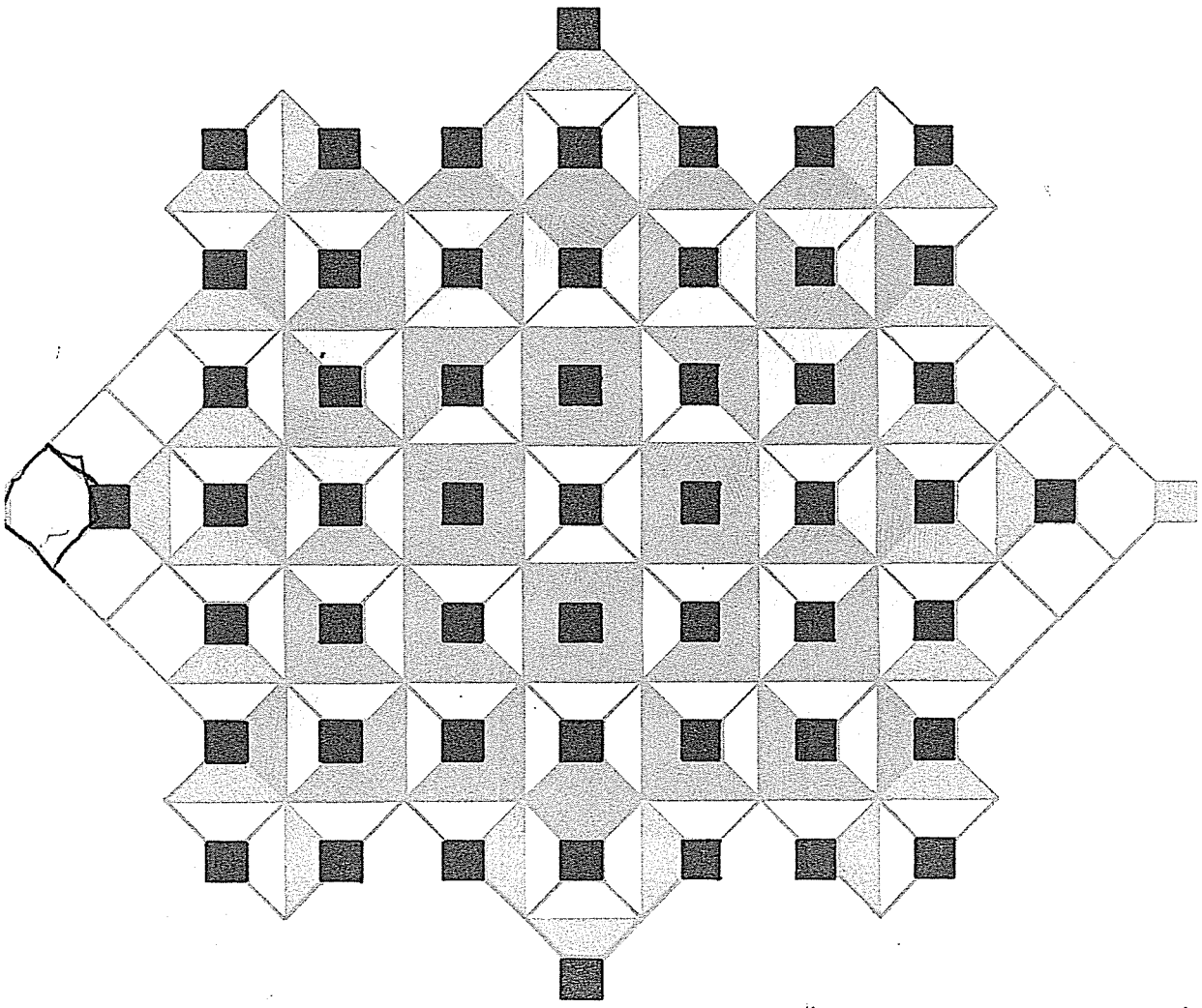
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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Editorial correspondence and articles should be submitted to one of the Editors, W. Eames or J. S. Griffith at the address below.

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Dr. J. S. Griffith
Department of Mathematical Sciences
Lakehead University
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P7B 5E1

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ONTARIO MATHEMATICS GAZETTE

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Editorial

At various times it has been suggested that a Canadian journal for mathematics educators be established. We see this as being supplemented by newsheets of regional interest, just as Abacus complements the Gazette. The various provincial journals would be amalgamated into one, which should be large enough to cover all areas of mathematics education. At present editors are sometimes short of good articles, or see one of the articles which has already appeared in their provincial journal appear in another one. For the past three years we have struggled with the Gazette. through a financial crisis, shortage of articles, and seen unnecessary duplication. We would be in favour of a national journal.

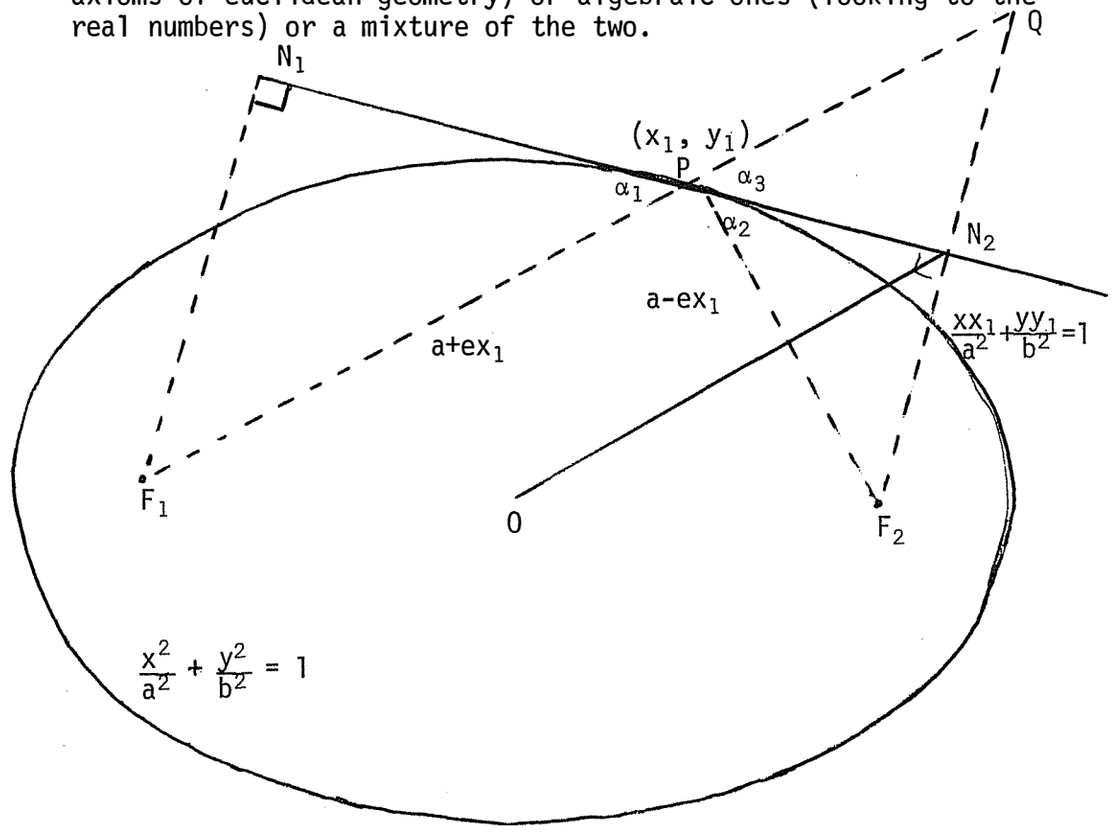
What about our readers? Have you any opinions about the matter or about the topic of the "New Math' which is discussed by articles in this issue. Editors appreciate knowing that the journal they produce is read and we would be pleased to received (and perhaps print) your comments.

The Editors.

MATHEMATICS OLYMPIAD

Garth Thomas, Chairman of the Olympiad Committee states that the Olympiad could be held as early as May 1, 1975, but that no date has yet been fixed. Principals who wish to nominate an outstanding student should write by April 1, 1975 to Dr. Thomas at the Department of Mathematics, University of Sasketchewan, Saskatoon S7N OWO. The Ontario Committee (ProfessorsDuff, Coleman and Fryer) usually nominate 50 students, choosing from the top students in the M.A.A./Society of Actuaries Contest.

resting mainly on a foundation of euclidean goemetry and the properties of number systems. The place of geometry (inasmuch as it is taught at all now, and not squeezed out by statistics or numerical analysis) itself will be likewise ambivalent; for some results we may still have recourse to practical investigation, thus recognizing geometry as a part of applied mathematics, whilst for others we may choose to employ either distinctively geometrical arguments (looking back to the axioms of euclidean geometry) or algebraic ones (looking to the real numbers) or a mixture of the two.



Take, for example, some of the traditional work on the ellipse: the method sometimes used to prove the property of auxiliary circle. We begin from the cartesian equation of the ellipse and use calculus to find the equation of the tangent at (x_1,y_1). We then call on a result from analytical geometry to write down the distances from the foci to the tangent as $(a+ex_1)/aR$ and $(a-ex_1)/aR$ (where $R^2=x_1^2/a^4+y_1^2/b^4$), and on one from synthetic geometry (focal distance/directrix distance=e) to find the distances from the foci to the point of contact. The rest is simple euclidean geometry: similar triangles give $\alpha_1=\alpha_2$, so $F_1Q=F_1P+PF_2=2a$ (using a different

definition of the ellipse) and $ON_2=a$.

You might think of this as a splendid example of 'integrated mathematics'. Of course, 100 years earlier it would have been done differently. Instead of having recourse to differentiation and equations of lines, the whole development would have been carried out on a euclidean basis, with some 'reasonable' assumptions about the limiting behaviour of chord and angles to cope with the tangent problem. Now I am going to be very heretical, and suggest that our forbears had a clearer sense of direction than many teachers have nowadays when they do this piece of work. For at least they were displaying consistency in their philosophy of mathematics teaching - a view that mathematics consists in working out the consequences of definitions and axioms - whereas it is not at all clear what is the purpose of the development just described in a mathematics course.

You must not get me wrong. Let me make it quite clear that there are two things that I am <u>not</u> saying:

(i) I am not (need I say?) advocating a return to the teaching of geometrical conics, or anything similar. Indeed, as already indicated, I reject the idea of developing mathematics, at school or later, as a set of distinct 'branches'.

(ii) I am not saying that students should not meet pieces of mathematics like this. It is fun, the results are attractive and not obvious, the way in which different items of knowledge are brought to bear on the theme is powerful, it has some potential applications in engineering - and, as those of you who read Polya or Courant and Robbins will know, the idea can be extended to throw light on a range of extremum problems which, in their turn, suggest some new geometrical results besides having a relevance to fundamental thinking in theoretical physics. So please go on teaching this, especially if you are master enough of your subject to use it as a base from which to broaden your pupils' mathematical experience (but please don't set a test on it afterwards!).

What I <u>am</u> saying is that bits of mathematics like this do nothing to help us with the business of mathematics curriculum planning. It would be an exaggeration to say that these properties of the ellipse will be relevant to 1% of our

students - as already suggested, the occasional mechanical or civil engineer might have some use for the information, but that is hardly a prescription for mass education. So let this stay around in our classrooms and our library books for its entertainment value, and let us give our students time to enjoy this sort of thing, but let us look elsewhere for guidance about the syllabus.

The first essential is to determine what is our primary aim in mathematical education at this level. Is it, as current practice would suggest, to produce mathematical craftsmen, skilled in the application of selected techniques and with a detailed knowledge of certain predetermined sequences of theorems? I suggest that such training is rapidly becoming irrelevant; in many traditional areas of mathematics the computer can already do better than the craftsman. What is needed rather is a new kind of mathematical technologist, distinguished not for greater technical skill but for insight, having an eye for structure in its widest sense, a capacity for recognizing new relationships and an awareness of their potential as models for situations in the real world.

I wish to stress that a new approach is as important for those who will apply mathematics in other fields as for the so-called pure mathematician. It is not long since it was assumed that, because measurements in the laboratory were of necessity represented by real numbers, therefore the appropriate mathematical training for a scientist consisted in a study of the continuum of real numbers - i.e. the calculus. But things are not always what they seem; and it is now recognized that in a particular investigation the appropriate model is quite likely to be found at some higher level of abstraction - in the complex numbers or a probability function, in a space of functions or a non-commutative algebra. It is in the secondary school, rather than at the level of post-graduate training, that such possibilities should be opened up.

This is not to suggest that a secondary mathematics course could or should be conducted at an exalted level of abstraction, or that there is no place in it for the development of manipulative skills. Apart from their intrinsic usefulness to a practising mathematician, these have an important place in teaching as the roughage of a mathematical diet, and in ensuring for the student a bed of security within which mathematical ideas can germinate and grow. To take an example, we shall still doubtless wish students to

know that $\int x^2 dx = {}^1/_3 x^3 + k$. But rather than pursuing the formal integration of algebraic expressions ad nauseam, the student can link the concept of integration to many other themes suitable to secondary school study - for example, he may look at the indefinite integral as an equivalence class; at integration as a linear operator; at similarities and differences between integration and summation of sequences; at generalizations such as the integration of vector functions and integration over sets other than intervals of the real numbers; at the assumptions of continuity implicit in using integration in physical and probability models; at numerical approximations to the values of definite integrals; and so on.

This was borne in upon us some years ago when the S.M.P. took the decision to introduce 'transformation geometry' into its syllabuses. To begin with we (as a group of teachers who had been offering our pupils 'diluted Euclid' previously) were full of enthusiasm for the new weapons which transformations gave us for attacking the familiar theorems and construction problems - making equilateral triangles with vertices on three given lines, and the like - and we were always asking each other the question "can you find a way of proving (or constructing) such-and-such by transformations?". It was only with the sobering effect of experience that we realized that the real pay-off lay not in 'doing euclidean geometry by transformation methods' (and in many cases the old euclidean proofs were better anyway) but in the recognition that transformations were in fact functions looked at in a geometrical context; that the combining of transformations was none other than the 'function of a function' situation; that the symmetry groups of geometrical figures illustrate in a small scale more general groups of functions; that transformations are isomorphic to sets of matrices under appropriate operations; and so on. The result of this was not to remove from the course the simple experimentally based work on particular geometrical figures, but to give it a new emphasis and direction - and it is emphasis and direction which are my theme in this lecture.

The labels which we conventionally attach to school mathematics do little harm so long as they are regarded as descriptive rather than prescriptive. This has been put succinctly in the description of geometry as "not so much a subject, more a point of view." Inasmuch as differentiation within mathematics is useful at school, it lies in the means

What are our mathematical models to be made with? I suggest that the answer to this question lies in the 'growing points' which we find near the hub of our diagram. I deliberately refrain from attempting a complete list, or even from claiming that mine is right at the present time - and it is certainly not static. If pride of place has to be given, then it is surely still to numbers - in particular to natural numbers and real numbers - which could be described as the 'bricks' of mathematics. But, even in our conservative British cities, there are many materials other than brick around these days; and if we may continue the metaphor, we might identify vectors, operators and matrices as the plate glass, the reinforced concrete and the plastic in the mathematical builder's yard.

Let us look at just one of these in more detail. I will deal with the concept of vector, but several others would do as well. How did the notion of vector first appear in mathematics? I suggest to you that it was not as an equivalence class of directed line segments, or as a mapping from $R^3 \to R^3$, or as an element of a linear space - it was because it was a tool which was needed in order to get to grips with the growing complexity of physical phenomena being investigated. It is more convenient to write

$$\underline{F} = m \frac{d^2\underline{r}}{dt^2}$$

than to write the separate equations

$$X = m \frac{d^2x}{dt^2} \ , \ Y = m \frac{d^2y}{dt^2} \ , \ Z = m \frac{d^2z}{dt^2} \ ;$$

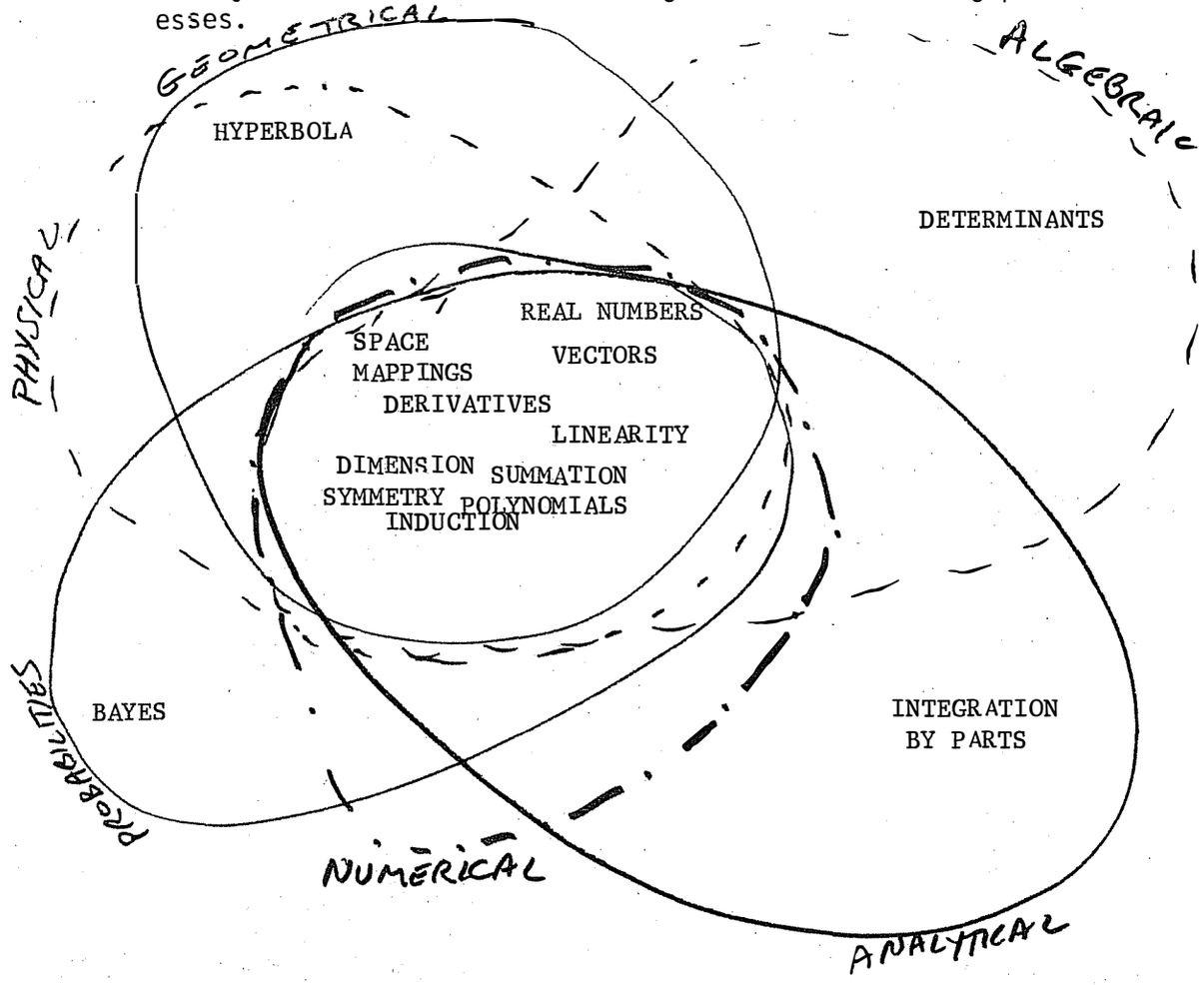
and the laws of electromagnetic induction make simple sense in the notation of vector products. It is sometimes argued that the advantage is more apparent than real, since we must ultimately revert to the separate equations to compute any usable results; but this is surely to miss the point, that it is the ease of <u>conceptual</u> handling that makes this such a valuable and adaptable building material. To take an example from another area of application of the vector idea: regarding a set of m equations in n unknowns as a linear mapping from a space of n dimensions to one of m dimensions does not of itself make the equations any easier to solve, but it provides us with a conceptual framework within which we can analyse a situation of apparently excessive complexity.

Having identified this conceptual role, let us come

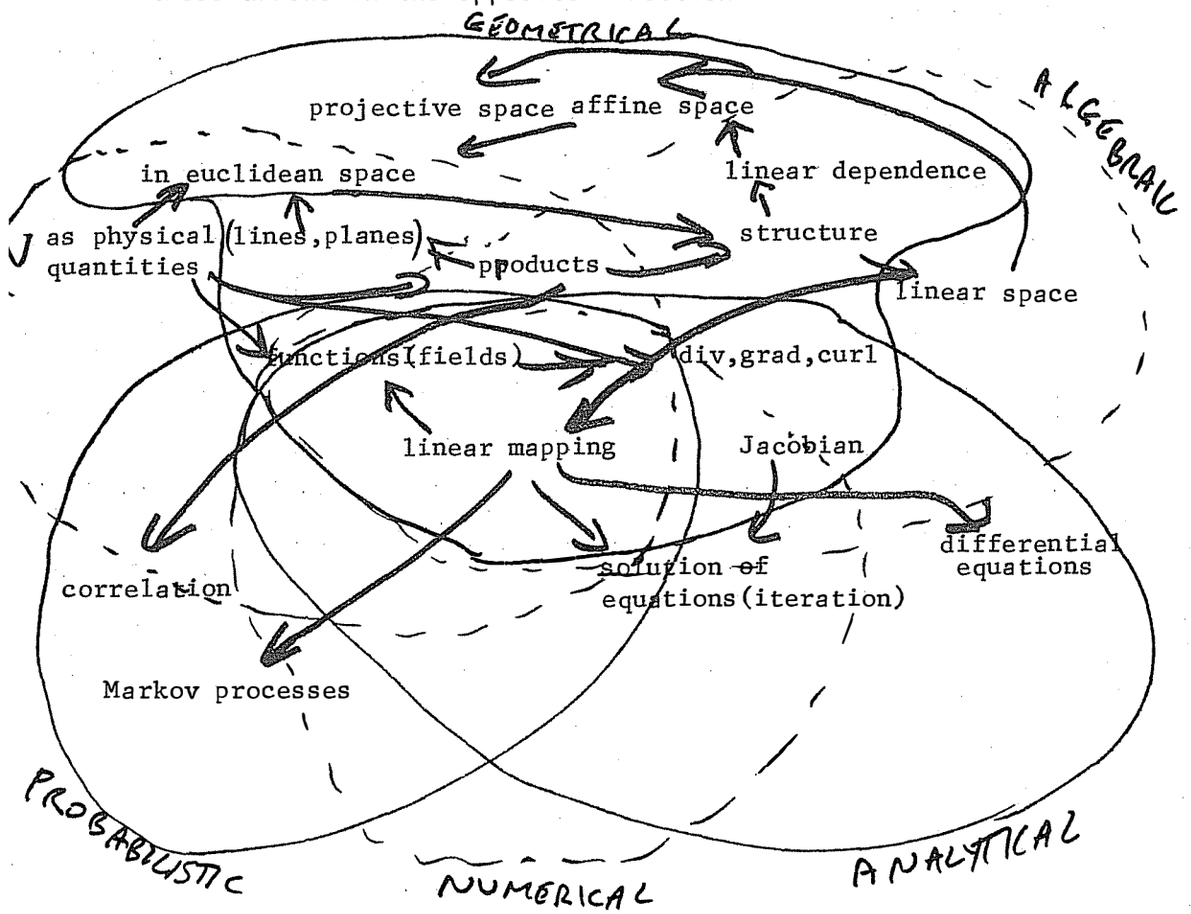
To make progress along these lines it is first necessary to break free from subject-based planning and to look at the mathematics relevant to this level of study as a whole. This is not to deny the usefulness at times of distinguishing various modes of thought - geometrical, numerical, physical, probabilistic, algebraic, analytical, and so on - but we aim at a view of mathematics in which, in moving from one mode to another, 'changing gear' is a more appropriate metaphor than 'crossing a frontier.' Some topics by their very nature fit into one particular slot - the hyperbola is essentially geometrical with physical overtones, determinants are algebraic, Bayes' theorem is probabilistic, integration by parts analytical. These, by and large, are inessentials of the curriculum: they do little to deepen our insight or to increase our capacity to solve problems outside the sphere of immediate application. In a curriculum pressed for space, a choice between topics such as these can be a matter of personal taste and the motivation of the individual student. The kernel of a mathematics programme, on the other hand, lies in those themes which increase in richness as they are studied from more and more points of view: space, real number, mappings, derivatives, linearity, vectors, dimension, summation, symmetry, polynomials, induction, etc. These belong to no one branch of mathematics, but are growing points at the base of many branches.

It will not surprise you if I confess at this stage that at heart I am an applied mathematician - I like to see mathematics being used. By this I would not wish it to be understood that I want to teach theoretical hydrodynamics in place of linear algebra, or to ban the mention of ideals in college algebra courses. But when half the world lies at the feet of the mathematician - and you will notice that I am being modest in my claims! - I find it difficult to enthuse over the discovery of even bigger prime numbers, or the refining yet further of the conditions under which a certain process converges. My concern, then, is for those students who are going to find mathematics grist to their mill; and for them the essence of mathematical training must be the creation and manipulation of models appropriate to their needs. (Some of us in England are at present trying to devise a collection of 'starting points' with presisely this in mind - leading into mathematics through problems raised by recognized needs, rather than through the accumulated wisdom of preceding generations of mathematicians.)

of communication rather than the internal structure. In 'geometry' we use the language of points, lines, planes and circles and convey much of the argument through diagrams; in 'algebra' we communicate formally through symbols; whilst in 'analysis' we have recourse to arguments from limiting processes.



back and see how this can help us in the process of curriculum planning. The chart below shows some of the ways in which, in this particular area of mathematical application, one thing can lead to another. There are, of course, many ways of organizing one's overview of a particular part of mathematics, and my own breakdown and order reflects my own taste in mathematics; you might well choose to put some of these arrows in the opposite direction:



Notice how such a network spreads right across our mathematical map. Vectors do not belong to physics, or to geometry, or to algebra,..., they belong to mathematics, and are (at the present time) one of the mathematician's central resources. The theme might be described as 'diversity without fragmentation' - and it is in these terms that we should be looking to plan curricula for our students in the 1980s.

Our constraint on the realization of this approach has been the limitation imposed by the conventional 'book of the course.' For one thing that is plain from the chart is that we cannot conceive of the learning of mathematics in terms of a linear model. The notion of the 'steady logical development' as the route to mathematical knowledge is a thing of the past. One of our most urgent problems is to decide how one may structure this view of mathematics into a system of usable resources for the wide variety of students who will want to call upon it. So let me close as I began: I do not bring you answers to our problems, but rather I hope to share with you some of our on-going thinking, so that we may help each other towards a programme which will meet the needs of the coming generation of students.

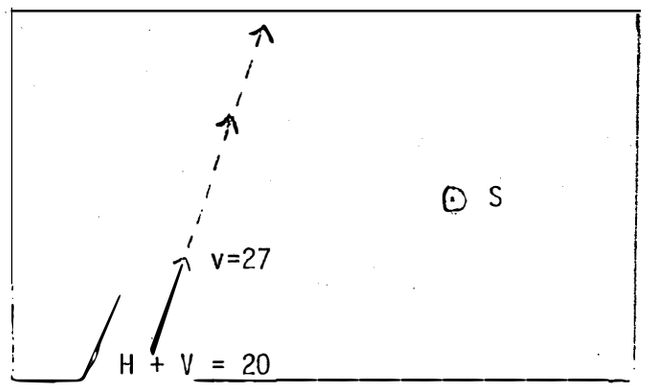
D. A. Quadling

Hunt the slipper

Douglas Quadling

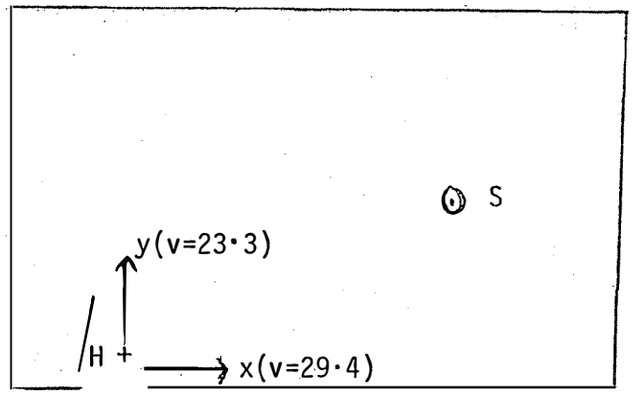
I hope you are all familiar with the ancient British party game with this title - but for the benefit of any who are not, here is a brief description of the rules. The game is traditionally played on the evening of Christmas Day, when the extended family is replete and in an uncritical frame of mind, sitting around the fireside. By some pseudo-random process, an individual is selected and temporarily expelled from the room. Those who remain conspire together to conceal a slipper in some improbable place in the room. (There is a sophisticated version of the game in which the slipper is replaced by a thimble - but this is obsolescent in view of the decline of thimble-usage in British households.) The exile is re-introduced to the company, and simultaneously, the conspirators begin to sing a song with many verses. The contest is now on: the exile hunts for the slipper, and the only help given to him is through the volume of singing, which is controlled so that it is inversely proportional to some power of the hunter's distance from the slipper.

The problem I wish to put to you is this: what tactics should the hunter pursue in conducting his search? Prolonged study has suggested to me that the average player employs none at all - he merely takes a step into the room in an arbitrary direction, and if the volume V increases then he proceeds across the room on a fixed course, usually missing the target by a wide margin:



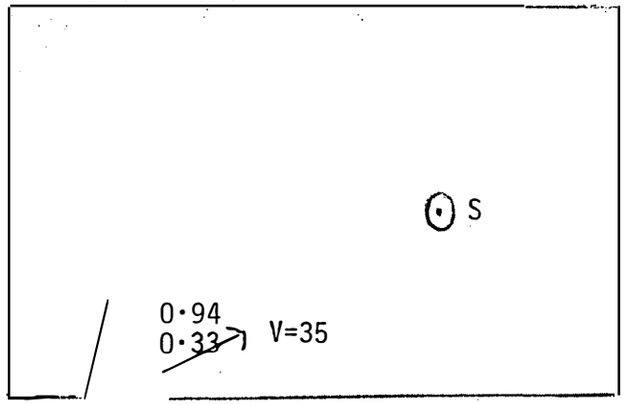
Suppose, however, that (even after Christmas dinner) the hunter is alert enough to look at the problem systematically. Then he might proceed as follows. He first takes one step HX along the wall of the room; the volume of singing

increases, say, from 20 units to 29.4, a rise of 9.4 units. He now returns to the doorway, and takes a step HY directly into the room; the volume now goes up to only 23.3, a rise of 3.3.



He decides on this evidence, perhaps, to return to the doorway and to take steps having x- and y-components proportional to 9.4 and 3.3 - say 0.94 and 0.33. In this way he hopes that the volume of sound might go up to

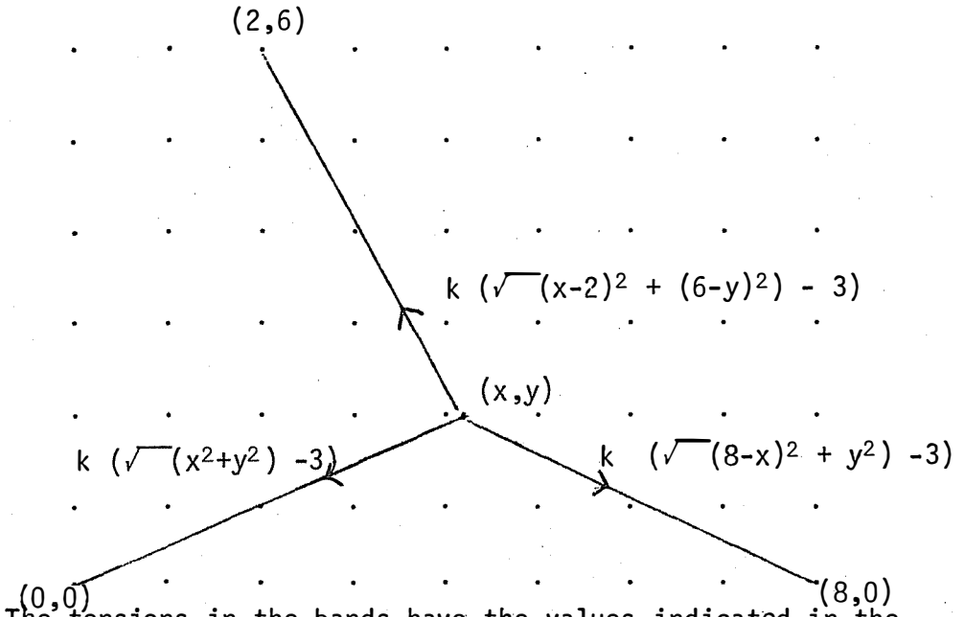
$$20 + 9.4 \times 0.94 + 3.3 \times 0.33 = 30.$$



The fact that, on completing the step, the volume turns out to be 35 rather than 30 units is a salutary reminder that things are very non-linear around the slipper, so he was perhaps wise not to take too big a step; but at least he achieved a very good direction and it should not be long before his search is rewarded.

This party game gives me my excuse for introducing to you the general problem of locating a point in a plane. Here is a simple practical problem that leads to a situation of a similar kind. We have a board with pegs fixed at

points with coordinates (0,0), (2,6) and (8,0). Elastic bands, of unstretched length 3 units, are hooked round the three pegs and joined together with a knot. The problem is to locate the coordinates (x,y) of the knot in the position of equilibrium.



The tensions in the bands have the values indicated in the figure (k being an irrelevant constant which merely says that the bands are identical), and considering the forces in the x- and y-directions gives the two equations

$$(\sqrt{((8-x)^2+y^2)}-3)\frac{8-x}{\sqrt{((8-x)^2+y^2)}} + (\sqrt{(x^2+y^2)}-3)\frac{x}{\sqrt{(x^2+y^2)}}$$

$$- (\sqrt{((x-2)^2+(6-y)^2)}-3)\frac{x-2}{\sqrt{((x-2)^2+(6-y)^2)}} = 0,$$

$$(\sqrt{((8-x)^2+y^2)}-3)\frac{y}{\sqrt{((8-x)^2+y^2)}} + (\sqrt{(x^2+y^2)}-3)\frac{y}{\sqrt{(x^2+y^2)}}$$

$$- (\sqrt{((x-2)^2+(6-y)^2)}-3)\frac{6-y}{\sqrt{((x-2)^2+(6-y)^2)}} = 0.$$

Nobody, I hope, is going to suggest that we attempt an algebraic solution of these fearsome equations. Perhaps the best way to start is to carry out the experiment; this indicates that the knot settles around (4,2). Actual substitution of these values for x and y in the left sides of the two equations yields -0.66 and 0 respectively, so this is in fact quite a good estimate. But our problem now is

the same as the slipper-hunters - how can we improve on this and get a better approximation.

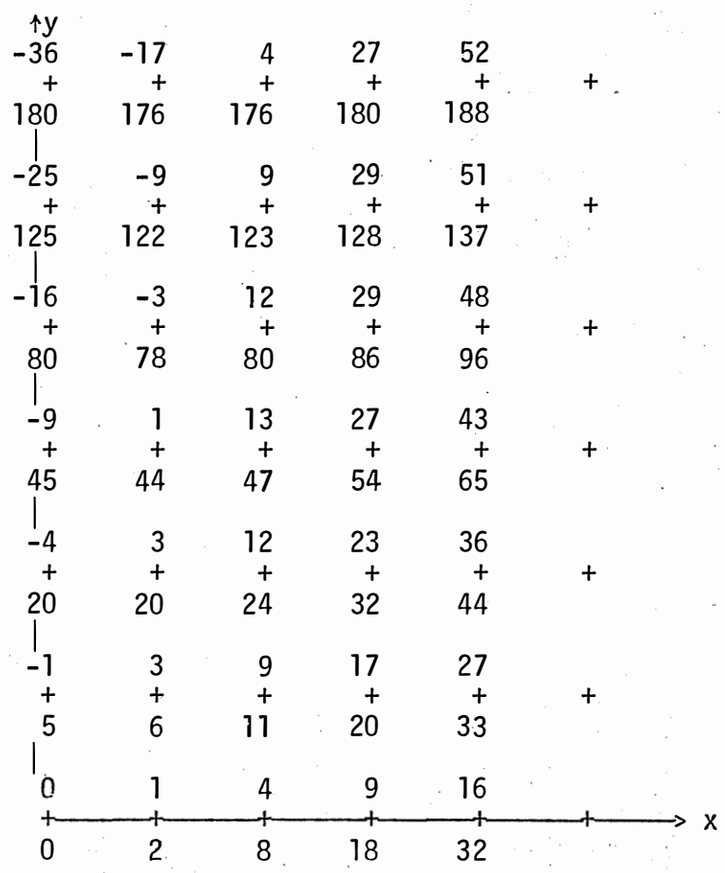
To simplify the discussion, I shall not in fact proceed with these two equations, but deal instead with a simple pair of equations in two unknowns, not related to a real problem; but the method which I shall describe would work in just the same way for the pair of equations above if I had the courage to carry out the manipulations. Let us then try to solve the following pair of equations for x and y:

$$x^2+3xy-y^2=30,$$
$$2x^2-xy+5y^2=130.$$

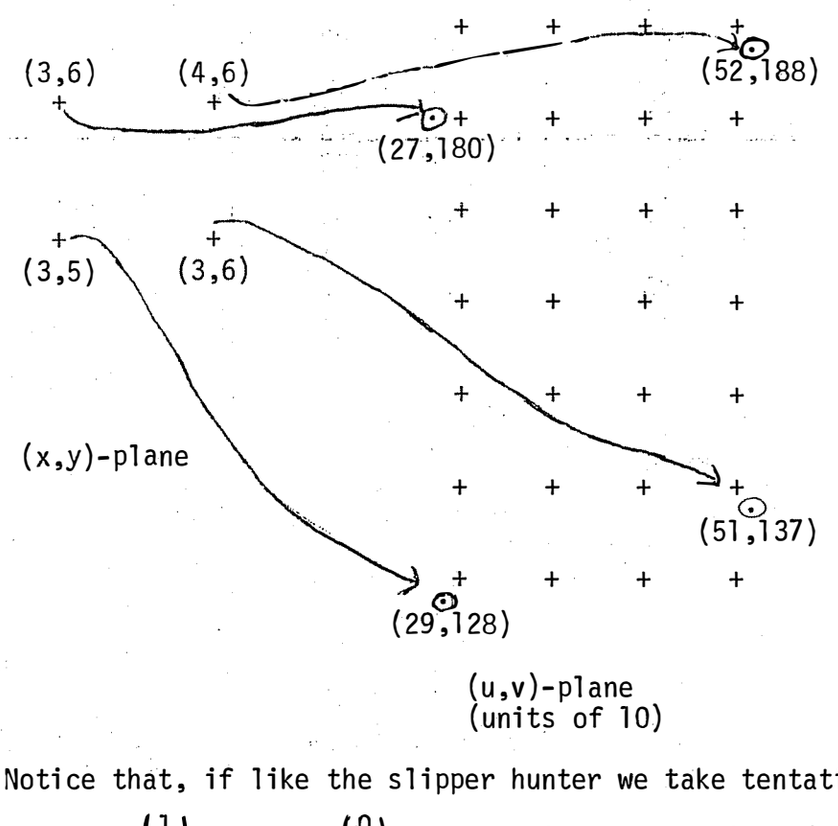
(There is, of course, an algebraic procedure for solving a pair of simultaneous quadratics; but we will ignore this, so as to get a general method that could be used for any pair of equations.)

To give me a line on where the solution might lie, I shall begin with a trial and error procedure. The figure below shows a part of the (x,y)-plane with crosses marked at the points with integral coordinates. Against each point are marked two numbers: the upper one is the value of $x^2+3xy-y^2$ (=u, say), and the lower is the value of $2x^2-xy+5y^2$ (=v).

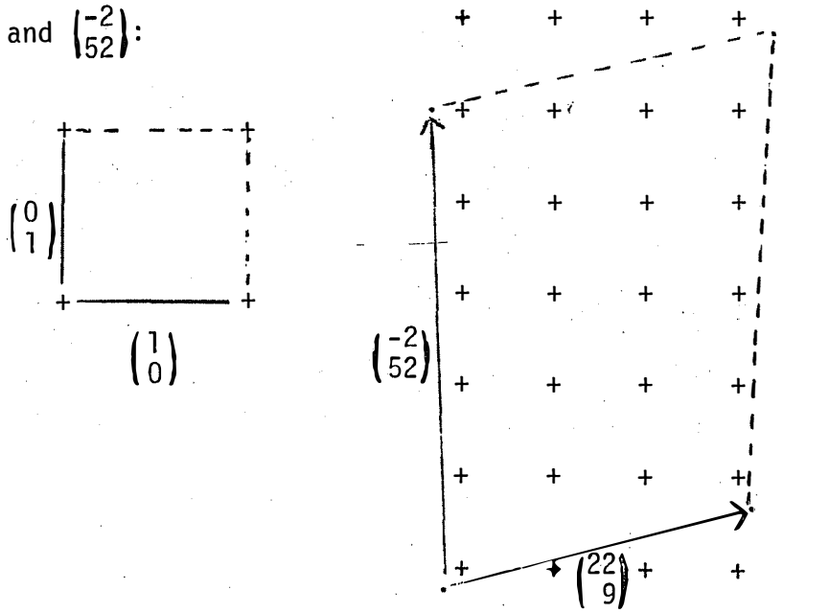
A quick scan suggests that, to get u=30 (the first equation) we shall need to be somewhere to the south-east of the point (3,5), whilst to get v=130 we want to be somewhat north-east of that point. Our solution must resolve this apparent conflict of interests so that both equations are satisfied simultaneously.



It is very tempting now to regard the two numbers against each point as coordinates of a point (u,v). When we do this, there emerges the idea of a mapping from values of (x,y) to values of (u,v):



Notice that, if like the slipper hunter we take tentative steps of $\begin{pmatrix}1\\0\end{pmatrix}$ and then $\begin{pmatrix}0\\1\end{pmatrix}$ from our 'first guess' of (3,5), then the corresponding steps in the (u,v)-plane are $\begin{pmatrix}22\\9\end{pmatrix}$ and $\begin{pmatrix}-2\\52\end{pmatrix}$:



This suggests that if from (3,5) we were to take a step of $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}$, that is $\delta x \begin{pmatrix}1\\0\end{pmatrix} + \delta y \begin{pmatrix}0\\1\end{pmatrix}$, then we might expect the corresponding (u,v) step to be approximately

$$\delta x \begin{pmatrix}22\\9\end{pmatrix} + \delta y \begin{pmatrix}-2\\52\end{pmatrix},$$

which can be written as

$$\begin{pmatrix}22\,\delta x + (-2)\delta y\\ 9\,\delta x + 52\,\delta y\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}22 & -2\\ 9 & 52\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}.$$

Now to solve our equations we wish to make u=30 and v-130. Since (3,5) maps onto (29,128) this means taking a step in the (u,v)-plane of $\begin{pmatrix}1\\2\end{pmatrix}$ - and our problem is to find the step $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}$ which will achieve this. This means that we require

$$\begin{pmatrix}22 & -2\\ 9 & 52\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \approx \begin{pmatrix}1\\2\end{pmatrix},$$

which suggests that

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \approx \begin{pmatrix}22 & -2\\ 9 & 52\end{pmatrix}^{-1}\begin{pmatrix}1\\2\end{pmatrix}$$

$$= \frac{1}{1182}\begin{pmatrix}52 & 2\\ -9 & 22\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}$$

$$= \begin{pmatrix}.0474\\ .0296\end{pmatrix}.$$

Since this step is taken from (3,5), this gives an approximate solution for our equations as

x=3.0474, y=5.0296.

At this point it may be helpful to pause and take stock. We began with a pair of algebraic equations in two unknowns. To the person with the traditional mathematical upbringing the 'obvious' step would be to destroy the symmetry by eliminating one of the unknowns (though in fact this is not the trick for dealing with a pair of equations of this particular type). We have proceeded quite differently - by re-casting the situation so that it appears as a mapping from points in an (x,y)-plane to their images in a (u,v)-plane;

to observe that the mapping, though not exactly linear, is not far off being so; to call therefore on our experience of transformations, and in particular to use the expression in matrix form; and to effect the 'inverse mapping' required with the aid of the 'inverse matrix.' To me this is the kind of thing that 'modern mathematics' is about: the establishment of a range of abstract central concepts (mapping, vector, linearity, matrix, inverse) which belong to no particular 'branch of mathematics' but which the student of the subject needs to learn to recognize when they are manifested in any context, however unexpected.

It must be admitted that, as with our slipper-hunter, this first shot at solving the equations was based on a very rough-and-ready assumption. We can see this by remarking that, had we extended the approximation over the whole square in the (x,y)-plane, the image of (4,6) would have turned out to be displaced from (29, 128) by an amount

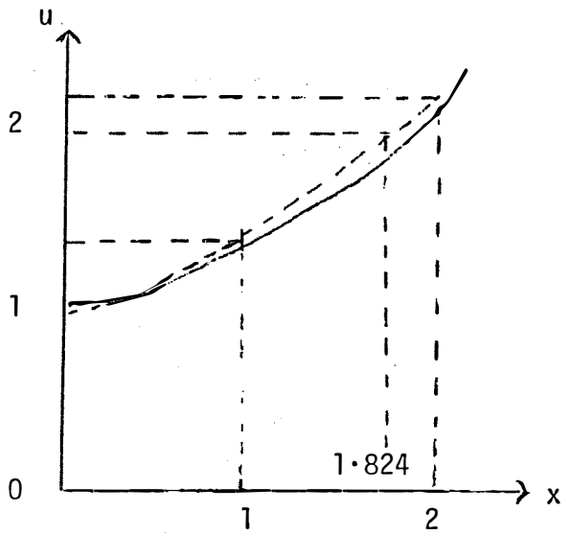
$$\begin{pmatrix} 22 & -2 \\ 9 & 52 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 61 \end{pmatrix} ,$$

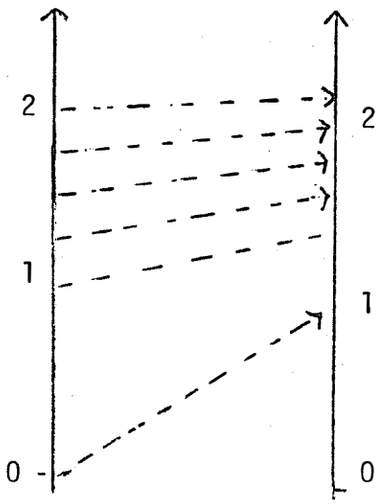
so that this image would have been at (49, 189) rather than (52, 188) as it actually is; the points in the (u,v)-plane would have been at the vertices of a parallelogram rather than an irregular quadrilateral. This reminds us once more that there is some non-linearity about.

Let us ask, then, how the solution might be improved. When we are faced with a problem in mathematics it is often helpful to proceed by analogy with some more familiar and perhaps simpler situation. In this case, it may help to look at what we have been doing in the context of an equation in just one unknown, for example

$$x + e^{-x} = 2.$$

Here again we may think of the function $x \longmapsto x+e^{-x}$ (=u, say), and we may begin with a trial-and-error procedure. It is easily calculated that $0 \longmapsto 1, 1 \longmapsto 1.368$, $2 \longmapsto 2.135$, and so the solution would appear to lie between 1 and 2. We can show what is going on with a graph; but notice that this is going to give us no idea how to proceed in the example with two unknowns. The equivalent of the diagram illustrating the mapping from (x,y) to (u,v) is rather an 'arrow diagram' showing the mapping from x to u.





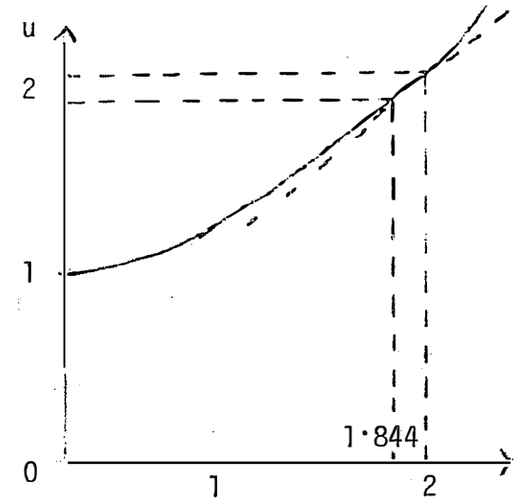
The closest analogy to what we have done so far is, of course, 'linear interpolation', represented on the graph by drawing the chord joining (1,1·368) to (2,2·135), or on the mapping diagram by a uniform scale from 1·368 to 2·135 (shown in the figures by broken lines). An increase of 1 in x corresponds to one of ·767 in u; so a decrease of ·135 in u requires a decrease of ·135/·767, or ·176 in x. This gives an approximate solution

$$x \approx 2 - \cdot 176 = 1 \cdot 824.$$

It should be noticed that, in the simpler example, the single number ·767 has replaced the matrix $\begin{pmatrix} 22 & -2 \\ 9 & 52 \end{pmatrix}$; instead of multiplying by the inverse matrix we merely divide by the scale factor.

In the one-variable case it is, of course, well known how we proceed. Instead of using the chord of the graph as the approximation, we use the tangent where x=2. This is the Newton-Raphson approximation: the gradient of the tangent at x=2 is $f'(2) = 1 - e^{-2} = \cdot 865$; using this (rather than ·767) as the 'scale factor', a decrease of ·135 in u requires a decrease of ·135/·865, or ·156 in x, giving an approximate solution

$$x \approx 2 - \cdot 156 = 1 \cdot 844.$$



For a comparison of the quality of our two approximations, we may note that $1{\cdot}824 \longmapsto 1{\cdot}985$, whereas $1{\cdot}844 \longmapsto 2{\cdot}002$; a considerable improvement.

The graph gives us a splendid picture of what is going on - but unfortunately it is of no help at all in extending the idea to our two-unknown problem, since we cannot draw a graph of (u,v) against (x,y). This is where the 'one view' approach to mathematics teaching is so limiting. Many courses in calculus begin by introducing the derivative as the gradient of a tangent, rush on to teach all sorts of technical skill - chain rule, product rule, integration by parts, etc., etc. - but leave the students high and dry when they come to need to see differentiation in the more general contexts of vectors, complex variable or functions of two variables. This presentation is a peg on which to hang a plea for a concept-based curriculum, which students can use as a foundation for whatever structures they later find the need to build.

What, then, is going on when we draw the tangent? The answer is that we are creating a <u>locally linear</u> situation. The mapping

$$x \longmapsto u = x + e^{-x}$$

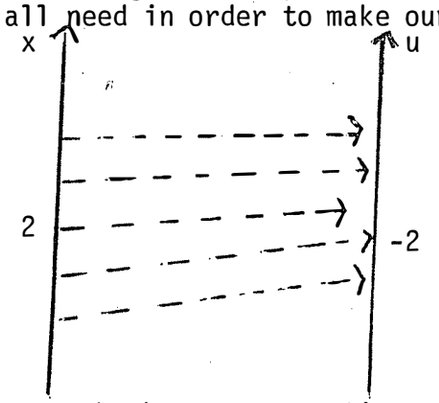
is certainly not linear; but, if we consider small displacements δx from $x=2$, the mapping

$$\delta x \longmapsto {\cdot}865\,\delta x$$

has two properties:

(i) the mapping is linear;
(ii) the image of δx under the mapping approximates to δu.

We can now look at this in our 'arrow diagram': to solve the equation we require $\delta u = -\cdot 135$, so we take $\cdot 865\ \delta x \approx -\cdot 135$, i.e. $\delta x \approx -\cdot 156$. Notice that the <u>calculation</u> is exactly the same as it was before; but what we <u>think when</u> we do it is very different. Perhaps we do not give enough attention to 'what people are thinking when they do mathematics' - the images which we all need in order to make our mathematics effective.



Let us then come back to our equations in two unknowns - but instead of considering the mapping over a whole square, look at the mapping in the immediate neighbourhood of the point (3,5). What is the equivalent to the local linear mapping $\delta x \longmapsto \cdot 865\ \delta x$ in the neighbourhood of x=2 in the one-unknown case?

The main difference now is that we can make displacements in different directions from the point (3,5) - like our slipper-hunter in the party game. So we might take a leaf from his book and feel our way in different directions; but instead of taking a whole step, let us 'put a toe in the water' and make a tentative step of $\begin{matrix}\delta x \\ 0\end{matrix}$. Since we are varying only x, we need think of the equations

$$(x,y) \longmapsto x^2+3xy-y^2=u \ , \ (x,y) \longmapsto 2x^2-xy+5y^2=v$$

as simply mapping x to (u,v) - y being in effect constant. Thus

$$x \longmapsto x^2+3xy-y^2 \quad , \quad x \longmapsto 2x^2-xy+5y^2 \ ,$$

and proceeding as before we replace these non-linear maps with the linear maps

$$\delta x \longmapsto (2x+3y)\delta x \ , \ \delta x \longmapsto (4x-y)\delta x$$

with x=3, y=5; that is, by

$$\delta x \longmapsto 21\ \delta x \ , \ \delta x \longmapsto 7\ \delta x$$

respectively. These expressions give approximations to δu and δv; and setting them back in a two-dimensional context, we get

$$\begin{pmatrix}\delta x\\ 0\end{pmatrix} \longmapsto \begin{pmatrix}21\ \delta x\\ 7\ \delta x\end{pmatrix} \approx \begin{pmatrix}\delta u\\ \delta v\end{pmatrix} .$$

(Notice how the 'partial derivatives' arise quite naturally from this point of view.)

Similar work with a trial step of $\begin{pmatrix}0\\ \delta y\end{pmatrix}$ leads to a consideration of functions

$$y \longmapsto x^2+3xy-y^2 \ , \ y \longmapsto 2x^2-xy+5y^2$$

with x constant. The linear maps are now

$$\delta y \longmapsto (3x-2y)\delta y \ , \ \delta y \longmapsto (-x+10y)\delta y$$

with x=3, y=5; that is,

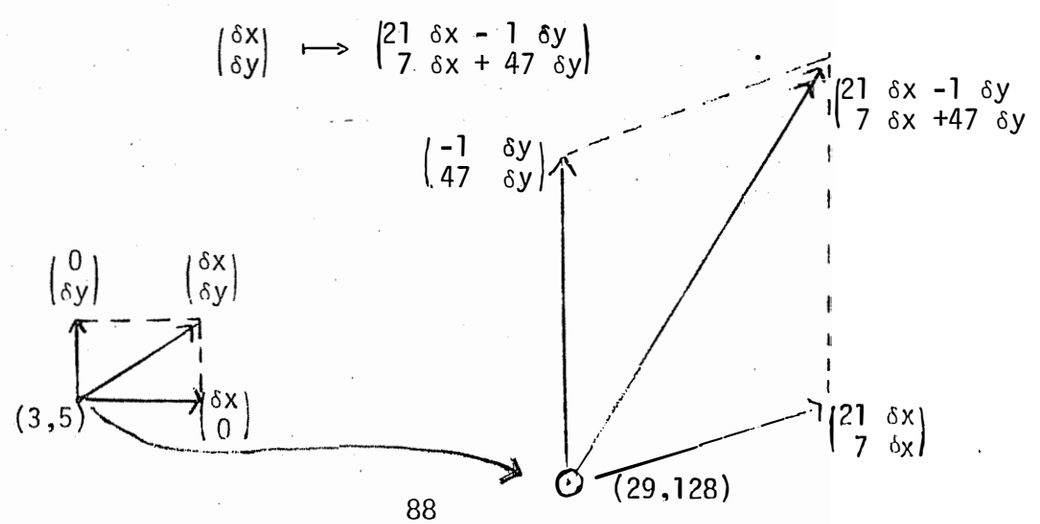
$$\delta y \longmapsto -1\ \delta y \ , \ \delta y \longmapsto 47\ \delta y,$$

or

$$\begin{pmatrix}0\\ \delta y\end{pmatrix} \longmapsto \begin{pmatrix}-1\ \delta y\\ 47\ \delta y\end{pmatrix} \approx \begin{matrix}\delta u\\ \delta v\end{matrix} .$$

These then are two particular aspects of the linear map which approximates to the mapping $(x,y) \longmapsto (u,v)$ in the neighbourhood of x=3, y=5. But, since this local map is linear, we know that the expression for the complete map is

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \longmapsto \begin{pmatrix}21\ \delta x - 1\ \delta y\\ 7\ \delta x + 47\ \delta y\end{pmatrix} .$$



This last expression can be put into the form

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \longmapsto \begin{pmatrix}21 & -1\\ 7 & 47\end{pmatrix} \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \approx \begin{pmatrix}\delta u\\ \delta v\end{pmatrix} \quad ,$$

which should be compared with the mapping in the one-dimen sional problem

$$\delta x \longmapsto \cdot 865\ \delta x \approx \delta u \ .$$

In that case, ·865 was the <u>derivative</u> at x=2. The corresponding interpretation in two dimensions is that

$\begin{pmatrix}21 & -1\\ 7 & 47\end{pmatrix}$ is the value of $\begin{pmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{pmatrix}$ at (x,y) = (3,5).

To solve our equations we want to find the point (x,y) which is mapped onto (30,130). From (29,128), therefore, we want to make a displacement of amount $\begin{pmatrix}\delta u\\ \delta v\end{pmatrix} = \begin{pmatrix}1\\ 2\end{pmatrix}$. To do this, we require to make

$$\begin{pmatrix}21 & -1\\ 7 & 47\end{pmatrix} \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \approx \begin{pmatrix}1\\ 2\end{pmatrix} ,$$

so that

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \approx \begin{pmatrix}21 & -1\\ 7 & 47\end{pmatrix}^{-1} \begin{pmatrix}1\\ 2\end{pmatrix} = \frac{1}{994}\begin{pmatrix}47 & 1\\ -7 & 21\end{pmatrix} \begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}.0494\\ .0352\end{pmatrix} ,$$

giving the solution x = 3·0494, y = 5·0352.

The matrix $\begin{pmatrix}21 & -1\\ 7 & 47\end{pmatrix}$ is the Jacobian matrix at the point (3,5). Notice how it is this, rather than the individual partial derivatives, which arises naturally. It is the matrix which corresponds to the local linear mapping: so if we want to combine two mappings, we combine their matrices (by multiplication); if we want to invert the mapping, as above, we invert the matrix.

Finally, for those who like to formalize their mathematics, we may look at the process we have investigated in general terms. It is convenient to make the small modification of taking the number terms to the left sides of the equations. For the one-variable problem we then have:

To solve F(x) = 0, with approximate solution x = a,

consider the mapping $x \longmapsto F(x)$, which gives rise to a local mapping around a described by $\delta x \longmapsto m\delta x$, where m is the value of du/dx evaluated at a. Since we require $\delta u = -F(a)$, the improved approximation is

$$x = a - \frac{1}{m} F(a).$$

With two equations in two unknows, this becomes:

To solve $p(x,y) = 0$, $q(x,y) = 0$, with approximate solution $(x,y) = (a,b)$, consider the mappings $(x,y) \longmapsto p(x,y)$ and $(x,y) \longmapsto q(x,y)$, which can be combined into a single mapping written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto F\left\{\begin{pmatrix} x \\ y \end{pmatrix}\right\} = \begin{pmatrix} u \\ v \end{pmatrix} .$$

This gives rise to a local linear mapping around (a,b), described by

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \longmapsto M \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \approx \begin{pmatrix} \delta u \\ \delta v \end{pmatrix},$$

where M is the Jacobian matrix at (a,b). We require $\begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = -F\left\{\begin{pmatrix} a \\ b \end{pmatrix}\right\}$, so the improved approximation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} - M^{-1} F\left\{\begin{pmatrix} a \\ b \end{pmatrix}\right\} .$$

What we have, in fact, is a straight extension, with appropriate re-interpretation, of the Newton-Raphson process to sets of equations in several unknowns.

But my purpose is not to emphasize the result itself, but to point out that, although tne problem is meaningful in purely classical terms, we would not have tackled it in this way without the conceptual background of mappings, of the derivative as a local linear mapping, of the representation of linear mappings by matrices, of vectors - indeed, of many of the concepts which I identified as 'growing points' in my earlier presentation. If our students can get these, through example and analogy, into their bones, it is not all that important where they put on their extra fat.

D. A. Quadling
is a member of the Cambridge Institute of Education,
Shaftesbury Road, Cambridge CB2 2BX

"MODERN" MATHEMATICS:

AN EDUCATIONAL AND PHILOSOPHIC ERROR?

René Thom

In the minds of most of our contemporaries, so-called modern mathematics holds a place of high prestige lying somewhere between cybernetics and information theory in the bag of tricks promoted by deceptive publicity as the essentials of modern technology, the indispensable tools for the future development of all scientific knowledge. And, on another level, since the modernization of school curricula, many parents, no longer capable of helping their offspring, have become concerned. They no longer hear the old familiar notions in the vocabulary of their children and thus feel lost when confronted with the new terminology. Some, perplexed, see this as one more symptom of the generation gap and have adopted an obstructionist stance toward the new ideas. Others, on the contrary, particularly those in the teaching profession, have accepted the new curriculum, ideas, and symbols with enthusiasm. What should we make of all this?

Curriculum revisions. Let us list briefly the changes made in the curriculum:

1. *Added material:* (a) "Elementary" set theory, the use of symbols ($\in$, $\subset$, $\cup$, $\cap$), the mappings of one set into another, and quantifiers. Most striking of all, sets now appear ubiquitously in the cirriculum from kindergarden through the final year of secondary education. We will return to this point later. (b) Development of algebraic notions; laws of composition on a set; concepts of group, ring, and field. (c) Introduction earlier of fundamentals of differential and integral calculus, derivatives, indefinite integrals, elementary functions such as logarithm and exponential.

2. *Eliminated material:* Traditional Euclidean geometry, in particular the intricacies of plane geometry.

In sum, the reader will note that the curriculum has been modified by a substantial addition of material introduced in the secondary school years. The tendency to emphasize algebra at the expense of geometry is even

greater in university teaching.

ALGEBRA AND GEOMETRY

The elimination of traditional Euclidean geometry is based on two arguments. The first is theoretical: the axiomatic work resulting from Hilbert's *Grundlagen der Geometrie* has shown that the alleged rigor of the *Elements* of Euclid is in large part illusory; it is compromised by frequent appeals to intuition. As a consequence, the argument runs, it is better to avoid Euclidean geometry by developing the ideas of algebra, in which a rigorous presentation is possible. The second argument is a practical one: classical plane geometry, with its elaborate study of the triangle's properties, is useless and pedantic. Who in his lifetime ever needs to use the "Simpson's line" or the "nine-point circle"?

Let us first discuss the argument about utility. It is said that algebra is more useful and necessary than geometry. There is no question of denying the general scientific utility of linear algebra or of certain notions of multi-linear algebra. As for general commutative algebra--polynomials, etc.--caution is in order. In ordinary life, who has ever needed to solve a second-degree equation or to use explicitly the notion of a module over a ring? The argument for the utility of algebra is not as compelling as it appears. As for differential and integral calculus--point (c) above--they are indispensable for any presentation of classical physics.

At an elementary level, certainly, the use of algebra leads to massive simplifications. Solving "through reasoning" the "word" problems one used to have as a twelve-year-old required an extraordinary dexterity of mind, whereas the algebraic solution was purely mechanical. Here the economy of thought introduced by algebra is undeniable. With more complex situations, however, the advantage of algebra tends to disappear. Descartes devised analytic geometry in order to reduce geometry to algebra. But it is a fact well known to all university applicants who have crammed for advanced standing in mathematics that the advantage of analytic methods over geometric ones for a qualitative theoretical problem is far from being decisive.

"Modernism." For professional mathematicians, the use of algebra as an instrument of proof is highly important and perhaps essential. Contemporary mathematicians, steeped in the ideas of Bourbaki (*1*), have had the natural tendency to introduce into secondary and university courses the algebraic theories and structures that have been so useful in their own work and that are uppermost in the mathematical thought of today. Yet one can ask with reason if the needs of specialists and their latest findings should be introduced into the school curriculum.

Mathematicians are not alone in succumbing to this temptation. I have read biology texts--both for beginners and advanced students--in which the double helix of DNA of Watson and Crick and the precise enzymatic mechanism of its replication are presented as definitive scientific truth. Innovations should not be introduced into the curriculum without a certain waiting period. In France, we should have been able to rely on the school inspection corps to assure the necessary curricular stability. However, for fear of having genuine skepticism interpreted as sclerosis due to old age, this institution has not functioned with all its desired efficacy. After all, texts must change and editors must live.

The problem of geometry. In the last analysis, the argument about the utility of material presented in the curriculum is perhaps not the decisive one. Let us ignore "culture"--"that which remains when all else is forgotten"--as a vestige of times past. Some still persist in thinking that, in one form or another, one of the goals of teaching is *selection*, that is to say, determining the aptitudes of each student and developing them to the maximum, with particular emphasis on the gifted student. I claim that it is impossible to carry out such a task in the framework of a discipline that does not include at least some gratuitous, nonuseful aspects. In order to judge fully the capabilities of a student, it is necessary to place him in an active role and to call on his individual initiative and enterpresing spirit. None of this is conceivable within a framework of "useful" studies, where all the elements, included because of their technical utility, are dogmatically taught and where scholarly excellence is defined as exact and rapid memorization of given material. Only those topics which have a quality of "play" have educational value, and of all such games, Euclidean geometry, with its constant references to underlying intuitively understood fundamentals, is the least gratuitous and the richest in meaning.

By this line of reasoning, the contemporary trend to replace geometry with algebra is educationally baneful and should be reversed. There is a simple reason for this: while there are geometry problems, there are no algebra problems. A so-called algebra problem can only be a simple exercise requiring the blind application of arithmetical rules and of a preestablished procedure. With rare exceptions, one cannot ask a student to prove an algebra theorem; either the requested answer is almost obvious and can be arrived at by direct substitution of definitions, or the problem falls into the category of theoretical algebra and its solution exceeds the capacities of even the most gifted student. Exaggerating only slightly, one can say that any question in algebra is either trivial or impossible to solve. By contrast, the classic problems of geometry present a wide range of challenges.

Geometry problems require a combination of time, effort, concentration, and powers of association of which few students are capable. Perhaps Euclidean geometry, like Latin translation, is one of those lofty, obsolete exercises that are limited to the elite and incompatible with mass education. If such is the case, expelling geometry from the curriculum becomes essentially a sociological question that I do not wish to discuss here. Still, it would be a grave error to hope to simplify the learning of mathematics by replacing geometry with algebraic structures that are then widely and prematurely taught without adequate motivation.

Rigor. Let us now turn to the objection to Euclidean geometry that criticizes the axiomatics of the *Elements* as being flawed and lacking in rigor. One can point out, first of all, that geometry books long ago gave up the heavy, indigestible rhetoric of Euclid. Some cherished the hope of substituting an acceptable version of Hilbert's *Grundlagen*. Not surprisingly, this hope was defeated by the dreadful complexity of this work. One cannot take a stand on this issue without first attacking the philosophical question of what conception of mathematical rigor one should adopt. Three attitudes are possible: (1) the formal view. In a formal system S, a proposition P is true if it can be deduced from the axioms of S by a finite number of steps permitted within the system of S. (2) The realist of Platonic view. Mathematical entities exist independently of thought, as Platonic ideas. A proposition P is true when it expresses a relationship actually existing between ideas, i.e. when it is an idea of higher order, structuring a group of ideas that are subor- to it. (3) The empirical or sociological view. A proof P

is accepted as rigorous if it obtains the endorsement of the leading specialists of the time.

Of these three attitudes, mathematicians today favor the first. At first sight, it is the most tempting; it does not raise the ontological difficulties of the second, and it is not as vague and arbitrary as the third. Bertrand Russell has said that "mathematics is the subject in which we never know what we are talking about nor whether what we are saying is true" (*2*). Unfortunately, the purely formal view is difficult to uphold, paradoxically for formal reasons. We know the difficulties presented by the formalization of arithmetic associated with Godel's Theorem. Professor Kreisel, in his recent article in *L'Age de la science* (*3*), put the formal view on trial. For myself, I am content with the following illustration: Let us suppose that we have been able to construct for a formal theory S an electronic machine M capable of carrying out at a terrifying speed all the elementary steps in S. We wish to verify the correctness of one formula F of the theory. After a process totaling 10^{30} elementary operations, completed in a few seconds, the machine M gives us a positive reply. Now what mathematician would accept without hesitation the validity of such a "proof," given the impossibility of verifying all its steps?

"Meaning" in mathematics. Any mathematician endowed with a modicum of intellectual honesty will recognize that in each of his proofs he is capable of giving a *meaning* to the symbols he uses. Because of this, his work differs from that of the theoretical physicist, who very frequently does not hesitate to put his trust magically in the virtues of blind formalism in the hope (often deceived) that the light at the end of the tunnel will dispel the intervening darkness.

If one gives up the formal definition of rigor, one must of necessity choose between the two remaining alternatives. Everything considered, mathematicians should have the courage of their most profound convictions and thus affirm that mathematical forms indeed have an existence that is independent of the mind considering them. This existence is without doubt different from the concrete existence of the external world, but it is still subtly and deeply related to it. If mathematics is only an arbitrary game which is the random product of cerebral activity, how can one explain its unquestioned success in describing the universe? Mathematics is found not only in the mysterious fixed order of physical laws but also, in a more hidden though equally certain manner, in the infinite

succession of animate and inanimate forms and in the formation and breaking up of their symmetries. Despite appearance, this is why the hypothesis stating what Platonic ideas give shape to the universe is the most natural and, philosophically, the most economical.

Yet, at any given moment, mathematicians have only an incomplete and fragmentary vision of this world of ideas. As a result, each proof is, above all, the revelation of a new structure whose elements lie disconnected in man's intuition until reason joins them together. In this sense, each proof is a Socratic experience requiring the re-creation in the reader of the psychological processes necessary to elicit the implicit truth, all the elements of which he possessed but which had remained hidden in an unformulated state. In this sense, there is no contradiction between the second and third views. The world of ideas is not revealed to us in one stroke; we must both permanently and unceasingly re-create it in our consciousness.

The opponents of the ontological view would do well to reflect on the following: There is no case in the history of mathematics where the mistake of one man has thrown the entire field on the wrong track. Frequently, mathematics has become lost in the formal development of insignificant, uninteresting theories. It has done so in the past, does so today, and will certainly continue to do so in the future. But never has a significant error slipped into a conclusion without almost immediately being discovered. How could one explain such a consensus if it did not correspond to a general opinion that is the result of the mind's struggle with permanent, timeless, and universal constraints? With this confidence in the existence of an ideal universe, the mathematician need not worry unduly about the limits of formal procedure; likewise, he can forget the problem of noncontradiction, for the reason that the world of ideas infinitely exceeds our "technical possibilities." It is in the intuition that the *ultima ratio* of our faith in the truth of a theorem resides. And, according to a now-forgotten etymology, a theorem is above all the object of a vision.

Each must decide for himself. There is no rigorous definition of rigor. We will therefore affirm that any proof is rigorous if it wins acceptance by all readers who are adequately educated and prepared to understand it. Furthermore, the evidence leading to persuasion results from having a sufficiently clear understanding of each of the symbols involved, so that their combination convinces the reader.

From this point of view, rigor (or its contrary, imprecision) is essentially a *local* property of mathematical reasoning. No elaborate axiomatic structure or refined conceptual machine is needed to judge the validity of a line of reasoning. It suffices merely to have an understanding of the meaning of each symbol involved and a clear idea of how to combine them.

Limits and necessity of axiomatization. Such a point of view suggests that we retreat somewhat from axiomatics. To formalize a theory means, starting with the material presented by the theory which is organized as intuitive "morphology" T, to give a formal set of symbols and rules generating a formal system S isomorphic to the morphology T; the isomorphism $S \to T$ being precisely the correspondence which attaches to any symbol s belonging to S its "meaning" i.e. its intuitive content in T (its semantic realization, logicians would say). Can one reasonably hope that the intuitive material of the theory T can be fully covered by the symbolic expressions of S? An example immediately comes to mind, that of natural languages. Linguists of the formalist school have been trying strenuously to reduce natural language grammar and syntax to axioms. In doing so they have come up with a certain number of formal procedures--generative and transformational grammars--whose validity, on the level of formal description of the sentences contained in the corpus, cannot be denied. But if these procedures are systematized into a series of rules which are then pursued blindly to their logical conclusion, the resulting sentences soon become so long and complex that they lose all meaning.

I see no reason why a similar phenomenon could not happen in mathematics; in extrapolating a formal mechanism to the limit of its generative capacities, it does not take long to assemble formulas that are so long and complex that all possibility of intuitive interpretation disappears. The "theorems" thus obtained will probably be formally correct but semantically insignificant. Thus for a given intuitive theory T one must expect to have to use not one but several "local" axiomatizations; each local axiomatization S has a contact zone Z_S in the morphology for which S is valid; but as soon as one constructs formulas in S which are too long or involved, the intelligibility disappears. At the boundary of the zone Z_S the semantic link between S and Z_S breaks down; this prohibits the extension beyond Z_S of the ismorphism $S \to T$, defined by the meaning. The idea that a theory T could be generated by just one formal system S is, a priori,

day thought to formalized thought takes place naturally through geometric thinking. This has always been the case in the history of human thought and, insofar as one believes Haeckel's Law of recapitulation, which states that in his development the individual passes through all the stages of the species, it should be the case in the normal development of rational thought.

SET THEORY

I come now to my first point, set theory. This is the essential litany intoned by those who advocate the so-called modern mathematics. Some affirm that the use of set theory permits the entire renovation of mathematics teaching and that, thanks to this change, the average student will be able to achieve mastery of the curriculum. Needless to say, this is pure illusion. As long as it is a matter of handling the obvious facts of naive set theory, of course anyone can get by. But this is neither mathematics nor even logic. As soon as one comes face to face with real mathematics (i.e. real numbers, geometry, functions), one rediscovers that there is no royal road and that only a minority of students are capable of fully understanding the material.

Everything considered, the excessive optimism bred by the use of set theory symbols has its roots in a philosophical error. It was believed that by teaching the use of the symbols $\in, \subset, \cup, \cap$ it was possible to make explicit the mechanisms underlying all reasoning and deduction. Twentieth-century man has enthusiastically rediscovered the syllogisms Darapti and Celarent taught by the medieval scholastics. But what a deterioration has taken place! When, in the nineteenth century, Boole wrote the celebrated treatise on algebra that bears his name, he did not hesitate to entitle it "An Investigation into the Laws of Thought." The naive belief that every deduction finds its model in set theoretic manipulations was shared by such modern philosophers as the neopositivists. Neither Aristotle nor the medieval scholastics shared this illusion. As J. Vuillemin reminds us (5), Aristotelian logic has its base in a rich and complex ontology of substance. Modern protagonists of set theory should realize that this theory is insufficient to account for even the most elementary deductive steps of ordinary thought. Permit me to give an example of this fact.

The copulas or *and* and. Classically, it is taught that the grammatical equivalent of the symbol $\cup$ (union) is *or* and that of the symbol $\cap$ (intersection) is *and*. Let us apply this rule to two simple sentences whose subjects are proper names:

(1) Peter or John is coming.
(2) Peter and John are coming.

The first sentence can be paraphrased, "Peter is coming or John is coming." Here there is complete agreement of the symbol *or* with the logical union , with the condition that the copula refers not the the subject but to the verb "to come."

The second sentence too can be paraphrased, "Peter is coming and John is coming." Having done this, one realizes that the original sentence is subtly ambiguous, for it implicitly contains what linguists term "presuppositions." For example, "Peter and John are coming" frequently presupposes, "Peter and John are coming *together*." While the phrase "Peter or John" alone has no semantic interpretation, it is possible to conceive of "Peter and John" as an entity formed by a pair of individuals, Peter and John, who, spatially, are together. This fact explains the different grammatical treatment of the verbs in (1) and (2): the copula *and* requires the plural because it presupposes a certain spatial contiguity of the subjects.

Let us consider some sentences in which copulas are used with qualities.

(3) Peter is short or intelligent.
(4) Peter is short and intelligent.
(5) Joan's hair is gray or brown.
(6) Joan's hair is gray and brown.

Sentences (4) and (5) are semantically acceptable whereas (3) and (6) may be dubious or unacceptable. One may extrapolate these remarks to the following principle:

Exclusion principle: If *X* and *Y* are two qualities, the sentences

A is *X* or *Y* } cannot both be
A is *X* and *Y* } semantically acceptable

When "*X* or *Y*" may be preceded by a subject, one would say that *X* and *Y* belong to the same semantic field: for example, "gray" and "brown" in sentences (5) and (6). In this case, "*X* and *Y*", is, in principle, meaningless. There is, nevertheless, an important exception, the case where *and* designates not logical intersection but spatial contiguity. Thus it is perfectly possible to say:

(7) This flag is white or blue.
(8) This flag is white and blue.

The fact that in (8) the copula does not have the meaning ∩ explains why "This flag is white and blue" implies that "This flag is white" is false.

Indeed, the conditions necessary for the expression "*X* or *Y*" to be meaningful are extremely restricted, thus "Joan has red or auburn hair" is clearly more acceptable than "Joan has red or brown hair" because, in terms of the semantic category of hair colors, "red" and "auburn" are adjacent to one another whereas "red" and "brown" are not. The copula *or*,geometrically speaking, has the effect of lowering the threshold between the domains of attraction defined by the adjectives "red" and "auburn." When the semantic distance between two qualities *X* and *Y* is too large, in particular when these qualities belong to different semantic fields, as with a physical quality and a moral quality, then the phrase "*X* or *Y*" loses all meaning.

Although it is rather obvious, this fact seems to have escaped the authors of many set theory textbooks. They offer students exercises in Boolean algebra which discuss "cubes that are big or blue," and "Parisians who are bald or rich." Not only are these exercises outlandish and useless, but, if pursued too far, they can become harmful to the child's intellectual equilibrium. One of the fundamental constraints imposed by accurate thought is precisely the avoidance of mixing distinct semantic fields. This mixing has a name--delirium. In attempting to attach meaning to all the phrases constructed in ordinary language according to Boolean rules, the logician proceeds to a phantasmic, delirious reconstruction of the universe.

All these points show the narrow limits of set theory in describing ordinary thought. Everyday reasoning calls upon profound psychic mechanisms, such as analogy, which can never be reduced to the level of set theoretic operations.

An important factor in such cases is the organizational isomorphism between semantic fields which are homologically associated.

In fact, Boolean schematizations hardly apply without some defect except in cases described by spatial inclusions of subsets in space, as in Venn diagrams. In such a case, no one will take the trouble to put the reasoning in a syllogistic form. The fox knows that if the hens are in the hen-house and the hen-house is in the yard, then the hens are in the yard; he does not bother with set theory. Everyone uses set theory from the moment he exists, just as M. Jourdain in Moliere's *Le Bourgeois Gentilhomme* uses prose without knowing it. Some say that it is better to use it knowingly. The advantage here, if there by any at all, applies to the rhetoric. It is only to the extent that the technique of mathematical proof is a type of rhetoric that it becomes worthwhile to proceed by local formalizations--which actuall y are local "spatializations"--and to apply the set-theoretic formalism to them. The persuasive force of the logical scheme comes from spatial inclusions, and not vice versa. This indicates to us the attitude that reasonable educational thought should take toward set theory. In its simple, concrete form, it should be introduced in kindergarten, which is its natural habitat. In the early years of secondary school, students should learn the use of the symbols $\in, \cap, \cup, \subset$ later they should be introduced to the quantifiers, and that should be the end of it.

It is not certain that, even in pure mathematics, each deduction can have a set-theoretic model. Poorly resolved paradoxes that undermine formal set theory are there to remind the mathematician of the dangers that await him in the injudicious use of these seemingly innocent symbols. Perhaps, even in mathematics, quality subsists, and resists all reduction to sets. The old hope of Bourbaki, to see mathematical structures arise naturally from a hierarchy of sets, from their subsets, and from their combination, is, doubtless, only an illusion. No one can reasonably escape the impression that the most important mathematical structures (algebraic structures, topological structures) appear as fundamental data imposed by the exterior world, and that their irrational diversity finds its only justification in reality.

REFERENCES AND FOOTNOTES

[1] Nicolas Bourbaki is the non-de-plume adopted in the 1930s by a group of outstanding young French mathematicians who undertook the monumental task of reorganizing mathematics in terms of basic structural components. This enterprise is an ongoing effort, whose members must resign at age fifty according to Bourbaki's bylaws.

[2] Bertrand Russell, quoted in A. Hooper, 1948. *Makers of Mathematics*. N. Y.: Random House, p. 384.

[3] G. Kreisel. 1970. The formalist-positivist doctrine of mathematical precision in the light of experience. *L'Age de la science* 3: 17 - 46.

[4] R. Thom. 1970. Topologie et linguistique. In *Essays on Topology and Related Topics*, André Haefliger and Raghavan Narasimham, eds. (Mémoires dédiés à Georges de Rham.) New York: Springer-Verlag, pp. 226 - 48.

[5] J. Vuillemin. 1967. *De la logique à la théologie.* Paris: Flammarion.

Rene Thom is a distinguished French Mathematician and is described in the 'American Scientist" (from which this article is reprinted with permission) as France's leading geometer. The article is a translation of Thom's article in L'Age de la science 3: No. 3, 225-36.

THE "NEW MATH" IN SCHOOLS -

SHOULD WE EXTEND IT, AMEND IT, OR ABOLISH IT?

W. W. Sawyer

It is dangerous and difficult to speak about the 'New Math' because this term does not describe a single unified thing. Ten or fifteen years ago various ideas - good, bad and indifferent - invaded the sphere of mathematics education. At present there is a worldwide reaction against the 'New Math' and there is a danger than the backlash may treaten the good along with the bad. I therefore list some individual points, and my attitude to them.

1. That mathematics should be taught through understanding and not purely by rote. This is not new; it has been the practice of all good teachers and goes back at least to Socrates around B.C. 400.

2. That calculus should be taught in secondary schools. Calculus again is not new; it dates from about 1660 A.D. and in Europe it has been taught in secondary schools for about half a century now. I have found students in my Grade 8 mathematics club can understand and enjoy the beginnings of calculus.

3. That matrices and linear algebra should be in the secondary school curriculum. This I am strongly for. You will find matrices being used today in almost every form of study of the real world and they provide a very efficient method of investigation.

Now for two views that I oppose:

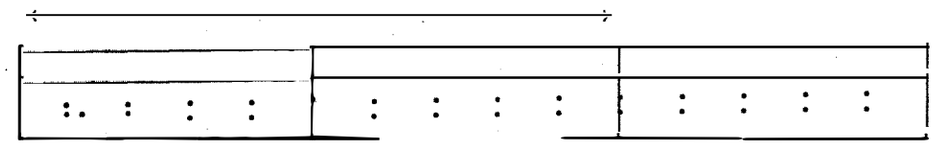
4. That all teaching should mimic the procedures of graduate school; that the main aim is to present mathematics as the study of abstract systems; that the language of set theory and axiomatics should be used form Grade 1 on; that the curriculum should be organized around the systems, the natural numbers, the integers, the rationals, the reals.

5. That everything dones before the 'New Math' was obsolete and valueless.



with five tens making 50.

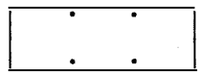
Another example of the formal, abstract approach deals with equivalent fractions. The argument is that we do not alter a number if we multiply it by unity. So, since $5/5 = 1$ we do not alter a fraction if we multiply it by $5/5$. Accordintly $2/3 = 2/3 \times 5/5 = 10/15$. I have discussed this argument with intelligent 5th graders, and it means absolutely nothing to them. When they ask me for help, I go back to the time-honoured diagram



with the explanation that $2/3$ means we are to break some object into 3 equal parts and have 2 of these parts. If each of these parts is broken into 5 equal pieces, we have broken our object into 15 pieces and must have 10 of these to get the same amount as before. There is nothing wrong with this argument and it does mean something to children.

I have stressed actual objects in the first example above. It is well known to teachers of young children that even drawings, which are semi-concrete, do not always convey any message. In the Ontario Mathematics Gazette for September 1972 I reported on my conversations with children who were finding difficulties with mathematics. In the vast majority of cases the cause was lack of concreteness. Sometimes the cure was as simple as having a child handle bundles of sticks instead of looking at a picture of bundles of sticks.

Emma Castelnuovo, one of the leading writers on mathematics teaching, tells a story from her early experiences. She was teaching 11-year olds. They had squared paper, and she told them to draw some rectangles with the base 3 times as long as the height. They did so.



She then asked them, "Suppose I knew the distance round the perimeter of this rectangle. What would I do to calculate the height?" The children called out, "Divide by 2," "Divide by 4", "Divide by 3". Evidently they had no idea. As she said, the children looked at their drawings but they did not see what was there. This is a familiar experience of a teacher; the class have before them all the evidence needed to answer a question - and nothing happens. What is to be done?

Emma Castelnuovo then gave the children toothpicks and said, "Use these to make a rectangle with the base 3 times as long as the height." She then repeated her question, and this time there was no child unable to answer it correctly.*

This story touches the central problem of teaching, which is to find a way into the children's actual thinking, instead of presenting ideas that sound good to us and mean nothing to them.

Return to the Past?

It is natural that the public, on becoming aware of the defects of the 'new math', should think we ought to go back to the way math used to be taught. If anyone thinks we can find in our past a golden age of mathematics teaching, he is suffering from an illusion. Twenty years ago, just as today, the majority of adults were ignorant of mathematics and were frightened of it. They perhaps knew their tables better but they were not, as a rule, able to apply mathematical thinking to practical problems in personal or national life.

Something much better than the past is certainly possible. All the evidence shows that young children are interested in numbers and enjoy work with numbers, but that this interest and pleasure are gradually destroyed through mishandling by the adult population. The minority who survive as mathematicians are those who have teachers or parents able to preserve and encourage the children's thinking about mathematics.

There will be no sudden advance in mathematical education, because progress depends on gradually spreading out to the entire population the way of looking at mathematics that this minority has at present.

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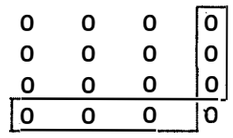
* Emma Castelnuovo: Didattica della matematica (La Nuova Italia Editrice, 1963) page 89.

There are two guiding principles. (1) Mathematics should come to the children as a development of their own thinking, not as something told them from outside. (2) In order to make this possible, topics must be presented in a form that the children can think about; like Emma Castelnuovo's toothpicks, the treatment must be concrete and pictorial.

A Sample Topic.

As an example I will discuss the teaching of the integers. The method is one I originally worked out with Grade 5 students, but it could be used anywhere between Grade 5 and Grade 9. We began with a study of "Think of a number" tricks, and visualized these by means of a bag containing the number stones given by the number thought of. Gradually this bag became drawn more and more carelessly until it looked like the letter x. A calculation such as $3(2x+5)$ is then dealt with by asking, "What is 3 times as much as 2 bags and 5 stones?" This goes on until the children are quite happy with x as a sign for an arbitrary number.

We then make a table of the square numbers and the children discover properties of this table. At a certain stage it is noticed that 4^2 exceeds 3^2 by the sum of two 3s and 1. Can the children draw a picture to illustrate this? We arrive at the diagram



This leads to the general idea that you can get from the square of any number to the square of the next number by sticking on two flaps, together with a single object at the corner. We draw pictures of a square x by x and make this into a square $(x+1)$ by $(x+1)$ by adding the 2 flaps - "x standing up and x lying down" - and one in the corner. This is recorded as $(x+1)^2 = x^2 + 2x + 1$. We carry this idea further, with drawings done all the time. We find, for instance $(x+3)^2 = x^2 + 6x + 9$ and, from this and other results, observe a rule; the 6 in $6x$ is twice 3, the 9 at the end is 3^2. We are now ready to see what happens if you add any number, a, to x and we arrive at the result $(x+a)^2 = x^2 + 2ax + a^2$.

You will notice that I have done a moderate amount of

algebra before starting on the integers. Most textbooks do it the other way round, the integers within arithmetic. But this leads to two difficulties. First of all there is motivation. Why should the student want to know about the integers? The official answer seems to be, "Well, it is the next abstract structure after the natural numbers and the rationals." It is assumed that every student is dying to know all the abstract structures, which is not my own experience. Secondly, there is the question of how you deal with $(-1)\cdot(-1) = +1$. Addition and subtraction of integers are not too bad, but multiplication of integers is hard to explain, as has been recognized for years. Now I do not believe that anyone would ever have discovered $(-1)^2 = +1$ if the algebra of positive numbers had not developed to a considerable stage before negative numbers were dreamed of.

With the order I am discussing, boththe need for negative numbers, and the discussion of $(-1)^2$ arise naturally.

There is a well known type of simple problem; John is 19, his father is 39, when will John's father be exactly twice as old as John? If we wait x years for this to happen, we find easily $x = 1$.

Now suppose the question had been; John is 21, his father is 41, when is the father twice as old as John? If we wait for $\underline{x}$ years, John becomes $21+x$, his father $41+x$ and we want $41+x = 2(21+x)$, which boils down to $0 = x+1$. Now we have two choices. We can say, "We made a mistake in assuming it lay in the future. We should go back to the start, assume it was $\underline{x}$ years ago, and do the whole question again." No one wants to do this; it involves work! The other way is to say, "Can we interpret our answer $0 = x+1$ as $x = -1$, which means that it happened 1 year ago?" Of course we can, and we do not have to do any more work, so using negative numbers seems to be a very useful device. But can we still use all the results we found earlier? What happens to our rule for $(x+a)^2$ if someone choosing "any number a" is liable to hit on $a = -1$?

The term $2ax$ does not cause much controversy. If $a = -1$ it seems quite reasonable that $2a$ should be -2. This also fits in nicely with the table

$$(x+2)^2 = x^2 + 4x + 4$$
$$(x+1)^2 = x^2 + 2x + 1$$
$$(x+0)^2 = x^2 + 0x + 0$$
$$(x-1)^2 = x^2 - 2x + ?$$

The numbers with x, 4,2,0,-2 run in a smooth pattern. What are we to put for the question mark, where a^2 comes? We can try putting $x = 10$. This gives us

$$(10-1)^2 = 100-20+?$$

As we know $(10-1)^2$ is 81, it is clear we have to put +1 in the place where $(-1)^2$ falls.

It is of course necessary to explore many other situations in which negative numbers are used. Finding that in all of these $(-1)\cdot(-1) = +1$ leads to a result which we know on other grounds to be correct, the children increasingly come to have confidence in this procedure.

Relaxation.

In various sports you continually hear that players do well only when they are relaxed. When they are anxious they become ineffective.

If we look at mathematical education through the centuries, what has been its main effect on the majority of the population? Beyond all doubt, to create anxiety. The result of this anxiety is that, whenever any question arises which involved quantitative thinking, the average person runs away from it.

Our main aim in teaching mathematics therefore should be to keep children able to think about quantitative and geometrical situations in a relaxed way, without feeling they are doing anything out of the usual.

I do not think anyone can maintain that the formal, abstract approach to mathematics has decreased anxiety. If anything, it has tended to make simple things complicated. I do not see that you help anybody by saying, "Addition is quite simple. It is just the union of disjoint sets."

On the other hand, I believe the approach I have just outlined brings out the underlying simplicity of algebra, and that it can be understood by almost any child or adult.

That adults can understand it is important. For the

attitude of children to any subject is largely formed in early years, under the influence first of parents and then of elementary school teachers. As things are, it is reasonable to assume that the majority of parents and teachers will have been made to feel that mathematics is both mysterious and frightening, and of course they are bound to pass this feeling on to the rising generation.

Perhaps the most urgent reason for trying to make a radical improvement in secondary school textbooks is that the secondary school students of today are the elementary school teachers of tomorrow. The most important objective of secondary mathematical education is that all students should leave the school able to face a mathematical problem in a relaxed and confident manner, so that as future parents or teachers, they will help to spread this attitude through the whole of society. I do not think the present textbooks are in any way suited to this purpose. This is not always the fault of the textbook authors. Several of them disagree with the formal, abstract approach, but they had to put it in if the books were to be approved.

What about commutative and distributive?

In order to teach something, it is not necessary to name it. For instance, it is entirely reasonable and desirable that in quite early grades teachers should raise questions such as "In finding $1+2+3+4$, does it matter in what order we add the numbers?" or "In the multiplication $2\times3\times4$, does it matter whether we work this out as 2×12 or 3×8 or 6×4?" (In my Grade 5, all the children thought these would give different answers, in which case it would be impossible to find the volume of a brick!) Children will then come to be aware of what commutativity really means, and will be prepared for situations in algebra where $(2y)\cdot(3x)$ works out to $6xy$. But all of this should be handled lightly and informally, not as a revelation of advanced mathematics.

The stage where it becomes sensible to lay stress on such properties is when they become of practical importance. When the students reach matrices, they will reach calculations in which some but not all of the procedures of elementary algebra apply. They will find that $(X+I)^2 = X^2 + 2X + I^2$ is still correct but $(X+Y)^2 = X^2 + 2XY + Y^2$ is, as a rule, incorrect for matrices X and Y. Now it becomes of practical importance to know which properties of operations still hold and which must be discarded. In the last year or two at school, there are several themes of advanced mathematics that

can be discussed and illustrated by themes now familiar to the student. But the advanced concepts must always be leading to tangible results. It is no good talking about fields or groups if you are not going to apply them and show what they do for you.

To sum up, my recommendations are the following:
(1) Stop aping graduate school. Try to be simple, down to earth, concrete, pictorial.

(2) Talk informally about the effect of order of operations in addition and multipoication, but do not treat commutativity and associativeness formally until matrix theory is reached. In the same way, let children realize that, say, seven 8s is the sum of five 8s and two 8s but do not treat this as a property of an abstract system. It will turn up quite naturally in algebra, with $5x + 2x = 7x$, and the distributive property can be stressed formally when matrices are reached.

It may be objected that with this procedure the bulk of the population may never reach modern abstract mathematics at all.

There are several answers to this. First of all no one has ever shown in what circumstances the bulk of the population could make use of modern subjects such as topology or abstract algebra. On the other hand, traditional elementary algebra stands at the beginning of almost every science: by this I mean the ability to understand what a formula means and to combine formulas with insight and common sense: I do not mean traditional bad teaching of algebra by rote. If you want to know something about electricity you must be able to cope with Ohm's Law, $A = V/R$: if you want to understand machines, you must understand Hooke's Law and Newton's Laws of Motion, all expressed by algebraic symbols; if you want to study economics, you will have to cope with graphs and with algebraic equations - though whether, in the present state of economic theory, you will be any better for knowing what is in the textbooks, is open to doubt.

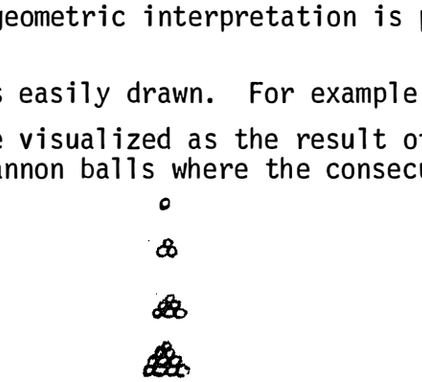
Another argument is that, if elementary mathematics were taught attractively, as it can and should be, far more students would want to continue into Grade 12 and 13 mathematics, where they would meet and be able to understand the more abstract approaches, because they would know enough particular cases that would serve as illustrations of the general principles.

Now of course it is true that the most recent branches of mathematics are being applied in various fields, but this research is carried on by a minute fraction of the population. No curriculum intended for students in general, or even for future-university students as a whole, can possibly meet the needs of the child who will one day do research work in mathematics, physics or engineering. Individual reading, started at the earliest possible date, is the only possible prescription for such exceptional students. We should not hold them back, but neither should we sacrifice all the rest of the population. In any case, I am by no means convinced that premature abstraction is desirable even for a future professor of mathematics.

TRIANGULAR ARRAYS

Bonnie H. Lotwiller and David R. Duncan

The triangular and tetrahedral numbers are two sets of non-negative integers of considerable historical interest. Pythagoras himself was acquainted with triangular numbers. The n^{th} triangular number (denoted t_n) is defined to be $\sum_{i=1}^{n} i$. A common geometric interpretation explains their "triangular" name; for instance, t_4 $(1 + 2 + 3 + 4 = 10)$ can be depicted as:



The n^{th} tetrahedral number (denoted T_n) is defined to be $\sum_{i=1}^{n} t_i$. Again, a geometric interpretation is possible although it is less easily drawn. For example, T_4 $(1 + 3 + 6 + 10 = 20)$ can be visualized as the result of stacking four "layers" of cannon balls where the consecutive layers appear as follows:

Table I is a partial listing of the triangular and tetrahedral numbers. We define $t_0 = T_0 = 0$.

Table I: The Triangular and Tetrahedral Numbers

Triangular Numbers:	0	1	3	6	10	15	21	28	36	45	55	...
Tetrahedral Numbers:	0	1	4	10	20	35	56	84	120	165	220	...

Triangular and tetrahedral numbers appear in many surprising places; one such example will be illustrated and justified in this article.

First, write the multiples of 1, 2, . . ., 6 in triangular arrays reminiscent of Pascal's triangle. For each of these arrays, find the sum of the terms in each row; re-write these sums in forms which involve the consecutive triangular and tetrahedral numbers. Tables II and III partially illustrate the results of these computations.

Table II: Multiples of 1

M_1	Row Sum	Relation to triangular numbers	Relation to tetrahedral numbers
0	0	$0(1) = 0t_1$	$3(0) = 3T_0$
1 2	3	$1(3) = 1t_2$	$3(1) = 3T_1$
3 4 5	12	$2(6) = 2t_3$	$3(4) = 3T_2$
6 7 8 9	30	$3(10) = 3t_4$	$3(10) = 3T_3$
10 11 12 13 14	60	$4(15) = 4t_5$	$3(20) = 3T_4$
15 16 17 18 19 20	105	$5(21) = 5t_6$	$3(35) = 3T_5$
.	.	.	.
.	.	.	.
.	.	.	.

Table III: Multiples of 6

M_6	Row Sum	Relation to triangular numbers	Relation tetrahedral numbers
0	0	$0(1) = 0t_1$	$18(0) = 18T_0$
6 12	18	$6(3) = 6t_2$	$18(1) = 18T_1$
18 24 30	72	$12(6) = 12t_3$	$18(4) = 18T_2$
36 42 48 54	180	$18(10) = 18t_4$	$18(10) = 18T_3$
60 66 72 78 84	360	$24(15) = 24t_5$	$18(20) = 18T_4$
.	.	. .	. .
.	.	. .	. .
.	.	. .	. .

For a specific m we have

M_m	Row Sum	Relation to triangular numbers
$0 \cdot m$	0	$m \cdot t_0$
$1 \cdot m$ $2 \cdot m$	$3 \cdot m$	$m \cdot t_1$
$3 \cdot m$ $4 \cdot m$ $5 \cdot m$	$12 \cdot m$	$m \cdot t_2$
$6 \cdot m$ $7 \cdot m$ $8 \cdot m$ $9 \cdot m$	$30 \cdot m$	$m \cdot t_3$
$10 \cdot m$ $11 \cdot m$ $12 \cdot m$ $13 \cdot m$ $14 \cdot m$	$60 \cdot m$	$m \cdot t_4$
.		
.		
.		

Also observe that the first entries in the rows are $t_0 \cdot m$, $t_1 \cdot m$, $t_2 \cdot m$, $t_3 \cdot m$, $t_4 \cdot m$, ... The following conjectures are now made:

(1) $\sum_{i=t_j}^{t_j+j} (m)(i) = mjt_{j+1}$ (suggested by column 3 of Tables II, III)

(2) $\sum_{i=t_j}^{t_j+j} (m)(i) = 3mT_m$ (suggested by column 4 of Tables II, III)

Proof of conjecture 1 (recall that $\sum_{i=N}^{N+n} i = (n+1)N + \frac{n(n+1)}{2}$.

$$\sum_{i=t_j}^{t_j+j} i$$

$$= \left[(j+1)\, t_j + \frac{j(j+1)}{2}\right]$$

$$= \left[j\,(t_j + \left(\frac{j+1}{2}\right) + t_j\right]$$

$$= \left[j\left(\sum_{i=1}^{j} i + \left(\frac{j+1}{2}\right)\right) + \sum_{i=1}^{j} i\right]$$

$$= \left[j\left(j\left(\frac{j+1}{2}\right)\right) + j\left(\frac{j+1}{2}\right) + j\left(\frac{j+1}{2}\right)\right]$$

$$= \left[j\left(j\left(\frac{j+1}{2}\right) + (j+1)\right)\right]$$

$$= \left[j\left(\sum_{i=1}^{j} i + (j+1)\right)\right]$$

$$= \left[j\left(\sum_{i=1}^{j+1} i\right)\right]$$

$$= jt_{j+1}$$

or, as $t_{j+1} = \sum_{i=1}^{j+1} i = \frac{(j+1)(j+2)}{2}$

and $= t_j + (j+1)$

then $\sum_{i=t_j}^{t_j+j} i = t_j + (t_j+1) + (t_j+2) + \ldots (t_j + t_j)$

$$= (j+1)t_j + (1 + 2 + \ldots j)$$

$$= (j+1)t_j + \frac{j}{2}(j+1)$$

$$= (j + 1)t_{j+1} - \frac{(j + 1)(j + 2)}{2}$$

$$= jt_{j+1}$$

Proof of conjecture 2.

To establish this conjecture, note that the left-hand sides of the two conjectured equalities are identical. Consequently, it suffices to show that $3mT_m = mjt_{j+1}$ or equivalently $3T_j = jt_{j+1}$.

$$3T_j = 3 \sum_{i=1}^{j} t_i$$

$$= 3 \sum_{i=1}^{j} \sum_{k=1}^{i} k$$

$$= \frac{3}{2} \sum_{i=1}^{j} i\,(i + 1)$$

$$= \frac{3}{2} \sum_{i=1}^{j} i^2 + \frac{3}{2} \sum_{i=1}^{j} i$$

$$= \frac{3}{2}\left[\frac{j(j + 1)(2j + 1)}{6}\right] + \frac{3}{2}\left[\frac{j(j + 1)}{2}\right]$$

$$= \frac{j(j + 1)}{4}\left(2j + 1 + 3\right)$$

$$= j\left[\frac{(j + 1)(j + 2)}{2}\right]$$

$$= j \sum_{i=1}^{j+1} i$$

$$= jt_{j+1}$$

From this we find

$$3(T_j - T_{j-1}) = 3t_j$$

$$= jt_{j+1} - (j - 1)t_j$$

so that

$$(j + 2)t_j = jt_{j+1}$$

or $2t_j = j(j + 1)$

which is true.

Q.E.D.

The conjectures proved provide interesting information concerning the triangular and tetrahedral numbers. In addition, their proofs require the understanding and manipulation of summations, a topic of some difficulty to many students.

Bonnie Litwiller and David Duncan are Associate Professors of Mathematics at the University of Northern Iowa.

The evaluation of curricula

D. Alexander, J. Beamer, W. Higginson

At the International Congress of Mathematicians, Vancouver, August 21 - 29, 1974 a three-day symposium was organized by the I.C.M.I. (International Commission on Mathematical Instruction) to discuss evaluation of Modern Mathematics Curricula. Reports were presented by representatives of the United Kingdom, Russia, Poland, Brazil, Japan, India, U.S.A., Canada, Germany, Denmark and Hungary with a general discussion on the topic on the third day. An official report will be sent in due course to all affiliated organizations (in Canada the Canadian Mathematical Congress). This present report reflects the impressions of the Canadian representatives.

There appeared to be general agreement that initial evaluations conducted in the experimental phases of the new curricula had been favourable while the evaluations became less significant as the curricula were expanded to encompass more schools. Reference was made to three factors which could explain this: 1) the reduction in the selectivity of the secondary and tertiary student bodies which coincided with the introduction of the new programs in most countries; 2) the lack of adequate teacher training for implementing the new programs; 3) the inability to define "modern mathematics curricula" (is it primarily content or method and if so what content and what method?).

The only large-scale study reported was the National Longitudial Study of the U.S.A. where the most important factor determined was the teacher. No characteristics were identified for the effective teacher and the effectiveness of the teacher was not consistant in successive years. It is expected that the National Assessment of Educational Progress will contain some evaluation of mathematics curricula in the U.S.

In the U.K. a study is just being undertaken which will be an observational type rather than statistical. There is a definite feeling that large-scale statistical studes are doomed to failure because of the many uncontrollable variables involved.

Dr. Christiansen (Denmark and UNESCO) emphasized the need for evaluation of curricula on the basis of well-established goals (i.e. the goals of a developing country like Brazil, where illiteracy is a major problem, are entirely different from those of the U.S.A.) He also referred to his experience with UNESCO as impressing him with the impossibility of successfully 'transplanting' curricula. "Each country must find its own salvation."

The Polish answer to teacher training was enlightening. At 4 p.m. on a given day each week every elementary teacher will be required to watch an inservice T.V. program and assignments will be marked weekly by the local superintendant. This is one way of solving the teacher training problem. (It was not clear what happens to delinquents.)

Although no conclusions were reached on the evaluation of the modern mathematics curricula or on how such evaluation should take place, nor indeed even whether it should take place, the symposium did provide a forum for an exchange of information and opinions of an international level and was valuable for this alone.

The ICMI is a Commission of the International Mathematical Union. Canadian representation is through the Canadian Mathematical Congress. Dr. A. J. Coleman, President of the CMC, hopes to institute a mechanism for making this representation more effective in the future through the formation of a national committee or commission for Mathematics Education. It is hoped that he will find the support, political and financial, for such a committee.

O. A. M. E.

SECOND ANNUAL CONFERENCE

May 23 - 25, 1975

Presentation, Workshops and educational displays for elementary and secondary school teachers, members of faculties of education, universities and community colleges,

Location--University of Western Ontario, London, Ontario

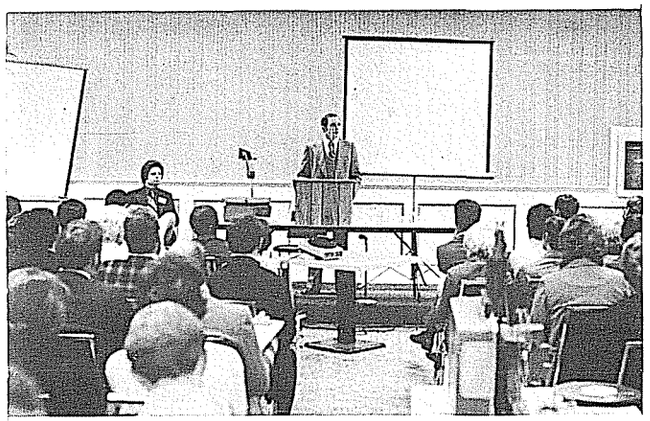
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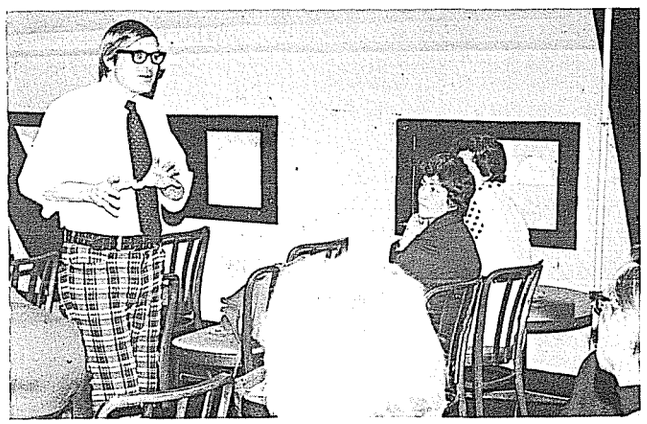
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Four members of the Planning Committee at the Registration desk. Standing are T. Mousseau (Principal of Falls View School, Niagara Falls) and Jack Le Sage (Ministry of Education) while seated are Bruce Jay (former Mathematics Consultant, Niagara Falls) and M. MacGregor (Secretary-Treasurer, O.A.M.E.).



Mr. Mike McKenna (Ministry of Education) addresses the Opening Session.



Mr. Reg. Shedbolt (Superintendent with Frontenac County Board of Education) addresses a session on Saturday morning.

Leadership Seminar

Bruce Jay

The general feeling was one of enthusiasm and approval for the opportunity of taking part in the seminar. There were about 120 participants from 45 Boards of Education. Would it be possible to find out why there was no representation from the other Boards? The size of the group seemed to be about right for this type of seminar and the hotel chosen offered suitable facilities for this size of group.

The group leaders did a very fine job and, in general, were well received. There was a variety of material and most participants were happy with the selection of leaders and topics. Of course, we can't expect to please everyone - there are even some people who don't like home-made apple pie. We are fortunate to have people who are both willing and able to accept responsibility for leading such a seminar. They are a dedicated lot. Many felt that it was a happy circumstance by which we obtained Dr. L. Calahan as a leader and that he did a terrific job. Some of the participants didn't seem to realize that this was a leadership seminar and made little use of available time to discuss qualities of leadership, etc. Maybe there should have been a place and time for this on the programme - on an optional basis.

Most participants appreciated the efforts of O.A.M.E. in having this seminar, and from the number who obtained membership in the Association, it would appear that they wish to benefit by further opportunities.

Bruce Jay - Programme Coordinator

H E L P !

Are you a member of O.A.M.E.?

Who would you choose to be Vice-President (President elect)?

Who do you know who would be a good Executive member?

Who would you like to represent you as a Councillor?

The Nominating Committee needs your suggestion.

Your O.A.M.E. Executive and Council will only be as good as your nominations make it.

Please use the following form or its duplicate.

I nominate_____ (name)

_____ (professional address)

for the position of_____ (Vice-President, Executive or Councillor)

Name of Nominator_____

Professional Address_____

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If possible this nomination should be accompanied by information which supports the nomination.

Return to: D. W. Alexander

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BRIEF TO MINISTER OF EDUCATION

Copies of a brief on the qualifications and preparation of Teachers of Mathematics (K-8) are available from M. MacGregor, 247 Bright Street, Sarnia, N7T 4E9. This brief was prepared from comments from OAME members and Faculties of Education who were sent a preliminary draft. The brief has now been approved by the OAME Council and sent to the Minister.

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