8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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FRACTIONS AS PART OF A WHOLE

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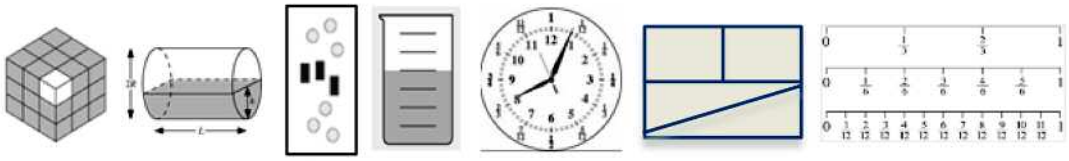


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A fraction is a quantity we use to measure objects with greater precision. For example, when we measure the length of an object, it will not always be a whole number, so we need to measure with number units that are less than 1; that is, numbers that are parts of 1, such as half of 1, a third, a fourth, a twelfth or even a thousandth. It turns out that the concept of "part of a whole" is a widely used image in length and area contexts. What mathematical contexts in your kindergarten to grade 6 mathematics program meaningfully connect to the concept of fraction, as part of a whole? A clock? A graduated cylinder? Puzzle pieces? What are the ways that your students' understanding of fraction as part of a whole can be developed?

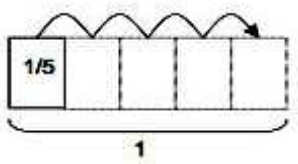


- Examine these different representations of fractions.
- Which representations show a fraction quantity greater than ¼ of the whole, 1? How do you know? Could any of these representations show a fraction quantity greater than ¼ of the whole, 2? Explain your thinking.
- How could any of these models be used to construct a set of unit fractions, like fourths or sevenths?

In this issue, the Research Summary and reasoning tasks (with multiple solutions) focus on fractions, as part of a whole within set model and measurement contexts. Reasoning tasks in this issue are designed for use with a three-part problem-solving lesson.

RESEARCH SUMMARY – FRACTIONS AS PART OF A WHOLE (SET AND MEASUREMENT MODELS)

Fractions are quantities or numbers that are "in-between" whole numbers. The transition from using whole numbers to fractions is difficult for learners as a fractional quantity is described using *one number* but *two digits*, and understanding a fraction requires one to explicitly comprehend a relationship between two quantities—the whole and its parts. A fraction is a symbolic expression of the form a/b, where a and b are whole numbers and b is not zero (whole numbers are 0, 1, 2, 3, and so on). Battista (2012), Bruce, Chang, & Flynn (2013), Lanon (2012), Small (2014) and Watanabe (2006, 2007) concur that fractions describe quantities as portions or parts of wholes that are a result of equal partitioning of a whole and there is a specific relationship between the parts and the whole.



Partitioning starts with the whole and divides it into equal parts. *Iteration* starts with a part and repeats it to make the whole. The processes of iteration and partitioning are relational; that is, iterating a piece to make a whole defines a partitioning of the whole; partitioning a whole into equal pieces defines an iteration of a part that makes the whole. More specifically, for the fraction, *a*/*b*, *a* (numerator) identifies how many equal parts are in the fractional quantity specified by *a*/*b*, while *b* (denominator) identifies how many equal parts are in the whole. The order of the numbers is important. Thus fractions are ordered pairs of numbers, as 3/4 is not the same as the fraction 4/3.

Unit and Non-Unit Fractions

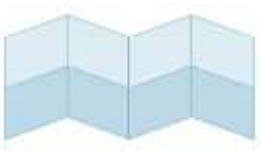
Watanabe (2006, 2007) identified the Japanese emphasis of the *unit fraction* (e.g., 1/3, 1/5, 1/10). That is to say that a fraction may also be considered to be a multiple of a unit fraction (i.e., *m*/*n* means m times 1/*n*). More specifically, to use iteration to create a fraction, *n*/*d* of a shape, one part is iterated *d* times to make the whole, of which the part is iterated *n* times. For example, 3/4 would be considered as 3 times the unit obtained by partitioning 1 into 4 equal parts; that is, ¼. Therefore, this fraction can be named as 3 one-fourth units.

$$\frac{3}{4} = \underline{3} \text{ (numerator)}$$
$$= 3 \times \frac{1}{4} \text{ units}$$
$$\underline{\quad\quad} \text{ and 4 (denominator)}$$

Non-unit fractions, like 3/8, 4/5 are made up of iterations of unit fractions. (e.g., 4/5 = 1/5 + 1/5 + 1/5 + 1/5 = 4 x 1/5)

Fraction as Part of a Whole Models

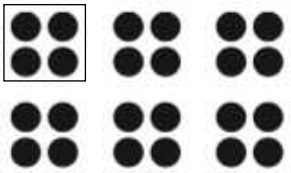
Measurement - Area Model





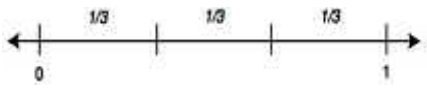
Generally, students understand fraction of whole geometric regions before they understand fractions of sets of objects. In the area model, the whole is defined by the area of the defined region; the parts are defined by equal area; the fraction means the part of the area covered as it relates to the whole unit. (e.g., 1 whole is folded into 4 or 8 equal size parts, where ½ or 4/8 of the pieces are shaded. For shape B, 3/6 of shape B are 3 triangles, while ½ of the shape is 1 trapezoid)

Set of Objects Model



In the set model, the whole is defined by the number of objects (size of objects is irrelevant) in the set; the parts are defined by equal number of objects; the fraction means the count of objects in the subset as it relates to the defined whole. (e.g., 1/6 of the 6 groups of dots are enclosed by a rectangle or 4/24 of the dots are enclosed by a rectangle).

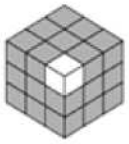
Measurement - Linear Model



In the linear model, the whole is defined by the unit of distance or length; the parts are defined by equal distance/length; the fraction means the location of a point in relation to 0 and other values on the number line. (e.g., 1/3 of 1. Length of 1 linear unit is divided into 3 equal lengths, 3 one-thirds)

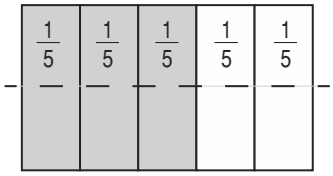
Measurement - Volume Model

In the volume model, the whole is defined by the total number of cubic units; the parts are defined by the number of cubic units; the fraction means the number of cubic units as it relates to the whole cubic units. (e.g., 1 cubic unit is part of the whole, 1/27 cubic units)



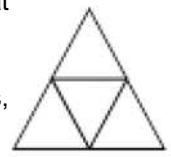
Equivalent Fractions

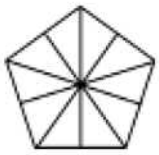
Two or more fractions are equivalent if they are the same quantity; meaning, that the same quantity can be named in different ways. By partitioning into two parts, each having five iterations of the unit fraction, fifths, then the same quantity 5/5 is equivalent to 10/10, while 3/5 = 6/10 and 2/5 = 4/10.

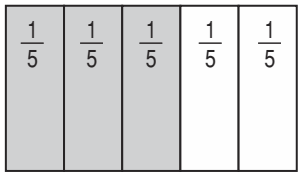


LINKS TO MANIPULATIVES – PAPER FOLDING

Folding paper shapes, like rectangles shows fractions, as part of a whole; that is, dividing the area or length of a whole into equal size parts. These equal size parts comprise unit fractions, such as halves, thirds, fourths and fifths.

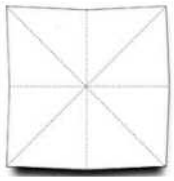






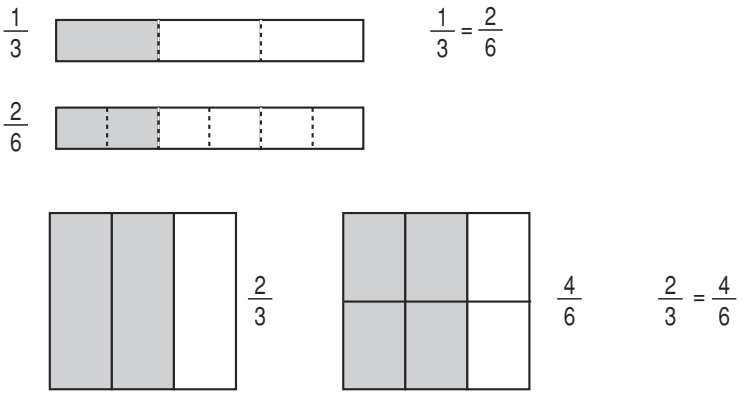
Making Fractions in Different Ways

- Provide students with several congruent pieces of paper; that is, same size and same shape (e.g., square). Have them fold each piece of paper into fourths in different ways.
- If one fold gives halves; halving the halves makes fourths, then what paper folding is done to the fourths to make sixteenths?



Making Equivalent Fractions

- Provide strips of paper that are congruent. To show that a third is equivalent to two sixths, have students fold a strip of paper into thirds (1 row x 3 column) and then again into half.
- Provide same size square or rectangular pieces of paper. To show that two-thirds is equivalent to fourth sixths, fold the paper into thirds (1 row x 3 columns), then rotate the paper 90˚ and fold the thirds into half, to make sixths.



LESSON LEARNING GOALS

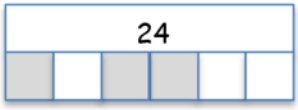
Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Posing Fraction Problems

Pose and solve a fraction problem using this diagram.



*A **fractional part** is in relation to its identified **whole** (e.g., if the whole has 6 parts, then 1 part is 1/6 of the whole; if whole has 48 parts, then 1 part is 1/48 of the whole)*

Solution 1

What fraction of all small rectangles are grey?

-6 small rectangles and 3 of them are grey
- 3/6 or ½ of 6 small rectangles are grey

Solution 2

What fraction of the whole is white?
-long white rectangle is worth 24 units, so each small rectangle is worth 4, so whole is 48
- 36/48 or 3/4 of the whole

Solution 3

What fraction of the whole is taken up by small rectangles?

-24/48 or ½ of the whole are small rectangles

DURING (Working on It) – 15 to 20 minutes

- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Buying Shoes Problem

Arlene spent 4/6 of her money on a laptop computer. How much money could she have had at first and how much could the computer have cost?

What information are we going to use to solve this problem?
-known - fraction represents cost of laptop computer
- unknown – actual cost of computer, amount of money Arlene had before she bought the laptop computer

` AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies,

Unit Fraction
-Indicates the fractional unit or the number of equal parts of the whole. (1/6 of the whole, 6/6)
*-Equal parts made through **partitioning** (or dividing)*



Solution 1

Estimate 1/6 is about $100

4/6 is $100 x 4 or four 1/6 or $400 → a laptop computer costs about $400

6/6 is $100 x 6 = $600
→the whole or the amount of money Arlene had at first.

Solution 2

Choose $600 to be the whole (money Arlene had first)

$600 divided into 6 equal parts
1 part, 1/6 is $100 = $600÷6
4/6 means four parts that are 1/6 each worth $100
→ 1/6, 2/6, 3/6, 4/6
→ $100, $200, $300, $400
→ cost of laptop is $400

Estimating Whole, Parts of a Whole *by Iterating Unit Fractions to 1 Whole*

Partitioning into Equal Parts *Counting Unit Fractions*

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical co-construction.

Grade 4	Grade 5	Grade 6
-represent fractions using concrete materials, words, and standard fractional notation, and explain the meaning of the denominator as the number of the fractional parts of a whole or a set, and the numerator as the number of fractional parts being considered	-represent fractional amounts with like denominators including proper and improper fractions and mixed numbers using a variety of tools and using standard fractional notation	-represent fractional amounts with unlike denominators, including proper and improper fractions and mixed numbers, using a variety of tools and using standard fractional notation

Fraction as Part of a Whole

A fraction is a quantity representing a part of a whole number. To partition a whole means to divide it into equal size parts (e.g., regions, lengths, groups of objects)

*$\frac{4}{6}$ -> **numerator** – number of equipartitions being counted*
*-> **denominator** – fractional unit used to partition the whole*

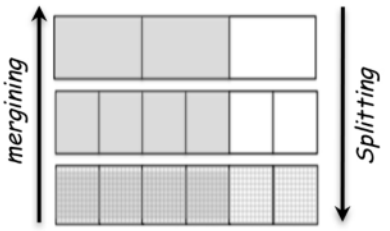
The quantity represented by the fraction is related to the size and number of parts of the whole. A fraction is always defined in related to its identified whole.

4/6 of 1 = 4/6 of 6/6 = 4/6 of 100/100
4/6 of 1 ≠ 4/6 of 2; 4/6 of 1 < 4/6 of 2



Equivalent Fractions

If the whole is 1 = 3/3 = 6/6 = 600/600, then 2/3= 4/6 = 400/600 because the same quantity can be described using different fractional units by:
*-**merging partitions** together (fewer #, larger partitions)*
*-**splitting partitions** further (greater #, smaller partitions)*



Solution 3

Choose \$600 to be the whole (money Arlene had first)

\$600 divided into 6 equal parts
\$600÷6 = \$100 for each part
4/6 means four parts that are 1/6 each worth \$100
→ 1/6 x 4 = 1/6+1/6+1/6+1/6 =
→ \$100 x 4 = \$400 → cost of laptop is \$400

Solution 4

4/6 of 1 whole, 6/6 is 4/6
4/6 of 1 whole, 3/3 is 2/3
-a laptop computer costs in the hundreds of dollars, so the whole, 1, is 100/100 (splitting partitions from 1/1 to 100/100)
→ 4/6 of the whole, 100/100, is equivalent to 400/600
→ cost of laptop is \$400 with \$600 to start

Partitioning into Equal Parts
Clustering Unit Fractions

Equivalent Fractions
Splitting Partitions from 6ths to 600ths

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria

Highlights/Summary

Determine fraction quantities using methods:

- ***counting unit fractions** – dividing whole (fraction, 6/6; and related quantity, \$600) into equal parts (unit fraction, 1/6 and 1/100) in order to count each unit to the non-unit fraction (4/6, 400/600)*
- ***clustering unit fractions** -non-unit fraction, like 4/6 is a cluster of 4 unit fractions of 1/6 each; so the corresponding quantity of a unit fraction is also clustered (e.g., 1/6 of \$600 is worth \$100, so 4/6 o \$600 is worth 4x\$100=\$400)*
- ***equivalent fractions** – the same quantity can be described using different fractional units by merging (fewer number and larger partitions → 4/6 to 2/3) or splitting partitions (greater number, small partitions → 4/6 to 400/600).*

AFTER (Practice) – 5 to 10 minutes

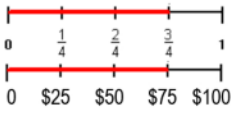
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Buying Shoes Problem

Jason spent 3/4 of his money on shoes. How much money did he have at first?

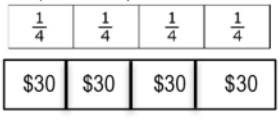
Solution 1

He had \$100 at first.
-\$100 divided by 4 is \$25
-Three ¼, unit fraction is ¼, 25, ¾ which is the equivalent to \$25, \$50, \$75.
-\$75 is ¾ of the money that he spent on shoes.



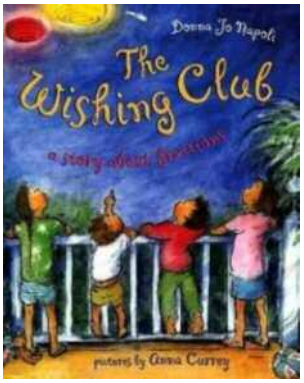
Solution 2

Choose \$120 to be the whole
-\$120 divided into 4 equal parts is \$30 for 1 part
-1 part, ¼ is \$30 = \$120÷4
-¾ means 3 parts of ¼ each
→ ¼ x 3 = ¼+ ¼ + ¼
→ \$30 x 3 = \$90 cost of shoes



Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO LITERATURE – THE WISHING CLUB: A STORY ABOUT FRACTIONS



Donna Jo Napoli tells a story about four children who make wishes on a star, but their wishes were only partially fulfilled. While Joey who is two years old gets half of the cookie he wishes for, Petey who is four, gets a fourth when he wishes for a dollar, and eight-year-old, Sally, and her twin Samantha each get 1/8 of the bag of marbles they wished for. Twin, Samantha recognizes a pattern. Next, they all work together to combine their wishes to receive the pet for which they have all been hoping.

- How could the children prove that they will receive one whole pet when they each wish for the same pet? Justify.
- What other combinations of ages in a family allow them together to make a wish and get a complete animal? Explain your thinking.

LET'S DO MATH – PATTERN BLOCK FRACTIONS

What are some fractions you can see in the pattern blocks?

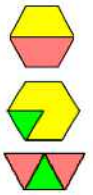
Show your work.

Solution 1

A trapezoid is $\frac{1}{2}$ of a hexagon

A triangle is $\frac{1}{6}$ of a hexagon

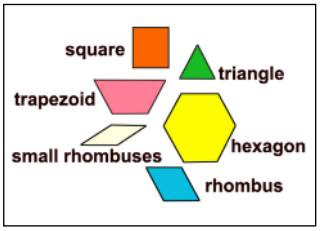
A triangle is $\frac{1}{3}$ of a trapezoid



Solution 2

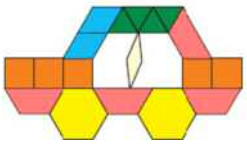
Two rhombuses is $\frac{4}{3}$ of a trapezoid

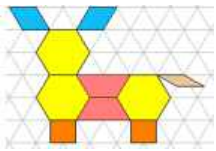


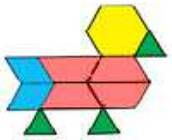


During (Working On It)

Which pattern block is used the most in each of the pattern block images of a car, dog and duck?







After (Consolidation)

Anticipating Student Responses:

Solution 1

Set Model Count (Part of Whole (car, dog and duck))

For all three pattern block images, 10 trapezoids are used altogether, which is greater than any other pattern block shape used.

Car -> 4 red trapezoids

Dog -> 2 red trapezoids

Duck -> 4 red trapezoids

Solution 2

Set Model Comparative (Part of a Whole (car, dog or duck))

Car -> 5 orange squares out of 18 pattern blocks or orange squares are $\frac{5}{18}$ pattern blocks

Dog -> 3 yellow hexagons out of 10 pattern blocks or yellow hexagons are $\frac{3}{10}$ pattern blocks

Duck -> 4 red trapezoids out of 10 pattern blocks or red trapezoids are $\frac{4}{10}$ pattern blocks

Solution 3

Area Model (Part of a Whole (car, dog or duck))

1 triangle covers an area of $\frac{1}{2}$ square unit; Area (2 triangles) -> 1 square unit; Area (1 square) -> 1 square unit.

Car -> total area is about $21\frac{1}{2}$ square units; 2 hexagons or 4 trapezoids is 6 square units ($\frac{6}{21}$)

Dog -> total area is about 16-$\frac{1}{2}$ square units; 3 hexagons is 9-$\frac{1}{2}$ square units ($\frac{9}{16}$)

Duck -> total area is about 12-1/2 square units; 4 trapezoids is 6 square units ($\frac{6}{12}$)

Coordinating Discussion for Student Learning:

Why might Solution 1 be chosen first for student discussion, followed by solution 2 and 3?

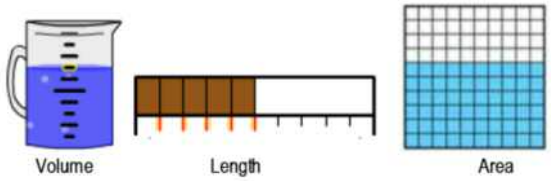
- *Solution 1* – counting the number of pattern blocks used to make the car, dog and duck altogether to determine which pattern block type is used the most altogether
- *Solution 2* – counting the number of pattern blocks used to make the car, dog and duck separately, to determine which pattern block type is used the most to make the car, dog and duck
- *Solution 3* – measuring the area (in square units) of each pattern block type in relation to the total area of the car, dog and duck.

LET'S DO MATH – COMPARING FRACTION MEASUREMENTS

Before (Getting Started)

Which measurement model shows the greatest measure of a whole?

The whole for each measurement is shown as the unit of 1.



Solution 1	*Solution 2*	*Solution 3*
The pitcher shows 7 parts filled. The length is about half shaded. The area is about half full. The pitcher shows the most.	The pitcher shows $\frac{7}{10}$ filled. The length shows $\frac{5}{5}$ shaded.	The pitcher shows $\frac{7}{10}$ filled. The length shows $\frac{5}{10}$ shaded. The area shows $\frac{60}{100} = \frac{6}{10}$ shaded. In order, volume, area, length.

During (Working On It)

Name each fraction image and identify the greatest and the least fractions.

After (Consolidation)

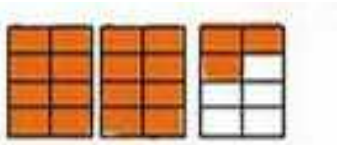


Image A

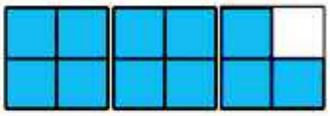


Image B



Image C

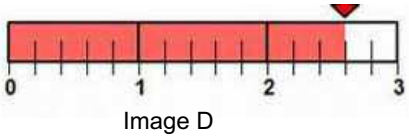


Image D

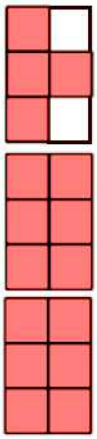


Image E

Anticipating Student Responses:

Solution 1

A	$\frac{19}{8} = 2\frac{3}{8}$
B	$\frac{11}{4} = 2\frac{3}{4}$
C	$\frac{5}{2} = 2\frac{1}{2}$
D	$\frac{13}{5} = 2\frac{3}{5}$
E	$\frac{16}{6} = 2\frac{4}{6}$

Solution 2

A is $2\frac{3}{8}$

B is $2\frac{3}{4}$

C is $2\frac{1}{2}$

D is $2\frac{3}{5}$

E is $2\frac{4}{6}$ or $2\frac{2}{3}$

Solution 3

A < C because the part in A is not half.
B, D, and E > half.
So A is the least fraction.
B > D because $\frac{3}{4} > \frac{3}{5}$ (same numerator)
B > D comparing $2\frac{3}{4}$ and $2\frac{2}{3}$ B is $\frac{1}{4}$ from 1 E is $\frac{1}{3}$ from 1
$\frac{1}{4} < \frac{1}{3}$ so B is larger
So B is the greatest fraction.

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solution 1, and 3?

- *Solution 2* - shows the mixed number for each image so we can use it to agree on the fractions we are dealing with. Our fractions are all between 2 and 3.
- *Solution 1* - shows the improper fractions which can be counted, in A like 1 one-eighths, 2 one-eighths, 3 one-eighths, …, 7 one-eighths, 8 one-eighths, 9 one-eighths, …,15 one-eighths, 16 one-eighths, 17 one-eighths, 18 one-eighths, 19 one-eighths. A mixed number is written beside each fraction.
- *Solution 3* - shows ordering and reasoning to identify the least and greatest fractions

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how fraction reasoning tasks (e.g., set, linear, area, volume, capacity) develop students' understanding and use of methods of representing and comparing fractions with like and unlike denominators.
- To suit your students' learning needs, select and design reasoning tasks that are relative to grade specific curriculum expectations in the Number Sense and Numeration.
- Practise noticing the different methods and the breadth of the mathematics your students use to solve fraction as part of a whole reasoning tasks.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit aspects of different methods for representing fractions as part of a whole using a set model or measurement model.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

- Battista, M. (2012). *Cognition-based assessment & teaching of fractions.* Portsmouth: Heinemann.
- Beyranevand, M. (2014). The different representations of rational numbers. *Mathematics Teaching in the Middle School, 19*(6), 382-385.
- Bruce, C., Chang, D., & Flynn, T. (2013). *Foundation to learning and teaching fractions: Addition and Subtraction.* Toronto: Ontario Ministry of Education.
- Empson, S., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals.* Portsmouth: Heinemann.
- Fennell, F., McCord Kobett, B., & Wray, J. (2014). Fractions are numbers, too! *Mathematics Teaching in the Middle School, 9*(8), 486-493.
- IES. (2010). *Developing effective fractions instruction for kindergarten through 8th grade.* U.S. Department of Education.
- Lamon, S. (2012). *Teaching fractions and ratios for understanding.* New York, Routledge.
- Mills College Lesson Study Group. (2011). *Developing Number sense for fradtions: An electronic resource guide for lesson study.* CA: ills College
- Norton, A., & McCloskey, A. (2008). Modeling students' mathematics using Steffe's fraction schemes. *Teaching Children Mathematics,* 48-54.
- Wilkins, J., Norton, A., & Boyce, S. 2013). Validating a written instrument for assessing students' fractions schemes and operations. *The Mathematics Educator, 22*(2), 31-54.
- Small, M. (2014). *Uncomplicating fractions to meet Common Core Standards in math, K-7.* New York: Teachers College.
- Watanabe, T. (2006). The teaching and learning of fractions: A Japanese perspective. *Teaching Children Mathematics*, 368-374.
- Watanabe, T. (2007). Initial treatment of fractions in Japanese textbooks. *On Learning Problems in Mathematics, 29*(2), 41-60.

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