8 PAGE
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ELEMENTARY
MATH

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MATHEMATICS
EDUCATION

Abacus

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COMPARISONS IN GEOMETRY AND SPATIAL SENSE

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ABACUS CO-EDITORS



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What are the least number of bottle-moves to make, so that no same colour bottles are side by side? Visualize only one glass being moved one position at a time.



start end

- What is the purpose of visualizing the "end" result first? *To determine how many bottle-moves, it is important to know what the end result could look like.*
- What ways could the bottles be moved? *Imagine the sequence of each bottle move and the resulting position of the adjacent bottles.*
- How does comparing the "start" position and the "end" position of the bottles help to determine the least number of moves? *Visualize one bottle at its start and its end positions and estimate the number of moves to get to the "end" position.*

In this issue, the Research Summary and contextual problems (solved with different methods and models of representation) focus on comparisons of two-dimensional shapes and three-dimensional figures using geometric attributes, properties, within the context of spatial visualization and reasoning. The problems and reasoning tasks in this issue are designed for use within a three-part problem-solving lesson and as ten-minute mini-lessons.

RESEARCH SUMMARY – COMPARISONS IN GEOMETRY AND SPATIAL SENSE

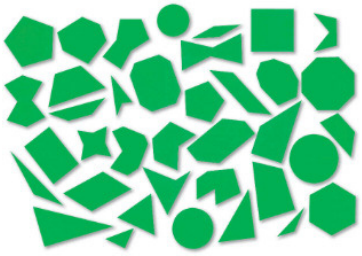
In the general sense, comparison is the act of examining objects to determine if things (e.g., objects, phenomena, ideas) are similar or different. To compare something is to look for similarities among two or more elements. If there are no similarities, then one would conclude there are differences. However, to contrast something is to look for the differences among two or more elements.

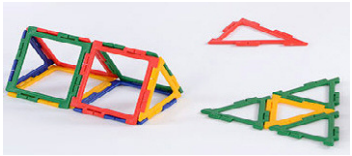
This notion of comparison within geometric and spatial contexts is explicitly identified in the developmental progression research of Pierre van Hiele and Dina van Hiele-Geldof. The van Hiele theory of geometric thought (1984) is a five-level hierarchy of understanding about spatial ideas, in terms of thinking processes used in geometric contexts. More specifically, applicable to elementary mathematics are these Van Hiele levels described in these ways:

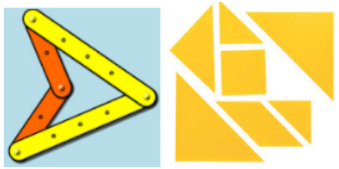
- how we think (i.e., visualization, analysis (Why? What if?), informal deduction)
- types of geometric ideas we think about (objects of thought) – identification of shapes, classes of shapes, properties of shapes, relationships between shapes
- what learners can do (products of thought) –classes or groupings of shapes that seem to be "alike", relationships between properties of geometric shapes

Some typical learning activities to develop these geometric thinking processes and products of thought include:

- sorting and classifying shapes by their geometric attributes using materials like power polygons
- composing and decomposing shapes using materials like pattern blocks, regular and irregular polygons, and tangrams
- constructing two-dimensional shapes and three-dimensional figures using materials (e.g., GeolegsTM, geometry tools kit, GeostripsTM, polydronTM frames, plastic straws, and pipe cleaners
- relating nets to three-dimensional figures
- categorizing two-dimensional shapes and three-dimensional figures by their geometric properties







Problem 1: How could these triangles be categorized? What would the categories be called? Which geometric properties are used for each category?



Number and Type of Interior Angles - acute, right, or obtuse angles

Side Lengths - equilateral, isosceles, or scalene

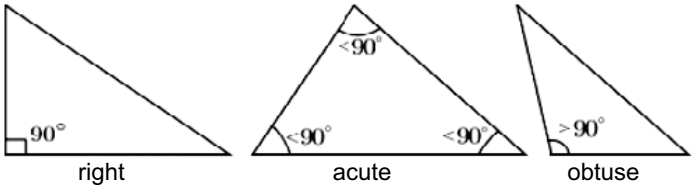
Sum of the 3 interior angles

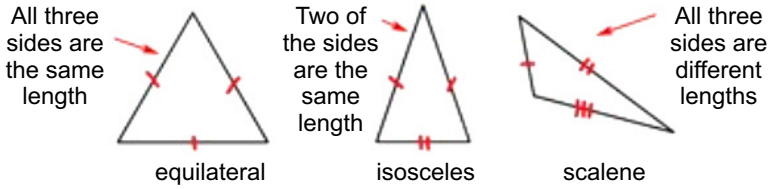
Reflect on your geometric thinking process. You made comparisons of each triangle according to the geometric attributes for each triangle.

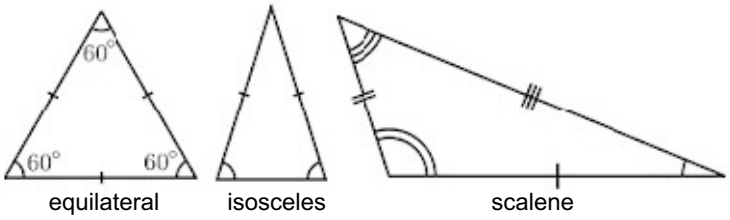
The size or angular measure of each interior angle in a triangle can be compared to a referent angle, a 90 degree angle. The angle measure is identified as being *greater than, equal to, or less than 90 degrees.*

The side lengths of each side within 1 triangle is not compared to a referent, but is instead grouped as being congruent (same side length) or not congruent to one or more other triangles.

In fact, the comparison of these different types of triangles could include the identification of both types of geometric attributes (size of angle and side length). For example, the equilateral triangle has three congruent interior angles and three congruent side lengths.







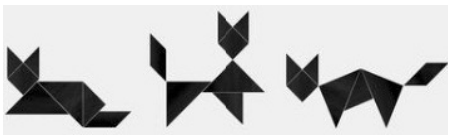
LINKS TO MANIPULATIVES – TANGRAMS



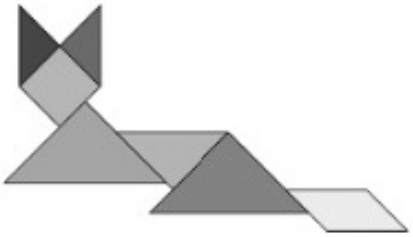
A set of tangrams has been typically used to compose a two-dimensional shape from a given outline or image.

Make different tangram cats using one tangram set.

- Compare the different tangram cats by the relative locations of each tan within the tangram set.
- Use measurement attributes (e.g., area) of the ears, body, and tail of each tangram cat for comparison.



Make the same tangram cat, but rotate, reflect, or translate 2 or 3 tans, to show the tangram cat in a different body position, such as bending down, walking, or laying down.



Discuss what comparison attributes were used to determine if two tangram cats are in a different body position or if the tangram cats are actually different because their body parts are different sizes.

LESSON LEARNING GOALS
Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem) and After (Practice) parts of the lesson

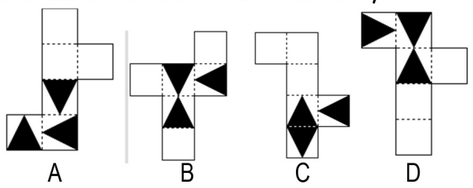
Grade 3	Grade 4	Grade 5
construct rectangular prisms using given paper nets and describe geometric properties (i.e., faces, edges, vertices) of the prisms	draw and describe nets of rectangular and triangular prisms; construct prisms and pyramids from given nets	construct nets of prisms and pyramids, using a variety of tools

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Packaging Problem 1

Unique Packaging Company designs custom cardboard box graphics. Which net, A, B, C or D matches the box shown? Why?





Solution 1* – *Elimination

It can't be A or C as the triangle points must meet. Not C because the triangles are not side by side. Fold D -the triangles are side by side.

Solution 2* – *Multiple Criteria Comparison

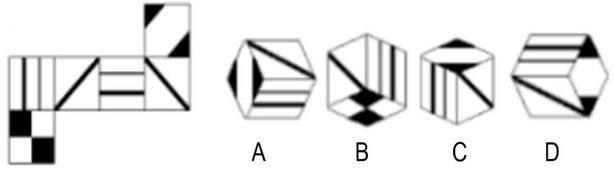
2 triangles must be a translation from face to face and 2 triangles must be a reflection around the same point on the triangles.

	Reflection	Translation
A	no	no
B	yes	no (rotation)
C	no	no
D	yes	yes

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Packaging Problem 2

Unique Packaging Company produced a new box pattern in the form of a net. Which cardboard box matches? Why?



` AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

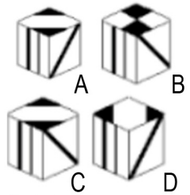
Use the paper model as a concrete model to rotate and make direct comparisons.

Imagine folding the cube net and notice the relative locations of the 6 different designs on 6 faces of the cube.

Direct comparison

Solution 1

Compare the boxes by first turning them so that they are drawn in the same orientation



Rotate paper model to match 3 design sides and their relative locations.

→ Box D matches net.

Rotate Paper Model to See Matching Design by 3 Sides

Indirect comparison

Solution 2

"Pointing to the right" diagonal line meets the vertex on white square and triangle, so eliminate box A.

"Pointing to left" diagonal line does not meet vertex on triangle, meets vertex on black square, so eliminate C.

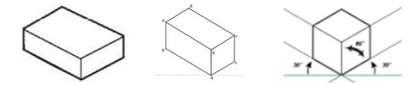
Pointing to left diagonal line and vertical lines are adjacent, so not B.

→ Box D matches net.

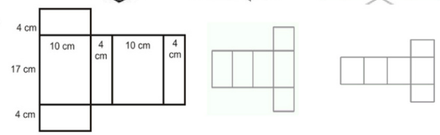
Use 1 Criteria at a Time to Eliminate Box Choices

Properties	Rectangular Prism	Cube
	6 rectangular faces 3 sets of 2 opposite congruent faces	6 square faces 6 congruent

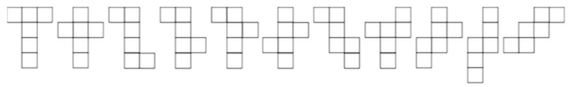
Isometric View



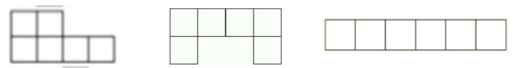
Rectangular Prism Nets



Cube Nets



Some nets will not fold into a cube. Why?
Faces overlap when folded; faces need 1 fold line only.



Indirect comparison

Solution 3



Referent position to name attributes of cube faces in relation to opposite faces (squares, triangles).

(i) "Diagonal to right" meets vertex of triangle and vertex of the white square.
(ii)"Diagonal to left" does not meet vertex of triangle and meets vertex of black square
(iii) horizontal lines adjacent to triangles and diagonal to right.

	(i)	(ii)	(iii)
A	no	?	yes
B	?	yes	no
C	no	?	no
D	yes	yes	yes

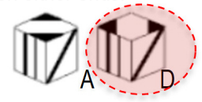
Use Criteria Chart to Make Comparisons Simultaneously

Indirect comparison

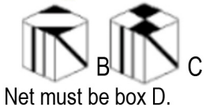
Solution 4



Vertical lines adjacent to diagonal to the right, triangle on either ends.



Diagonal to right meets vertex of triangle.



Net must be box D.

Group By Referent Position and Use Criteria to Compare

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria

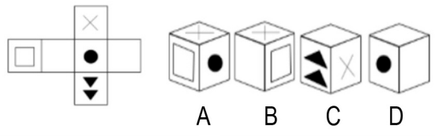
Highlights/Summary

Describe comparisons of rectangular prism nets to rectangular prisms using relative locations of rectangular faces and fold lines by:
- Direct comparison of a folded paper model of the net to diagrams of the 3D figure
- Indirect comparison (visualization) using 1 criterion at a time to eliminate choices; multiple criteria for simultaneous comparisons; grouping by a referent

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Packaging Problem 3

Which box design matches the net?



Solution 1

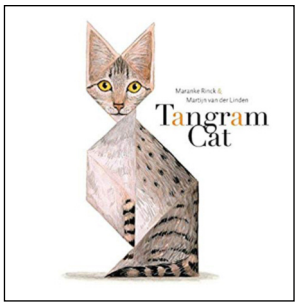
Circle is opposite square -> not A
X is opposite triangles -> not C
Circle is adjacent to x and triangles -> not D
->By process of elimination, the net matches box B

Solution 2

i)Circle adjacent to X and triangle
ii) square opposite circle
iii) square adjacent to X and triangles
-Not A, C, D
-B meets all criteria

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.



LINKS TO LITERATURE *TANGRAM CAT* BY MARANKE RINCK AND MARTIJN VAN DER LINDEN

A boy makes a cat, using a tangram set. He imagines that the cat needs a playmate. The boy aims to make another tangram cat, but it turns out to be a dog, which chases after the cat. He tries again, but his shape turns out to be a crocodile. After he makes several more tangram animals, the boy makes himself, a tangram boy, so they can play together.

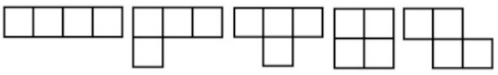
- As the teacher reads the story, have the students compare the location (e.g., beside, to the right of, under) and transformations (i.e., translation, rotation, reflection) of each tan that makes up the tangram cat and animals that the boy made. Record their comparison methods and explanations. Have the original tangram cat visible for comparison reference.
- Use GeoGebraTM at **https://www.geogebra.org/m/zrGERP88** to make tangrams for whole class discussion. Take a screen shot to capture each tangram animal made.

LET'S DO MATH – POLYOMINOES REVISTED

Before (Getting Started) – *Tetrominoes Problem 1*

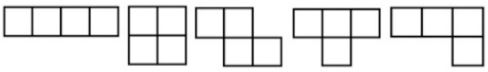
A tetromino is composed of 4 squares that are joined at its sides only. What different 2D shapes can be made? How do you know you have created all of them?

Solution 1 – (4, 3, 2) Squares in a Row



Row of 4 squares
Row of 3 squares on top; changing position of 4th tile
Row of 2 squares on top; 2 rows x 2 columns, 2 rows & 1 column translated;
Row of 1 square on top; 4 rows of 1 square is the same as 1 row of 4 squares

Solution 2 – Rectangle, Square, Octagon



Rectangle → 1 row x 4 columns

Square → 2 rows x 2 columns

Octagon → z, t, l tetrominoes

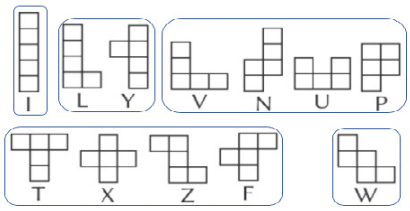
During (Working On It) – *Pentominoes Problem 2*

A pentomino is composed of 5 squares that are joined at its sides. What different 2D shapes can be made? How do you know you have created all of them?

After (Consolidation)

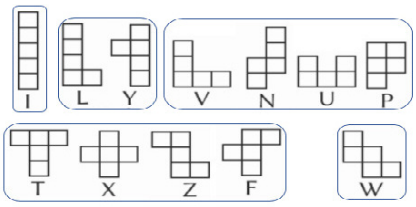
Anticipating Student Responses:

Solution 1 – Grouping Similar Shapes



Grouped by number of square tiles in a row and position of remaining tiles on the same side or different side of the row of square tiles.

Solution 2 – Sequenced Variation



Shapes are created by the greatest to least number of squares in a row (5, 4, 3, 2); translate remaining square tiles on the same side, then opposite sides.

Coordinating Discussion for Student Learning:

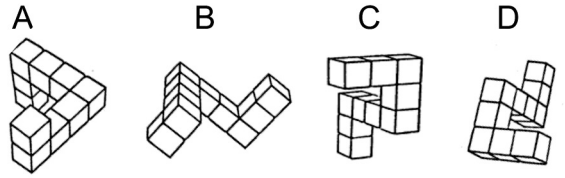
Why might solution 1 be chosen first for student discussion followed by solution 2?

- *Solution 1* – grouping by shapes with same number of squares in a row
- *Solution 2* – sequence the variation of the pentomino by the number of squares in a column (e.g., 5, 4, 3 and 2 squares) and the position of remaining 2 tiles, then 3 tiles

LET'S DO MATH – CUBES

Before (Getting Started) – Cubes Problem 1

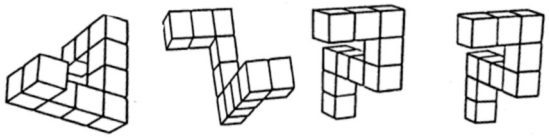
Which composite cube figures match? Why?



Solution 1 – Counting Cube

Total # cubes are the same
→ A (10); B (10); C (10), D (10)

Cubes Vertical (V) and Horizontal (H)
→ A (3V,7H); B (3V,7H); C (3V,7H); D (3V,7H)



Solution 2 – Referent Positioning

Orient all figures to the same position
– 3 horizontal cubes at the top, then compare the the number and orientation of the remaining horizontal and vertical cubes:
A and B match → 3H, 2V, 4H outwards front, 1H
C and D match → 3H,1V, 4H outwards back, 2V

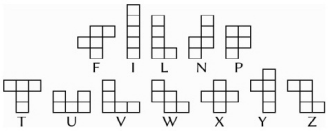
During (Working On It) – *Cubes Problem 2*

A pentacube is composed of 5 cubes that are joined at its faces. What different 3D figures can be made? How are the figures similar and different?

After (Consolidation) Anticipating Student Responses:

Solution 1 – Relating to Pentominoes

1 row of cubes -12 pentominoes

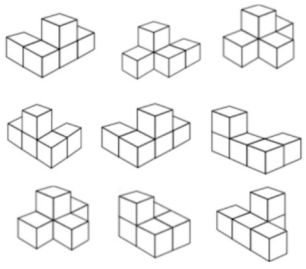


2 rows of cubes with at least 1 cube attached in 2 directions (attached to side of row and on top of a row)



Solution 2 – Grouping

2 rows with 1 cube on second layer and, their reflections if possible



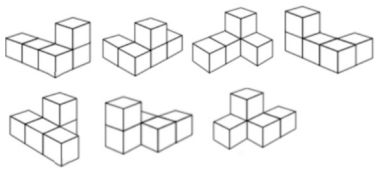
2 rows of cubes with 2 cubes in the 2nd row and their reflection



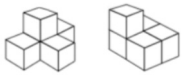
Solution 3 – Sequenced Variation by Position

1 row of cubes that are like the 12 pentominoes

2 rows of cubes with different bases (e.g., L-shape, T-shape, Z-shape)
Move 1 cube to make figures with 2 layers and change its position



No reflection



Increase the number and position (in a row or reflection) of cubes on the second layer



Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first, then solutions 2 and 3 for student discussion?

- *Solution 1* – uses the 2D shapes of pentominoes to represent 3D figures
- Solution 2 – groups pentacubes according to the number of rows or layers of cubes (i.e., 1 row, 2 rows, 3 rows)
- Solution 3 – sequence pentacubes using the base of 2D pentominoes for layer 1 (e.g., L-shape, T-Shape, Z-shape) and removing 1 cube from the 1st layer and varying its position on the 2nd layer, then 2 cubes from the1st layer and varying its position on the 2nd layer and so on.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how geometry comparison problems develop students' understanding of the relationship between geometric attributes and properties of two-dimensional shapes and three-dimensional figures within spatial visualization contexts.
- To suit your students' learning needs, select and/or design geometry and spatial sense problems or reasoning tasks that are based on grade specific curriculum expectations in the Geometry Spatial Sense. In order to design developmentally appropriate and challenge learning tasks, consult mathematics learning trajectories for geometry and spatial sense concepts, such as the Ontario geometry and spatial sense curriculum expectations across the grade and published mathematics learning trajectories (e.g., Daro et al, 2011; Clements and Sarama (2014), Maloney et al (2014)).
- Design geometric spatial visualization problems and reasoning tasks that develop students' experience with different applications in real life situations, such as descriptions of objects' relative location in the classroom, recording dance steps using transformation terms, creating and designing 2D and 3D puzzles. Search **www.geoogebra.org** for spatial visualization puzzles.
- Practise noticing the different comparison methods (grouping by similar geometric attributes and properties; selecting a referent two-dimension shape or three-dimensional figure for comparison; identifying the sequence of variation to describe a set of comparisons) and using related mathematical language that your students and you use to explain the comparisons (position - above, below, to the right of; congruent 2D shapes with a quarter turn clockwise; reflections of three-dimensional figures). Consider the mathematical reasons why your students choose to use or not these different comparison methods to solve these geometry and spatial sense problems. Is the mathematics inherent in the problems related to previously experienced and/or learned mathematics or is it not within their realm of experience?
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit the details of different comparison methods for geometric attributes and properties, location and transformation of shapes and objects.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

Clements, D., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories.* NY: Routledge.

Daro, P., Mosher, F.A, Corcoran, T. (2011). *Learning trajectories in Mathematics: A foundation for standards, curriculum, assessment, and instruction.*

Maloney, A.P., Confrey, J., Ngyuten, K.H. (Eds). (2014). *Learning over time: Learning trajectories in mathematics education. Information Age Publishing Inc.*

Ontario Ministry of Education. (2005). *Ontario Curriculum, Mathematics, Grades 1 to 8. Toronto:* Queen's Printer for Ontario.

Pittalis, M., & Christou, C. (2010). Types of reasoning in 3D geometry thinking and their relation with spatial ability. *Educational Studies in Mathematics, 75*(2), 191-212.

Van Hiele, P. M. (1984). *A child's thought and geometry.* In Geddes, D., Fuys, D. and Tischler, R. (Eds.). English Translation of Selected Writings of Dina van Hiele-Geldof and Pierre M. van Hiele, Washington, D.C., National Science Foundation.

Vandenberg, S.G & Ruse, A.R. (1978). A group test of three-dimensional spatial visualization. *Perceptual and Motor Skills, 47*(2), 599-604.

Wilson, P.H., Mojica, G.F., & Confrey, J. (2013). Learning trajectories in teacher education: supporting teachers' understandings of students' mathematical thinking. *The Journal of Mathematical Behavior, 32*, 103-121.

CLOSING

Our next issue series will focus on absolute and relative comparisons in number. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a three-part problem-solving lesson design framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or maryloukestell@gmail.com