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FRACTIONS

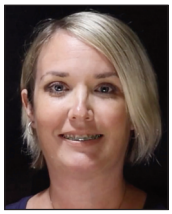
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MULTIPLICATION OF FRACTIONS: USING WHAT STUDENTS ALREADY KNOW

IN THIS ISSUE

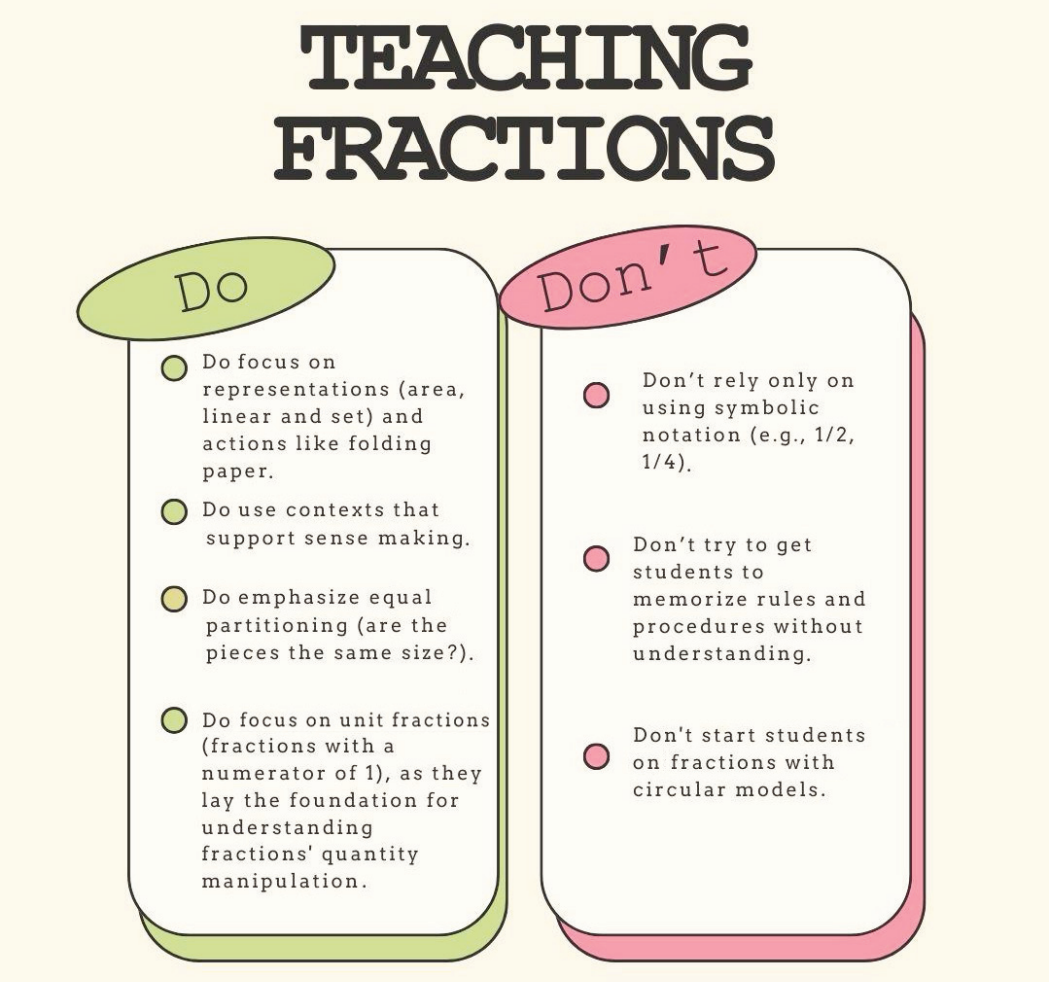
In this issue, our primary focus is on fractions, particularly on the multiplication of fractions and fractional magnitude.

Magnitude is just a fancy word for the size or amount of something. Magnitude is about quantity, and we know from research that students who struggle with fractions often lack an understanding of fractions as quantities (Bruce et al., 2023). Understanding magnitude helps students figure out how big or small their answer should be when they multiply fractions.

This edition features an article authored by Dr. Margaret Karrass, delving into the experiences of students transitioning out of the school system, and their challenges with fractional magnitudes. This piece is particularly interesting because of the students' worked examples that are explained and interpreted by Dr. Karass. Indeed, these examples provide much insight into the kinds of questions we could be asking in elementary schools, and students' ideas of fractional magnitude. We present several easy-to-understand research-grounded graphics that outline the **do's** and **don'ts** of working with fractions, and shed light on the **common struggles** students face with fractions.

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Additionally, we provide a lesson idea about introducing multiplication of fractions by building upon students' existing knowledge of multiplication of natural numbers and encouraging written explanations. The lesson focuses on how students can articulate solutions to multiplication of fractions in a manner that concentrates on conceptual understanding and is accessible to classmates.





LESSON IDEA: INTRODUCING MULTIPLICATION OF FRACTIONS

BUILDING ON WHAT THEY ALREADY KNOW
Ontario Curriculum Expectations:

Grade 4	Grade 5	Grade 6
B2.7 represent the relationship between the repeated addition of a unit fraction and the multiplication of that unit fraction by a whole number, using tools, drawings, and standard fractional notation	B2.8 multiply and divide one-digit whole numbers by unit fractions, using appropriate tools and drawings	B2.9 multiply whole numbers by proper fractions, using appropriate tools and strategies

We have found that there are two ways that children will start to think about fractions:

1) **Part–whole situation:** The numerator represents the count, and the denominator represents the unit (i.e., how many pieces that whole is split into).
2) **As a quotient:** The numerator is a dividend of the denominator. Although both are important and should be honoured, to build on what they already know, when introducing multiplication of fractions, we focus on the part–whole situation.

Specifically, we want students to build on the idea of multiplication: ***groups OF***. For example, 2×3 can be thought of as two groups **OF** three or three groups **OF** 2. Transposing the notion of **OF**, students can think of $\frac{1}{3} \times 9$ as $\frac{1}{3}$ OF 9.

To develop an introductory lesson that focuses conceptual understanding for Grades 4–6, we asked students in Grades 7–9.

Teacher prompt:
Convince others how you know you're right.

Our lesson with students in Grades 7–9 started by posing the following question and telling them they need to provide us with a solution strategy that a Grades 4–5 student would understand by building on what they already know:

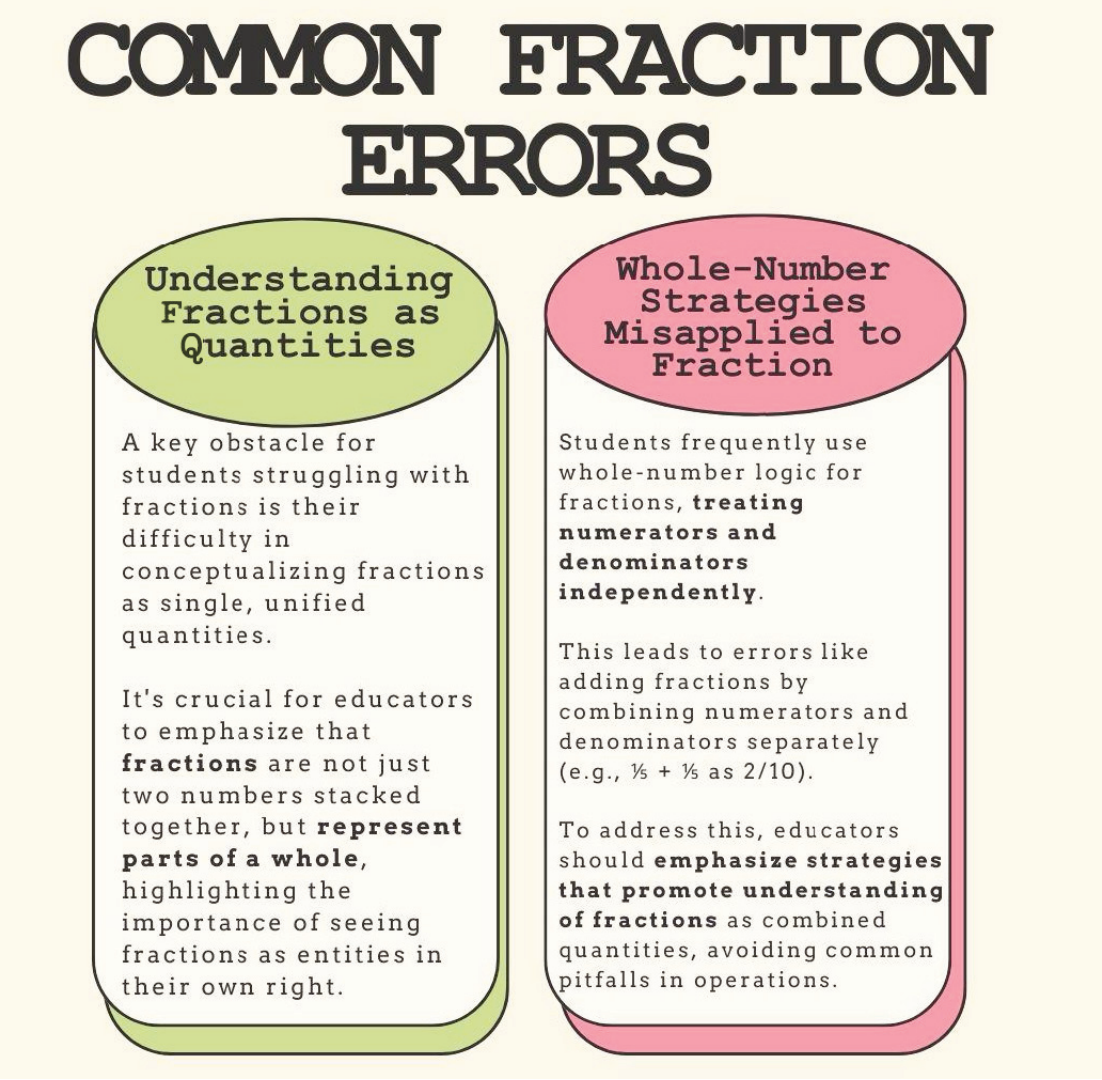
Task:
What is $\frac{2}{3} \times 9$?

However, when this question stumped students, we folded them back to a question with which they were more familiar:

What is $\frac{1}{2} \times 9$?
(and verbally said, "What is $\frac{1}{2}$ ***of*** 9?")

After our discussions, the Grades 7–9 students wrote up solution strategies that we could provide to Grades 4–5 students. Note that we told the older students they would not be there with us to explain their reasoning, so they needed to provide all ideas and details in writing.
Check out the student samples! Here are some tips on how you could print the samples and use them in your classroom.

- Print copies of the solutions for your entire class.
- First ask students to think about $\frac{2}{3} \times 9$ (don't spend too much time on this; i.e., when you ask if anyone has ideas, only take up one solution verbally and move on).
- Tell students you have a better question to ask: Write $\frac{1}{2} \times 9$ on the board and verbally say, "$\frac{1}{2}$ of 9." Ask students to think about that question and give them 2 minutes.
- Tell students that some older children were working on $\frac{1}{2} \times 9$ and wrote up some solutions for them that would explain how they could understand what is going on with multiplication of fractions.
- Pass out the printed solutions.
- Prompt students to pick their favourite solution and to be prepared to share why.
- When students are sharing reasons for why they picked the solution, make sure to highlight:



$\frac{1}{2} \times 9$

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

Think of adding $\frac{1}{2}$ 9 times. Every 2 $\frac{1}{2}$'s are equal to 1, since two halves make up a whole.

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$1 + 1 + 1 + 1 = 4$

there are four wholes in total. the remaining $\frac{1}{2}$ stays.

Final Answer:

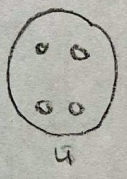
$4\frac{1}{2}$

METHOD #2

Split nine into two equal groups.



4



4

1
0

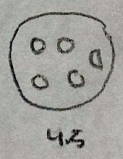
$9 \times \frac{1}{2}$

↳ Finding $\frac{1}{2}$ of 9

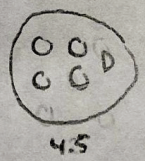
For the remaining 1, you'll have to split it in half (0.5 + 0.5).

Therefore:

$\frac{1}{2}$ of 9 = 4.5



4.5



4.5

$\frac{1}{2} \times 9$

$\frac{1}{2}$ of something means split that something in half. That means we can take half of 9 and get an answer.

$9 \div 2 = 4.5$

That means the answer is 4.5.

$\frac{9}{2} \times \frac{1}{2}$

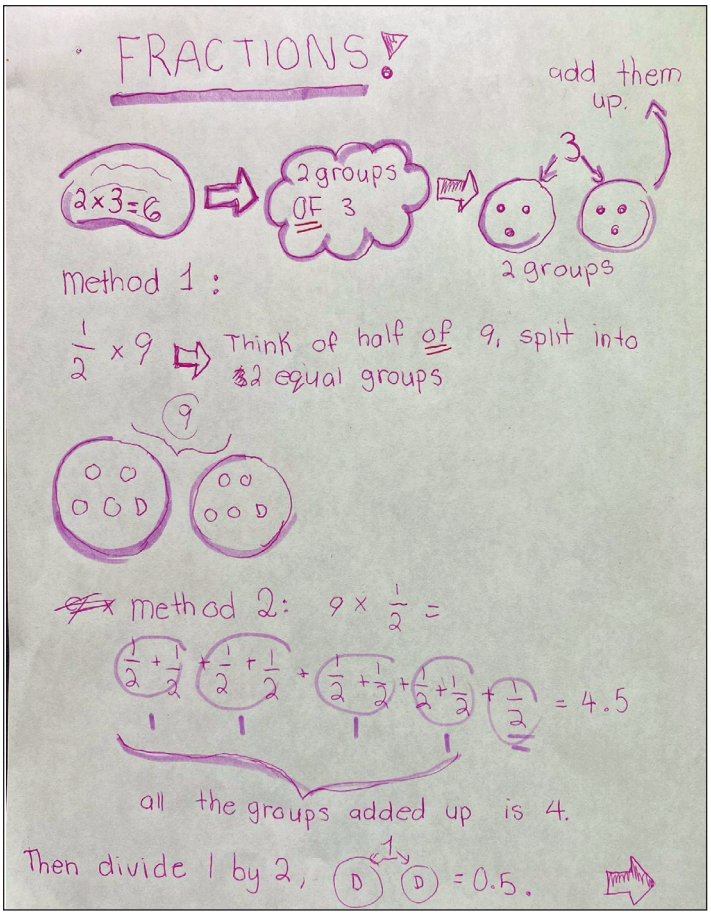
One way to do this question is $9 \times \frac{1}{2}$ or 9 groups of $\frac{1}{2}$. A more simple example is 2×3. for 2×3 you could do 2 groups of 3 or $3+3$. If we do this same stratigy for $9 \times \frac{1}{2}$ we well get somting that looks like this.

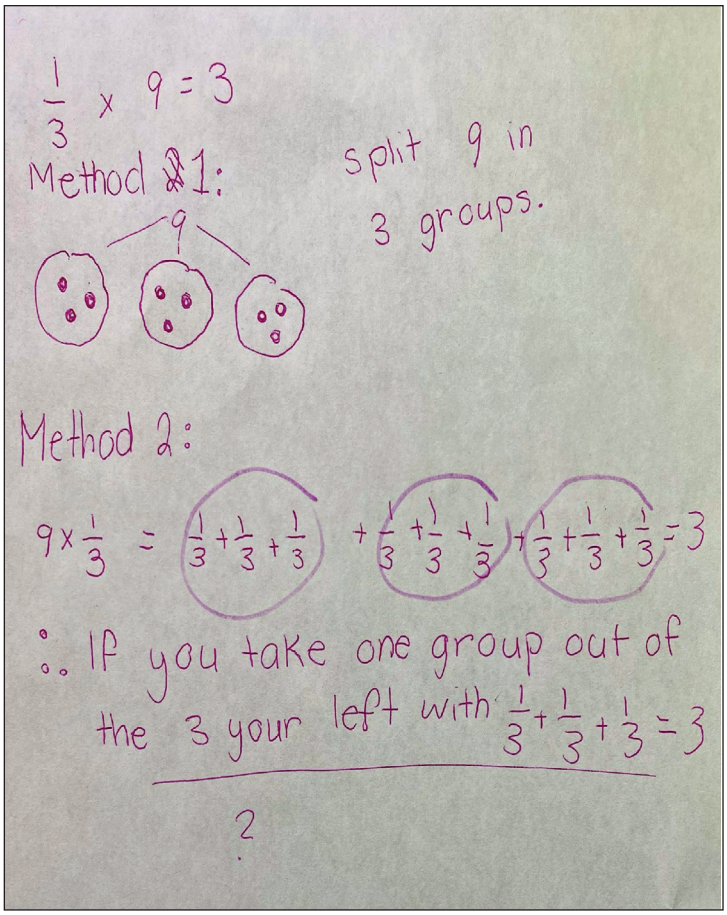
$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{9}{2}$

this is 9 groups of $\frac{1}{2}$

9 groups of $\frac{1}{2}$ is $\frac{9}{2}$ and we can turn $\frac{9}{2}$ into 4.5. That means $9 \times \frac{1}{2} = 4.5$

FRACTIONS





MULTIPLE WAYS TO ANSWER A FRACTION QUESTION

By Margaret Karrass

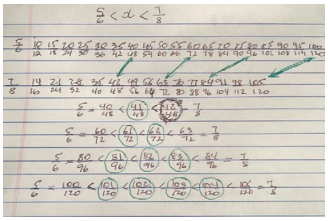
The transition from high school to university is a rite of passage and presents many challenges for students. Regardless of the program, students are expected to possess strong numeracy skills. I teach a numeracy course focused on helping students improve their ability to understand, use, and interpret numbers and mathematical concepts in everyday life. Numeracy encompasses a range of abilities, including understanding magnitudes, manipulating fractions, and using fractions in measuring and estimating.

Research shows a significant association between the level of numeracy and the ability to work with fractions, specifically, understanding the magnitude (Mousley & Kelly, 2017). Understanding fraction magnitude is a necessary and unavoidable stage of understanding fractions (Gabriel, 2016; Siegler et al., 2013). According to Bentley and Bossé (2018, p. 242), "… college students share similar misconceptions regarding fraction arithmetic with elementary students."

According to Fazio et al. (2015), a magnitude comparison task may help students understand fraction magnitudes. To understand how students in the numeracy course that I teach conceptualize fraction magnitudes, I asked them to find a number between $\frac{5}{6}$ and $\frac{7}{8}$. The students were given up to ten minutes to write down and submit their best explanations.

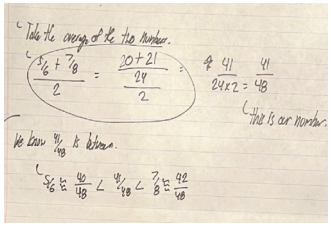
Thirty solutions were collected. The four samples from Ashok, Ben, Cici, and Deena (not their real names) discussed in this paper are representative of the strategies used to complete the task.

To find a number between $\frac{5}{6}$ and $\frac{7}{8}$, Deena listed (in two rows) fractions equivalent to $\frac{5}{6}$ and $\frac{7}{8}$, respectively, and then identified pairs of fractions (one element from the top row and one element from the bottom row) that shared a common denominator. Next, she compared the numerators of the equivalent fractions and found numbers that would fall between these numerators. Deena chose four pairs of fractions equivalent to $\frac{5}{6}$ and $\frac{7}{8}$ to illustrate that the larger the denominators, the greater the range of numbers between the numerators. That is, for the denominator equal to 48, $\frac{5\times8}{6\times8}=\frac{40}{48}<\frac{41}{48}<\frac{42}{48}=\frac{7\times6}{8\times6}$, she identified one number between the original fractions, but for the denominator equal to 120, Deena named four numbers between $\frac{5}{6}$ and $\frac{7}{8}$: $\frac{5\times20}{6\times20}=\frac{100}{120}<\frac{101}{120}<\frac{102}{120}<\frac{103}{120}<\frac{104}{120}<\frac{105}{120}=\frac{7\times15}{8\times15}$.



Had she continued producing equivalent fractions with greater and greater denominators, Deena would have been able to name more and more numbers between the two originally given fractions. While completing this task, Deena demonstrated skills in comparing fractions and generating equivalent fractions. She also demonstrated an understanding of fractions' magnitude.

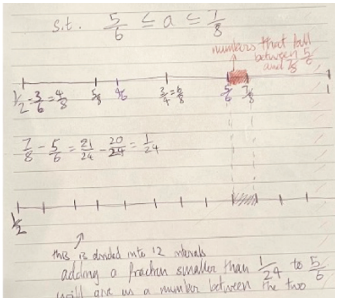
Ben decided to average out the two fractions, reasoning that there must exist a number halfway between $\frac{5}{6}$ and $\frac{7}{8}$. The average of any two numbers is half of their sum.



To compute the sum of $\frac{5}{6}$ and $\frac{7}{8}$, Ben found the least common multiple (LCM) of six and eight.

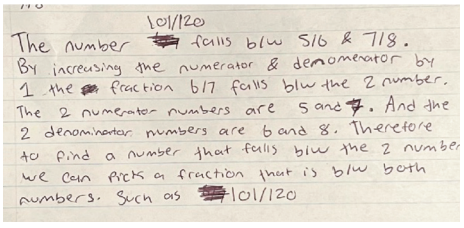
LCM(6,8) = 24, since $6 \times 4 = 24 = 8 \times 3$. He then produced and added the equivalent fractions $\frac{20}{24},\frac{21}{24}$, added them and divided the sum $\frac{41}{24}$ by 2, or equivalently multiplied $\frac{41}{24}$ by a half, $\frac{41}{24}\div2=\frac{41}{24}\times\frac{1}{2}=\frac{41}{48}$.

Ben produced only one solution and did not specify whether there is more than one number between $\frac{5}{6}$ and $\frac{7}{8}$. It is not clear whether Ben looked for the least common multiple of the two denominators because he conceptualized fractions' magnitudes or simply applied an algorithm for finding the LCM of two numbers. Such a procedure could have been learned by rote.



Cici's solution includes computational elements and graphical representation. From the diagram, we see that she was able to determine that $\frac{5}{6}<\frac{7}{8}$. She then considered the magnitude of the distance between the two fractions and computed the difference by finding the least common multiple *LCM*(6,8) = 24 and subtracting $\frac{7}{8}-\frac{5}{6}=\frac{21}{24}-\frac{20}{24}=\frac{1}{24}$.

She then concluded that since the entire distance between $\frac{5}{6}$ and $\frac{7}{8}$ is $\frac{1}{24}$, if we take a number smaller than $\frac{1}{24}$ and add it to $\frac{5}{6}$, we will obtain a number between $\frac{5}{6}$ and $\frac{7}{8}$.



In addition to offering a geometric interpretation of the solution, Cici conceptualized a number between two fractions as a sum of the smaller of the two given fractions and any number smaller than the total distance between the two given fractions.

Ashok offered the next example of the solution to the given task. He suggested adding one to the numerator and one to the denominator of the fraction $\frac{5}{6}$, producing $\frac{5+1}{6+1}=\frac{6}{7}$, claiming that it is between $\frac{5}{6}$ and $\frac{7}{8}$.

We can easily verify it by the following set of steps: *LCM*(6,7,8) = 168

$\frac{5\times28}{168}=\frac{140}{168},\frac{6\times24}{168}=\frac{144}{168},\frac{7\times21}{168}=\frac{147}{168}$.

Since $\frac{140}{168} < \frac{144}{168} < \frac{147}{168}$, indeed $\frac{5}{6} < \frac{6}{7} < \frac{7}{8}$; however, he did not offer any supporting arguments.

Siegler et al. (2011) write that knowledge of natural numbers serves as a conceptual barrier to students' learning about fractions because the natural numbers' counting principle does not always apply to fractions. It seems that Ashok applied the natural numbers' counting principle and obtained his result. It worked, but there is no explanation as to why it worked, or whether his method would work if the fractions were, say, $\frac{1}{7}$ and $\frac{1}{6}$.

The strategies for finding rational numbers between two other rational numbers presented here are examples of the solutions produced by a group of first-year university students enrolled in a numeracy course. Understanding the experiences and challenges faced by first-year university students can offer valuable insights into the types of questions that can be explored in elementary schools to support students in their understanding of fractional magnitude.

According to Siegler (2006), thinking is variable. We often know multiple ways of solving a problem. We chose a strategy depending on the characteristics of the problem and personal preferences. Therefore, not only will different people use different strategies for solving the same problem, but the same person may use different strategies for solving similar problems. Even the same problem presented on two separate occasions may be solved differently. I have outlined various solution strategies with the goal of supporting other teachers in anticipating the diverse approaches students may take when faced with similar questions that are adapted to elementary school contexts. I hope this contribution has underscored the significance and complexity of fractional magnitude.

CONCLUDING THOUGHTS

This issue has served as a powerful reminder of the challenges posed by fractions, and the importance of students developing confidence in understanding fractional magnitudes. Dr. Karrass' offerings have provoked us to think about asking elementary students to order ½, ⅓, and ⅔ from least to greatest. The list students provide can demonstrate great insight into their experiences of fractional magnitudes. For example, if a student provides the exact order of ½, ⅓, and ⅔, then it may be a similar case to Ashok. To address this, it would be helpful to have students create these fractions using paper, and then they can compare their magnitudes and provide a new order. We are eager to hear your thoughts on the reflections prompted by this issue, and are considering a follow-up edition focused on fractions to ensure all students become confident learners in this area.

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