividend for 2nd

Operational
Product of 1st Multiplication Expression Used as Product for Missing Factor for 2nd

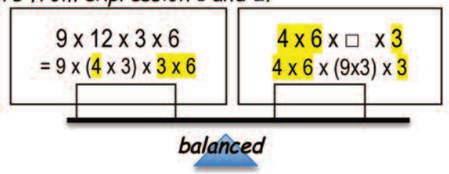
Equivalence or Equality

An equation indicating that two quantities (and/or expressions) are equivalence (is the same as, balance) (e.g., 9x12x3x6 = 4x6x□x 3)

Compensating Method to determine Equivalence



Relational Method to determine Equivalence
Use associative property for multiplication to compare factors from expression 1 and 2.



Commutative Property for Multiplication – *product is the same regardless of the order that the factors are multiplied (e.g., 3x3x4x3x3x6 = 4x6x3x3x3x3)*
Associative Property for Multiplication – *product of a set of factors is the same regardless of how the factors are grouped (e.g., 9x(4x3)x3x6 = 4x6x(9x3)x3)*

Solution 3
9x12x3x6 = 4x6x □x3
3x3x4x3x3x6 = 4x6x□x3
3x3x4x3x3x6 = 4x6x3x3x3x3
□ = 3x3x3 = 27

Decompose factor and group together the same factors for each multiplication expression to make visible the unknown, □.

Solution 4
9x12x3x6 = 4x6x □x3
9x(4x3)x3x6 = 4x6x□x3
9x(4x3)x3x6 = (4x6)x(9x3)x3
(4x6)x(9x3)x3 = 4x6x□x3
□ = 9x3 = 27

Decompose 12 to make the factor, 4 visible, then changed the order of the factors and grouped the factors that are the same as expression 2 unknown factor.

Compensating
Make Terms Same to See the Unknown

Relational
Compare Pairs of Terms Across 2 Expressions

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria

Highlights/Summary

Determine unknown number in multiplication equations using different methods:

- ***operational*** *– calculating the operations for both number expressions*
  (e.g., 9x2x3x6 = 1944; 1944 divided by 36=27)
- ***compensating*** *– making pairs of numbers the same by decomposing factors and associating factors to make same factors*
  (e.g. 9x12x3x6 = 3x3x4x3x3x6 = (4x6)x(3x3x3)x3)
- ***relational*** *– comparing pairs of numbers in expressions to know the number relations*
  (e.g., 9x(4x3)x3x6 = 4x6x(9x3)x3)

AFTER (Practice) – 5 to 10 minutes

- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

What is the Unknown Factor?
What is □? Show your work.
9 x 20 x 3 x 6 = 5 x 6 x □ x 3

Solution 1
9x20x3x6 = 5x6x□x3
9x5x4x3x6 =5x6x□x3
9x4 = 36 = □

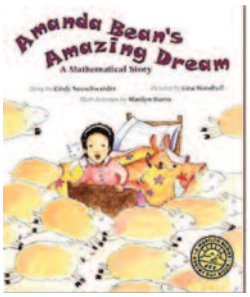
Look for same factors for expressions; different factors is the value of □.

Solution 2
9x20x3x6 = 5x6x□x3
9x(5x4)x3x6 = 5x6x□x3
9x(5x4)x3x6=(9x4)x(5x6)x3
(5x6)x(9x4)x3 = 5x6x□x3
□=9x4=36
Associate (or group) factors in expression 1 that are the same as factors in expression 2 and associate remaining factors to be value of □.

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical co-construction.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO LITERATURE: *AMANDA BEAN'S AMAZING DREAM*



Cindy Neuschwander's picture book, *Amanda Bean's Amazing Dream: A Mathematical Story* is about Amanda Bean who loves to count anything and everything one by one and is not convinced that she needs to learn about multiplication. Amanda has an amazing dream about sheep on bicycles, speed-knitting grannies, balls of yarn and long-sleeved sweaters, but she just can't count them fast enough. She takes the advice of a talking sheep to learn to multiply.

This story is well suited to introducing multiplication, using a few conceptual representations (i.e., equal size groups, arrays of objects), which are studied in grades 2 and 3. Some ways to have students think about multiplication using the story context:

(a) Determine the number of objects using different calculations (addition and multiplication) number expressions and discuss if these expressions are equivalent using an equation form. (e.g., 6 + 6 + 6 = 6 x 3; 3 x 6 = 3 + 3 + 3 + 3 + 3 + 3)

- On the first page, the illustration of the building shows six large windows with six by three array of smaller panes. How many panes altogether?
- How many brownies or cookies are on the bakery shelves?
- How many tiles are on the counter?

(b) Ask comparative questions that focus students' thinking about 2 sets of quantities and their relationship to one another (e.g., equal, not equal, greater than, less than.)

- Are there more window panes on the first building or the second building?
- Are there more brownies than cookies on the bakery shelves?
- Are there more sheep than grannies? More sheep than balls of yarn?

LET'S DO MATH – MULTIPLICATION OPEN NUMBER SENTENCES

Before (Getting Started)

4 x 9 = 2 x □ What is the value of □? Show your work.

Solution 1	*Solution 2*	Solution 3
LS => 4 x 9 = 36 RS => 36 | 2 = 18 2 x 18 = 36 □ = 18	4 x 9 = 2 x □ halve and double (4 | 2) x (9 x 2) = 2 x 18 ⇒ □= 18	4 x 9 = 2 x □ (2 x 2) x (3 x 3) = 2 x (2 x 3 x 3) = 2 x 18 □= 18

During (Working On It)

12 x 9 = 54 x □ What's the value of □? Show your work.

After (Consolidation)

Anticipating Student Responses:

Solution 1	Solution 2	Solution 3
LS => 12 x 9 = 108 RS => 108 | 54=2 54x2=108 □= 2 Calculate product to determine unknown factor x 54 by dividing	12 x 9 = 54 x □ 2 x 6 x 9 = 54 x □ 6 x 9 = 54, so □=2 Decompose factors to make 1 factor the same for both expressions	12 x 9 = 54 x □ (2x2x3) x (3x3) = (2x3x3x3) x 2 □=2 Decompose all factors into prime factors and group them to make the original factors

Coordinating Discussion for Student Learning:

Why might solution 3 be chosen first for student discussion, followed by solution 1 and 2?

- *Solution 1* – operational – multiplied the terms in the 1st multiplication expression (LS) and used that product as the dividend for the term in the 2nd multiplication expression (RS), where the quotient is the value of □
- *Solution 2* – compensating – decomposed one factor into a set factors so that there is at least one same factor as the known factor in the 2nd multiplication expression (RS)
- *Solution 3* – relational – decomposed one factor in a set of prime factors (LS) and group them so that some of the factors have the product of the known factor in the 2nd multiplication expression (RS) using the Associative Property for Multiplication

LET'S DO MATH – MULTIPLICATION TRUE/FALSE NUMBER SENTENCES

Before (Getting Started)

True or False? 120 x 300 = 400 x 90

Solution 1	*Solution 2*	*Solution 3*
120 x 300 = 12 x 10 x 3 x 100 = 36 x 1 000 = 36 000 400 x 90 = 4 x 100 x 9 x 10 = 36 x 1 000 = 36 000 120 x 300 = 400 x 90 ⇒ True	120 x 300 = 400 x 90 12 x 3 x 1000 = 4 x 9 x 1000 36 x 1000 = 36 x 1000 True Making factors same	(4 x 3 x 10) x (3 x 100) = (4x100) x (3x3x10) True Associative Property for Multiplication

During (Working On It)

True or False? 120 x 14 = 114 x 20

After (Consolidation)

Anticipating Student Responses:

Solution 1	*Solution 2*	*Solution 3*
120 x 14 = (120 x 10) + (120 x 4) = 1 200 + 480 = 1680 114 x 20 = 114 x 2 x 10 = 228 x 10 = 2280 1680 ≠ 2280 120 x 14 ≠ 114 x 20	120 x 14 = 114 x 20 20 x 6 x 14 = 6 x 19 x 20 20 x 6 = 6 x 20 and 14 < 19 120 x 14 < 114 x 20 120 x 14 ≠ 114 x 20	120 x 14 = 114 x 20 (20 x 6) x (2 x 7) = 20 x (19 x 2 x 3) 6 x 2 x 7 ≠ 19 x 2 x 3 84 ≠ 114 120 x 14 < 114 x 20 120 x 14 ≠ 114 x 20

Coordinating Discussion for Student Learning:

Why might solution 3 be chosen first for student discussion followed by solution 1, and 2?

- *Solution 1* – operational –> calculated each expression and compared the products
- *Solution 2* – compensating -> decomposed one factor into a set factors so that there is at least one same factor as the known factor in the 2nd multiplication expression (RS) and compare remaining factors
- *Solution 3* – relational -> decomposed one factor in a set of prime factors (LS) and group them so that some of the factors have the product of the known factor in the 2nd multiplication expression (RS) using the Associative Property for Multiplication

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how equivalence, reasoning tasks (e.g., True/False Equations, Open Number Sentences) develop students' understanding of number relationships and the use of different strategies for multiplication of 1, 2 and 3-digit whole numbers and relational thinking.
- To suit your students' learning needs, select and design reasoning tasks that are relative to grade specific curriculum expectations in the Patterning and Algebra strand (equivalence) and the Number Sense and Numeration strand (multiplication 1 digit, 2 digit and 3 digit whole numbers; mental math multiplication strategies; arithmetic properties), being mindful of the transition from calculating and comparing the results of multiplication expressions towards analyzing number relationships between factors when determining the equivalence of multiplication expressions.
- Incorporate five to 10 minutes of daily practice of multiplication strategies and relational thinking practices through daily mental math mini lessons, two times a day.
- Practise noticing the different methods and the breadth of the mathematics your students use to solve equivalence reasoning tasks, such as, carrying out the operations for each expression, comparing factors in multiplication expressions, adjusting the factors to make them similar in order to see the equality or inequality (True/False Equations) and unknown (Open Number Sentences).
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit addition and relational thinking strategies for solving equivalence reasoning tasks.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

Heirdsfield, A. (20110. Teaching mental computation strategies in early mathematics. *Young Children, 66*(2), 96-102.

Jacobs, V., Franke, M. l., Carpenter, T., Levi, L. Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258-288.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10-32.

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Molina, M., Castro, E., and Mason, J. (2008). Elementary school students' approaches to solve true/false number sentences. PNA 2(2), pp. 75-86.

Stephens, M. (2006). Describing and exploring the power of relational thinking. Conference *Publication at* ***http://www.researchgate.net/publication/236661303***, pp. 479-486.

Stephens, M. & Wang, X. (2008). Investigating some junctures in relational thinking: A study of Year 6 and Year 7 students from Australia and China. *Journal of Mathematics Education, 1*(1), 28-39.

Stephens, M. & Ribeiro, A. (2012). Working towards algebra: The importance of relational thinking. *Relime, 15*(3), 373-402.

CLOSING

Our next issue will focus on Equivalence and Relational Thinking within decimal number multiplication contexts. As well, the ways that teachers can use these mathematical ideas and pedagogical strategies within learning and teaching mathematics within a three-part problem solving lesson framework are outlined.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email