8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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A Publication of the OAME/AOEM

COMPARISONS IN PATTERNING AND ALGEBRA AND DATA MANAGEMENT

ABACUS CO-EDITORS

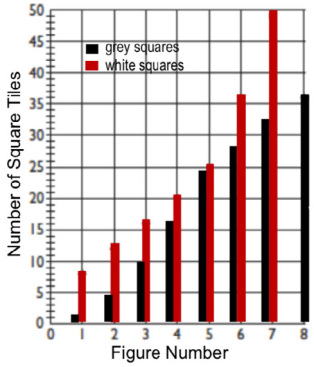


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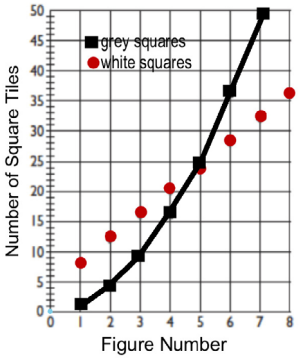
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We often see bar graphs and line graphs with contextual detail identifying the meaning of the data. In this example, the number of square tiles and figure number are given as labels. These two graphs have no title and contextual detail about data. These graphs could represent the same data or very different data. What other details could be used to determine how these graphs are similar and different?



Consider these ideas.

- scale units
- shape of data in the graph
- data values in a table of values
- purpose of using a bar graph and a line graph
- meaning of the grey and white square tiles
- reason for plotting two sets of data on the same graph
- meaning of the figure number and the number of square tiles and how they are related in the table of values and graphs.

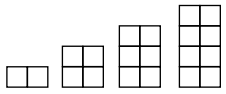


In this issue, the Research Summary and contextual problems (solved with different methods and models of representation) focus on absolute and relative comparisons of growing number patterns and data represented in a table of values, bar graph and/or line graph. These problems and reasoning tasks are designed to make connections between mathematical concepts in patterning and algebra and data management within a three-part problem-solving lesson structure.

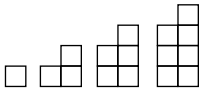
RESEARCH SUMMARY – COMPARISONS IN PATTERNING AND ALGEBRA AND DATA MANAGEMENT

Comparing Growing Number Patterns

Pictorial growth patterns, shown with concrete materials or diagrams of pattern blocks or square tiles are common ways for students to explore growing number patterns. Students tend to look for relationships from one figure to the next regarding what is staying the same and what is changing. "If students only think about the change from one position of the pattern to the next, they will have difficulty accurately predicting values for terms far down the sequence … If we focus our instruction on the relationship between two sets of values, rather than on "what comes next," growing patterns become powerful tools for exploring algebraic relationships for all levels of learners. Patterns offer a way for students to think about the functional relationships between two sets of data. In math, a function is a consistent relationship between two sets of data." (Beatty, 2014, p. 20)



For a function, each element of one set is associated with a unique element in the other set, and a function rule describes how the elements in one set are associated with the elements in the other set. In fact, there is a unique number of square tiles associated with each figure number. The figure number illustrates the relationship between one set of data (figure number), and another set of data (the number of square tiles at each figure number).



In this example, the comparison of odd and even numbers is represented using square tiles. The arithmetic sequence for odd and even numbers share the same common difference of 1, however the first term is different (1 for odd numbers; 2 for even numbers). Such explicit or functional reasoning is used by students to determine the number of square tiles for any figure number.

Therefore, as students learn to describe patterns using a pattern rule (recursive or explicit), they are developing ways to think about and use generalizations about larger figures and any figures in the pattern and function.

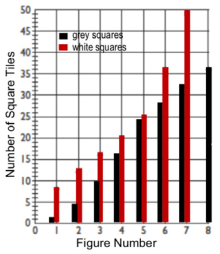
Comparing Data in Tables of Values and Graphs

Figure	Data A	Data B
1	1	8
2	4	12
3	9	16
4	16	20
5	25	24
6	36	28

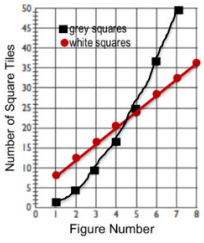
These sorts of number pattern representations provides ways for the students to make connections between concrete models, drawn models, pattern rules, number and algebraic expressions and equations, in relation to a table of values and graphs, especially bar graphs, line graphs and scatter plots.

When growing number patterns data are organized a **table of values**, the exact data values are displayed numerically. But, tables do not show visual comparisons. In fact, tables take longer to read and understand data relationships and examine overall trends and data comparisons

If you were asked to create a graph from a given set of data, how would you know which type of graph to use? The type of graph chosen depends on the type of data given and the kinds of data interpretations that can be convey to the reader. Bar graphs and line graphs are commonly used to display data.



- **Bar graphs** are used for comparing quantities in different categories or groups or comparing items over time and track changes. Bar graphs help us to see relationships quickly. However, bar graphs can be difficult to read accurately. A change in the scale in a bar graph may alter one's visual perception of the data. Therefore, when trying to measure change over time, bar graphs are best when the changes are larger.



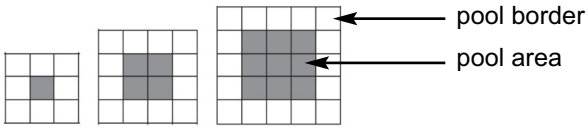
- **Line graphs** are used to display data or information that changes continuously over time. Line graphs allow us to see overall trends, such as an increase or decrease in data over time. Also, it shows how one variable relates to one or numerous different variables. A line graph reveals trends or progress over time and can be used to show many different categories of data. Line graphs are used to track changes over short and long periods of time. When smaller changes exist, line graphs are better to use than bar graphs.

LINKS TO MANIPULATIVES - SQUARE TILES - MULTIPLE USES FOR REPRESENTATION

Square tiles are often used to show growing number patterns. It is also used commonly as a concrete material to show the collection data and the creation of pictographs, line plots and bar graphs. However, square tiles are especially useful in representing different data values and their number relationships in concrete and real life contexts.

See below the different context in which square tiles are used. The table of values and the diagrams of the square tiles show the same number relationships with different contexts: area, geometric growing patterns and cost of picture framing and mounting.

Area is shown in square units.						
Figure #	1	2	3	4	5	6
Area of Pool	1	4	9	16	25	36
Area of Pool Border	8	12	16	20	24	28



Pool 1 Pool 2 Pool 3

Geometric patterns are shown as number of squares						
Figure #	1	2	3	4	5	6
Geometric Pattern 1	1	4	9	16	25	36
Geometric Pattern 2	8	12	16	20	24	28

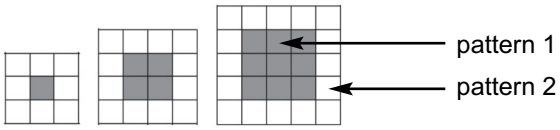
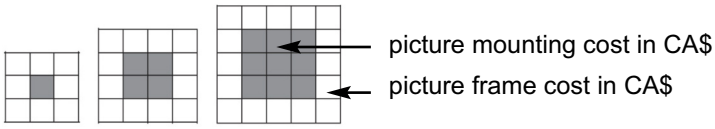


Figure 1 Figure 2 Figure 3

Picture and frame cost is shown in CA$						
Figure #	1	2	3	4	5	6
Mounting Picture Cost	1	4	9	16	25	36
Picture Frame Cost	8	12	16	20	24	28



Frame 1 Frame 2 Frame 3

LESSON LEARNING GOALS

Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem) and After (Practice) parts of the lesson

BEFORE (Getting Started) – 5 to 10 minutes

- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Square Tiles Problem 1

Term 1 is 5, for the sequence 5, 9, 13, 17, …
How many square tiles for the 10th term?

Solution 1 *– Table of Values*

Since term 1 is 5 and the next term increases by a unit of 4, for each term number, from 0 to 10.

Term	0	1	2	3	4	5	6	7	8	9	10
Tiles	1	5	9	13	17	21	25	29	33	37	41

Solution 2 *- Pattern Rule*

If term 1 is 5 and the pattern increases by a unit of 4, then term 0, is 5-4=1,. Pattern Rule is start at 1, increase by a unit of 4.
1 + (4x10) = 1+40 = 41 (for 10th term)

Term	0	1	2	3	4
Tiles	1	5	9	13	17

term number

unit of change *or constant difference between term values*

term values

DURING (Working on It) – 15 to 20 minutes

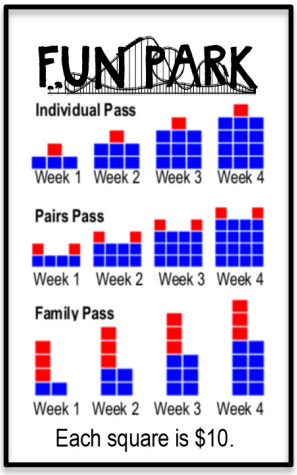
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Square Tiles Problem 2

FUN PARK opens 10 weeks. The "FUN" poster gives an example of cost of the summer pass for the first 4 weeks.

- Individual - 1 person cost
- Pairs - 2 people cost
- Family - 1 person cost (minimum of 3 people)

Which pass should a family of 3 people buy and for how many weeks?



AFTER (Consolidation) – 20 to 25 minutes

- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

arithmetic number sequence *- is a sequence of numbers that have a constant difference between consecutive numbers.*

table of values *- a graphic organizer or chart that includes 2 sets of numbers that related as independent and dependent variables (term number, term value)*

Solution 1

Individual Pass
$40, $70, $100, $130, $160, $190, $220, $250, $280, **$310**

Pairs Pass
$60, $100, $140, $180, $220, $260, $300, $340, $380, **$420**

Family Pass
$50, $70, $90, $110, $130, $150, $170, $190, $210, **$230**

For 10 weeks, 3 people:
- 3 x **$310** = *$930*
- (1 x **$420**) + (1 x **$310**) = $420 + $310 = *$730*
- 3 x **$230** = *$690.00*

Buy the family pass for 10 weeks because it is the least cost for 3 people, $690 for 10 weeks.

Solution 2

Prices over 10 weeks for
- 3 x 1 individual pass = 3 x $310 = $930
- 1 pairs + individual = $420 + $310 = $730
- 3 x 1 family pass = 3 x $230 = $690

Week	Indiv	Pairs	Family
1	$40	$60	$50
2	$70	$100	$70
3	$100	$140	$90
4	$130	$180	$110
5	$160	$220	$130
6	$190	$260	$150
7	$220	$300	$170
8	$250	$340	$190
9	$280	$380	$210
10	$310	$420	$230

Weekly Cost Comparisons:
5-> Pairs+Ind = Family**+$10**
6-> Pairs+Ind = Family
7-> **$10**+Pairs+Ind= Family
10->**$40**+Pairs+Ind= Family
Buy the 10 week family pass as it is least price for weeks 7 to 10.

Arithmetic Number Sequences

Table of Values Absolute Differences

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical discussion.

Grade 5	Grade 6	Grade 7
-build a model to represent a number pattern presented in a table of values that shows the term number and term -make a table of values for a pattern.	-identify geometric patterns and represent them numerically -make table of values, given pattern rules in words, list the ordered pairs and plot points on a graph	-represent linear, growing patterns using concrete materials, table of variables, graph and pattern rule. -represent general term using algebraic expressions

Recursive Pattern Rule *describes beginning of the pattern at term 1 and how it changes from term to term, only if the term values have a common difference.*

- Individual Passes -> Start at $40, increase by $30
- Pair Passes -> Start at $60, increase by $40
- Family Passes -> Start at $50, increase by $20

Explicit Pattern Rule *is a pattern rule that tells you how to find the term value for any term in the pattern without listing all the terms before it.*
(Start at term 0 and increase by the unit of change.)

w represents the numbers of weeks at the park

- Individual Passes -> $10 + (# weeks x $30) -> $10 + (*w* x $30)
- Pair Passes -> $20 + (# weeks x $40) -> $20 + (*w* x $40)
- Family Passes -> $30 + (# weeks x $20) -> $30 + (*w* x $20)

Purchasing Options (for any number of weeks, *w*)

- 3 x Ind -> $10 + (*w* x $30)
- Pair+Ind -> [$20 + (*w* x $40)] + [$10 + (*w* x $30)]
- 3 x Fam -> $30 + (*w* x $20)

Solution 3

Prices over 10 weeks for:

- 3 individual passes
- 1 pairs pass + individual
- 3 family passes

Week	Indiv x 3	Pairs + Ind	Family x 3
1	$120	$100	$150
2	$210	$170	$210
3	$300	$240	$270
4	$420	$310	$330
5	$480	$380	$390
6	$570	$450	$450
7	$660	$520	$510
8	$750	$590	$570
9	$840	$660	$630
10	$930	$730	$690

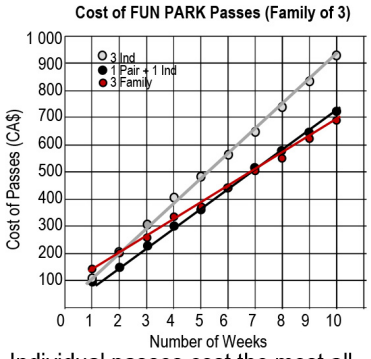
Weekly Cost Comparisons:
The relative differences between the cost of passes, as either Pair+Ind or 3 Family passes is equivalent (1X) at day 6 with very small variations before are after day 6. So, buy 3 Family passes for 10 days, as the family loves FUN PARK.

Table of Values Relative Differences

Solution 4

Plotted data as a line graph from the table of values to show changes of prices over 10 weeks for 3 different ways to purchase FUN PARK passes:

- 3 individual passes
- 1 pairs pass + individual
- 3 family passes



-Individual passes cost the most all the time. Family passes are the least expensive by week 6 by a slim difference.

Comparative Line Graphs

AFTER (Highlights/Summary) – 5 minutes

- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria).
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria.

Highlights/Summary

Compare growing number patterns by the:

- *arithmetic number sequence from 1st to the 10th term and associated term values*
- *table of values, analyzing the absolute or relative differences of the term values at each term number*
- *graphing multiple sets of data as multiple line graphs, so the shape of the data can be compared, especially at specific data points.*
- *explicit pattern rule, which identifies the term value at term number 0 and the type (increase or decrease) and unit of change.*

AFTER (Practice) – 5 to 10 minutes

- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Square Tiles Problem 3

For 2 people, which FUN PARK pass has the least cost for 5 weeks. Use the recursive pattern rule provided.

- Individual Passes -> $10 + (# weeks x $30)
- Pair Passes -> $20 + (# weeks x $40)
- Family Passes -> $30 + (# weeks x $20)

Solution 1

Individual
[$10+(# weeks x$30)] x 2
= [$10 + (5x$30)] x 2 people
= $160 x 2 people = $320

Pair Passes *(Least Cost)*
$20 + (# weeks x$40)
= $20 + (5x$40) = $220

Family Passes -
[$30 +(# weeks x$20)] x 2
= [$30 + (5x$20)] x 2 people
= $130 x 2 people = $260

Solution 2

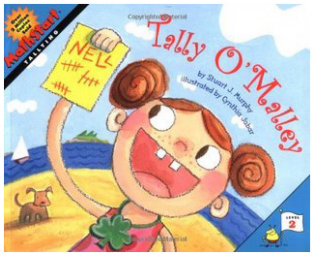
Individual
$20+(# weeks x$60)
= $20 + (5x$60)]
= $320

Pair Passes *(Least Cost)*
$20 + (# weeks x$40)
= $20 + (5x$40) = $220

Family Passes -
$60 +(# weeks x$40)
= $60 + (5x$40)
= $260

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

LINKS TO LITERATURE – *TALLY O'MALLEY* BY STUART MURPHY



The O'Malley family is driving to the beach. "They had been on the road for almost three hours when Eric grumbled "Aren't we ever going to get there?" So, they decided to play short tallying competitions; first with car colours, followed by t-shirt colours.

- Read the book and challenge the students to make tallies as you read. At the end of each tally game, compare the number tallies in the book to the tallies the students recorded using absolute (difference) and relative comparisons. How many times are there as many tallies recorded in the story as there are tallies recorded by the students?
- Read the book and have the students draw a class pictograph using the tallies from the story.
- Take a school community walk and have the students take tallies of objects, such as maple trees, colour of houses. Compare the students' tallies that they recorded during their community walk.

LET'S DO MATH - COMPARING RATES

Before (Getting Started) -Trotting Horses Problem

What does the graph tell you about the horse's trotting behavior?

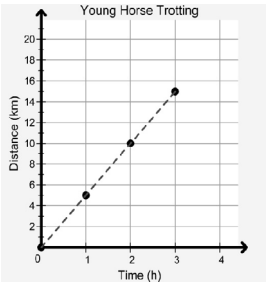
Solution 1

Match the graph with a table of values matches the line graph.

Time (h)	Distance (km)
1	5
2	10
3	15

Solution 2

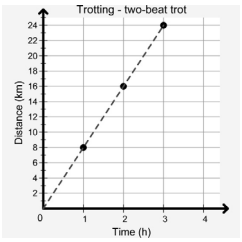
- In 1 hour, the horse trotted 5 km.
- In 2 hours, the horse trotted 10 km.
- In 3 hours, the horse trotted 15 km
- The horse trotted at a steady pace because the line in the line graph is straight.

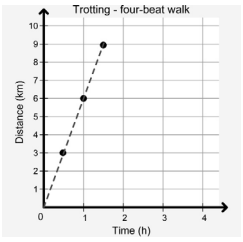




During (Working On It) – Comparing Rates

Trotting horses race either using a four-beat walk or two-beat trot. How do the speeds of the different racing techniques compare?





After (Consolidation)

Anticipating Student Responses: Comparing Rates as Speed

Solution 1

- I plotted the line graph for the two-beat trot and it is less steep than the line graph for the four-beat walk, so the fourbeat walk has a greater speed.
- BUT, the scales are different so I had to re-think.
- I plotted 2 sets of data on the same graph and the two-beat trot is a steeper line, so it is faster.

Solution 2

- For the two-beat trot, the horse walks 8 km in 1 h and 16 km in 2 hours. The speed is 8 km/h.
- For the four-beat trot, the horse walks 3 km in 30 min (1/2 hour) and 6 km in 1 h. Its speed is 6 km/h.
- The two-beat trot has a greater speed, with an absolute difference of 2 km/h.

Coordinating Discussion for Student Learning:

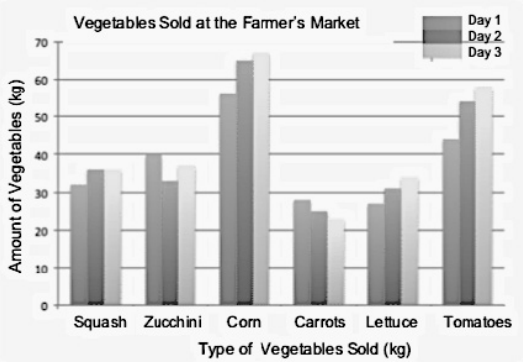
Why might solution 1 be chosen first for student discussion followed by solution 2?

- *Solution 1* – direct comparison of the two line graphs with same scale units
- *Solution 2* – absolute comparison of unit rate, speed

LET'S DO MATH - COMPARING VEGETABLES

Before (Getting Started) - Vegetable Sales Problem

What do you notice about the vegetable sales?



Solution 1 – Total Vegetable Sales Each Day

Day 1 (32+40+56+28+28+42) kg = 226 kg

Day 2 (35+32+64+26+31+52) kg = 240 kg

Day 3 (35+35+66+22+32+58) kg = 248 kg

- The greatest amount of vegetables were sold on Day 3 because 248 kg > 240 kg > 226 kg

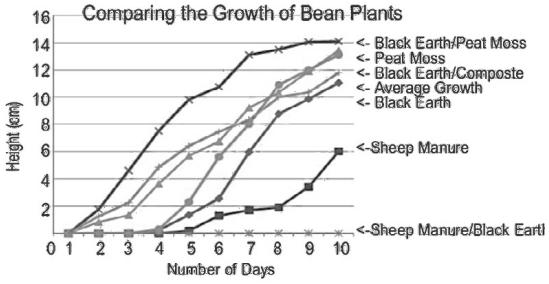
Solution 2 – Greatest Amount of Vegetables

- Each bar shows the amount of different vegetables sold on day 1, 2 and 3.
- Line up the bars at 0, then compare the lengths of same colour bars; day 3 bars have the greatest lengths.
- Day 3 had the greatest amount of vegetables sold.

During (Working On It) - Bean Plant Problem

Grade 4 students grew 7 green bean plants Each plant received the same amount of light, water and soil, but a different soil mixture.

- How do the heights of the bean plants compare?
- How might the type of soil mixture influence the growth of the bean plants?



Adapted from Fu, Truong, Lai, & Saleem (2010)

After (Consolidation)

Anticipating Student Responses

Solution 1 – Range of Heights

- greatest plant height each day is grown in the black earth/peat moss
- least plant height (no growth) each day is the plant grown in the sheep manure/black earth
- plant heights differs from 0 cm to 14 cm; a range of 14 cm
- black earth/peat moss is best mixture for greatest plant growth

Solution 2 – Absolute Differences

On day 10, black earth/peat moss is the best mixture for plant growth:

- 1 cm more than peat moss and black earth/composte
- 2 cm more than the average
- 3 cm more than black earth
- 8 cm more than sheep manure
- 14 cm more than sheep manure/black earth

Solution 3 – Relative Differences

On day 10, black earth/peat moss is the best mixture for plant growth:

- about the same as peat moss, black earth/composite and black earth
- about 2X more than sheep manure
- about 3X more than sheep manure/black earth

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first, then solution 2 for student discussion?

- *Solution 1* – compared the bean plants with the greatest and least heights on day 10 and calculates the range of the plant heights in cm
- *Solution 2* – compared the absolute height of the black earth/peat moss plant to each other plant to determine the absolute height differences in cm.
- *Solution 3* – compared the relative heights in terms of whether the heights or triple, double or about the same height as the plant in the black earth/peat moss.

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how growing number pattern and data comparison problems develop students' understanding of the relationship between independent and dependent variables
- To suit your students' learning needs, select and/or design growing number pattern and data comparison problems or reasoning tasks that based on grade specific curriculum expectations in the strand, Patterning and Algebra. In order to design developmentally appropriate learning tasks, consult mathematics learning trajectories, such as the Ontario patterning and algebra and data management curriculum expectations across the grade (MOE, 2005). Take notice of the mathematical understanding needed to represent, compare and describe growing patterns and to read, interpret and make inferences of relational data, especially with bar and line graphs.
- Design absolute and relative pattern and data comparison problems and reasoning tasks that develop students' experience with different applications in real life situations, such as comparing the growth of plants over time, the cost of fruit drinks at two different stores, and the number of large and small stone slabs used to build 25 inukshuks.
- Practise noticing the students' use of different comparison methods (operational thinking, comparative thinking) in order to compare data values or term values. Be consistent in using comparative mathematical language, such as equivalent, greater than, less than, two times as many, three times as many, half as many. Consider the mathematical reasons why your students choose to use or not these different comparison methods to solve these problems. Is the mathematics inherent in the problems related to previously experienced and/or learned mathematics or is it not within their realm of experience?
- Think ahead of the kinds of mathematical annotations you will record on and around the growing number patterns (geometric pattern, table of values, arithmetic number sequence, recursive or explicit pattern rule and graph) in order to describe the absolute and relative comparisons of the growing number patterns.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

- Beatty, R. (2014). *Exploring the Power of Growing Patterns. What Works? Research Into Practice.* Toronto: Ontario Ministry of Education.
- Beatty, R. & Bruce, C. (2012). Fr*om patterns to algebra: Lessons for exploring linear relationships.* Toronto: Nelson Canada.
- Blanton, M. (2008). *Algebra and the elementary classroom: Transforming thinking transforming practice.* Portsmouth, NH: Heinemann.
- Fulton, B., & Lombard, B. (2001). *The pattern and function connection*. CA: Key Curriculum Press.
- Fu, J., Truong, M., Lai, T., & Saleem, Z. (2010). *Comparing the growth of mung bean plants (vigna radiata) in terms of height in different soil substances*. Pierre Elliott Trudeau High School;
- Kaput, J., Carraher, D., & Blanton, M. (Eds). (2008). *Algebra in the Early Grades*. New York, NY: Lawrence Earlbaum Associates.
- Ontario Ministry of Education. (2005). *Ontario Curriculum, Mathematics, Grades 1 to 8.* Toronto: Queen's Printer for Ontario.
- Russell, SJ. et al. (2018). *But why does it work? Mathematical argument in the elementary classroom.* Portsmouth NH: Heinemann.
- Small, M. (2014). *Uncomplicating algebra to meet common core standards in math, K-8.* Toronto: Nelson Canada.
- Small, M. (2010). *Big ideas from Dr. Small:* Grades 4-8. Toronto: Nelson Canada.
- Warren, E., & Cooper, T. J. (2008). Generalizing the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics, 67*, 171–185.
- Warren, E., & Cooper, T. (2006). *Repeating and growing pattern rules: Relationships between geometric and numerical representations.* Paper presented at the American Educational Research Association Annual Conference, San Francisco, CA.

CLOSING

Our next issue series will focus on the use of quick images and mathematics games that can be used for mini lessons and practice lessons for composing whole numbers and the addition of whole numbers. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a three-part problem-solving lesson design framework that focuses on practise.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email kathkubo@rogers.com or maryloukestell@gmail.com