8-PAGE INSERT ELEMENTARY MATH

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EQUIVALENCE AND RELATIONAL THINKING – MULTIPLICATION WITHIN DECIMAL NUMBER CONTEXTS

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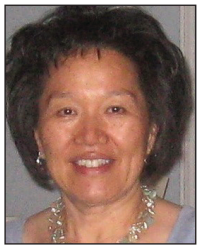
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ABACUS CO-EDITORS



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In the previous *Abacus* issue, the balance scale model was used to represent the concept, equivalence, within the context of whole number multiplication expressions. How might the balance scale model represent decimal number multiplication? For the equations below, which calculation methods and/or arithmetic properties could be used to model the comparison of two decimal and/or whole number multiplication expressions?

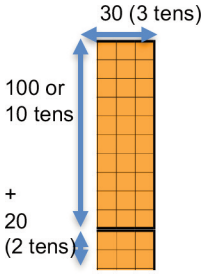
- 6 x 4.5 = 3 x 9
- 5 x 6.4 = x 3.2
- 0.6 x 0.5 = x 1
- 1.5 x = 3 x 0.8
- 12 x 2.5 = (3 x 4) x
- 50 x 1.4 = 10 x

In this issue, the Research Summary and reasoning tasks (with multiple solutions) focus on equivalence and relational thinking, within decimal multiplication contexts. The reasoning tasks in this issue are designed for use with a three-part problem-solving lesson or a Mental Math mini lesson structure. Consider including in your mathematics program planning daily learning and practice for solving decimal number multiplication true/false number sentences and open number sentences.

RESEARCH SUMMARY – EQUIVALENCE AND RELATIONAL THINKING FOR DECIMAL NUMBER MULTIPLICATION

A typical calculation method for decimal number multiplication is often without the use of a mathematical explanation. “Multiply normally, ignoring the decimal points. Then put the decimal point in the answer - it will have as many decimal places as the two original numbers combined.” This idea can be explained mathematically, by considering a decimal number as being comprised of a quantity in relation to a number unit; that is factors of tens (e.g., tenths, hundredths, thousandths).

Re-Interpreting Decimal Numbers in Relation to Whole Numbers

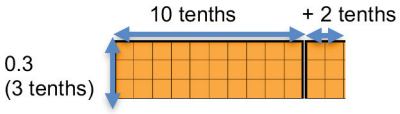


Multiplying Whole Numbers (120 x 30)

- 120 ones x 30 ones = 12 tens x 3 tens
- 120 x 30 = (12 x 10) x (3 x 10) = (12 x 3) x (10 x 10) = 36 hundreds = 3600

Group numbers that represent quantity (12 x 3=36) and numbers that represent the number unit (10x10 = 100; hundreds)

Multiplying Decimal Numbers (1.2 x 0.3)



- 0.3 ones x 1.2 ones = 3 tenths x 12 tenths
- 0.3 x 1.2 x 0.3 =(3 x 0.1) x (12 x 0.1)
  = (3 x 12) x (0.1 x 0.1) = 36 x 0.01 =36 hundredths
  = 0.36

Group numbers that represent quantity (12 x 3=36) and numbers that represent the number unit; that is, factors of tenths (e.g., 0.1 x 0.1 = 0.01; hundredths).

Fundamental Properties of Decimal Number Multiplication

As stated in the previous *Abacus* issue, Mason, Stephens and Watson (2009) argued that “appreciating [mathematical] structure is powerfully productive [and that] attention to structure should be an essential part of mathematical teaching and learning” (p. 10). The following arithmetic properties also apply to decimal number multiplication, as they comprise key mathematical structures that underpin students’ multiplicative and relational reasoning.

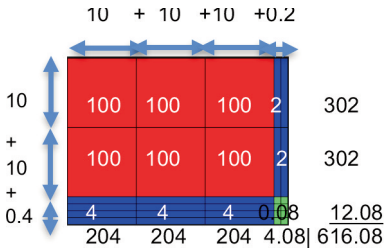
(a) Distributive Property for Multiplication

The product of the sum of two numbers is the same as the sum of their products.

20.4 x 30.2 = (20 + 0.4) x (30 + 0.2)

= (20 x 30)+(20 x 0.2)+(0.4 x 30) + (0.4 x 0.2)

= 600 + 4 + 12 + 0.08 = 616.08

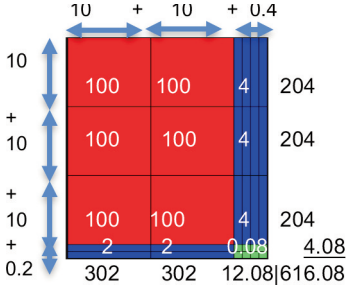


(b) Commutative Property for Multiplication

Numbers can be multiplied in any order and get the same product. (Different orientation)

a x b x c = b x a x c

20.4 x 30.2 = 30.2 x 20.4



(c) Associative Property for Multiplication

The product of a set of numbers is the same, regardless of how the numbers are grouped; associate different pairs of factors

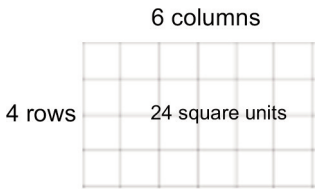
(a x b) x c = a x (b x c)

20.4 x 30.2 = 204 tenths x 302 tenths = (204 x 0.1) x (302 x 0.1)
= (204 x 302) x (0.1 x 0.1) = [(200 + 4) x (300 + 2) x (0.01)]
= (60 000 + 1 200 + 400 + 8) hundredths
= 61 608 hundredths = 616.08

LINKS TO MANIPULATIVES - SQUARE GRID RECTANGLES

- In grade 3, students learn about multiplication as arrays up to 7x7 and in grade 4, students learn about multiplication as arrays with 2 digit by 1 digit factors. Arrays, as the counting of number of rows x columns in square grid rectangles to determine the product (total objects in the rows x columns).

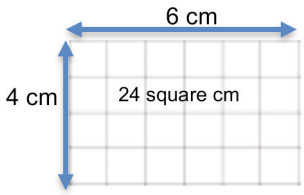
  Area (rectangle) = 4 rows x 6 columns = 24 square units



- From grade 4 to 5, students' understanding of multiplication as the area of a rectangle, in terms of linear dimension x linear dimension is developed by constructing the linear units of the side lengths to be 1 cm for each unit. That way, the students can measure the exact length of the side lengths of each square unit to discern multiplication relationships of linear dimensions and area (rectangle).

  Area (rectangle) = 1 cm x 1 cm = 1 cm^2 (1 square centimetre)

  Area (rectangle) = 4 cm x 6 cm = 24 cm^2 (24 square cm)



- Decomposing the rectangle into smaller rectangles using the Distributive Property for Multiplication helps the student to think about decimal multiplication as partial products (decomposition the rectangle into smaller rectangles).

LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

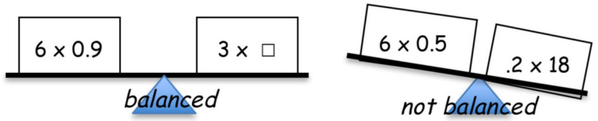
Grade 5	Grade 6	Grade 7
-determine the missing number in equations involving multiplication of one- and two-digit numbers –multiply 2 digit by 2 digit whole numbers	-solve problems that use two or three symbols or letters are variables to represent different unknown quantities –multiply decimal numbers to tenths by whole numbers	- evaluate algebraic expressions by substituting natural numbers for variables -evaluate expressions that involve whole numbers and decimals

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

What is the Unknown Factor?
What is □? Show your work.
6 x 0.9 = 2 x □

factor *- a number or expression that is multiplied by another to yield a product*



Solution 1
6 x 0.9 = 5.4
What is the result of the product of 1st expression, 5.4 divided by 3 or
3 x □ = 5.4 □ = 1.8

Solution 2
6 x 0.9 = (3x2) x 0.9
= 3 x (2 x 0.9)
= 3 x 1.8
□ = 1.8

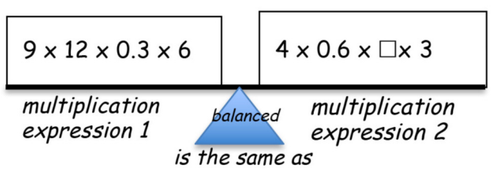
Solution 3
6x0.9 = 2 x □
(2x3) x 0.9
= (3x2) x 0.9
= 3x (2x0.9)
= 3 x 1.8 □ = 1.8

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

What is the Unknown Factor?
What is □? Show your work.
9 x 12 x 0.3 x 6 = 4 x 0.6 x □ x 3

What information are we going to use to solve the problem?

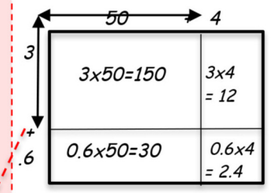


□ *means the unknown number*

`AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Partial Products
(or ***Distributive Property for Decimal Multiplication***
-Decompose the rectangle with linear dimensions of 3.6 and 54 linear units into 4 rectangles.
-Decompose factors into addends of tens and ones. (3.4=3+0.4; 54=50+4)
-Determine area of each of the 4 rectangles.



Same Product *- the result of multiplying numbers in the first expression is the same quantity as the result of multiplying numbers in the 2nd expression*

Solution 1
9 x 12 x 0.3 x 6
= 9 x 6 x 12 x 0.3
= 54 x 3.6
= (50+4) x (3+0.6)
= (50x3) + (4x3) + (50x0.6) + (4x0.6)
= 150 + 12 + 30 + 2.4
= 194.4
194.4 ÷3 ÷4 ÷0.6
194.4÷3=64.8; 64.8÷4=16.2
16.2÷0.6=(162x0.1)÷(6x0.1)
(162÷6)x(0.1÷0.1)=27

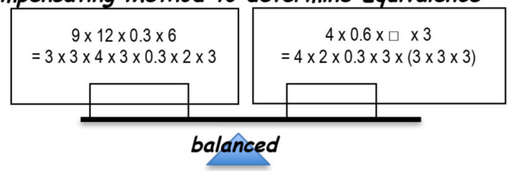
Solution 2
9 x 12 x 0.3 x 6
= 9 x 6 x 12 x 0.3
= 54 x 3.6
= (50+4) x (3+0.6)
= (50x3) + (4x3) + (50x0.6) + (4x0.6)
= 150 + 12 + 30 + 2.4
= 194.4
4 x 0.6 x □ x 3 = 194.4
(4 x 0.6 x 3) = 72
72 x 20 x 0.1 = 144 too small
72 x 30 x 0.1 = 216 too big
72 x 27 x 0.1 = 194.4
□=27

Operational
Product of 1st Multiplication Expression Used as Dividend for 2nd

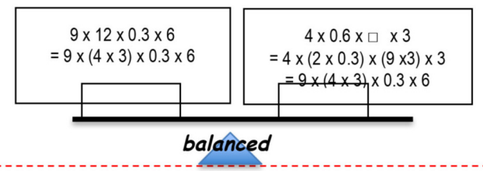
Operational
Product of 1st Multiplication Expression Used as Product for Missing Factor for 2nd

Equivalence or Equality
An equation indicating that two quantities (and/or expressions) are equivalence (is the same as, balance) (e.g., 9 x 12 x 0.3 x 6 = 4 x 0.6 x □ x 3)

Compensating Method to determine Equivalence



Relational Method to determine Equivalence
Use associative property for multiplication to compare factors from expression 1 and 2.



Quantity x Number Unit .3x6 = (3x0.1)x6 = (3x6)x0.1 = 18 tenths = 1.8

Commutative Property for Multiplication *– product is the same regardless of the order that the factors are multiplied (e.g., 3x3x4x3x0.3x6 = 4x0.6x3x3x3x3)*

Associative Property for Multiplication *– product of a set of factors is the same regardless of how the factors are grouped (e.g., 9x(4x3)x0.3x6 = 4x0.6x(9x3)x3)*

Solution 3
9x12x0.3x6 = 4x0.6x □x3
3x3x4x0.3x3x6 = 4x0.6x□x3
3x3x4x0.3x3x6 = 4x0.6x3x3x3x3
□ = 3x3x3 = 27
Decompose factor and group together the same factors for each multiplication expression to make visible the unknown, □.

Solution 4
9x12x0.3x6 = 4x0.6x □x3
9x(4x3)x0.3x6 = 4x0.6x□x3
9x(4x3)x0.3x6 = (2x2)x(2x0.3)x(9x3)x3
(9x6)x(4x3)x0.3 = (9x3x2)x((2x2)x3)x0.3
□ = 9x3 = 27
Decompose 12 to make the factor, 4 visible, then changed the order of the factors and grouped the factors that are the same as expression 2 unknown factor.

Compensating
Make Terms Same to See the Unknown

Relational
Compare Pairs of Terms Across 2 Expressions

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria

Highlights/Summary
Determine unknown number in decimal multiplication equations using methods:
- ***operational*** *– calculating the operations for both number expressions*
  (e.g., 9 x 12 x 0.3 x 6 = 194.4; 194.4 ÷3 ÷4 ÷0.6 = 27)
- ***compensating*** *– making pairs of numbers the same by decomposing factors and associating factors to make same factors*
  (e.g., 9x12x0.3x6 = 3x3x4x3x0.3x6 = (4x0.6)x(3x3x3)x3)
- ***relational*** *– comparing pairs of numbers in expressions to know the number relations*
  (e.g., 9x(4x3)x0.3x6 = 4x0.6x(9x3)x3)

AFTER (Practice) – 5 to 10 minutes

- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

What is the Unknown Factor?
What is □? Show your work.
9 x 20 x 0.3 x 6 = 5 x 6 x □ x 3

Solution 1
9x20x0.3x6 = 5x6x□x3
(3x3)x5x4x0.3x6
=5 x 6 x 3 x (3 x4 x 0.3)
3 x 4 x 0.3 = 3.6 = □
Look for same factors for expressions; different factor is the value of □.

Solution 2
LS= 9 x 20 x 0.3 x 6
= (3x3) x (5x4)x 0.3 x 6
= (5 x 6) x 0.3 x (4 x 3 x 3)
RS= (5 x 6) x 3 x □
= (5 x 6) x 3 x (0.3 x 4 x 3)
□= 0.3 x 4 x 3=3.6
Associate (or group) factors in expression 1 that are the same as factors in expression 2 and associate remaining factors to be value of □.

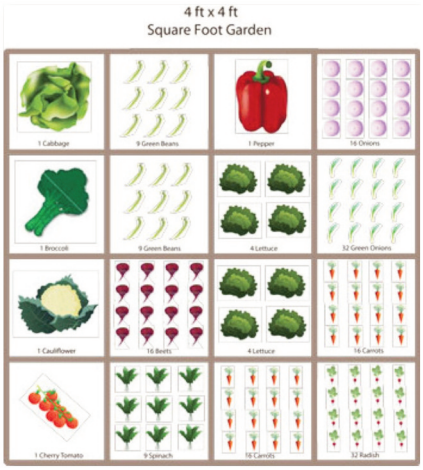
Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical co-construction.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO LITERATURE: PROBLEM CONTEXTS

- Rather than math picture books, think about the visuals associated with problem contexts that represent arrays (e.g., seating arrangement in a concert hall; musicians walking by in a parade, tulips growing in gardens in the Netherlands).
- Area measurement contexts could focus on the amount of space (product) given a set of linear dimensions (factors) of a rectangle, determining the amount of space for a garden.





LET'S DO MATH – MULTIPLICATION DECIMAL NUMBERS OPEN NUMBER SENTENCES

Before (Getting Started)

4 x 0.5 = 2 x What is the value of ? Show your work.

Solution 1
LS ⇒ 4 x 0.5 RS ⇒ 2 x □
= 2 ⇒ 2 x 1 = 2
So □ = 1

Solution 2
4 x 0.5 = 2 x □
halve and double
(4 |2) x (0.5x2) = 2 x 1 → □ = 1

Solution 3
4 x 0.5 = 2 x □
(2 x 2) x (0.5) = 2 x (2 x 0.5)
= 2 x 1 □= 1

During (Working On It)

12 x 0.9 = 36 x What's the value of ? Show your work.

After (Consolidation)

Anticipating Student Responses:

Solution 1
LS ⇒12 x 0.9 RS ⇒ 36 x □= 10.8
= 10.8 ⇒ 108 ÷ 3 = 36
⇒ 10.8 ÷ 0.3 = 36
□= 0.3
Halve and double or triple and take one third
12 x 3 = 36 and 0.9 x 1/3 = 0.3

Solution 2
12 x 0.9 = 36 x □
RS: 36 x □ = (12 x 3) x □
= 12 x (3 x 0.3) = 12 x 0.9 = LS
so □=0.3
Decompose factors to make 1 factor the same for both expressions

Solution 3
12 x 0.9 = 36 x □
(2 x 2 x 3) x (0.3 x 3) = (2 x 2 x 3 x 3) x 0.3
= 36 x 0.3 □=0.3
Decompose all factors into prime factors and group them to make the original factors

Coordinating Discussion for Student Learning:

Why might solution 3 be chosen first for student discussion, followed by solution 1 and 2?

- *Solution 1* - operational – calculated value on left side and then used tripling and taking one-third to determine the value to go in the box. Multiplied to check LS = RS

- *Solution 2* - relational – decomposed one factor into a set factors so that there is at least one same factor as the known factor in the 2nd multiplication expression (RS)
- *Solution 3* – relational – decomposed one factor in a set of prime factors (LS) and group them so that some of the factors have the product of the known factor in the 2nd multiplication expression (RS) à Associative Property for Multiplication

LET'S DO MATH – MULTIPLICATION TRUE/FALSE NUMBER SENTENCES WITH DECIMAL NUMBERS

Before (Getting Started)

True or False? 120 x 4.5 = 600 x 0.9

Solution 1
LS = 120 x 4.5 = 60 x 9 = 540
RS = 600 x 0.9 = (100 x 6) x 0.9
= 6 x (100 x 0.9) = 6 x 90 = 540
LS=RS
True

Solution 2
120 x 4.5 = 600 x 0.9
LS = (12 x 10) x 4.5
= (6 x 2 x 5 x 2) x 10 x 0.45
= (120) x (5 x (2x 0.45))
= (120 x 5) x (0.90) = 600 x 0.9
True Making factors same

Solution 3
120 x 4.5 = 600 x 0.9
(2 x 2 x 3 x 10) x (10 x 0.45)
= (2 x 3 x 10 x 10) x (2 x 0.45)
= 600 x 0.9
True
Associative Property for Multiplication

During (Working On It)

8.2 x 30 = 16.4 x 10 True or False?

After (Consolidation)

Anticipating Student Responses:

Solution 1
LS = 8.2 x 30 RS = 16.4 x 10
= 82 x 3 = 164
= 246
Not Equal False

Solution 2

8.2 x 30	16.4 x 10
= 8.2 x 3 x 5 x 2	= (8.2 x 2) x 5 x 2
	= 8.2 x 2 x 5 x 2
	= NOT equal to LS

False

Solution 3
8.2 x 30 = 16.4 x 10
= (8.2 x 2) x (5 x 2)
= 8.2 x (2 x 5 x 2)
= 8.2 x 20 Not equal
False

Coordinating Discussion for Student Learning:

Why might solution 3 be chosen first for student discussion followed by solution 1, and 2?

- *Solution 1* – operational –> calculated each expression and compared the products
- *Solution 2* – relational –> decomposed one factor into a set factors so that there is at least one same factor as the known factor in the 2nd multiplication expression (RS)
- *Solution 3* – relational –> decomposed one factor in a set of prime factors (LS) and group them so that some of the factors have the product of the known factor in the 2nd multiplication expression (RS) –> Associative Property for Multiplication

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

Consider how equivalence, reasoning tasks (e.g., True/False Equations, Open Number Sentences) develop students' understanding of number relationships and the use of different strategies for multiplication of decimal numbers and relational thinking.

To suit your students' learning needs, select and design reasoning tasks that are relative to grade specific curriculum expectations in the Patterning and Algebra strand (equivalence) and the Number Sense and Numeration strand (multiplication 1 digit, 2 digit and 3 digit whole numbers and decimal numbers to tenths, hundredths and thousandths; mental math multiplication strategies; arithmetic properties), being mindful of the transition from calculating and comparing the results of multiplication expressions towards analyzing number relationships between factors when determining the equivalence of multiplication expressions.

Incorporate 5 to 10 min of daily practice of multiplication strategies and relational thinking practices through mental math mini lessons, two times a day.

Practise noticing the different methods and the breadth of the mathematics your students use to solve equivalence reasoning tasks, such as, carrying out the operations for each expression, comparing factors in multiplication expressions, adjusting the factors to make them similar in order to see the equality or inequality (True/False Equations) and unknown (Open Number Sentences).

Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit multiplication of whole and decimal numbers and relational thinking strategies for solving equivalence reasoning tasks.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

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Jacobs, V., Franke, M. l., Carpenter, T., Levi, L. Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258-288.

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CLOSING

Our next year's issues will focus on Fractions Re-Visited and examine what we are learning about spatial reasoning that deepens our concepts of fractions and prepares us to hear children's mathematical thinking. The more we know about Mathematics for Teaching the better prepared we are to help children re-present their thinking even if it does not match ours. As is custom, we will model how teachers can use these mathematical ideas and pedagogical strategies within learning and teaching mathematics within a three-part problem solving lesson framework are outlined.

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Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email