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INSERT
ELEMENTARY
MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Abacus

Volume 60 • Number 1 • September 2021

MENTAL MATH

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A Publication of the OAME/AOEM

ABACUS CO-EDITORS



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We are so excited to share the first issue of the *Abacus*, with us as coeditors! It is a great honour to take on this new role and fill in the big shoes of past editors. The *Abacus* has a long history of providing high-quality professional resources for elementary educators, and as new coeditors, our first priority is to maintain this outstanding reputation and offer more digital ways (videos) to engage.

We know that the *Abacus* has consistently promoted mathematics as a subject about ideas and how educators can support all children to be confident learners. As has been the focus of past issues of the *Abacus*, we will be supporting and sharing strategies about how teachers can elicit, document, and orchestrate rich mathematical discussions with their students. YES! We will be sure to include lessons, videos, strategies, and concrete activities that you can use the very next day in your classroom.

IN THIS ISSUE

We decided to focus our first issue on mental math for the following three reasons:

a) We've been doing research in classrooms for several years now and have gathered a lot of insights into how teachers can orchestrate number talks to develop mental math strategies. Specifically, we will be sharing our experience learning how standard algorithms crept their way into our number talks from a student named "Arosha."

b) Our research projects included collaborations with teachers and researchers, two of whom have graciously agreed to contribute a short piece to this issue (thank you, Matthew and Heather!).

c) The new 2020 Ontario curriculum has a section emphasizing mental math, so we wanted to take this opportunity to showcase what mental math can look like in a classroom. We hope you're inspired to try out some of the ideas presented in this edition.

MAKING THE TEACHING OF ADDITION VISUAL (BY HEATHER BOURRIE AND MATTHEW CHEUNG)

Pairing number talks with recommendations for teachers to make students' thinking visible inspired short mini-lessons on addition and subtraction, where small circular counters were spatially arranged based on recommendations in subitizing research. The goal of these mini-lessons is to make individual students' solution strategies visible so that they can be discussed by the whole class. Essentially, one could say these mini-lessons are visual number talks.

Subitizing is generally described as "instantly seeing how many" (Clements, 1999, p. 400). Subitizing is deeply linked to visualization, and it has been explicitly suggested that subitizing should be capitalized on for the teaching of addition (Clements et al., 2019). There are two types of subitizing: conceptual and perceptual. Perceptual subitizing occurs when a student sees 4 instantly, and conceptual subitizing happens when a student sees 4 by seeing 2 and 2. Specifically, perceptual subitizing is "[r]ecognizing a number without consciously using other mental or mathematical processes and then naming it" (Clements, 1999, p. 401). Drawing on the work of Markovits and Hershkowitz (1997), Clements (1999) states that "[c]onceptual subitizing is a component of visualization in all its forms... and [c]hildren refer to mental images when they discuss their strategies" (p. 403).

Here, we describe a mini-lesson about 18 + 12 and how it unfolded in a Grade 2 classroom, where subitizing was used with the goal of making ideas visible.

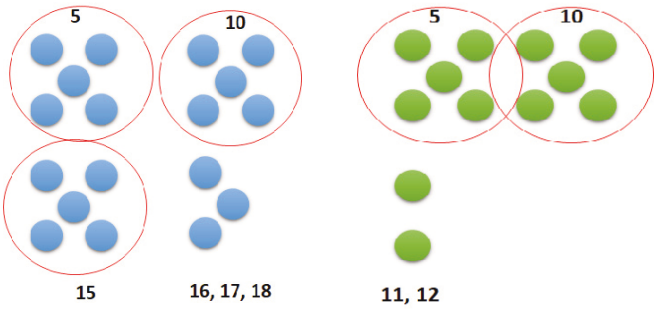
Beginning of the Mini-Lesson ("Minds On")

A student comes to the projector that has 18 blue counters and 12 green counters in a random pile. The class was asked how many blue and green counters there are. The student begins to move the counters one by one to create two strings of 3 counters each. The teacher interrupts the student and says, "Can I offer you an idea? I would really like 5's," and the teacher arranges 5 of the blue counters as you would see on the face of a die. Before the student begins to rearrange the other counters, the teacher addresses the entire class with, "Does everyone agree this is 5?" as she circles the 5 counters she rearranged with her finger. A chorus of students calls out "yes."



> ***Link to literature: Why 5 as on a face of a die?***
> Studies (e.g., Mandler & Shebo, 1982) about subitizing have concluded that students find it easier when dots appear as they do on dominoes.

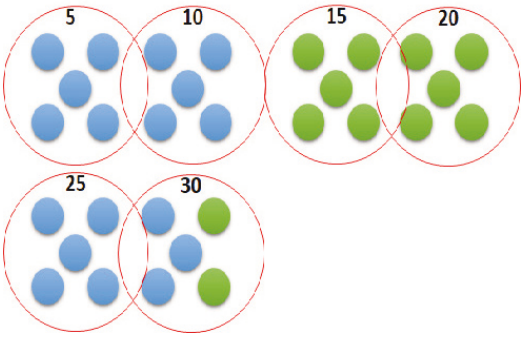
The student rearranges the rest of the blue counters and declares 18. The teacher asks, "How do you know that?" The student explains by saying "5, 10, 15, 18," while simultaneously pointing at the counters. Another student figures out in a similar manner that there are 12 green counters.



Addition: 18 + 12

The students are then told that the goal of the teaching episode is to figure out how many counters in total are on the projector (i.e., to solve 18 + 12), and figure out as many strategies as they can.

The teacher invites another student to the projector to use counting on as a strategy to solve 18 + 12. The student starts by recognizing the 18 blue counters. Then the teacher and student count one by one together, using their hands and fingers—"19, 20, 21,…"—until 30 (note that the counters are not used). The teacher then prompts for connection making by pointing to the counters individually as she says, "19, 20, 21,…," until 30. Two other students raise their hands to indicate they used a solution strategy of counting on by 1's.



The teacher asks, “Who used a different strategy?” A student offers a strategy that counts by 5’s and 10’s. The student starts by pointing and saying, “5, 10, 15, 20, 25 [pause], then I moved these 2 and put them together” as she moved the 2 green counters to be with the 3 blue counters to form a 5, as you would see on the face of a die (see image above). This student is using subitizing while completing an addition question.

> ***Link to literature – Combining counters to make 5***
> Conceptual subitizing applies the perceptual process repeatedly and quickly uniting recognized numbers. There is evidence of conceptual subitizing when the student puts 2 green counters together with the 3 blue counters.

Advice for Future Implementation and Conclusion

Our teaching example illustrates how a lesson about addition can be made visual by: a) using two different colours of small circular counters, b) spatially arranging the counters so that students can flash their eyes and see the number 5, and c) moving the counters based on students’ ideas so that students can compare two different ways to add.

MENTAL MATH MEETS STANDARD ALGORITHMS

We discuss students’ use of standard algorithms during number talks. Number talks are “5- to 15-minute classroom conversations around purposefully crafted computation problems that are solved mentally” (Parish, 2010, p. xviii). Number talks are a great way to get students talking about how numbers relate to each other and develop operational strategies. In our own teaching and research, students often indicate that they are using standard algorithms by: 1) describing a visualization of two numbers spatially placed one on top of the other, or 2) using the word “borrow” or “carry.”

As researchers and teachers ourselves, we have led many number talks, and often students indicate they are using standard algorithms (even though we have never explicitly taught standard algorithms), which can lead to incorrect solutions. This is not surprising, and there has been much debate about teaching algorithms. Researchers (Kamii & Dominick, 1998) caution that introducing algorithms too early can be harmful and may interfere with understanding. Basically, memorizing procedures without number sense can limit what students can do, and they become prone to making mistakes (Boaler, 2015). In Canada, an emphasis on memorizing and standard algorithms can be seen as declining standardized test scores have provoked public calls to go “back to the basics” (e.g., Morrow, 2014). Carpenter, Franke, Jacobs, Fennema, and Empson (1997) did extensive research and concluded that:

> …[s]tudents who used invented strategies before they learned standard algorithms demonstrated better knowledge of base-ten number concepts and were more successful in extending their knowledge to new situations than were students who initially learned standard algorithms (p.148).

In the end, even if teachers do not explicitly teach algorithms, we feel it is important to prepare how to respond to students who indicate they are using standard algorithms during number talks.

Subtraction Lesson Plan for a Number Talk about 17 – 9

Getting Started

- Step 1: Make sure to VERBALLY (no writing) prompt students. For example, you can say: "Today's number talk is about 17 – 9."
  Follow-up prompts: What are you picturing In your head when you think about 17 – 9? Which number are you seeing in your mind? How do you take away nine things? How would you explain what you did and your thinking to friend?
- Step 2: Provide a couple of minutes of think time. During think time, display an image of 17, using counters. (See Figure 1 for an example.)
- Step 3: Have students explain their strategy to a partner.

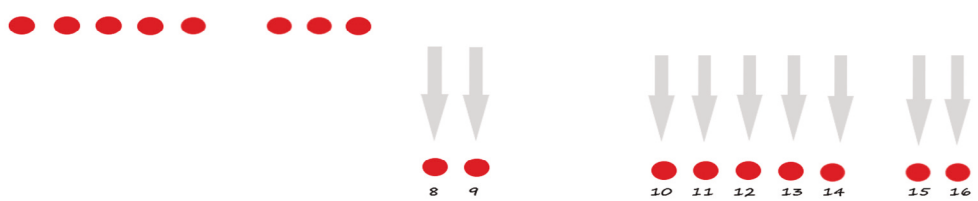


Figure 1: *Image of 17 for teacher to display*
*Teacher note: It is important to leave a large gap between the 10 and 7.

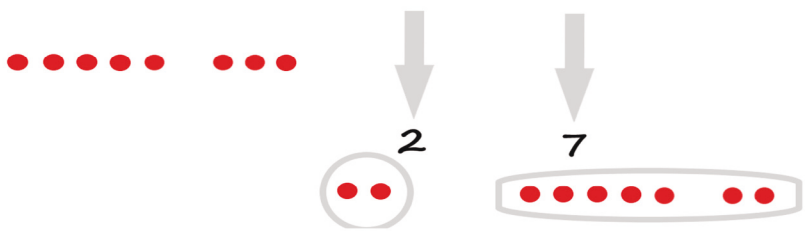
Quick Tips
- *Avoid showing and telling and having students mimic.*
- *Do start with students' ideas to questions.*

Anticipate Solution Strategies for 17 – 9

Solution 1: *Counting down by 1's, using fingers. Start by saying 17. Then, as you pull each counter, say how many are left (see video).*



Solution 2: *Splitting /decomposing 9*



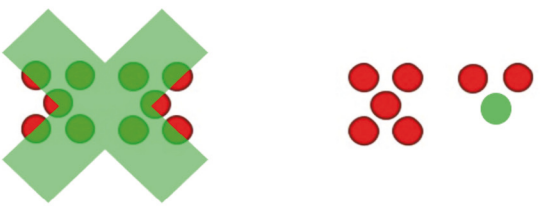
Quick Tip
Ask clarifying questions when students are sharing their ideas:
- *Why did you subtract 7 first? Why not 4?*
- *Tell me more.*
- *How does that make sense?*
- *How do you know that?*
- *Do you really like 10?*

Video resources: These anticipated solutions are static representations of what children may offer… Please watch the videos (link below) to see how students may share their solution strategies. The attached "open-go" number talks video can be easily used in your classroom. NO PREP!
www.vimeo.com/551721440/158770a096

- Curriculum Expectations (Ontario Ministry of Education, 2020) (not to be put on the board with students)

Previous Grade	Current Grade	After Grade
Grade 1: B2.3 use mental math strategies, including estimation, to add and subtract whole numbers that add up to no more than 20, and explain the strategies used	**Grade 2:** B2.3 use mental math strategies, including estimation, to add and subtract whole numbers that add up to no more than 50, and explain the strategies used	**Grade 3:** B2.3 use mental math strategies, including estimation, to add and subtract whole numbers that add up to no more than 1000, and explain the strategies used

Solution 3: *Using 17 – 10*



Quick Tip (re-voicing)
Take advantage of opportunities to teach students mathematical terminology.

For example, if students use the "split," the teacher can re-voice, using the word "decomposed."

Quick Tip
Mental math can lay the foundation for problem solving and algebraic thinking.

Solution 4: *Preference for addition*

9 + __ = 17



DURING
- 5–10 minutes
- "Can someone come up and tell us what they did to figure out 17 – 9?"
- As the student explains, the teacher records by moving counters and make annotations as needed.
- Invite another student to share. "Is there someone who did something different?"
- Once again, the teacher records by moving counters and making annotations as needed.

Quick Tip
Mental math is much more than mere computations.

ENDING (Consolidation)
- Have students analyze and compare solutions.
- "How are these solutions different or the same?"
- "If your solution was not talked about, which of the solutions was most similar to yours? Which one did it resemble the most?
  OR Which one was most different from your solution?"

Quick Tip
Mathematical Annotations: Includes mathematical vocabulary, symbols, elaborations of mathematical details from solutions, and labels describing approach/strategy. All annotations are records of students' mathematical discussion.

AFTER (Next steps and practice)
- If appropriate, the teacher can challenge students to use a different strategy next time (e.g., "If you used your fingers, for our next talk, I challenge you to make use of the number 10.").
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved.
- Students are to create their own written document of the strategies shared.
- Students can make their own word problems for the question.

Note: Template adapted from OAME ABACUS Template created by Kathy Kubota-Zarivnij and Mary Lou Kestell.

THE STORY OF AROSHA: A GRADE 2 STUDENT'S EXPERIENCES WITH SUBTRACTION

To investigate how to respond to students' use of standard algorithms, we tell the story of Arosha (pseudonym) and subtraction.

Setting: Arosha was part of a Grade 2 classroom (students were 7–8 years old). We tell the story by offering five scenes (see annotated solutions in Table 1).

Scene 1 (41 – 16): On the first occasion, in November, Arosha shared her image of 41 – 16, which made use of the standard algorithm to incorrectly obtain 35. She placed 41 above 16, subtracted 1 from 6, and 1 from 4.

Scene 2 (17 – 9): During a number talk in February, students were asked to compute 17 – 9 and were presented with 17 round counters on the projected whiteboard. The round counters were arranged in 3 groups of 5's, as you would see on a die face, and 2 extra counters (see Table 1). Notice that there was space between 2 groups of 5 that made 10 and 7. Arosha explained how her final answer was 8 by saying:

> "So I took away the 7 from the 9, and I did 7 minus 17, and it equals 10, and then I had 2 more left from the 9, and I knew that 8 plus, I mean 10 subtract 2 equals 8, and 8 plus 2 equals 10.

When asked why she took away 7 first, Arosha said: "To make a friendly number 10." During this number talk in February, no students shared ideas about the standard algorithm, nor were they asked by the teacher whether they used the standard algorithm.

Scene 3 (91 – 79): On the third occasion, in May, Arosha guided her class on how to solve 91 – 79, using the standard algorithm, and the teacher created an image based on what Arosha was describing, which included number symbols, and placed 91 above the 79. Arosha said:

> "So you can't take away 9 from 1, so you cross out this (pointing to the 9 on the projected whiteboard) and borrow a 10 and put it over here."

Scene 4 (25 – 16): One week later, students were prompted to build and compare two images representing 25 – 16. One involved using base-10 rods and the circular counters, and the other emphasized the standard algorithm, as it included number symbols and inscribed 25 above the 16. Important to note was that these two images were presented side by side (see Table 1). Originally, 25 was represented by 2 base-10 rods and 5 small round counters, arranged as 5, as on a die face. When a classmate could not take away 6 counters from 5 counters, a base-10 rod was exchanged for 10 counters. A different student performed the standard algorithm and guided the class by "borrowing" and "crossing out." To connect the two solutions, the teacher-researcher pointed to the algorithm image and asked "Where's the 15?" as they pointed to the rods and circular counters. Arosha came up to the front of the classroom and, using her finger, circled the 15 individual blue counters.

Scene 5 (17 – 9 revisited): On one of our last visits in June, 17 – 9 was revisited with the purpose of finding out how many students attempted to use the algorithm to solve this question. Four students, including Arosha, indicated that they used this method. Encouraged to share this method with her peers, Arosha walked the class through an application of the algorithm—"cross out" the 1 in the 10's column ("borrow" a 10), and getting 17 minus 9 again—and declared the answer to be 8. The teacher stated:

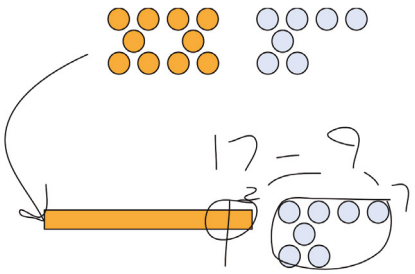
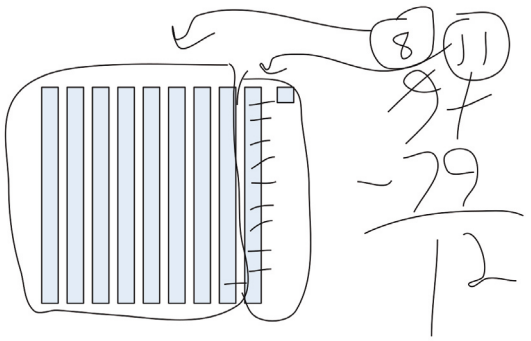
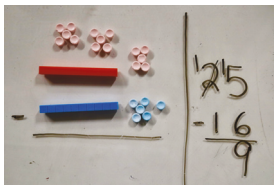
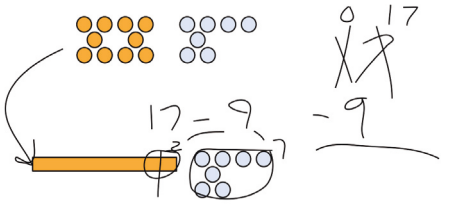
> "I have a question. Did you end up at the same place? Because you started off at 17 minus 9, you borrowed, and then you ended up with 17 – 9. Did you go like this (gestures a loop)?"

Arosha agreed. When the teacher asked Arosha how she got 8, Arosha answered: "I took 7 from 9" (note this is similar to a response she gave in February). Initially, Arosha recites the algorithm for subtraction. However, when prompted about how she actually got the answer 8, Arosha folds back to prior understandings by decomposing the 9 into 7 and 2 and subtracts 7 from 17 to get 10, then subtracts 2 more.

MOVING FORWARD AND REFLECTING ON AROSHA AND HER EXPERIENCES WITH SUBTRACTION IN GRADE 2

Focusing on the story of Arosha leaves us thinking about what teachers can do when algorithms emerge during a number talk. Arosha used the standard algorithm incorrectly for 41 – 16 early in the school year, but later was able to make sense of how it worked; she identified the 15 within the standard algorithm for 25 – 16 in the image of blocks. Our interactions with Arosha suggest that she may have developed a preference for standard algorithms—toward the end of the school year, she attempted to use it to solve 17 – 9. It's worth noting that she did not make a mistake in her final answer, and was able to use her original strategy from earlier in the year to decompose 9 into 7 and 2 to compute 17 – 9. The point is that we should anticipate seeing standard algorithms in our elementary classrooms, especially if we use students' ideas as a basis for our teaching. And even if we do not plan on explicitly teaching algorithms, we feel avoiding algorithms is very unlikely. However, there is value in students knowing why standard algorithms work and being able to demonstrate that understanding in multiple ways, using multiple tools (e.g., blocks, small circular counters). So when algorithms appear, we believe that the key is to support students with several images.

Table 1

Scene 1 (41 – 16)	
Scene 2 (17 – 9)	
Scene 3 (91 – 79)	
Scene 4 (25 – 16)	
Scene 5 (17 – 9 revisited)	

SUGGESTED READINGS FOR PROFESSIONAL LEARNING ABOUT MENTAL MATH

Olsen's (2015) article about mental math is an excellent read, and was written for teachers. Very practical advice is offered, and Olsen discusses the five reasons that teachers should develop mental math strategies. Here's our interpretation of the five reasons:

1. Mental math is connected to conceptual understanding and problem solving.
2. It's useful; there's a real-life application.
3. You can notice important mathematical properties (e.g., commutative, distributive, and associative properties).
4. Selecting and using mental math strategies supports decision-making skills.
5. It's about a disposition/mindset about learning; we get better over time.

REFERENCES

Boaler, J. (2015, January 28). Fluency without fear: Research evidence on the best ways to learn math facts. *Youcubed.* www.youcubed.org/evidence/fluency-without-fear/

Clements, D.H. (1999). Subitizing: What is it? Why teach it? *Teaching Children Mathematics, 5*, 400–405. doi.org/10.5951/tcm.5.7.0400

Clements, D.H., Sarama, J., & MacDonald, B.L. (2019). Subitizing: The neglected quantifier. In A. Norton & M.W. Alibali (Eds.), *Constructing Number* (pp. 13–45). doi.org/10.1007/978-3-030-00491-0_2

Fennema, E., Carpenter, T.P., Franke, M.L., Levi, L., Jacobs, V.R., & Empson, S.B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. J*ournal for Research in Mathematics Education, 27*(4), 403–434. doi.org/10.5951/jresematheduc.27.4.0403

Kamii, C., & Dominick, A. (1998). The harmful effects of algorithms in grades 1–4. In L.J. Morrow & M.J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics: 1998 NCTM yearbook* (pp. 130–140). National Council of Teachers of Mathematics.

Markovits, Z., & Hershkowitz, R. (1997). Relative and absolute thinking in visual estimation processes. *Educational Studies in Mathematics, 32*(1), 29–47. www.jstor.org/stable/3483009

Olsen, J.R. (2015). Five keys for teaching mental math. *The Mathematics Teacher, 108*(7), 543–547. doi.org/10.5951/mathteacher.108.7.0543

Parrish, S. (2010). *Number talks: Helping children build mental math and computation strategies, grades K–5.* Math Solutions.

Pourdavood, R., McCarthy, K., & McCafferty, T. (2020). The impact of mental computation on children's mathematical communication, problem solving, reasoning, and algebraic thinking. *Athens Journal of Education, 7*(3), 241–253.

Threlfall, J. (2002). Flexible mental calculation. *Educational Studies in Mathematics, 50*(1), 29–47.

CLOSING REMARKS

We are very interested in how you teach mental math and what we can learn from other teachers! Please complete the anonymous survey: https://forms.gle/1poFNcfTFzqP5neLA

We will share ideas from the survey in the next issue.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email abacus@oame.on.ca