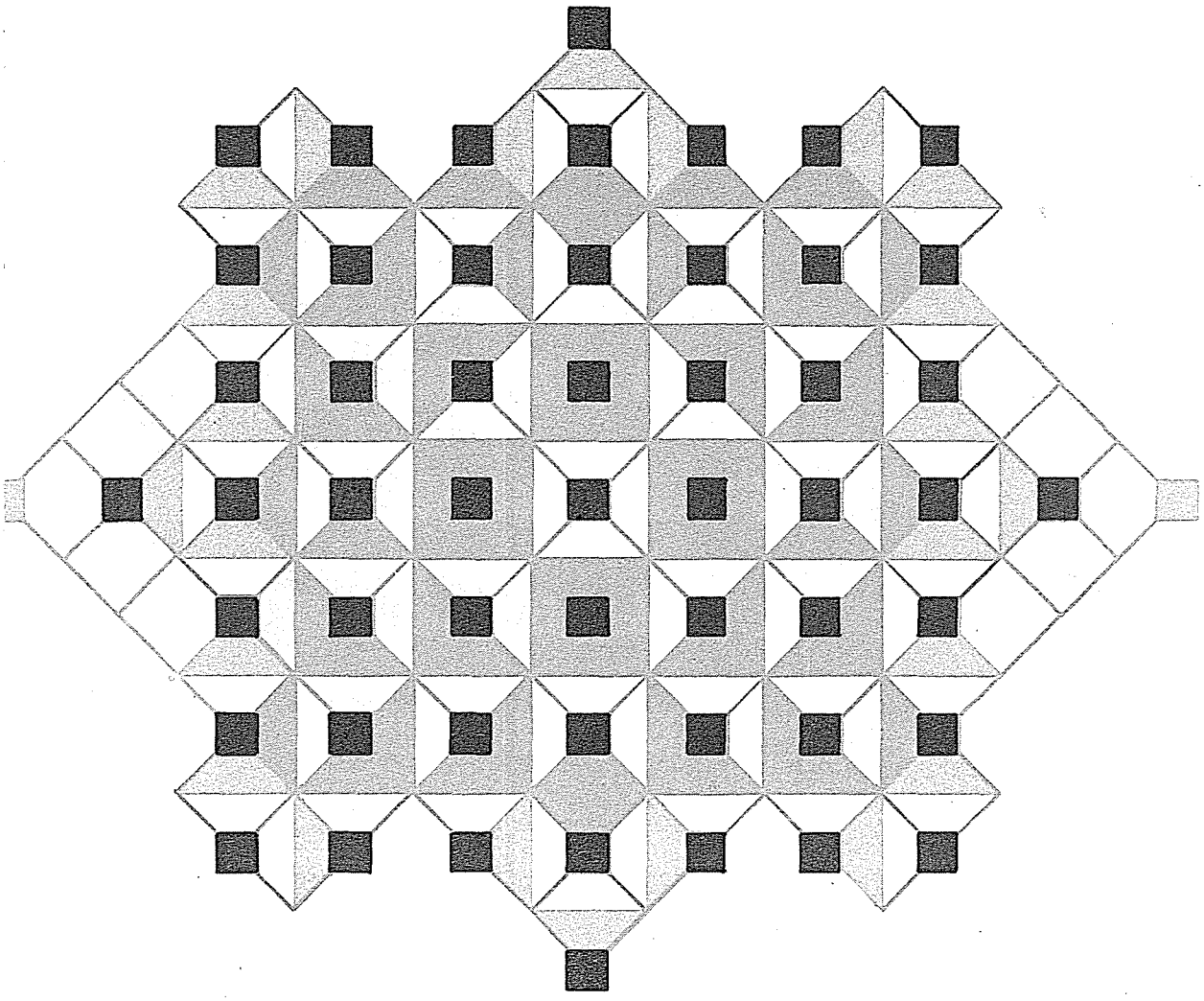
ontario mathematics gazette

VOLUME 13, NUMBER 1 SEPTEMBER, 1974



SPONSORED BY: THE SAMUEL BEATTY FUND
THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Editorial correspondence and articles should be submitted to one of the Editors, W. Eames or J. S. Griffith at the address below.

EDITORIAL BOARD

Dr. W. Eames
Department of Mathematical Sciences
Lakehead University
THUNDER BAY 'P'
P7B 5E1

Mr. D. A. Fox
Ministry of Education
1825 Woodward Drive
OTTAWA
K2C OR2

Mr. N. J. Gillespie
Ministry of Education
2025 Sheppard Avenue
Suite 3201
WILLOWDALE
M2J 1W4

Dr. J. S. Griffith
Department of Mathematical Sciences
Lakehead University
THUNDER BAY 'P'
P7B 5E1

Mr. J. W. LeSage
Ministry of Education
2025 Sheppard Avenue
Suite 3201
WILLOWDALE
M2J 1W4

Mr. M. J. Liebovitz
Ministry of Education
279 Weber Street N.
WATERLOO

Mr. M. J. McKenna
Ministry of Education
15 Church Street
Suite 402
ST. CATHARINES
L2R 7A1

Mr. J. F. S. Milliken
Ministry of Education
1349 Lasalle Blvd.
SUDBURY

Mr. A. Norrie
Peel County Board of Education
90 Dundas Street West
MISSISSAUGA
L5B 1H5

Mr. H. B. Rapley
Ministry of Education
News Chronicle Building
Water Street
THUNDER BAY 'P'
P7B 2S6

Mr. R. J. Stevenson
Ministry of Education
759 Hyde Park Road
LONDON

ONTARIO MATHEMATICS GAZETTE

Contents

	Page
Editorial	1
Proofs that Euclid did not have by John C. Egsgard	2
Discovering Difference Patterns by B. H. Litwiller and D. R. Duncan	12
Stimulus & Innovation in Secondary School Mathematics by Jim Swift	19
Samuel Beatty : 1881-1970 by G. de B. Robinson	27
A New Look at those Coloured Rods by W. Osborne	30
Transformation Geometry with Elementary Pupils by Anna M. Pataky	38
Opinion	56
Book Reviews	57
Officers, Councillors and Chapters of the O.A.M.E.	61

Editorial

September - the beginning of a new school year - new pupils, new problems, and, at least for us, the same old resolutions. Try to keep up with the administration so we can occasionally see the top of our desks; try to be bright, alert, and interesting, at least in front of our classes; try to remember everyone's name, even the name of that fellow who sits at the back, half asleep, in the calculus class of 80. Lots of resolutions, which we make every year, and assiduously try to carry out. *This year we simply must do something to make the Gazette more attractive, and more relevant to teachers in the classroom.* Same resolution as last year; something came of it - a new cover (did you notice?), individual titles, and our attempts to get articles from speakers at the various meetings are beginning to materialize. Let's see what this year brings - and keep our fingers crossed. The Gazette continues to need articles, suggestions, book reviews, opinions, especially from the classroom teacher who is - sometimes literally - in the front trenches of the battleground.

The Editors.

Proofs that Euclid did not have

John C. Egsgard

Euclidean geometry has been a mainstay of the study of mathematics over the years. There are two principal purposes for studying Euclidean geometry:

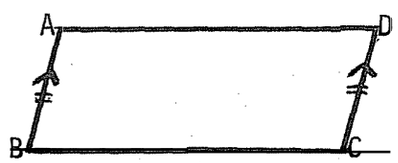
(1) to learn geometric facts;
(2) to understand the meaning of proof.

Either of these purposes can be fulfilled without the other by various means. Note that I am not stating that geometry is the study of proofs. (I believe that geometry is the study of spatial relationships of all kinds in 1, 2 or 3 dimensions.) What I am saying is that Euclidean geometry has played an important role in mathematics because it has been a tool that has enabled students to understand the nature of proof. The understanding of proof is vital to an appreciation of mathematics especially if we admit the definition of one mathematician that "mathematics is the science of proving things to be true". If we accept the fact that a student of mathematics should know what is meant by a proof and be able to create proofs then I believe that we can no longer confine ourselves to the study of the proofs that Euclid had. Mathematics has gone a long way since Euclid! In high school we should make use of as many of the newer ideas as possible such as one found, for example, in analytic geometry, vector geometry and transformation geometry. Yet I wish to make myself clear. I am not advocating the replacement of Euclidean geometry by other kinds of geometry but rather I am stating that Euclidean geometry should be accompanied by other kinds of geometry and solve them in other ways. But before I look at specific problems I shall explain what I think proof is and give some reasons why students should know several methods of proof.

A proof is a process through which true statements are derived from assumptions and/or from facts previously proved using these assumptions. I believe that it is a mistake in the elementary mathematics of high school to try to make these assumptions as few as possible. Some people confuse the study of the foundations of geometry with the study of proof. Hilbert's attempt to put Euclidean geometry on a firm axiomatic basis has no place in the high school curriculum except for the extremely bright student who has finished the standard high school course!

There are several reasons why methods other than Euclid's should be used in geometry. First, the student will have more tools to do the job. Sometimes there will be many methods of proof but one method will be easier than the others. At other times a problem can only be solved if one specific tool is used. Second, the students will get a better understanding of what proof is. Some methods of proof (for example see the Euclidean proof of problem 1) are based on many weeks of work and many assumptions and theorems. Other methods of proof (for example see the vector proof of problem 1) follow immediately from the basic assumptions and can be used after a few days study. The contrasts revealed in these cases give a fuller understanding of what a proof is. A third reason for studying several methods of proof is that the elegance of some proofs emphasize the beauty of mathematics. Finally the student (and teacher) realize that there is no one correct way of doing a mathematics problem. In any case the teacher should praise any proof by any method no matter how complicated it is. Then he should encourage the student to look for a shorter, more elegant proof.

Problem 1



Given: AB//DC
AB≅DC

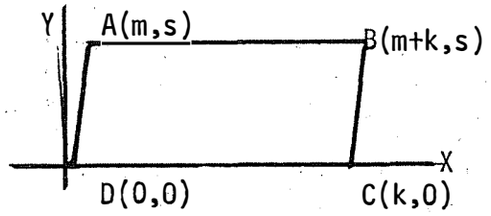
Prove: AD//BC
AD≅BC

Euclidean Proof of Problem 1

This proof is well known. It uses the alternate angles properties of parallel lines and the side-angle-side congruent triangles assumption. The problem and its proof occur after several months of the study of proof by Euclidean methods.

Analytic Proof of Problem 1

This proof is based on a suitable selection of the position of the axes of reference and of the coordinates of the vertices of the quadrilateral. Students have difficulty realizing the difference between the proof of a "special case" and the selection of suitable axes.



Proof: Slope AD $= \frac{\Delta y}{\Delta x} = \frac{s-0}{m-0} = \frac{s}{m}$

Slope BC $= \frac{s-0}{\cancel{m+k-k}} = \frac{s}{m}$

equal slopes $\Rightarrow$ AD//BC

$m(AD) = \sqrt{(\Delta x)^2+(\Delta y)^2} = \sqrt{s^2+m^2}$

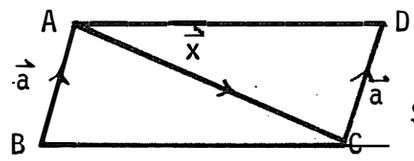
$m(BC) = \sqrt{s^2+m^2}$

$\Rightarrow$ AD≅BC

Note that once the coordinates of A,D, and C are given the coordinates of B follow at once.

Vector Proof of Problem 1

This proof follows immediately from elementary vector facts because the definitions of equal vectors depends on parallelism. Note that the two given facts $\overrightarrow{AB}//\overrightarrow{DC}$ and AB≅DC are included in the one statement $\overrightarrow{AB} = \overrightarrow{DC}$. Similarly we need only prove one vector fact.



Given: $\overrightarrow{AB} = \overrightarrow{DC}$

Prove: $\overrightarrow{AD} = \overrightarrow{BC}$

Solution: Let $\overrightarrow{BA} = \overrightarrow{CD} = \vec{a}$
$\overrightarrow{AC} = \vec{x}$

$\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD} = \vec{x} + \vec{a}$
$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \vec{a} + \vec{x}$

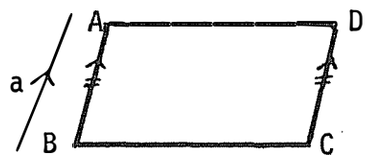
Since vector addition is commutative, $\vec{x} + \vec{a} = \vec{a} + \vec{x}$
$\therefore \overrightarrow{AD} = \overrightarrow{BC}$

First Transformation Proof of Problem 1

(a) Using a translation

Recall that a translation defined by a vector $\vec{a}$ maps a point P into a point P' such that $PP'//\vec{a}$. A translation maps a line segment into a congruent segment that is

parallel to the given segment. Problem 1 follows immediately from this property of a translation.

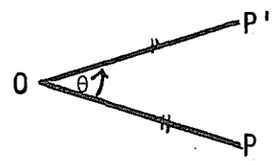


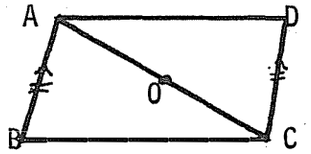
Proof: ∵ BA//CD, BA≅CD
there exists a translation defined by $\vec{a} = \overrightarrow{BA}$ such that

B → A
C → D
Hence BC → AD
Hence BC//AD and BC≅AD

(b) Using a rotation

Under a rotation of θ about a centre O then P → P' where m° < POP' = θ and OP≅OP'
A rotation is a sense preserving rigid motion hence a line segment maps into a congruent segment.
For a half turn (θ = 180°) a line not through O maps into a parallel line not through O.



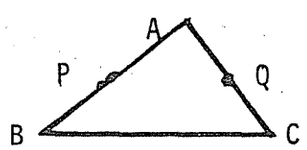


Proof: Let O be the mid point of AC. Under a half turn about O

A → C
C → A
B → B'
AB → CB' where AB//CB'
Since AB//CD
therefore B' ε CD.
Since CB'≅AB≅CD and sense is preserved,
therefore B' = D
∵ B → D
∴ BC → DA
∴ BC//DA and BC≅DA

Note that the centre of the half turn must also be the mid point of BD. Thus the fact that diagonals of a parallelogram bisect each other follows immediately.

Problem 2



Given: P, the mid point of AB
Q, the mid point of AC

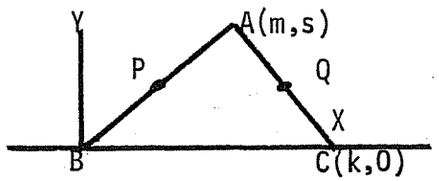
Prove: PQ//BC

$$m(PQ) = \frac{1}{2}\, m(BC)$$

Euclidean Proofs of Problem 2

There are two well known Euclidean proofs of problem 2. One proof, the most complicated, uses the properties of a parallelogram. PQ is extended its own length to D and PDCB is shown to be a parallelogram using problem 1. Another proof uses the side-angle-side similar triangles theorem to prove ΔAPQ is similar to ΔABC.

Analytic Proof of Problem 2



The triangle is positioned as shown. Recall that the mid point of $(x_1,y_1),(x_2,y_2)$ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

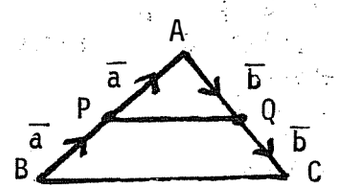
Proof: $P = \left(\frac{m}{2},\frac{s}{2}\right)$, $Q = \left(\frac{m+k}{2},\frac{s}{2}\right)$

Since y coordinate of P = y coordinate of Q, thus PQ// the x axis and hence PQ//BC.
Also $m(BC) = k$ and $m(PQ) = \frac{m+k}{2} - \frac{m}{2} = \frac{k}{2}$

$$\therefore\ m(PQ) = \frac{1}{2}\, m(BC)$$

Vector Proof of Problem 2

The problem is proved if we can show $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{BC}$.



Proof: Let $\overrightarrow{BP} = \overrightarrow{PA} = \vec{a}$
and $\overrightarrow{AQ} = \overrightarrow{QC} = \vec{b}$

$$\therefore\ \overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = 2\vec{a} + 2\vec{b}$$

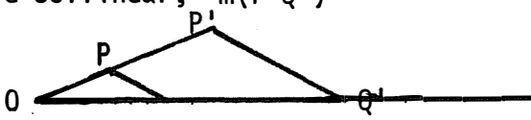
$$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ} = \vec{a} + \vec{b} = \frac{1}{2}(2\vec{a}+2\vec{b})$$

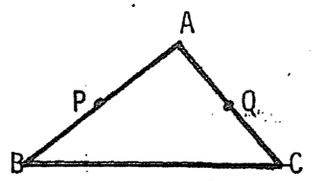
$$\therefore\ \overrightarrow{PQ} = \frac{1}{2}\overrightarrow{BC}$$

Transformation Proof of Problem 2 Using a Dilatation

Under a dilatation of factor k and centre O where $P \to P'$ and $Q \to Q'$ then O,P,P' are collinear, $m(P'Q') = |k|m(PQ)$ and P'Q'//PQ.

The proof follows immediately from these properties.

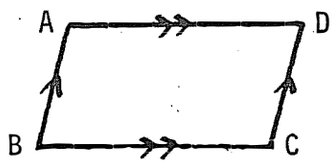




Proof: Observe that A,P,B are collinear, A,Q,C are collinear,
$m(AB) = 2m(AP)$
$m(AC) = 2m(AQ)$.
Hence there exists a dilatation of factor 2 with centre A such that $P \to B$ and $Q \to C$

Thus $PQ \to BC$.
Hence $PQ // BC$ and $m(BC) = 2m(PQ)$ or $m(PA) = \frac{1}{2} m(BC)$.

Problem 3



Given: AB//DC, AD//BC

Prove: AB≅DB, AD≅BC

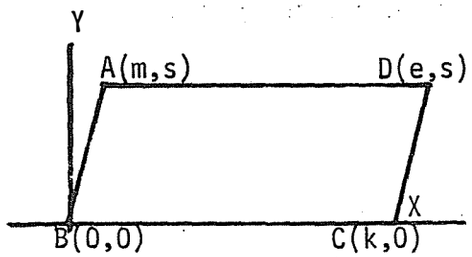
This problem has been selected because it does not have a vector proof as simple as the proof for problems 1 and 2 indicating that no one tool always gives the easiest proof.

Euclidean Proof of Problem 3

This well known proof is no more difficult than the corresponding proof for problem 1 and uses the same Euclidean properties and assumptions.

Analytic Proof of Problem 3

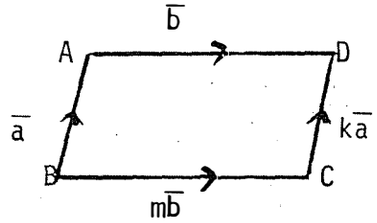
The proof is similar to that for problem 1 except that only the y coordinate of D is known. The first step is to find the x coordinate of D.



Proof: AB//DC
$\Rightarrow$ slope AB = slope DC
$\Rightarrow$ $\frac{s-0}{m-0} = \frac{s-0}{e-k}$
$\Rightarrow$ $m = e-k$
$\Rightarrow$ $e = m+k$

Vector Proof of Problem 3

The key to this proof is in recognizing that $\vec{a}$ and $k\vec{a}$ where k is a real number can be represented by parallel segments.

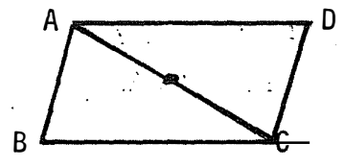


Proof: Let $\overline{BA} = \overline{a}$ and $\overline{AD} = \overline{b}$. Hence $\overline{CD} = k\overline{a}$ and $\overline{BC} = m\overline{b}$, $k, m \in R$. The problem is solved if we can prove $m = 1 = k$

$\overline{BA} + \overline{AD} + \overline{DC} + \overline{CB} = \overline{0}$
$\therefore \overline{a} + \overline{b} + (-k\overline{a}) + (-m\overline{b}) = \overline{0}$
$\therefore (1-k)\overline{a} + (1-m)\overline{b} = \overline{0}$

But $\overline{a}$ and $\overline{b}$ are not parallel, hence $(1-k)\overline{a} = \overline{0}$ and $(1-m)\overline{b} = \overline{0}$. $\therefore 1-k = 0$ and $1-m = 0$. $\therefore k = 1 = m$.

Transformation Proof of Problem 3 Using a Rotation

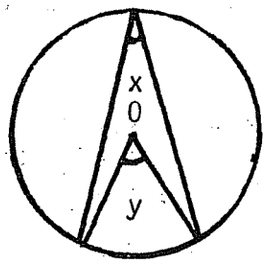


Proof: There exists a rotation of 180° about O, the mid point of AC such that
A → C
C → A
B → B'
D → D'
AB → CB'
CB → AB'

Because AB and CB do not pass through the centre of rotation O, AB//CB' and CB//AB'. But AB//CD and CB//AD.
Hence B' ε CD and B' ε AD.
Hence B' = D.
Hence AB≅CD and CB≅AD.

For each of the last four problems I will give only one proof. Each proof has been selected for its elegance.

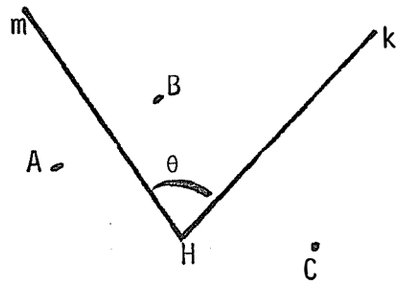
Problem 4



If O is the centre of the circle then $y = 2x$.

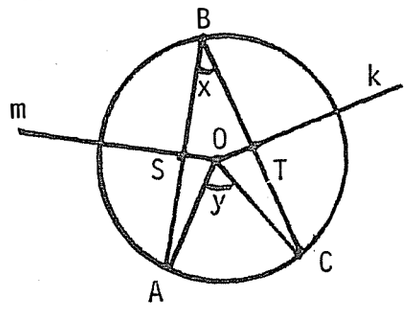
Transformation Proof of Problem 4 Using Reflections

We shall use the following properties of reflections.



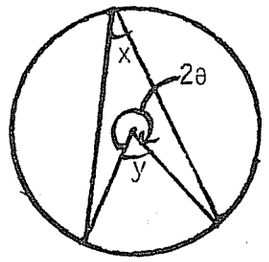
(1) If $A \to B$ on reflection in line m and $B \to C$ on reflection in line k then $A \to C$ under a rotation of 2θ about H where θ is the angle between the two mirrors and H is the point of intersection of the mirrors.
(2) If $A \to B$ on reflection in line m then m is the perpendicular bisector of segment AB.

Proof:

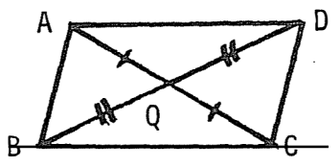


Let m and k be the perpendicular bisectors of AB and BC respectively. m and k intersect at O, the centre of the circle. But $A \to B$ under reflection on m and $B \to C$ under reflection on k. $\therefore A \to C$ under a rotation 2θ about O.

Thus $y = 360 - 2\theta$. But from quadrilateral OSBT
$90 + x + \theta + 90 = 360$
$\therefore x = 180 - \theta$
$\therefore y = 2x$



Problem 5



Given: $BQ \cong QD$
$AQ \cong QC$

Prove: ABCD is a parallelogram.

Transformation Proof Using a Rotation

Proof: Since Q is the mid point of both AC and BD there exists a rotation of 180° about Q such that

$A \to C$ and $B \to D$,
$C \to A$ and $D \to B$,
$\therefore AB \to CD$
and $AD \to CB$.

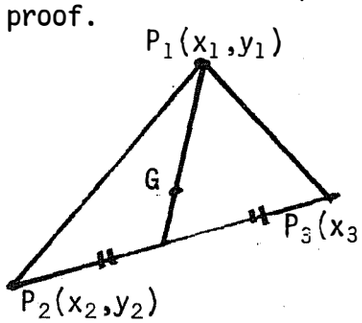
But lines not through Q map into parallel lines under a rotation of 180° about Q.
$\therefore$ AB//CD and AD//CB.

Problem 6

The medians of a triangle are concurrent and the point of intersection divides each median in the ratio 2:1.

Analytic Proof of Problem 6

Note that no special choice of axes is used in this proof.



Proof: Let P_1,P_2,P_3 have coordinates as shown. Consider the median P_1D. Since D is the mid point of P_2P_3

$D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$.

If G ~~divides median~~ P_1D in the ratio 2:1 then

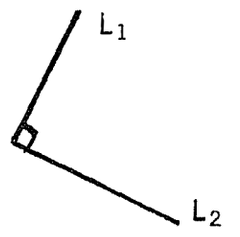
$$G = \left(2\,\frac{x_2+x_3}{2} + 1\cdot x_1,\ 2\,\frac{y_2+y_3}{2} + 1\cdot y_1\right)$$
$$= \left(\frac{x_2+x_3+x_1}{3},\ \frac{y_2+y_3+y_1}{3}\right).$$

The symmetry of the way x_1,x_2,x_3 and y_1,y_2,y_3 appear in the coordinates of G show that the same coordinates will be obtained for each of the points that divide the other 2 medians in the ratio 2:1. Hence the medians intersect in G, which divides each median in the ratio 2:1.

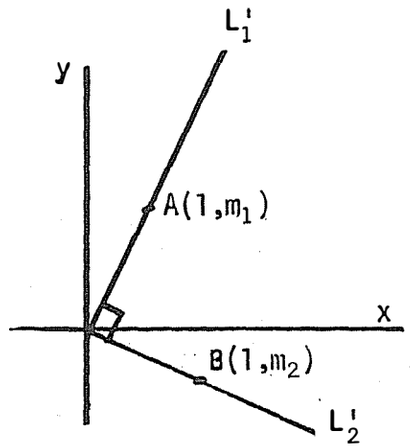
Problem 7

Given: lines L_1 and L_2 with slopes m_1 and m_2.
Prove: $L_1 \perp L_2 \Rightarrow m_1m_2 = -1$.

Transformation-Analytic Proof of Problem 7



Let $L_1 \to L_1'$
$L_2 \to L_2'$
under the translation that maps the point of intersection of L_1 and L_2 into the origin



$\therefore$ slope L_1' = slope L_1 = m_1
slope L_2' = slope L_2 = m_2
Thus L_1' and L_2' have equations $y = m_1x$ and $y = m_2x$
If A and B are points on L_1' and L_2' where $x = 1$ then $A = (1,m_1)$ $B = (1,m_2)$. But
$(m(OA))^2 + (m(OB))^2 = (m(AD))^2$
$\therefore (1^2+m_1^2) + (1^2+m_2^2) = 0^2 + (m_2-m_1)^2$
$2 + m_1^2 + m_2^2 = m_2^2 - 2m_2m_1 + m_1^2$
$\therefore m_2m_1 = -1$

The converse can be shown to be true by starting at the last line of the proof and working backwards. A few changes will need to be made.

The number of problems that can be solved using the tools of Euclidean geometry as well as other tools such as vectors, transformations and analytic geometry is almost limitless. I have selected but a few in the hopes that I could show the importance of teaching high school students to use proofs that Euclid did not have.

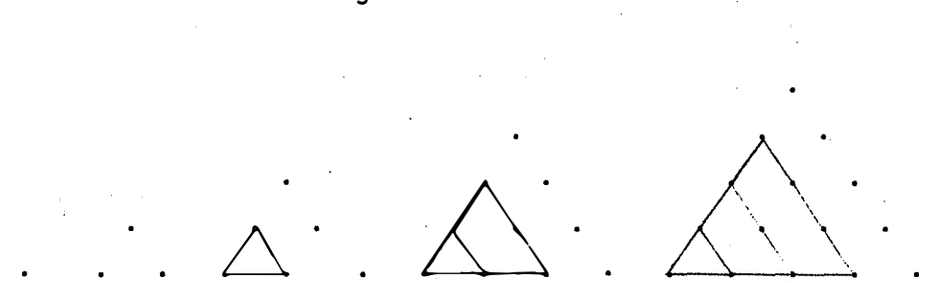
John Egsgard is the head of mathematics at Twin Lakes Secondary School, Orillia, Ontario. He has been active in mathematics teachers organizations in Canada and the United States, and has co-authored several textbooks. He is the president of the Ontario Association for Mathematics Education.

Discovering Difference Patterns

B. H. Litwiller and D. R. Duncan

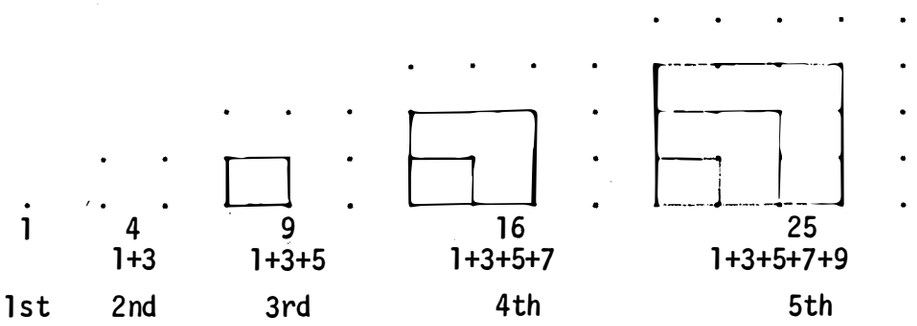
Numbers which represent geometric shapes have interested mathematicians both past and present. In particular "polygonal numbers" have attracted much attention; these numbers include triangular numbers, square numbers, pentagonal numbers, hexagonal numbers, ... For instance, those numbers which represent triangular arrays are said to be triangular numbers. The illustration below shows a geometric pattern for the first five triangular numbers.

Triangular Numbers

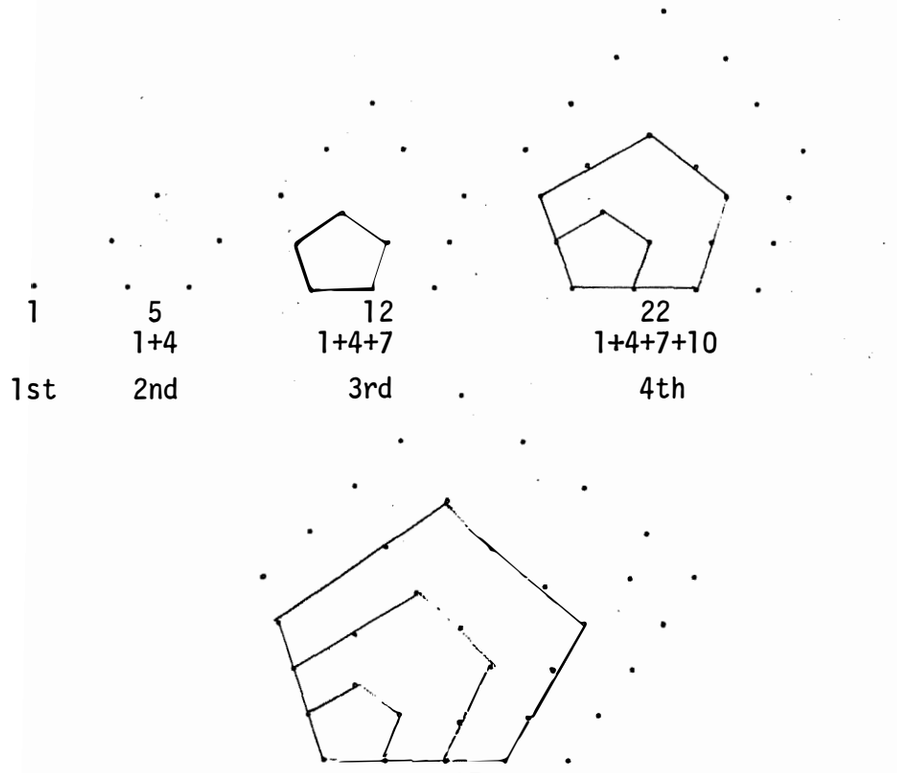


Similarly, geometric patterns which illustrate square and pentagonal numbers are as follows:

Square Numbers



Pentagonal Numbers



1 — 1st

5 — 1+4 — 2nd

12 — 1+4+7 — 3rd

22 — 1+4+7+10 — 4th

35 — 1+4+7+10+13 — 5th

Other polygonal numbers can be depicted in a similar manner.

The polygonal numbers may also be illustrated algebraically as follows:

The nth triangular number: 1+2+3+4+...+n (Every positive integer is added)
The nth square number: 1+3+5+7+...+(2n-1) (Every second positive integer is added)
The nth pentagonal number: 1+4+7+10+...+(3n-2) (Every third positive integer is added)
The nth hexagonal number: 1+5+9+13+...+(4n-3) (Every fourth positive integer is added)

The nth p-gonal number: $1+[1+(p-2)]+[1+2(p-2)]+[1+3(p-2)]+\ldots+[1+(n-1)(p-2)]$

A partial table of polygonal numbers is given here for the convenience of the reader.

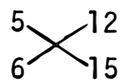
POLYGONAL NUMBERS

Natural	1	2	3	4	5	6	7	8	9	10
Triangular	1	3	6	10	15	21	28	36	45	55
Square	1	4	9	16	25	36	49	64	81	100
Pentagonal	1	5	12	22	35	51	70	92	117	145
Hexagonal	1	6	15	28	45	66	91	120	153	190
Heptagonal	1	7	18	34	55	81	112	148	189	235
Octagonal	1	8	21	40	65	96	133	176	225	280
Nonagonal	1	9	24	46	75	111	154	204	261	325
Decagonal	1	10	27	52	85	126	175	232	297	370

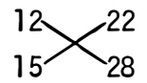
Many interesting patterns may be isolated using the entries of this table. For instance, the differences in the products of the diagonals of consecutive 1×1 squares (across the table) are the triangular numbers.



$(1 \times 6) - (1 \times 5) = 1$

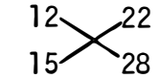


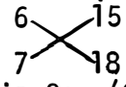
$(5 \times 15) - (6 \times 12) = 3$



$(12 \times 28) - (15 \times 22) = 6$

The difference in the sums of the diagonals of any 1×1 square in the table is the natural number which designates the column in which the left side of the square is located.

For example, in the square 

where the 12 and 15 are in the 3 column, the difference in the sums of the diagonals is 3. $(12 + 28) - (15 + 22) = 40 - 37 = 3$. In the square 

the 6 and 7 are in the 2 column and the difference is 2: $(6 + 8) - (7 + 15) = 24 - 22 = 2$.

To look at the relation of the differences in the sums

of the diagonals of 1×1, 2×2, 3×3, ..., 7×7 squares, let (x,y) be the columns in which ther vertices of the squares fall, that is, the vertices fall in column x and column y. For example, $(2,3)$ indicates a 1×1 square whose vertices fall in columns 2 and 3 and $(5,8)$ indicates a 3×3 square whose vertices fall in columns 5 and 8.

Figure I depicts a progression of 1×1 squares whose diagonal sum differences are summarized in Table I; Figure II illustrates a progression of 2×2 squares.

Figure I: 1×1 Squares

Natural	1	2	3	4	5	6	7	8	9	10
Triangular	1	3	6	10	15	21	28	36	45	55
Square	1	4	9	16	25	36	49	64	81	100
Pentagonal	1	5	12	22	35	51	70	92	117	145
Hexagonal	1	6	15	28	45	66	91	120	153	190
Heptagonal	1	7	18	34	55	81	112	148	189	235
Octagonal	1	8	21	40	65	96	133	176	225	280
Nonagonal	1	9	24	46	75	111	154	204	261	325
Decagonal	1	10	27	52	85	126	175	232	297	370

Figure II: 2×2 Squares

Natural	1	2	3	4	5	6	7	8	9	10
Triangular	1	3	6	10	15	21	28	36	45	55
Square	1	4	9	16	25	36	49	64	81	100
Pentagonal	1	5	12	22	35	51	70	92	117	145
Hexagonal	1	6	15	28	45	66	91	120	153	190
Heptagonal	1	7	18	34	55	81	112	148	189	235
Octagonal	1	8	21	40	65	96	133	176	225	280
Nonagonal	1	9	24	46	75	111	154	204	261	325
Decagonal	1	10	27	52	85	126	175	232	297	370

Similar figures can be drawn for 3×3, 4×4, ... For a square of given type (say type $(2,4)$), the diagonal difference is <u>independent</u> of the placement of the square.

This is illustrated in Figure III. The diagonal differences for squares A, B, and C are respectively:

$$(3 + 9 + 22) - (5 + 9 + 10) = 10$$
$$(6 + 18 + 40) - (8 + 18 + 28) = 10$$
$$(7 + 21 + 46) - (9 + 21 + 34) = 10$$

Figure III: Squares of Type (2,4)

Natural	1	2	3	4	5	6	7	8	9	10
Triangular	1	3	6	10	15	21	28	36	45	55
Square	1	4	9	16	25	36	49	64	81	100
Pentagonal	1	5	12	22	35	51	70	92	117	145
Hexagonal	1	6	15	28	45	66	91	120	153	190
Heptagonal	1	7	18	34	55	81	112	148	189	235
Octagonal	1	8	21	40	65	96	133	176	225	280
Nonagonal	1	9	24	46	75	111	154	204	261	325
Decagonal	1	10	27	52	85	126	175	232	297	370

Table I: Differences

1 × 1 Squares (See Figure I)	Differences in the Diagonal Sums	Second Set of Differences
(1,2)	1	1
(2,3)	2	1
(3,4)	3	1
(4,5)	4	1
(5,6)	5	1
(6,7)	6	1
(7,8)	7	1
(8,9)	8	1
(9,10)	9	

2 × 2 Squares (See Figure II)	Differences in the Diagonal Sums	Second Set of Differences
(1,3)	6	4
(2,4)	10	4
(3,5)	14	4
(4,6)	18	4
(5,7)	22	4
(6,8)	26	4
(7,9)	30	4
(8,10)	34	

3	3 Squares	Difference in the Diagonal Sums	Second Set of Differences
	(1,4)	20	
			10
	(2,5)	30	
			10
	(3,6)	40	
			10
	(4,7)	50	
			10
	(5,8)	60	
			10
	(6,9)	70	
			10
	(7,10)	80	

4	4 Squares	Difference in the Diagonal Sums	Second Set of Differences
	(1,5)	50	
			20
	(2,6)	70	
			20
	(3,7)	90	
			20
	(4,8)	110	
			20
	(5,9)	130	
			20
	(6,10)	150	

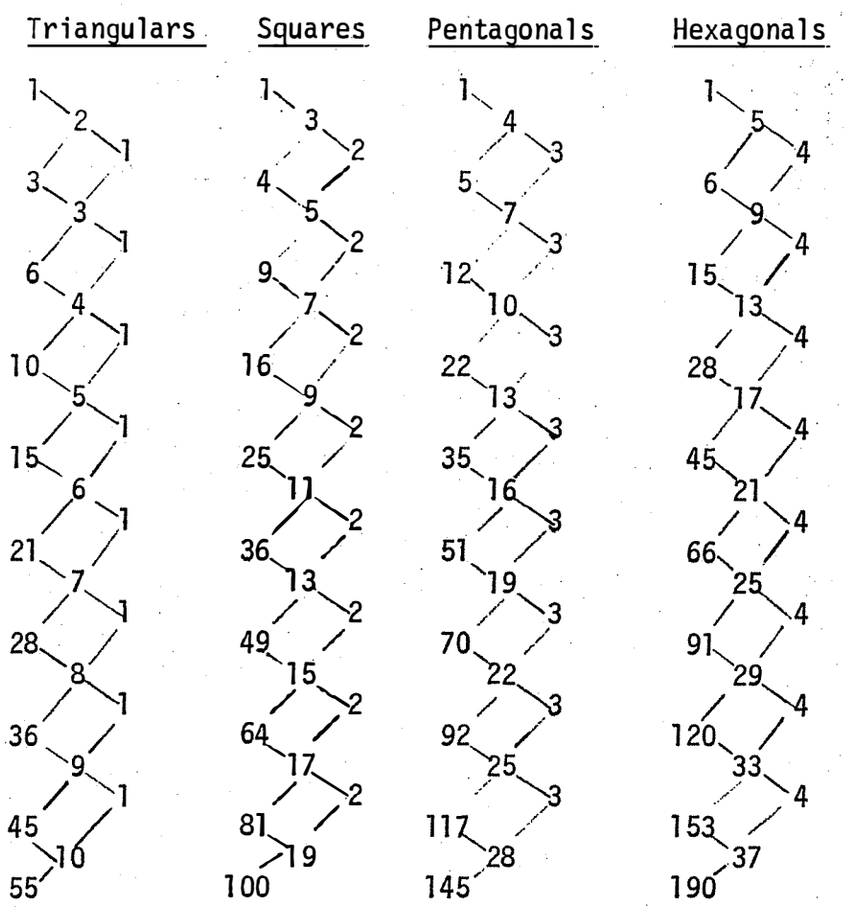
5	5 Squares	Difference in the Diagonal Sums	Second Set of Differences
	(1,6)	105	
			35
	(2,7)	140	
			35
	(3,8)	175	
			35
	(4,9)	210	
			35
	(5,10)	245	

6	6 Squares	Difference in the Diagonal Sums	Second Set of Differences
	(1,7)	196	
			56
	(2,8)	252	
			56
	(3,9)	308	
			56
	(4,10)	364	

7	7 Squares	Difference in the Diagonal Sums		Second Set of Differences
	(1,8)	336	336	
				84
	(,29)	420	420	
				84
	(3,10)	504	504	

The differences in the consecutive diagonal sums differences form the sequence 1, 4, 10, 20, 35, 56, 84. (These numbers are sometimes called tetrahedral numbers.)

The following diagrams illustrate another type of difference between polygonal numbers. In each case, two sets of differences are found.



Similarly, the second set of differences for heptagonals, octagonals, nonagonals, and decagonals are 5, 6, 7, and 8 respectively.

These are but a few of the many patterns which may be discovered by examining a table of polygonal numbers. The reader and his students are invited to seek more patterns among these numbers.

The team of Litwiller and Duncan are becoming well-known to readers of the Gazette. This is the third article they have written for us - the others being in volume 11, p. 208 and p. 279.

Stimulus & Innovation in Secondary School Mathematics

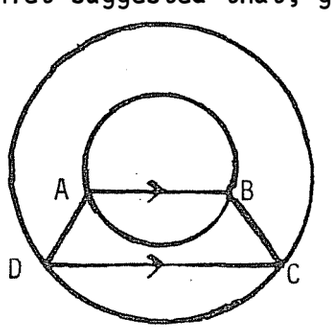
Jim Swift

The minds of change have been blowing very steadily, if not furiously, over the world of mathematics for many years now. It is illuminating to look over ones teaching at this time and trace the influences, which, over the years, have helped to shape and reshape its content, attitudes and technique. What emerges is a pattern of strong dependence on stimulating people, books, magazines etc. - stimuli that become more obvious and effective the more you increase your awareness that such stimuli exist all around you.

This article will trace the development of 3 ideas which are currently in use in high school math courses, and in so doing, examine the stimuli which have influenced that development. The three ideas are taken from the areas of Geometry, Statistics, and Growth functions. As mentioned, the emphasis will be on the influence others have had on the development and implementation of these ideas. It is hoped that the value of such influence will be seen and that ways of increasing ones exposure to such influences will be sought.

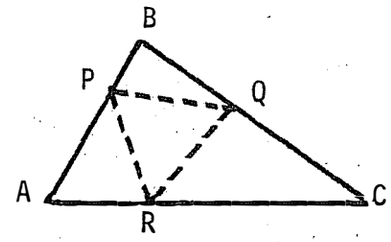
1. Attitudes to Geometry

Like most math teachers, my ideas on Geometry in the first years of teaching were those of my own teachers. Even when my class of 14 year olds made an animated film of the classical proof of Pythagoras, complete with rotating and shearing triangles, it did not really occur to me that this was something different. It took a pamphlet from a close friend, writing for SMP in 1963, to suggest that there were ways of looking at Geometry other than using theorem situations à la Durrell. This pamphlet suggested that, given the figure alongside, the two circles being concentric, an appeal to symmetry is a valid way of satisfying oneself that AD=BC.

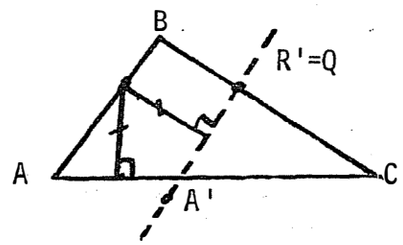


It was not until 2 years later, in 1965, that this alternative attitude was put seriously into practice with trials of SMP materials in Kenya. This was a great source of influence, since these trials, and subsequent development of materials in East Africa, brought frequent contact with the SMP writers and those from the many USA projects. East Africa in the late 60's became a point of confrontation between two very different attitudes to Geometry - the very precise deductive approach of Brumfield and others and the more intuitive approach using the transformation work of David Morris and others of the SMP. It was at this time that Brian Bolt, the SMP writer, mentioned earlier, visited us in Kenya and watched, very critically, some of my lessons in transformation. Essentially his comments were that I was still hung up on PROOF. Discovery of facts and proof of such facts turned out to be two sides of the geometrical coin. The first is not to be played down at the expense of the second. In following up these ideas, I came across this problem from a Cambridge Scholarship Paper.

Given a triangle ABC and a point P on AB. Find Q,R on BC, CA resp. such that PQR is equilateral.



A student from Charterhouse, I believe having been taught Geometry by a transformation approach, provided this method of construction.



Construct the image A'C' of AC under a 60° rotation about P. Then under this rotation R→R'
and R→Q
so R'=Q
but R' ε A'C'
and Q ε BC.
So R'=Q=A'C' ∩ BC. Having found Q, R follows!!!

At this point, the stewing process, which had been going on for some years, produced evidence of its existence. Small items, individually not apparently significant, went into the stew pot, without any obvious effect. But as more and more of these items were included, the result was a mixture, which only needed a catalyst, like the problem above, to produce a very powerful source of stimulus.

This process seems to be one of the very important aspects of innovation. It has been all too easy to jump on the band wagon of change, but if this happens without the foundations of a "stewpot" the results can often degenerate to the point of merely being a shallow copy without substance or inspiration. Given a good new idea, it is absorbed into the stewpot and when it emerges it has the unique flavour of an idea of your own. (On second thoughts a curry pot might serve as a more appropriate model. I was told that the best currys at the New Three Bells in Nairobi came from pots which were not often cleaned out, merely added to.)

In 1966, David Morris came to Kenya to work in an Editorial capacity on the School Math Project of East Africa. It was he who consolidated the process initiated by Brian Bolt, and his problem above. Transformation Geometry became an even more beautiful vehicle of discovery of new facts and insights in the realm of Geometry. Discovery brings its own special brand of stimulus and leads to even deeper insights.

Currently this has found fulfillment in Eschermania, a grade 10 unit on symmetry whose end product is the production of Escher type patterns and other units involving transformations. David Morris was again the source of this idea. One cannot work side by side with an Eschermaniac for long without infection, as the math staff at A. Y. Jackson have discovered to their cost. A closer look at this topic must be left until a future opportunity.

2. Statistics

If we were to ask how many people have been put off statistics by equating it with mere calculations of mean, mode, median, range, standard deviation, etc., etc., etc., the number would be legion. It was all the more stimulating, therefore, to come across two counterexamples in as many weeks. This happened in 1967. The first was a statistics workshop led by Douglas Quadling, and the second was a TV film seen whilst passing a TV shop on Sidcup High St. near London, England - a somewhat unexpected source of stimulation.

[Douglas Quadling must be one of the most stimulating Math teachers. At the NCTM convention last November we talked at different times of many subjects. There is no doubt at all that I was learning from him all the time, just as there is little doubt that he would not have considered himself *teaching* at that time. I find it very significant that there seems

to be no English verb to describe Douglas' activity during that time - the activity of simply being there and being himself, which is in itself a most powerful source of learning. All those I have mentioned so far in this article are mentors in one way or another. Now will someone please make a verb out of "mentor"!!!]

The workshop was stimulating in it being a vehicle for exposure to Quadling's ideas on Statistics. This was followed up by making a point of seeing all the series of the TV films made by the BBC under the title Mathematics in Action - Statistical Inference. These films have been the core of a unit on Statistics at the grade 13 level for the past 3 years.

Prompted by Mosteller's books, which reinforced all the ideas I had been using, a unit on sampling was put together for a class at the grade 11 general level. The essence of the unit is to discover what information can be discovered about a population, by looking at one sample from that population. The problems are all problems to which a Binomial model can be fitted. That is, every member of the population is in one of 2 disjoint and complementary sets. For example:
- a population of blue bead and red beads;
- a population of marked squirrels and unmarked squirrels in a wood;
- a class whose students weigh more than 120 lb., or who do not weigh more than 120 lb.

For purposes of illustration populations of marked and unmarked beads will be used. Po(60) refers to a population of which 60% are marked, and 40% unmarked.

The unit has the following stages:
(a) observation of 100 samples, size 10, taken from each of Po(5), Po(10), Po(20) ... Po(80), Po(90), Po(95).
(b) Plot frequency chart for each of the 11 sets of samples above.
(c) Comparison of the charts with the sample table (size 10) below [actually $100(p+q)^n$, for p=1-q=.05,.1 ... n=10]

frequency		No. of marked beads										
		0	1	2	3	4	5	6	7	8	9	10
population	Po(5)	60	32	7	1							
	Po(10)	35	39	19	6	1						
	Po(20)	11	27	30	20	9	3					
	Po(30)	3	12	23	27	20	10	4	1			
	Po(40)	1	4	12	22	25	20	11	4	1		
	Po(50)		1	4	12	21	24	21	12	4	1	
	Po(60)			1	4	11	20	25	22	12	4	1
	Po(70)				1	4	10	20	27	23	12	3
	Po(80)						3	9	20	30	27	11
	Po(90)							1	6	19	39	35
	Po(95)								1	7	32	60

(d) For which populations is a given sample unlikely (probability ≤ .05)? Example: a sample of 3 marked from 10 is unlikely to come from Po(5) or Po(>50).

(e) An Example: If there are known to be 30 marked beads in the population, and if the percentage of marked beads lies between 10% and 50%, then it is not unlikely that the total number of the population lies between 300 and 60.

(f) Development of (e) to hypothesis testing.

This unit is now in the process of being developed for use in grade 10. It was again the result of many ideas which had gone into the curry pot, from Quadling and the films to Mosteller's books and talks at the NCTM convention.

3. Growth functions

In June 71, looking around for a more realistic view of population growth, for use in grade 13, I came across an article entitled "An end to all this?". This summarized Jay Forrester's work on System Dynamics which led to the commissioning of the "Limits to Growth" study by the Club of Rome. Since then a study of the problematique has become the focus of the final unit in a grade 13 calculus course. The "Limits to Growth" model and the Dynamo language, in which it has been written, are studied. Many people have had a direct input into this unit, including the co-founders of the Club of Rome, Alexander King and Aurelio Peccei, through their seminars at Seneca College. The impact of exposure to this problem has been great, and contact with other interested people such as Professor S. Madras of York University and Ranjit Kumar of Seneca College has helped us keep abreast of current developments. The

latest development, a study entitled "Options for Mankind", is to be published soon. Its impact, too, will no doubt be as great. In many ways, this unit, and its adaptation for grade 12 general, may be the most significant thing we teach.

Postscript

After presentation of these thoughts at Kingston, it is probable that I became even more open to stimulation, for it came with great frequency and strength. But one part stood out as few things have ever done. It is a rare occasion in one's life when a new door opens and one sees not just a new room, but a whole new world. I had met APL and enjoyed using it, but Dr. Iverson, in just two hours of using the language, opened up a new world of mathematics. In any language, thoughts are limited by the language and the symbols we use. Just so in Math, and in Kingston I had a glimpse of the possibilities and potential of this new language.

Two examples might illustrate a point.

(a) Consider two vectors one row, one column. $A = \begin{pmatrix}2\\1\\3\end{pmatrix}$ $B = (4\ 1\ 2)$ The "product" $\begin{pmatrix}2\\1\\3\end{pmatrix}(4\ 1\ 2)$ involves just one operation, x, and yields $\begin{pmatrix}8&2&4\\4&1&2\\12&3&6\end{pmatrix}$

This would be generated by A°.xB in APL which generalizes this "outer product" through applications to other operations. For example ⌈, p⌈q ↔ greater of p and q so if A ← 2 1 3 and B ← 4 1 2

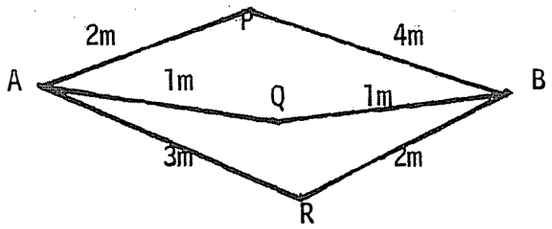
A°.⌈B yields $\begin{pmatrix}4&2&2\\4&1&2\\4&3&3\end{pmatrix}$

just as A°.xB yields $\begin{pmatrix}8&2&4\\4&1&2\\12&3&6\end{pmatrix}$

(b) The product $(4\ 1\ 2)\begin{pmatrix}2\\1\\3\end{pmatrix}$ involves two operations:

first x with corresponding elements yielding 8 2 6 then + between each pair 8+2+6 = 16. This would be written +/AxB or A+.xB. So we might have ⌈/A+B or A⌈.+B which would give ⌈/6 2 5 which is 6.

This might arise in a problem to find the greatest distance from A to B given the 5 towns as shown.



It may be a sign of the times when a grade 11 user of APL recently asked me what Σ meant, as it appeared to him to be the same as +/.

This is the current topic under development. Essentially the plan is to make extensive use of the APL language, not just as a computer language, but in a grade 13 calculus and statistics course, to enrich the mathematics, and aim for deeper insight into the ideas.

There must be numerous APL users amongst the readers of this magazine. It would be good to make contact with any who have similar ideas.

Conclusion

This article would not be complete without some reference to the ongoing stimulus which is obtained from being open to view in teaching in an open area type school. This opens up the possibility of much more effective feedback, on your techniques and content, than is ever possible within the confines of the four walls of a single classroom. At times you are both stimulated and stimulator, because feedback is always a 2 way process.

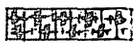
The imperfections of insularity are all too obvious in today's world. To open up to the influences around is to deepen and to enrich. There is a wealth of inspiration in the schools, universities and society of Ontario. Perhaps this magazine, like the Math Gazette, or Mathematics Teaching, or The Math Teacher, will become even more of a forum of ideas and influences.

Jim Swift graduated from University College, London, in 1957, followed by one year at the Cambridge Department of Education. Taught at Chiselhurst and Sidcup Grammar School 1958-60, Head of Mathematics Department at Alliance High School, Kikuyu, Kenya from 1960 to 1968, then worked full-time on writing, editorial, and in-service training work for

A New Look at those Coloured Rods

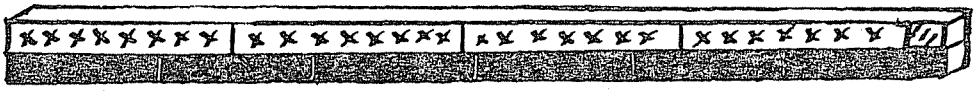
W. Osborne

In the chart recently sent round by the Curriculum Development Branch (reproduced at the end of this article) there appears a column with the heading, "Alternative Number Development Using Rods". This may cause some puzzlement if you haven't ever come across those ingenious mathematical models of the world of number. There are various different types on the market but, basically, they all seem derivations of those invented by M. Cuisenaire. It may be that someone is more used to Catherine Stern's "Structural Arithmetic" which uses much larger rods (3/4 in. thick as opposed to Cuisenaire's 1 cm cross-section). Another may have been weaned on "Colour Factor Rods" with a rod range from 1 to 12, instead of just 1 to 10. There is some difficulty, possibly, if a child changes school and suddenly finds the "10" rod has changed from orange to blue or becomes known as amber, so there may be a point in families, or boards trying to standardize the material among its schools. However, there will still be a variety of approaches, even with the same rods, so that the kind of rods used is of minor importance. The brief description of the rods which follows is based on the Cuisenaire Rods, but the method is not wholly Cuisenaire's.

Rod	Length	Suggested Colour	Code Letter	French Alternative	Code Letter
	1 cm	white	w	blanc	b
	2	red	r	rouge	r
	3	light green	l	vert	v
	4	mauve	m	pourpré	p
	5	yellow	y	jaune	j
	6	dark green	d	emeraude	e
	7	grey	g	gris	g
	8	tan	t	tanné	t
	9	blue	b	azur	a
	10	amber	a	orange	o

The above colour names avoid the problem of having black, brown and blue rods, which make code letters impossible. The French version is less satisfactory only in that I can't think of a French synonym for orange and the use of "o" is undesirable due to its confusion with zero.

FIRST STEPS: No matter at what age the rods are introduced, there must be a period of free play, experiment, investigation, call it what you will. A High School student working on a project to build a mathematical model of the number system might take five minutes to explore the possibilities, while a Kindergarten child might take five months, or more, before it can be seen that he is ready to take the next step. The rods can be introduced to ten-year-olds as readily as six-year-olds, as they are a great way of recapping or making sense of what has already been done in the field of numbers.



42÷7

FRACTIONS: The rods up to this stage have probably developed number names rather than colour names. The yellow rod is possibly becoming synonymous with the five rod. However, this can be avoided with forewarning and by practice in building ladders with other bases. The ladder built using red as the base is r,m,d,t and a and the white rod becomes 1/2, the yellow 5/2 or 2 1/2, etc. Similarly, the thirds family can be built using the light green as the unit. A mauve measured by a blue is 4/9. Equivalent fractions can be built up by seeing the red rod is half the size of the mauve as well as the white being half of the red and the light green is half the size of the dark green, so we get 1/2 is the same as 2/4 and 3/6 etc.



This is only meant to be a glimpse of what can be done

ALTERNATIVE NUMBER DEVELOPMENT USING RODS - taken from the Chart, "Guidelines in School Mathematics", sent around by our Curriculum Development Department

I. USING COLOUR NAMES
1. Free play
2. Substitution, equality, ordering
3. Addition and difference (Nothing - no difference)
4. Addition equivalences, commutativity, associativity and permutations
5. Primes and factors of composite rods
6. Fractions and equivalent fractions
7. Use of symbols for operations and colours

II. USING NUMBER NAMES
1. Any rod as the measuring rod
2. Exploration of free writing with brackets introduced to meet ambiguities
3. Numbers 11 to 20 defined. Families of addition and difference. Zero symbol

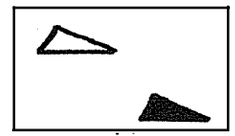
III. PRODUCTS AS X RESULTS
1. By games, chart or products and facts
2. Division as fraction (Inverse element)

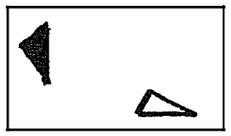
IV. FURTHER NUMBERS
1. Definition, doubling, etc., to obtain large numbers
2. Products to 10×10 and their study. Squares
3. Division using sign and remainders
4. Methods of addition and subtraction 'Vertical' setting

V. EXTENSIONS
1. Prime factors
2. Powers
3. Distributivity
4. Notation in binary and other basis
5. Permutations and factorials
6. Further number patterns and sequences

lesson the restriction of distance of movement was the edge of the projector surface. In the physical education lesson the restriction was the size of the playroom. The children appeared equally at ease in the small and large spatial planes. Subsequently the children adapted matter-of-factly to new and varying-size plane surfaces within which they were to work. Some of these were desk tops, flannel boards, plastic boards and bulletin board areas. The cut-out patterns or geometric shapes first introduced were triangles of construction paper, felt, plastic or bristol board. With younger children more substantial triangular solids, perhaps wooden, are more appropriate because of the kinesthetic factors.

Without further formal lessons I began to pose "problems" for the children via captions on the flannel boards and plastic boards, e.g. "How would you move the white triangle to cover the black one completely? What moves would you make?"





Although the materials were extremely simple, they proved to be dynamic and evoked enthusiastic response. We did not require that each child do the problems i.e. they become free-choice activity, but we found that all children responded quickly with attempts to solve them and almost without exception were correct in solutions to problems requiring as many as four or five successive transformations.

As my class was accustomed to frequent challenge and perceived challenge as part of the learning process, they often composed questions for classmates and myself to solve. During the next few days they began to create their own problems employing the materials mentioned above, as a challenge for others (see pupil-derived transformation problems - Appendix B). The challenge technique will succeed, in my opinion, only if the teacher is willing to abandon the role of lecturer or dispenser of knowledge and will assume an active co-learner role. I found that the children's problems equalled and even surpassed in difficulty those I designed and that the children became quite demanding of excellence and precision of execution in the solutions that their classmates demonstrated. I found myself placed in the role of

referee on several occasions because the precision sought by the child's mind was not easily managed as he manipulated a small cut-out shape; it is very difficult to make a slide without turning the triangle.

John and I discussed the situation facing us. In two days the children had adroitly mastered the three simple transformations on a plane field. The were becoming frustrated because of the accelerated precision demands they were placing upon one another. We decided that it was imperative to design a format which would allow sufficient scope. It was, I admit, a more crucial problem than it appeared at first for the obsession of the children for precise thought and execution was contradictory to their physical inability to measure precisely i.e. the conceptualization level of the mind was much higher than physical skill could accommodate. The answer lay, we felt, in a material that contained precision measurement implicit within it. After considering several alternatives we settled upon dot paper but John pointed out that the concept of point, which he defined as a spatial location so precise as to have no measurement, was a concept that the children must still evolve. His solution was a simple but, I think, a brillian one. He proposed that we use small circles rather than dots as the points. He explained that the most inept pupils could manage to connect the circles while the challenge to strike the centre of the circles when making the connection would be an omnipresent challenge as physical capacity developed. He maintained, and I agree, that this might be the most effective route to develop the concept of point. How right he was has been demonstrated in the subsequent execution.

The "geopaper" was enclosed by a heavy black boundary (Appendix C), the purpose of which was to provide a manageable finite environment within which children would feel secure to carry out investigatory manipulations. The viability of these security boundary lines also has been demonstrated conclusively in the freedom with which even the most insecure, disorganized and disturbed children seize the material. Within two days of our planning discussion, the material was in the hands of the children. The pupils were ecstatic! The "geopaper" was mimeographed so that we could provide an unlimited supply. In addition, we prepared overhead transparencies of the geopaper for large-group teaching purposes.

Using the geopaper format we designed an experiential program (ESFT) for each of the three basic transformations.

asked them if they would like to try some of the same type of thing. They responded eagerly so he had them demonstrate their versions of the simple transformations. Not surprisingly, they performed in much the same fashion that the older children had a month before. We have, therefore, substantial grounds to suspect that transformation work could be quite natural for the very young child.

The ease with which children absorb the concept of transformation can cause some rather terrifying moments for the teacher if she has not planned ahead. Such a moment occurred for the grade five teacher who had many of my pupils the following year. The pupils confronted him with the request for specific information as to when they would be getting on with transformation program. Needless to say he and the rest of us scrambled to prepare programming for them. Perhaps the best advice I can give to the teacher who wishes to carry on work of this nature is that she turn the responsibility for posing problems over to the students at the first opportunity. Their problems and challenges for one another are sometimes even more relevant then what adults can provide.

We have, of course, subsequently explored transformation work with younger children to a certain degree. We know, for example, that the ESFT program is satisfying to grade three children and we know that grade two children enjoy simple transformation work in art. Aside from transformations we have found that geopapers of the types that we use are satisfying to children of all ages. It has been at the grade four level, however, that we have found transformation program to be of real consequence to children.

I believe that I have demonstrated beyond question that transformation work can provide a rewarding program for the ten-year old. There are several important considerations that must be kept in mind. The pupil of this age is not a mathematical theorist. You must not attempt to put mathematics into the child; rather put the child into mathematics. He cannot grasp such concepts as 'image' because he has not had experience with it; program must be such as to give him experiences to build the concept. It is necessary that the child be allowed tangible investigations or explorations into his spatial environment. I feel it imperative that the child should experience for himself the phenomena and the relationships. Much emphasis must, therefore, be placed upon the tactile, the concrete and the manipulative. Demonstrations and explanations by the teacher do little to increase understanding and may actually impede it. What a

child experiences for himself he does not forget - he is involved in a real situation and hence in the thinking processes. Tangibility is of the essence. The pupil must define the concept to himself. It is only by acting as the agent of change that we achieve full understanding of the change concept. Requiring the child to decide upon the change type guarantees that he is the agent.

Exploration of space and spatial relationships allows the child security in his environment through the exploration and manipulation of that very environment. Transformations for the ten-year old must be a first-person study. The teacher must be prepared to drop the traditional role of demonstrator. She should concern herself, instead, with selecting relevant areas for the learner to probe and with using the interest and attention of the pupils as her barometers of selection. Thorough understanding of transformation comes only when the individual (teacher included) acts as the agent of change; physical participation is a prerequisite for the learner. A second level of understanding comes when, having been the agent of change, the child is able to participate vicariously while another individual acts as the agent of change. The third level of conceptualization is the stage at which one can recognize the changes in state or transformation by an agent or agents unknown. Transformation experiences in art and physical education are of particularly supportive value because of their emphasis upon the pupil being the agent of change.

An effective program must accommodate the needs of all groups in the classroom; it must be sequential and developmental in nature yet it must be flexibly applied according to individual needs and interests. The answer seems to lie in a basic class program supplemented by an individual rate experiential program. Obviously, an observant and imaginative teacher helps too!

Quality and quantity of materials used are of utmost important. The "geopaper" provides a consistent safe-sized environment within which to explore. The heavy border provides orientation lines which give locational ease. The precise black and white field encourages accurate work. The circle-style points allow even the beginner to make point connections and to see the degree of precision with which he has worked. It is important to provide the "geopaper" in large quantity. In my program it was not at all unusual for two hundred sheets to be profitably used in a single day.

This sort of scope for experimentation pays off in innumerable ways. There are other materials and techniques but they should be used, I feel, to reinforce, not introduce, the participatory aspects we emphasize. Geoboards, red plexiglass and mirrors are different kinds of exposures to phenomena in which the children should already have been participants.

In discussing transformation program with educators, John and I have been asked several times, "Does it help them with their addition and subtraction?" Our first temptation is to pose several alternative questions:

(a) Does it help in painless child birth?
(b) Does it improve their performance in speaking Spanish?

A less reactive response is that the program was never intended for such purposes so we cannot honestly answer the question. However, the question was followed up by asking it of the children two years after they had taken the program i.e. when they were twelve years old. Their answer was an unequivocal, "yes!" They stated that it had taught them to be more precise, that they had come to enjoy mathematics, that it had helped them to understand "things" and that their performance in all portions of arithmetic had consequently risen very substantially.

Another frequently asked question is, "But can't you do it another way?" Our reply is that perhaps we can, but we cannot, at this moment in time, answer for the effectiveness of some alternative technique. Using the approach, the techniques and the materials that we have employed, we know that a satisfying and significant program can be carried on.

Several times since we began our experimentation we have been approached by people who state that they wish to do transformation work with elementary pupils. In each case we have countered with the question, "Why do you wish to do it?" It is our pedagogical position that if you haven't very definite aims in mind it is questionable whether you should consider embarking on a program. I have stated in this position paper what our goals were and how we set about achieving these goals. That elementary children can operate in the transformation field at an exceptionally high level is now clearly established. Whether elementary children should be working in this field is the question to which educators should now direct their attention.

APPENDIX A

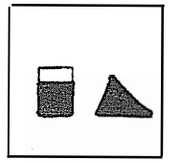
Transformations - Exposure 1

Class Junior 1 - Average distribution

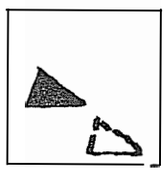
Purpose To introduce concept of transformation to pupils for the first time and to introduce the field of transformation geometry to several teachers without them becoming overawed or feeling the subject to be difficult or sophisticated. The teachers were not told until after the session that it was transformation geometry though no doubt there was some awareness of it. None of the teachers had previously attempted or witnessed transformation work with children.

Lesson The major portion of the lesson used cut-outs on an overhead projector as a means of working in silhouette to remove extraneous detail.

Step 1 The pupils were asked to decide whether the two shapes on the screen were the same or different. The decision was that they were different. When asked to state clearly the basis of their decision, they agreed unanimously with the pupil who said that one of the shapes had three points while the other shape had four points. The purpose of this warm-up question was partially to focus some attention upon the triangular shape with which I intended to deal.



Step 2 The pupils were informed that we now wished to concern ourselves with only 'this shape'. They were invited to demonstrate a solution to the problem of moving the shape to another location <u>without moving the paper cut-out shape from the projector surface.</u> Quite a number of pupils volunteered and the first moved it to the position indicated by the broken line. A second and third pupils were asked to show the same type of move which they did with ease. One of the pupils stated that he, "kind of slid it over". On being asked to be more specific he



General Comments

1. The lesson was of twenty-five minutes duration and contained much more than would be handled in a teaching period but this was necessary, I feel, to achieve the second ~~stated~~ purpose successfully.

2. The youngsters participated freely, remained interested despite the tiring that naturally would accompany such a heavy load.

3. The work was well within the scope of the children and frequently they gave the impression of wanting to move along to two or three-step problems.

4. The work and concepts could well be valid at the grade one level.

5. Many children eagerly went to the manipulative material at the first opportunity after the class session.

6. The teacher-observers all seemed secure enough to be willing to work with their own classes in the field and some to teach demonstration lessons for any other interested teachers.

7. The overhead, silhouette technique is apparently as dynamic and comprehensible as anticipated.

8. The triangle is well suited as the demonstration shape.

9. Careful programming must follow to prevent only spatter-awareness in a simple, consequential but potentially complex field.

10. The term "TRANSFORMATION" was not used at any time.

11. The pupils were not disturbed in any apparent way by the presence of several extra adults, this I suspect being in part due to their absorption in the topic.

12. A first lesson should include only two transformations, I feel, to reduce the load. Naturally, a very young group would perhaps have only one transformation in the first session. The advantage in two in the same period is that the pupils can decide which transformation of two is being used i.e. the implication that transformations of various types can occur.

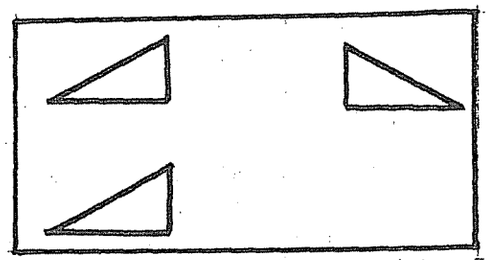
13. It seems unwise to introduce transformations before concept of shape has been synthesized reasonably.

14. The manipulative aspect of the presentation seems to have been a major factor in the high-percentage 'take'.

APPENDIX B

Pupil-Derived Transformation Problems

Problem 1 This puzzle was composed by one of the boys for another boy in his class. On the plastic board there were three green triangles which he had placed according to the diagram below.



The Problem The puzzle was stated as follows by the boy, "I want all of the triangles to end up here (he pointed to the approximate position of the triangle in the top right-hand corner). You may move each piece only once."

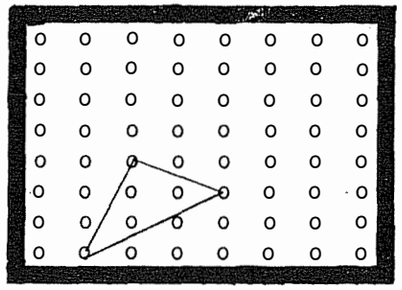
His Solutions

No. 1
- Flip the triangle at top right of page
- Slide the triangle at top left of page on top of the triangle at the top right
- Slide the triangle at bottom left on top of triangle at top right

No. 2
- Slide the triangle at bottom left up to the triangle at top right
- Flip triangle at top right on top of it
- Slide the triangle at top left on top of the other two

The boy's final comment to me was, "I'm going to keep trying to see how many more solutions I can find."

APPENDIX D

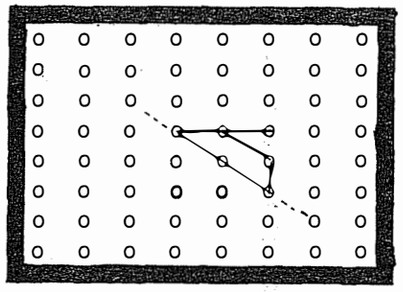


SLIDES 1·24

Rules: 1. You may use only slides.
2. Slides may be only straight across or straight up or straight down.
3. Points of the shape must always end exactly in the circles.

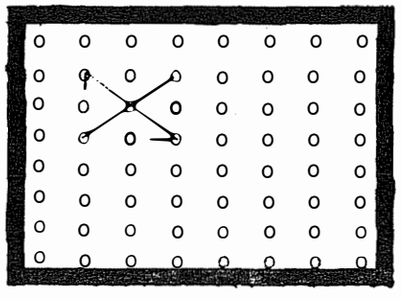
Problem: - Move the shape to at least two other places on the page.
- Put the problem number on the bottom of your worksheet.

FLIPS 2·55



Rules: 1. You may use only flips.
2. Flips may be made only on edges.
3. Points of the shape must always end exactly in the circles.

Problem: - Move the shape to at least two other places on the page.
- If a flip edge is marked, use it first.
- Try to use more than one flip edge.
- If there are circles inside your shapes fill them in.
- Number your shapes.



TURNS 3·62

Rules: 1. You may use only turns.
2. Turns may be made at the corners only.
3. Points of the shape must always end exactly in the circles.

Problem: - Move the shape to at least two other places on

the page.
- If a turning corner is marked, use it first.
- Try to use more than one turning corner.
- If there are circles inside your shapes fill them in.
- Number your shapes.

APPENDIX E

TRANSFORMATIONS

PROBLEM 1B

1. Get a piece of blue background paper.
2. Get one piece of small white paper and one piece of red paper.
3. Decide on a shape. Here are your rules:
   - at least four edges
   - one edge must be a curved line
4. Get a pair of scissors.
5. Cut two white shapes that are exactly the same.
6. Cut 15 shapes exactly the same as your original white shape. Return the scissors.

YOUR PROBLEM

Decide on a starting place on the background sheet and place your first white shape there.
You may use only flips and turns with the rest of your shapes.
You may make no more than 8 moves of the same kind.
Your shapes may not overlap.
Turns must be corner turns.
Flips must be made on edges only.
Get paste when you are satisfied with the moves and positions that you have.
Paste down your shapes.
Put your name on your work in its proper place.

Anna Pataky was born in Budapest and educated in Canada. She received a B.A. from York University. All her teaching has been done in the North York system, she was a consultant for several years with the North York Board and is currently a Teacher Consultant (Grades 1-6). The above article is the text of a talk she gave at the recent N.C.T.M. meeting in Toronto.

Opinion —

Currently the topic of Euclidean Geometry is required in both 5 year grades 10 and 11. As I seem to understand, the reasons of its inclusion is firstly that the facts about relationships can be useful in some practical situations, and secondly that the organization of the topic is meant to be a tool for the development of clear logical thinking.

If these indeed are the reasons then as to the second reason I would argue that the system seems to have used the proverbial awkward sledgehammer to drive the proverbial thumb-tack.

Many students become aware only gradually that the geometry is an example of formal logic. For many it takes a great deal of practice to recognize what is an "acceptable" proof and what is not. They seem so very concerned with the "facts" of the theorems, without reaching the introspective stage at which they are able to see through the meanings attached to the words to the actual procedure of solution.

To help overcome this difficulty I would argue as a first step that in grade 10 rather than the time being dedicated totally to Euclidean Geometry (they know some of the "facts" already) that formal logic be taught to a level at which students will be able to distinguish between a valid proof and one which is not valid.

It also seems likely that even a 1/2 credit or full credit course in symbolic logic is beneficial. The applications to math are obvious (verification of solutions, proofs of identities, etc.). The application to all other areas in which one must present a coherent connected argument (English, History, etc.) is equally valid.

D. E. Kennedy
John F. Ross C.V.I.

Book Reviews

For a list of books available for review, write the editors.

Mathsense 1, by Douglas S. Ailles, Gin & Co., pp. 142.

We need this book as a supplementary text in grades 4-6, can recommend it highly, and intend using it again next year. Some topics, the geometry, for example, were not suitable in our situation, but the treatment of fractions was excellent, and the drawings most enjoyable. It was particularly suitable for the upper mathematics groups in grades 4, 5, and could even be used as a supplementary text in grade 7.

Some of the grade 4 children's comments:

'I like your book very much. It teaches things you might not ever learn in Grade 4. I've learned a lot. Thank you very much.'

'I think your book is worth writing because someone is going to pass math with flying colours.'

'Some of the art work is rather modern. An excellent book.'

'It is a good book and I like math so it is good.'

Obviously, very highly rated by the consumers! We also enjoyed it!

L. Etienne
J. Milkie
C. D. Howe School
Lakehead Board of Education

Elementary Statistics Through Problem Solving, Avenoso & Cheifetz, Burns & MacEachern Ltd., 1974, 183 pages, $8.25.

The text provides a good introduction to the study of statistics. It could easily be used in high school at the senior level or at the post-secondary school level with a class of liberal arts students.

COUNCILLORS
Term Expiring 1975

James L. Boyd

Gordon Addison

Constance Cardenas

John Girhiny

Kenneth Kenney

Michael McKenna

Brock F. Rachar

Marianne Reid (Sister)

Jean Starrs

Neil Williamson

Don Roberts

Bruce Topping

Peter Weygang

David Wing

COUNCILLORS
Term Expiring 1977

Hugh Allen, Queen's University, Kingston, K7L 3N6

Dino Dottori

Douglas Evans

Louis Rancourt

Walker F. Schofield

Paul Sherk, Cedarbrae Collegiate, Scarborough, 86 Banmoor Blvd. M1J 2Z6

Michael R. Silbert

J. Edwin Silcox

Robert Smith

Sharon Smith

CHAPTERS

Lucien Lefebvre

Robert Tuck

Frederick Maskell

SUBSCRIPTION RATES:

$5.00 for one year's subscription

$2.50 for a single copy

O.A.M.E. members receive the Gazette as an adjunct to their membership.

Invoices will be sent 30 days prior to expiry of a subscription, and 60 days is allowed for payment.

Orders for subscription and single copies should be sent to the Secretary-Treasurer

M. B. MacGregor