8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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FRACTIONS AS A QUOTIENT AND UNIT FRACTIONS

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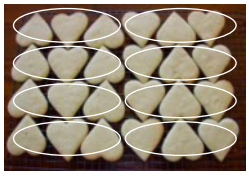


MARY LOU KESTELL works on K to 12 mathematics professional learning at the provincial level. She has spent 40+ years working in all aspects of Ontario mathematics education and is a past-president of both OAME and OMCA.



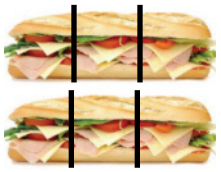
KATHY KUBOTA-ZARIVNIJ currently focuses on K to 6 mathematics professional learning at the provincial level. She is a long-time OAME board member and OAME past president. In relation to her work in mathematics education, she uses complexity thinking to interpret mathematics teaching and learning for students and teachers.

A fraction can represent the result obtained when two numbers are divided. This interpretation of fraction is sometimes referred to as the quotient meaning, since the quotient is the result of division.

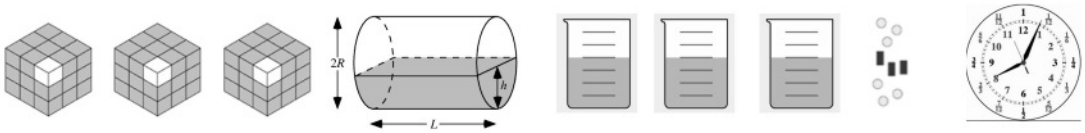


For example, the number of cookies each child receives when 24 cookies are shared among 8 children can be expressed as 24/8, 3/1 or 3, so that each child gets 3 cookies.

When 2 sandwiches are shared equally among 3 students, each students gets 2/3 of 1 sandwich for lunch. One sandwich is divided into 3 equal size parts, so 1 ÷ 3 = 1/3. And 2 sandwiches divided into 3 equal parts means 2/3.



Examine these different representations of fractions, as a quotient. Which representations show a fraction as a quotient? How do you know?



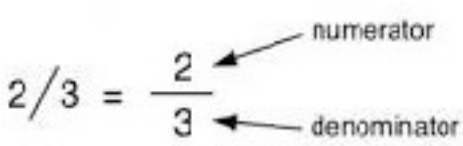
What are the ways that your students' understanding of fraction as a quotient can be developed?

In this issue, the Research Summary and reasoning tasks (with multiple solutions) focus on fractions, as a quotient and unit fraction contexts. Reasoning tasks in this issue are designed for use with a three-part problem-solving lesson.

RESEARCH SUMMARY – ABOUT FRACTIONS AS A QUOTIENT AND UNIT FRACTIONS

Fractions are expressed in the form:

- The top number, 2, is called the **numerator** and tells you how many pieces you have of the whole.
- The bottom number, 3, is called the **denominator** and tells you into how many equal pieces the whole was divided. It represents the **size** of the parts.



Division as Fractions

The fraction, 2/3, is considered the quotient that is the result of dividing the numerator by the denominator; that is 2 ÷ 3. More specifically, there are two models of division that can be used to connect whole number division to fractions.

- ***Partitive Division*** (fair sharing) focuses on portioning out equal amounts to each group, where the unknown (quotient) is the amount each group gets.

$$\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} \div \begin{pmatrix}\text{Number}\\ \text{of groups}\end{pmatrix} = \begin{pmatrix}\text{Amount}\\ \text{per group}\end{pmatrix}$$

I have 2 sandwiches and want to split them into 3 equal-size groups. How much of 1 sandwich will be in each group? (Answer: 2/3 of 1 sandwich)

- 1 sandwich divided into 3 parts, where 1 part is worth 1/3 of 1 sandwich;
- 2 sandwiches have 3 x 2 = 6 parts of 1/3 of 1 sandwich



- ***Quotative Division*** (grouping) focuses on making groups of a particular size, where the unknown (quotient) is the number of groups

$$\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} \div \begin{pmatrix}\text{Amount}\\ \text{per group}\end{pmatrix} = \begin{pmatrix}\text{Number}\\ \text{of groups}\end{pmatrix}$$

I have 2 sandwiches and want to make groups that contain 3 sandwiches. How many groups can I make? (Answer: 2/3 of 1 group)

- 1 group contains 3 sandwiches
- 1 sandwich is 1/3 of 1 group,
- 2 sandwiches is 2/3 of 1 group



Fractions as Division

Start with a visual model of a fraction and see division in the model.

- What is 2/3 of 24 cookies?
- Divide 24 cookies into 3 equal size groups. How many cookies in 2 groups?
  (24 ÷ 3) x 2 = 8 x 2 (3 groups, 8 cookies in each group, so 8 x 2 = 16 cookies in 2 groups)

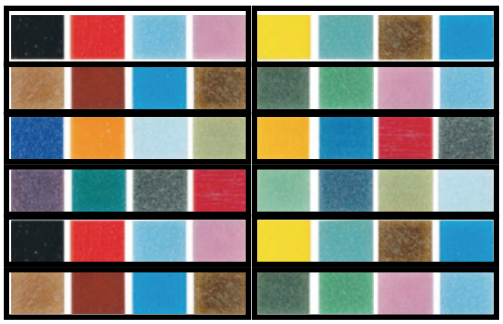


Unit Fractions as Division

Fractions can be considered as a multiple of a unit fraction. Using iteration to create a non-unit fraction is to say that a fraction may be considered a multiple of a unit fraction. For example, a non-unit fraction, 2/3 would be considered as 2 times the unit fraction, which is obtained by dividing 1 into 3 equal parts; that is, 1/3. Therefore, this fraction can be named as 2 one-third units. (e.g., 2/3 = 1/3 + 1/3 = 2 x 1/3)

LINKS TO MANIPULATIVES – COLOUR SQUARE TILES

Six packs of crayons (8 crayons per pack) are shared so that each friend gets 4 crayons.
How many friends receive 4 crayons?





6 packs x 8 crayons per 1 pack
6 rows x 8 columns = 48 crayons
48 crayons ¸ 4 crayons per 1 friend = 12 friends
48/4 = 48 ÷ 4 / 4 ÷ 4 = 12/1 = 12

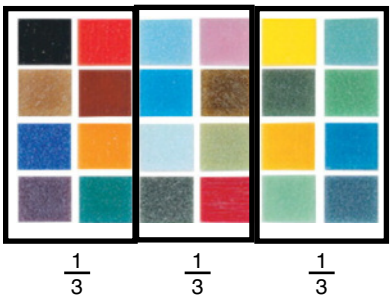
$$\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} \div \begin{pmatrix}\text{Amount}\\ \text{per group}\end{pmatrix} = \begin{pmatrix}\text{Number}\\ \text{of groups}\end{pmatrix}$$

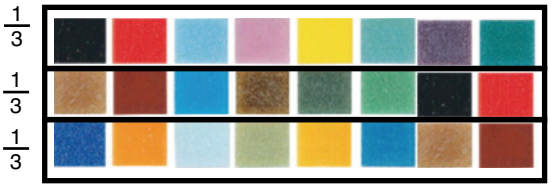
What is 2/3 of 24?

unit fraction of 2/3 is 1/3.
24 ÷ 3 = 8 (1/3 of 24 = 8)
2 x 1/3 = 2/3
2/3 of 24 = 2 x 8 = 16

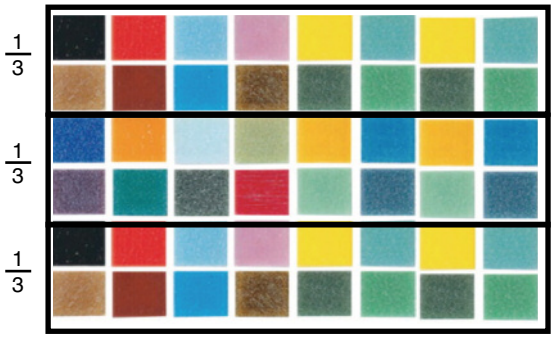
24 x 1/3 x 2 = 24/3 x 2
= 8 x 2 = 16

$$\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} \div \begin{pmatrix}\text{Number}\\ \text{of groups}\end{pmatrix} = \begin{pmatrix}\text{Amount}\\ \text{per group}\end{pmatrix}$$





2 square tiles is 1 group of tiles
3 rows of 8 groups of tiles
= 24 groups of tiles
2/3 of 24 is 2 rows of 8 = 16



LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem), and After (Practice) parts of the lesson

Grade 3	Grade 4	Grade 5
-divide whole objects and sets of objects into equal parts and identify the parts using fractional names without using numbers in standard fractional notation	-represent fractions using concrete materials, words, and standard fractional notation and explain the meaning of the denominator as the number of the fractional parts of a whole or a set and the numerator as the number of fractional parts being considered	-represent, compare and order fractional amounts with like denominators including proper and improper fractions and mixed numbers using standard fractional notation

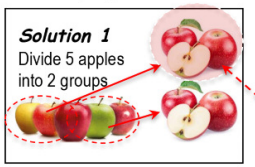
BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

What Fraction of the Whole?

5 apples are shared between 2 students equally.
How many apples did 1 student get? Explain.

Solution 1

Divide 5 apples into 2 groups



Partitive Division (Fair Sharing)
-whole (dividend) is 5 apples
-number of groups is 2
-amount in each group is $2\frac{1}{2}$

Solution 2

$5 \div 2 = (4+1) \div 2 = (4 \div 2) + (1 \div 2)$
$= 2 + 1 \div 2 = 2\frac{1}{2}$
$5 \div 2 = 5/2$
$5/2 = 4/2 + \frac{1}{2} = 2\frac{1}{2}$

Division as Fractions *(Quotient)*
$5 \div 2 = 5/2 = 2\frac{1}{2}$
quotient is $2\frac{1}{2}$

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

How Many Students?

In a recent class survey, two-thirds of the students responded, Yes. 20 students responded No.
How many students responded Yes?

What information are we going to use to solve this problem?
- *2/3 of total students say, Yes*
- *20 students say, No*
- *unknown – number of total students*
- *unknown – number of students who say, Yes*

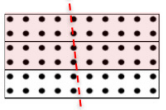
` AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Dividing whole (1) into equal parts
(1 ÷ 3 = 1/3)
(60 ÷ 3 = 20; so 1 group of 20 is 1 of the 3 groups of 60

The fraction unit is the count (1/3, 2/3, 3/3) which is proportional to unit fraction of the whole

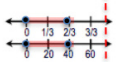
1/3 of 60 = 20, so counting the unit fraction (1/3) of the whole 60 is counting by 20s:
20 (1/3 of whole, 60)
40 (2/3 of whole, 60)
60 (3/3 of whole, 60)

Solution 1



-Thirds means divide the whole into 3 equal groups
-1 group is 20 (NO)
-2 groups of 20 is 40
-40 students say Yes

Solution 2



Counting 1/3, 2/3, 3/3 of total students is the same count as 20, 40, 60 students.

1/3 of students is 20
2/3 of the students, which responded, yes is 40 students.

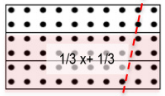
Grouping *Parts of a Whole*

Counting *a Unit Fraction*

multiplying non-unit fraction by the number of iterations (e.g., 2)
(2 x 1/3) of whole = 2 x (1/3 of whole)
= 2 x 20 = 40

Adding the unit fraction again and again to sum to a non-unit fraction ((1/3+1/3) of whole = 2/3 of whole)
= (1/3 of the whole) + (1/3 of the whole)
= 20 + 20

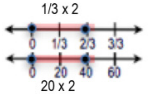
Solution 3



1/3 of the whole is 20
2/3 = 1/3 + 1/3
2/3 of the whole, 60
= (1/3 + 1/3) of the whole
= 20 + 20 = 40

40 students say Yes.

Solution 4



1/3 of the whole is 20
2/3 = 2 x 1/3
2/3 of the whole, 60
= (2 x 1/3) of the whole
= 2 x (1/3 of the whole)
= 2 x 20 = 40
40 students say Yes.

Iterating *a Unit Fraction*

Multiplying *a Unit Fraction*

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-

Highlights/Summary

Represent fractions by:
- *grouping parts of a whole – dividing whole into equal parts (1/3 = 1÷3)*
- *counting unit fraction – the fraction unit is the count (1/3, 2/3, 3/3) which is proportional to unit fraction of the whole*
- *iterating a unit fraction – adding the unit fraction again and again to sum to a non-unit fraction ((1/3+1/3) of whole = 2/3 of whole)*
- *multiplying a unit fraction – multiplying unit fraction by the number of iterations (2 x 1/3) of whole = 2 x (1/3 of whole)*

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

How Many Students?

In a recent class survey, three-fourths of the students responded, Yes.
20 students responded No.
How many students responded Yes?

Solution 1

¼ of the class is 20
¾ = 3 x (1/4 of class)
= 3 x 20
= 60 students

Solution 2

If ¼ is 20, then 4/4 (whole class) is 20x4=80

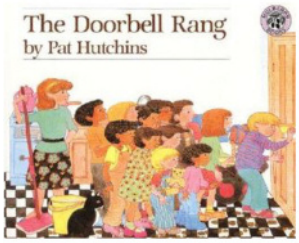
Total students – 20 students (No) = students (yes)

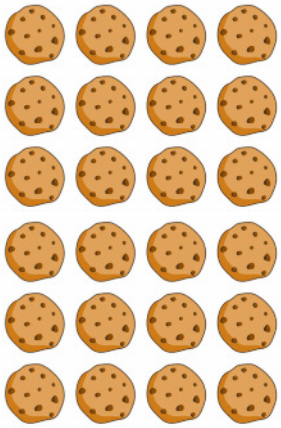
80-20=60 students (yes)

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO LITERATURE:





Ma makes a batch of a dozen chocolate chip cookies for her children Victoria and Sam. Each child has six cookies. But when the doorbell rings and friends drop by, the cookies must be shared. The doorbell rings again, and again, and again...until there is only one cookie for each child. How are the cookies being divided?

Suppose that you had this number of cookies, such as 12, 16, 24 or 32

- Show as many different ways as you can to share the cookies equally. Draw a picture and write an equation for each solution.
- If 1/4 of the total cookies (12, 16, 24 or 32) were eaten by 2 people, how many cookies did each person eat?

When Peter and his little brother arrive, there are now 6 children sharing the batch of cookies. Draw a picture to show that they each have 1/6 of the cookies.

If 2/3 of 24 cookies are eaten by 4 children, how many cookies did each child eat?

LET'S DO MATH –

Before (Getting Started)

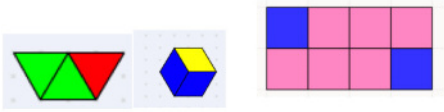
Model 3 different fractions with a numerator of 2. Compare your models.



Solution 1

I show $\frac{2}{6}, \frac{2}{5}$ and $\frac{2}{9}$.

The number of pieces the wholes are divided is represented in the denominator and in each 2 are coloured differently and the 2s appear in the numerator.



Solution 2

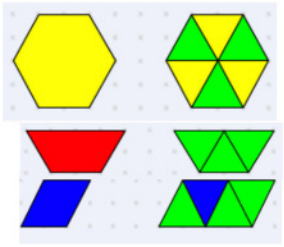
My figures show $\frac{2}{3}, \frac{2}{3}$ and $\frac{2}{10}$.

The first two are divided into 3 parts and I shaded 2 parts differently. The third image shows 2 of 10 squares shaded blue.

During (Working On It) – Pattern Block Area Problem

Examine these figures. The shapes on the left shows the whole and the shapes on the right show the whole divided into equal size parts.

What fractions are represented on the right side?



After (Consolidation)

Anticipating Student Responses:

Solution 1

The trapezoid is divided into 3 parts (3 triangles) so # parts is 3 (denominator).

3 triangles cover trapezoid, so the numerator is 3 and the fraction is $\frac{3}{3}$.

3 ÷ 3 = 1 whole so the whole trapezoid the is covered.

Solution 2

The hexagon is divided into 6 parts (6 triangles) so # parts is 6 (denominator).

3 triangles cover 1 hexagon (3 is numerator) the fraction is $\frac{3}{6}$.

3 ÷ 6 is 0.5 or $\frac{1}{2}$ so half the hexagon is covered.

Solution 3

The rhombus is divided into 2 parts (2 triangles), so # parts is 2 (denominator).

3 triangles cover it so the numerator is 3 and the fraction is $\frac{3}{2}$.

3 divided by 2 is 1.5 or $1\frac{1}{2}$ so the one and a half trapezoids are covered.

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion, followed by solution 2 and 3?

- *Solution 1* – 1 whole is covered
- *Solution 2* – $\frac{1}{2}$ of the whole is covered
- *Solution 3* – $1\frac{1}{2}$ of the whole is covered

LET'S DO MATH –

Before (Getting Started)

There are 12 packages of juice. 8 juice boxes are in 1 package.

¾ of the juice boxes are set aside for lunch.

How many juice boxes are left over?

Solution 1

12 packages x 8 juice boxes = 96 juice boxes

$\frac{3}{4}$ of 96 juice boxes means

(96 ÷ 4) x 3 = 24 x 3 = 72

72 juice boxes

Solution 2

$\frac{4}{4}$ of 12 = 3+3+3+3 packages

$\frac{3}{4}$ of 12 = 3+3+3 = 9 packages

9 packages x 8 juice boxes = 72 juice boxes

Solution 3

12 ÷ 4 = $\frac{12}{4}$ = $\frac{3}{1}$ (unit fraction)

$\frac{3}{4}$ of 12 = 3 x $\frac{1}{4}$ of 12

= 3 x $\frac{12}{4}$

3 x 3 = 9 packages of juice

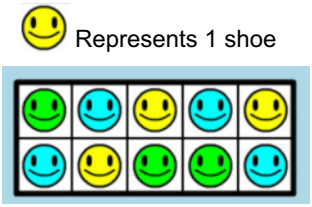
9 packages x 8 juice boxes = 72 juice boxes

During (Working On It) – Shoe Problem

Students are sending shoes are packed in boxes with 5 pairs of shoes in 1 box.

(a) How many boxes will be shipped if 84 shoes were gathered?

(b) If 5 boxes are shipped in 1 container, how many containers are needed for 84 shoes?



After (Consolidation)

Anticipating Student Responses:

Solution 1

I can count the fractions of a box like this one $\frac{1}{10}$, two $\frac{1}{10}$, three $\frac{1}{10}$ and so on to eighty $\frac{1}{10}$, which is 8 full boxes. Then there are 4 more $\frac{1}{10}$ s.

They will ship 8 and $\frac{4}{10}$ boxes.

This is the same as 8.4 boxes.

Solution 2

Think of the shoes in groups of 5.

If there are 84 shoes and 5 shoes in a half box, then $\frac{84}{5}$ represents the number of half boxes.

$\frac{84}{5}$ = 16.8 half boxes

2 half boxes = 1 whole box

16.8 half boxes = 8.4 whole boxes

Solution 3

There are 5 boxes to a container, 10 shoes per box, and that means 50 shoes per container.

Each shoe is $\frac{1}{50}$ of a container. With 84 shoes to ship, we could think of $\frac{84}{50}$ as the quotient we need to determine the number of containers.

$\frac{84}{50}$ = 1.68 containers.

This is about $1\frac{2}{3}$ of a container. 16 more shoes would make 2 containers.

Coordinating Discussion for Student Learning:

Why might solution 2 be chosen first for student discussion followed by solution 1, and 3?

- *Solution 1* – unit fraction counting
- *Solution 2* – unitize by fives to determine the number of half boxes and change to whole boxes
- *Solution 3* – fraction as a quotient (84 shoes to be packed in containers of 50 shoes)

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how fraction as division and unit fraction reasoning tasks develop students' understanding of the relationship between fractions and dividing a whole into equal size parts or groups.
- To suit your students' learning needs, select and design reasoning tasks that are relative to grade specific curriculum expectations in the Number Sense and Numeration strand (fractions as division, unit fractions), being mindful of the relationship between the concept of fractions as dividing a whole into equal parts and the symbolic notation showing equivalence between a fraction and a whole number division expression.
- Incorporate 5 to 10 minutes of daily practice of determining the equivalence between whole number division expression and a fraction, ongoing practice, two times a day.
- Practise noticing the different methods and the breadth of the mathematics your students use to solve fractions as division and unit fraction tasks, such as, determining the unit fraction, dividing a whole into equal size parts; interpreting division as being partitive and quotative in relation to its equivalent fraction.
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit key concept, fractions as division and unit fractions.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

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