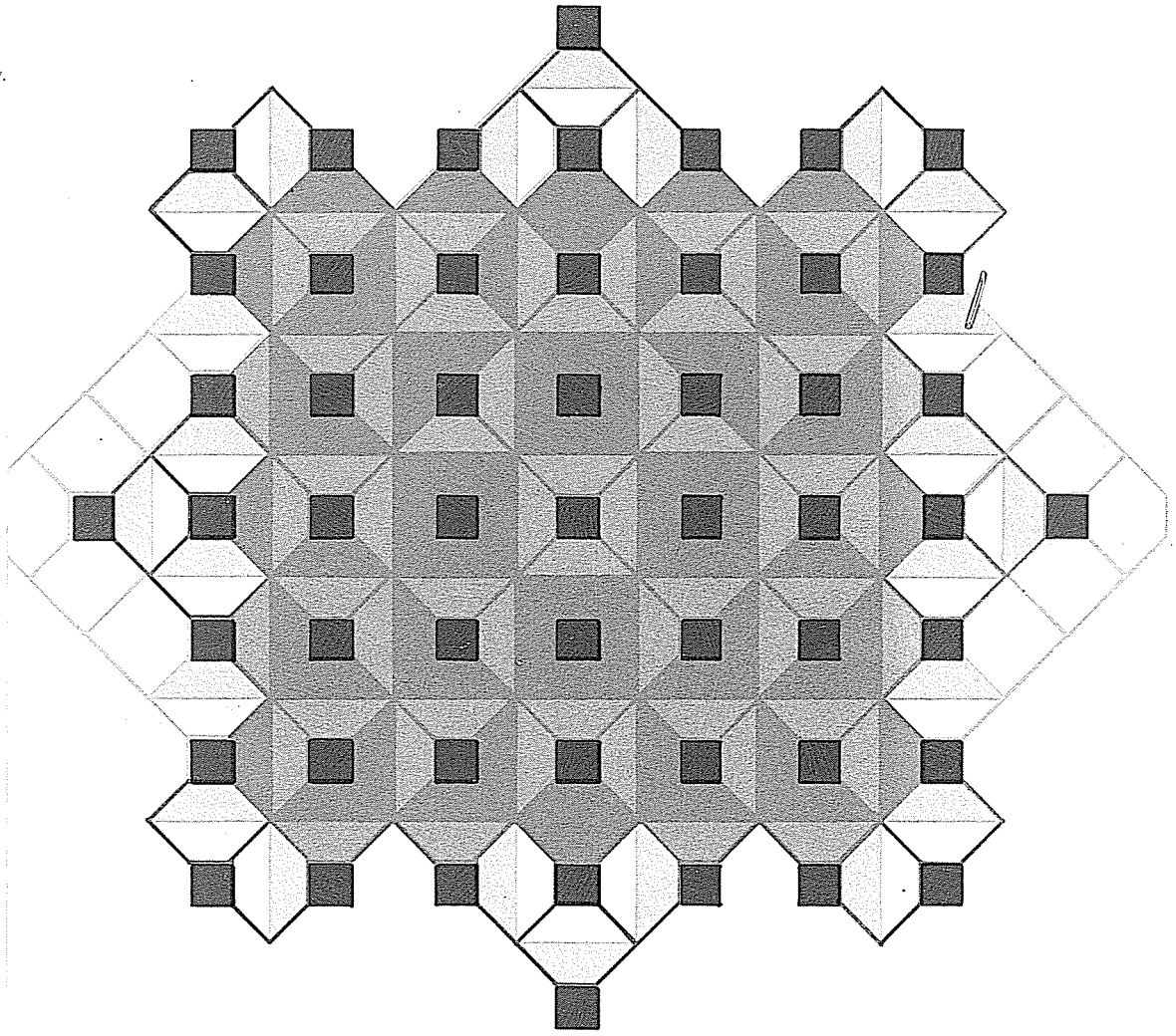
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L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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EDITORIAL

Last month your editor spent a pleasant day with the secondary school mathematics teachers of Huron County in Exeter. The conference was sponsored by O.S.S.T.F. and chaired by Graham Russell of Clinton. The theme was Transformations and Roy Yates of London and I were speakers.

It was exhilarating for me to get back into the classroom for a few hours, if only vicariously. Not only did I meet former associates but also young teachers who had been in my classes at Althouse College a few years ago.

It was particularly gratifying to see Marlene Storey, of Clinton, who was my student during my last year of teaching, give an excellent paper on the implementation of transformations in Grade 9 at her school. Marlene and the others I met made me feel that the torch is in good hands, to use a Remembrance Day analogy.

My presentation was a theoretical justification of transformations; Mr. Yates gave the practical side very capably. Perhaps you are asking yourself why the new Intermediate guidelines place so much emphasis on flips, slides and turns and the like. Is this just a temporary aberration, something that George Scroggie does, which will go away if we keep quiet?

My argument for the use of transformations in the schools has its roots in philosophy, history, psychology, practical paedogogy as well as the nature of mathematics itself.

The philosopher builds on very basic definitions and assumptions. The beginning of transformations is symmetry. Symmetry is basic in nature, being present as soon as we have cell division in life itself. Just as number forms the foundation of arithmetic, algebra, analysis, etc., symmetry and reflection can be used as the basic building blocks of geometry.

Historically, the use of symmetry and tessellations in art is very old, pre-dating the invention of the deductive method and Euclid's sequence of theorems. If you "sort of" believe in recapitulation theory (that the child's curriculum should bring him or her through the development of the human race), then you have a justification for transformations in the Intermediate Division, and earlier.

Transformations are psychologically sound. This kind of geometry is often called "motion" geometry. It can be illustrated by models and cut-outs, allowing the child to move things about and "feel" his way into a knowledge of the properties of figures. At first, the emphasis is on the special figures, such as the square and the equilateral triangle, which is best from the point of view of learning theory too.

In advanced mathematics transformations are important and contemporary. However, the idea is very old, going back to the cross-ratio theorem, perspective, descriptive geometry, projective geometry and so on. It was in 1872 that Felix Klein proposed that geometry be defined as the study of properties from the standpoint of groups of transformations, with emphasis on invariant properties.

From a practical teaching point of view, it is not just as a classifying tool that the transformation is useful. It illustrates powerfully something very important about mathematical problem solving. Mathematicians continually search for methods of illuminating complicated relationships. The grade 13 work on relations is an excellent example. We wish to study the properties of a relation which has a complicated formula. We match it with another relation having identical properties, by means of a transformation, thereby making our task simpler. Isn't that what mathematics is all about?

That is the gist of my argument. Transformations are now in the core of the Intermediate Division programme. It will not be possible for you to "let George do it". You will have to take part in the search for a workable sequence for elementary and secondary schools. Good luck! Thanks to my friends of Huron County for the opportunity to sum up my views on an important issue.

A History of Geometrical Methods J.L. Coolidge (Oxford, 1940)

Arn

WHAT'S NEW AT THE GAZETTE

This is the eighth issue of the Gazette since the editorship moved to London in the spring of 1975. It is also the last to be produced by the present editor. Our editor-designate, Bob Smith, will supervise the March, 1978 issue, and will, of course, be editor for the next two years.

Professor Robert Smith is an assistant professor in the Department of Mathematics, Faculty of Education, University of Western Ontario. He has had a distinguished career as an educator, having been an elementary school teacher in Brantford, principal of F.C. Bodley Public School and master at London Teachers' College, now Elborn College. He holds the Master of Education degree from the University of Toronto and studied at the University of London during 1975-76 in the Associateship Programme. Many of you will recall that Bob was programme chairman for the excellent annual conference held at Western in 1975 and co-editor of the conference proceedings.

It is not without regret that your editor lays down his pen and passes the responsibility along to his successor. However, I am confident that Bob will do very well and I offer him my help whenever he feels the need of it.

This is a good time for me to pay tribute to the excellent staff who have helped to keep the Gazette coming out on time with fresh and provocative material. Thanks are due to the Associate Editors who have been largely responsible for ensuring that we get plenty of high calibre articles. Pole Printing, especially, Jamie, Laurie McCormick, Lynn Evans and Eric Knight have been wonderful to work with. Dorothy Elliot and Nora Logan of North Lambton Secondary School have done the proofreading with care and skill. I am also grateful to the numerous contributors from all across the continent who have submitted articles. Morley and Mona McGregor have looked after the business arrangements and mailing with their usual quiet efficiency.

Lastly, thanks to you the readers for your continuing lively interest in our little magazine, and to O.A.M.E. for its loyal support over the past 2½ years.

Its time to say "thirty". Good luck! Keep the articles and letters coming in. You are reminded that the deadlines for submission of material are November 1, February 1, and August 1. From now on please submit articles to Professor Bob Smith at the address given inside the front cover, or to one of the associate editors.

Arn Harris

LETTERS TO THE EDITOR

Ottawa, Ontario
September 28, 1977

Dear Mr. Harris:

Today I received my copy of the September Gazette. It's excellent!

I do believe the colour scheme on the cover commemorates the Silver Jubilee of Queen Elizabeth II. The colours silver and royal purple, together with hues of the same, and white, are the same as those on the commemorative stamps of Her Majesty's Jubilee.

(It will be fun if I win a sweatshirt!)

Yours,
S.M. Kelly

Editors Note: An excellent suggestion. Why didn't we think of that?

Regina, Sask.
S4S 5M1

Dear Arn:

The September 1977 issue of the Ontario Mathematics Gazette, page 64, "Problems", contains a number of incorrect SI (metric) usages. First, it is not correct to put a period after an SI symbol. Symbols are not abbreviations. Second, when a numeral is used, then the symbol should be used, not the word, i.e. 5 l, not 5 litres (problem 8). Finally, a space is always left between the numeral and the symbols (problem 8, 3 m), but this is probably a typographical error since it occurred only once. In addition, I would like to add that the metric system seems to dictate an emphasis on decimal notation so we should attempt that usage as much as possible (problem 3, ½ metre should be 0.5 m and please note that we insist on 0 before the decimal in 0.5 — it makes good sense and is preferred style).

A Canadian mathematics journal should attempt to be "super" correct in SI, a relatively new area. After all, it is an example for many readers, I'm sure.

Yours sincerely,
Ed Klopoushak

Editors Note: Your concern has been passed on to our problems department. My excuse is that those were "pre-SI" problems. By the way, what ever happened to the hectare in Saskatchewan?

Department of Mathematics
University of Saskatchewan
Saskatoon, Canada
S7N 0W0
October 31, 1977

Editor OAME,
Mr. A.W. Harris

Dear Sir:

I am enclosing with this letter a copy of the report of the 1977 Mathematics Olympiad, along with copies of the examination, solution key and the certificate given to each participant. In the case of prize winners or those receiving Honourable Mention, the phrase, "Certificate of Competition" was replaced by the appropriate words ("First Prize, "Second Prize", etc.)

I have now completed my three year term as Chairman of the Olympiad Committee of the Canadian Mathematical Congress. Professor W.J. Blundon of Memorial University, Newfoundland, will be Chairman for the next three years. No doubt you will be hearing from him soon regarding the 1978 Olympiad.

I appreciate very much the assistance you provided in making information concerning the Olympiad available to teachers and principals. I am sure that, without your cooperation, many worthy students would not have been able to participate in the Olympiad. Thank you.

If you are no longer editor, please forward these documents to the new editor.

Yours sincerely,
G.H.M. Thomas, Chairman
1977 Olympiad Committee

To get the full value of a joy you must have somebody to **divide** it with.
Mark Twain

FATHER FAUGHT AWARD

Each year O.A.M.E./A.O.E.M. awards a plaque to the Ontario winner in the Junior Mathematics Contest. The winning school this year was Thornlea Secondary School. This team also stood first in all of Canada. Neil Williamson made the presentation. Don Attridge was the winning coach.



Left to Right: *Nevin Abernethy, Neil Williamson, John Sidwell, Saroja Polavarapu, Robin Davies, Philip Mansfield, Don Attridge.*

WHAT COMES FIRST: ½ OR 0.5?

J. John Kere, St. Catharines, Ontario

With the advent of the SI-system to the classrooms and the entire world, almost all measurements, with the exception of measurements of time and angles, can be expressed using decimal fractional numbers. Thus, the idea of teaching common fractional numbers extensively has lost its significance.

However, there are several reasons which support the need for keeping the common fractional numbers in any school program as one of the most important topics.[1] [2] . Among them are:

(1) Common fractional numbers come directlyafter natural numbers as an extension of the number system.
(2) Learning of common fractional numbers helps to deepen the meaning of operations with numbers.
(3) Common fractional numbers, to a realistic degree, are necessary for solving problems in the everyday world.
(4) Common fractional numbers, if learned **before** decimal fractional numbers in a methodical manner, construct a very necessary base for studying and understanding decimal fractional numbers.
(5) Common fractions are necessary in order to comprehend the study of algebraic fractions.

In the "Recommendations for Intermediate Division Mathematics in Ontario" on page 87 (Summary of specific Content Recommendations) for recommendation 6) Fractions: the last part reads:[3] ... "There is research being done now on **introducing decimals before fractions,** an idea that is not necessarily new:

"Decimal fractions, or Decimals, instead of being treated as a special class of fractions, are presented as **an easy and natural** extension of our ordinary system of numeration. Too often the result of deriving the rules for decimals from those for fractions is, that the pupil reduces all decimals occurring in his work to the fractional form and **operates** with or upon these, thus losing the enormous advantages of decimal calculation. By the method here followed all danger of this is avoided, while there is the additional advantage that **the teacher who prefers**--and some of our best mathematical masters do so--to introduce decimals before fractions **finds that his textbook leaves him at liberty to arrange the order as he chooses".**

(Public School Arithmetic, 1887, p. iii-iv)

This idea is proving to be most successful, and **many feel it is an improvement over the fraction-before-decimal approach". (Bold is mine).**

My intention is to present some ideas of the meaning of "... **introducing decimals before fractions..."**

At this point it may be necessary to distinguish between common fractional numbers and decimal fractional numbers. 3/10, 19/100, and similar, are NOT decimal fractions. A decimal fraction has two characteristics. Initially, the denominators of decimal fractions are always multiples of ten, and secondly, their symbols are written in the position system. Only under these two conditions is it possible and worthwhile to develop the special number idea of decimal fractional numbers. It is for this reason, that three tenths can be expressed as 3/10--a common fraction, and 0.3--a decimal fraction.

One must realize that in many situations common fractional numbers are needed for problem solving: [4]

I) **to find a fraction of a number.**

Examples of this include to find the measurement of time (⅓ of a year, ¾ of an hour); to express the measurement of a portion of an angle ⅔ of 90°, 1/10 of 180°); to find a fraction of a number (In class 25 students. 1/5 of them are honour students. How many honor students?

II) **to find the number, if the fraction of it is given.**

For example: a) Irene paid for a lunch $0.80. It is ¼ of her money. How much money has Irene? b) What is the number, if 1/8 of it is 7?

III) **to find what fraction makes a given number in the ratio to another number.**

For example: a) What fraction of a metre is 20 cm? b) In a class of 27 students 9 are boys. What fraction of the class represent boys?

Problems of this type should be solved in the formative years to deepen the understanding of the fraction concept, some time before a learner should face decimal fractions. Moreover, we have to communicate using common fractions before their symbols are introduced: ... each have one sixth of the pie, ... it took three quarters of an hour, ... farther than one half of kilometre, ... it may be longer than one third of a metre.

We are not going to say: Mary ate zero decimal five of the pie, or, Mary ate five tenths of the pie. Thus, in real life situations children use common fractions **before** decimal fractions.

Common fractional numbers WILL NOT disappear with the use of SI-system or because of calculators and computers. They are still in mathematics as one of the basic topics in countries (Scandinavian countries, U.S.S.R., Japan,5 to name some) where the metric system has been used

for 50 years.

Calculators can be used to "discover" decimal fractional numbers. Learners can use calculators to find, for example that 1/10 = 0.1, 7/100 = 0.07, etc. Input in calculator is NOT restricted just to decimal fractional numbers.

Since the above passage quoted from **Public School Arithmetic,** 1887, p. iii-iv, is from the books' Preface, it is hard to know, how in the text the **concept of fraction** has been developed: using common fractions or decimal fractions. Indeed, the quotation states "... the teacher who prefers ... finds that his **textbook** leaves him at liberty to arrange the order as he chooses." Thus, one can conjecture only that the suggestion is - to teach **the operations** (+, —, x, ÷) with decimal fractions before (they have been drilled too much) with common fractions.

Really, the **approach** of teaching fractions should be changed: The emphasis should not be put on acquiring a skillful cleverness in operations with common fractional numbers, but on the development and understanding of fraction concept. Ability to grasp and use very simple common fractions, to compare and to add and subtract them - with common denominators only - may be a good base for the introduction of decimal fractional numbers.

The statement, in the just mentioned quotation, that "Decimal fractions are ... **an easy and natural** extension of our ordinary system" is an exaggeration. The history of mathematics shows that common fractional numbers were developed and used a long time **before** decimal fractions were invented.

Several years after W.W. II, the U.S.S.R., which began using the metric system in 1924, reassessed their advance of teaching metrication in schools. In this connection there was experimentation done on teaching decimal fractional numbers **before** teaching common fractional numbers in various schools and in a multitude of dimensions for some years (K. DUKURS, J. MENCIS, ARITMETIKAS METODIKE, "LIESMA", Riga, 1964, page 285)6. As the results were reported in 1964, they showed that in principle one can teach decimal fractional numbers before teaching common fractions, but the practical approach in teaching **and learning** have been so and difficult that the idea was dropped completely.

In the latest, revised U.S.S.R. courses of study and textbooks (printed 1975)[7] common fractional numbers are **formally** taught in the beginning of grade 4 (they start school 2 years later than we do), and decimal fractions-in second part of school year. However, in grade 3 textbooks[8] there is some introduction of: i) Concept of basic fractions: as numerators 1 only, denominators-one digit numbers, and 10, and 100; ii) Concept of common fractions; iii) How to find a fraction of a number; and iv) To compare two common fractions, if denominators are the same. All is done at the very simple level by many illustrations and relevant applications.

The mentioned grade 4 textbook consists of two parts: Part I - Natural numbers and Common fractional numbers - ⅔ of the text. Part II - Decimal fractional numbers - ⅓ of the book - not counting review exercises. Starting from paragraph 1, the presentation of common fractional numbers is woven into the exploration of natural numbers, geometry and real world problems. The following topics of common fractions are introduced: Symbols for common fractions, to find fraction of a number, what fraction is one number in ratio to another; To compare two fractions, with the same denominators; Proper and improper common fractional numbers; Addition and subtraction of common fractional numbers, with the same denominators; Division and common fractional numbers. All presentations are done at the very simple level, emphasizing fractions with one digit numerators and as denominators: 2, 3, 4, 5, 8, 10, 12, 100.

One cannot change the fundamental idea, that decimal fractions in essence are common fractions written in a special way, which is favourable in operations with fractions. However, the symbols of common fractions support and help to learn the concept of a fraction and clarify the idea of "a fraction of a whole" and "a fraction of many". Before these ideas are absorbed by learners, they **can not grasp the meaning** even of simple problems with decimal fractions, for example, 0.5 x 16 = 8, 0.5 + 0.7 = 1.2.

To teach the concept of fractions by using a decimal fraction, one has to overcome a big obstacle - decimal fraction symbols **do NOT display** the concept, and do not help to visualize the meaning of

it. How can the symbol 0.5 **teach** the concept of one half? It can **represent** one half, because it is 5/10, which is ½.

Thus, ½ comes before 0.5.

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LET'S REALLY "GO METRIC" — REALLY?

Brock Rachar, Co-ordinator of Mathematics, London Board of Education

In the spirit of the article by Howard Munson in the September issue, we could continue to look at metrication as an all pervasive system. With him I am in full agreement that the move towards internationalization of a measurement system which is based upon the prevalent and comparatively easy decimal system, is the right move.

A politician of the 1920's, now forgotten, seems to have said that "when you get half way to Russia (the Russian system) you'll be half way to common sense". Few implied from that that he was advocating going all the way.

Many have tried to make hundredths of a circle a practical thing. Some early radios had tuning condensers arranged on a 0 to 100 scale control. Angular measure is of course decimalized beyond seconds ('') and more and more commonly even beyond the degree, where that sort of thing really matters. (e.g. 63.50° rather than $63^{\circ}30'$)

Then, what of the nature of geometric construction? A square corner (90°) being 25 is not bad, but what of the basic 60° angle of the equilateral triangle. 16.6 is not exactly neat. The angle sums of triangles and quadrilaterals at 50 and 100 respectively seem rather good, but it would be a shame to lose the highly factorable 360.

I believe also that one reason for using Celsius degrees for temperature is to avoid a duplication of terms. 1 centigrade = 10^{-4} right angles.

Let's next consider what happens to music. Natural resonances do not work in a decimal manner. Even the tempering of the 8-note scale to make the intervals regular, while a workable system for keyboard instruments does not satisfy the harmonic requirements of orchestras or human voices. And can you imagine a decimalized time signature.

The process of doubling and halving is almost certain to exist even in civilizations as yet unseen or dreamed of, but there is no guarantee that a ten-based system is inherent in the universe at large. There is the duodecimal system for one.

Perhaps we should be happy to change those things which would be better changed and leave the others (after looking at them).

I seem to have seen in a museum somewhere a large pocket watch showing 10 hours and 100 minutes per hour. This was an innovation of the French Revolution. Apparently what killed the system and its inventor (by guillotine in fact) was the 10-day week. What do you think?

BASE TEN, BASE WHAT?

Sam Pravica, Faculty of Education,
University of Western Ontario

My colleague Alex Norrie's article in the September '77 issue of the Mathematics Gazette prompted me to put pencil to paper and write some of my concerns relative to the application of the stages of development of children in curriculum. I was particularly interested in his diagram on page 17 showing a relationship of these stages and how they coincided with the stages of development in measurement. My objective in this article is to expand on Mr. Norrie's development but with a somewhat different tack by showing that without reflection and care the concepts relating to the stages of development may be misused. I will be referring specifically to the development of the Place Value concept in children.

Let me begin by repeating David Elkind's statement that "a basis in logic must be developed in the child. The logic base is dependent on many factors some of which are maturation, experience, social transmission, and equilibration". There is yet another factor which we must keep in mind when exposing children to new concepts and this, in the words of Jean Piaget (1973), is "the problem can be found at another level - the teacher is often tempted to present far too early notions and operations in a framework that is already formal".

In our enthusiasm to provide children with rich experiences we sometimes lose sight of the fact that we cannot force understanding nor can we force children to view concepts at a formal level when in fact they are operating at the concrete operational level. You will have no doubt recognized that providing the appropriate experience relative to the development of a concept gives no guarantee that the child's actions on the material produces understanding. This most often occurs when these experiences have not been matched to the child's stage of development. In the words of Jean Piaget, "logic does not arise out of language but from a deeper source and this is found in the general coordination of action". In other words, in the concrete operational stage, the child cannot develop understanding by verbal explanation alone, it is his actions on the material that produces meaning, and if this material is not presented at the appropriate level, neither language nor material manipulation will produce understanding; for example, presenting the child with the problem "which has a greater mass, a kilogram of iron or kilogram of lead?" produces in him a sense of disequilibrium. To solve this problem requires a fairly sophisticated mental process. My thesis is that it requires a similar degree of maturation and experiences to draw a parallel between the base ten system and bases other than ten.

How often have you had to struggle with the meaning behind the phrase "teach children bases other than ten so that they may better understand the base ten system" only to come to the realization that you failed to reach your objective and then rationalized that the fault was in not providing the appropriate experiences. Take heart, it is not your doing. It is more likely that the concept which you were trying to expose the children to was beyond their grasp.

There are a number of good explanations surrounding this phenomenon. I would like to discuss some of them in the following paragraphs.

Firstly, long before students can make sense out of the place value system in bases other than ten, they must be able to make the connection for example between the number 25 and its expanded form $20 + 5$. Given twenty-five objects the child can readily verify that there are in fact 2 groups of ten and 5 ones. However, it never ceases to amaze the teacher that after a clear demonstration that the child "understands" the grouping process he will still write $25 = 2 + 5$. The problem here is that of abstraction, or a mental process that requires operating at the formal level. The abstraction that a child must make here is similar to that encountered by a junior grade student when he is faced with attempting to understand that $2\frac{1}{2} = 2 + \frac{1}{2}$. To have students "understand" that $2\frac{1}{2}$

is a more convenient way to write 2 + ½ or, that 25 is a more convenient way to write 20 + 5 is at a level of abstraction beyond their operational level of understanding.

Secondly, in order that the child recognizes the operational equivalence between two independent place value systems, the student must be operating at the formal level relative to this concept. Carl Bereiter from the Institute for Studies in Education indicates that a reasonable test of a student's ability to handle abstraction at the formal level is related to how many operations the student is able to connect and hold in his conscious mind at one time.

In order that a child internalizes the parallelism between the base four system, say, and the base ten system he must hold a minimum of five operations at one time.

fig. 1. Equivalance Schema between the base four system and the base ten system.

Mental Process	base four	base ten
1	Groups objects into groups of four	Groups objects into groups of ten
2	Relates grouping to place value scheme	Relates grouping to place value scheme
3	Verifies that the processes are the same	

In the above diagram, it is shown that the child must recognize that the two independent mental processes in 1 are equivalent, the two independent processes in 2 are equivalent, and then, be able to draw the conclusion that the place value system for base four is equivalent to the place value system for base ten.

It is difficult enough for a person operating at the formal level to deduce that these systems are in fact equivalent, let alone the child operating at the concrete operational level.

There is a third indicator that I use which tells me that our expectations may have exceeded the ability of children to abstract the meaning of a concept, and that is simply, if most adults find the concept difficult to understand, then you can bet most children will find it difficult.

I have used the following exercise with adults (with backgrounds in mathematics at all levels) to verify within myself that in fact we are barking up the wrong tree when it comes to teaching bases other than ten to most children.

1. When asked to count from zero to twenty in base ten, adults have no difficulty whatsoever.

2. When asked to count from zero to twenty in base five, say, most adults will not know how to respond after five.
   They have a great deal of difficulty understanding that the counting process is independent of the place value system. Confused?! If you are, then I'm making my point!

3. When the same group of adults are asked to expand 234_{five} as a base five expansion, those who have some familiarity with bases other than ten will write, $234_{five} = 2 \times 5^2 + 3 \times 5^1 + 4 \times 5^0$
   If you cannot see anything wrong with this then I **have** made my point ... There is no numeral "5" in base five. The correct expansion is, $234_{five} = 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$.

As a matter of fact, the number 234 expanded in any base greater than four will look absolutely identical,

i.e. $234_{six} = 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$

or $234_{ten} = 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$

then what is the meaning of
$234_{five} = 2 \times 5^2 + 3 \times 5^1 + 4 \times 5^0$? The right hand side is of course the base ten equivalent of the number 234_{five}.

If the student has synthesized the place value concept then he should have little difficulty finding the base six representation of 234_{five}. Try it! Let me know how well you have succeeded.

For those of you who are familiar with Bloom's Taxonomy (1956) you will recognize that "synthesis" corresponds to the highest level of conceptualization (excluding evaluation). Avital and Shettleworth (1968) state that performance at the level of synthesis is "the putting together of given elements in an entirely new way".

I am firmly convinced that this is what we are demanding of children when we ask them to regroup objects in an "entirely new way" and draw inferences of equivalence between two place value systems. It is analagous to teaching students Latin in order that they better understand English grammar.

Does this mean we eliminate this activity from the curriculum? Not at all. I believe that it is an excellent enrichment activity in itself. Children enjoy working with bases other than ten, and who knows some bright light might see the connection.

My only intent in this article is to place the concept in proper perspective and to point out that the chances are very good that children exposed to bases other than ten will not develop a better understanding of the base ten system.

It is perhaps more valuable for teachers to experiment with bases other than ten and experience some of the frustrations and difficulties that children encounter when attempting to make some sense out of the base ten system. This would perhaps enable them to enrich their understanding of student difficulties and thereby enable them to develop better diagnostic procedures and provide more effective remedial activities.

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TEACHER SUPPLY — A PROBLEM AND AN OPPORTUNITY

Bernard J. Shapiro, University of Western Ontario

Dr. Shapiro is Dean of the Faculty of Education. In response to a request from the editor, he agreed to discuss the most vexing problem of today; lack of employment opportunities for young people, as it relates to teaching.

The current oversupply of teachers, that is the difference between the number graduated from teacher training facilities and the number of positions available for these graduates has received considerable public attention in recent months. In June, 1977, considerably more graduates were

produced than were required by Ontario schools, and this imbalance was relatively greater in the case of secondary programs. Further, the most recent Ministry of Education projections show that the demand for secondary teachers is expected to fall to near zero by 1981 or 1982 and then to increase modestly, with the elementary demand remaining approximately at the current level for several years and then increasing. Clearly, the need for new teachers is far less than the capacity of teacher education institutions to produce them. There is a 'prima facie' case for reviewing existing resources in teacher education institutions as well as the current deployment of effort within these institutions.

At the same time, there is the unexpected phenomenon of an increasing demand by students for admission to the Faculties of Education. At The University of Western Ontario, this aggregate demand is greater than it has ever been, having increased by 15% in each of the last two years. Moreover, the quality of these students has shown a marked increase along with their numbers. Thus, Faculties of Education find themselves the object of two opposite pressures. On the one hand, teachers' associations suggest that it is our responsibility to decrease the production of new teachers. On the other hand, qualified students insist on their right of access to our training programs. I wish to suggest that this difficult environment, in which the problem outlined above is only a single example, also creates certain planning opportunities, not opportunities for the dismantling of a public resource built up at such a cost and effort, but rather opportunities for the redirecting of this resource to the resolution of important problems in professional education.

The development of such opportunities depends in the first instance upon the willingness of the teacher education institutions to accept change, for substantial change is clearly required. I, for one, do not accept a strict manpower formulation of the role of Faculties of Education, for we have many obligations both beyond the schools (e.g., in the preparation of professionals who will direct the educational component of other social service and industrial agencies) and beyond pre-service programs (e.g., graduate training), to say nothing of our potential role in the continuing education of the general adult community. Nevertheless, teacher education institutions must - if they are to be publicly responsible - must reduce to some extent their activities in the preparation of beginning teachers. Although this will bring increased pressure to bear on our admissions processes, for we will be refusing admission to many, it will provide the opportunity to (a) clarify these admissions criteria, (b) focus on the need to admit only the best qualified to the teaching profession, and (c) go some way to reducing the extent of the current over-supply of teachers. It is in this context that the Faculty of Education at The University of Western Ontario has deliberately reduced the enrolments in its basic pre-service programs by 17% in the last two years and intends to make further reductions in the years ahead.

If this were the only opportunity of the current scene, the outlook for the teacher education institutions (and I believe the entire profession as well) would be rather bleak. To "decline with dignity" or to reduce ourselves to the "management of decline" is to forfeit the future in advance. Fortunately, therefore, this is not the only opportunity. The recent drop in the need for new teachers provides, in its own way, an exceptional opportunity for the teacher education institutions to redirect their resources and apply their resources towards the needs of the current body of experienced educators.

At the present time, there is a sense that the quality of teaching in the schools of the province can be improved, and this feeling is shared by many groups concerned with education, including the teachers' federations, parents, and taxpayers. There is always room for improvement. However, given the great expansion to the school system in the past twenty years, as well as the curricular changes and the steady expansion of knowledge in all areas, these in-service and graduate needs are now more pressing than ever before.

Further, in today's stringent financial climate, this need is more acute than before. Pressures to reduce public expenditures have forced school boards to curtail the number of consultative staff who assist classroom teachers just at a time when the Ministry of Education has placed increasing responsibility on local schools and their teachers for the implementation and evaluation of new curricula. Unfortunately, the model of teacher training which has operated in Ontario has neither fully prepared the teachers to respond to these new mandates nor provided for their learning while "on the job". Finally, it should be recognized that as school systems adopt many of the characteristics of a shrinking system, teacher mobility will decrease. In the past we often saw our best teachers leaving the classroom to pursue administrative duties. In the near future, the number of administrative posts available will be few, for our systems are not expanding and the

projected mobility of the current relatively young incumbents is rather limited. Thus, classroom teachers must become oriented toward their current role as a terminal position. If they are to continue to remain effective in this role, some systematic means must be provided to assist teachers to maintain and enhance their professional knowledge bases and to equip them with the skills required to meet rapidly changing conditions.

With their traditional concern for improvement in quality through teaching and research, I believe that this situation provides an opportunity for teacher education institutions to assume a greater responsibility than heretofore for the improvement of the quality of teaching by increasing their commitment to in-service training. Faculties of education, representing as they do a wealth of current subject matter orientation and knowledge of methodological techniques, are uniquely placed to respond to these field needs in a support capacity. Indeed, it can be argued that faculties of education - while not the system's only resource - is, perhaps, the single most important resource available to the province to (a) assist teachers in meeting their new and enlarged responsibilities, (b) provide a substitute support service to boards of education which have experienced major staff depletions, and (c) assist in the objective evaluation of new programs.

If, in the first instance, the opportunity to redirect resources within faculties of education toward the more salient need of in-service and graduate degree and diploma programs is dependent upon the willingness of such faculties to change, the actual change will require a good deal more than faculty willingness and imagination. These must be accompanied by a shift in the current method by which teacher education institutions are funded. At present, the funding arrangements encourage a continued focus on the pre-service preparation of beginning teachers. For each student admitted to a B.Ed. program, faculties receive specific Basic Income Units (BIU's) from the Ministry of Colleges and Universities. No such comparable funding is currently available for either graduate, certificate, in-service, or diploma programs. Thus, the present arrangements encourage faculties to continue their stress on pre-service education - a stress which is, as I have indicated above, counter-productive at the present time.

It is not often that government agencies are presented with the opportunity of improving the services of a public agency without increasing the costs involved. I believe, however, that relative to teacher education, such an opportunity is now available. Additional resources need not be allocated to teacher education institutions. All that is required is that faculties be empowered to use their current budget allocations to increase their in-service offerings while at the same time they reduce their pre-service enrolments. I suggest that such a policy shift on the part of the provincial government would redound to the benefit of the faculties, the teachers, the pupils, and the general public. It strikes me as an unbeatable combination.

A SURVEY OF PRACTICAL MATHEMATICS

John F. Leightell, Pickering College, Newmarket

Introduction

The last decade has produced radical changes in the teaching of mathematics at the secondary school level. Some traditional topics have been dropped and replaced with new ones. Other topics have been subjected to a new approach. Practical mathematics which was an important part of the 'old' curriculum seems to have disappeared entirely.

How much practical mathematics are today's students exposed to? How capable are they of coping with the fairly simple mathematics they might be expected to meet in everyday situations: These and similar questions led to the development of the survey discussed in this report.

The idea of the survey originated during a discussion about the 'numeracy' of todays students and their ability to function mathematically in real situations. There was, and still is, a great deal of criticism about the readiness of high school graduates to handle practical mathematics. These criticisms, however, are usually based on isolated incidents. They have little impact because they

are not based on a large number of similar situations. We felt that it might be informative to conduct a comprehensive survey to examine the ability of a fairly large group of students. If it were diverse enough it might provide some objective answers to the criticisms being voiced. We began by preparing a test based on a series of practical topics. In order to determine the topics to be included we asked a number of people (none of them connected with education) to list the situations where they encountered mathematics. The intent was to identify those skills necessary to the majority of people. From the responses we selected the more frequently occurring topics. These were (not in order of frequency): 1. Mathematics in the home (mensuration, home construction, etc.) 2. Reading charts and Graphs 3. Calculation of Averages in various situations 4. Automobiles (gas consumption, time, distance, speed calculations) 5. Shopping :making change, sales tax, unit pricing), 6. Finance (banking, credit buying, tax) 7. Pay cheques 8. Metrication.

Using this list as a guide we developed a series of items and organized them into a test. Since the school leaving age is 16, the test was designed to be used with grade 9 and 10 students. At this age they are presumably ready to enter the everyday world beyond the classroom and apply their skills and knowledge to everyday situations.

The test was administered originally to a group of 76 summer school students. The results were used to refine the test. Some items were rejected and others were modified and improved. A copy of the final version of the test, which contains 40 items, is appended at the end of the report.

Results

1. Reliability

The index of reliability was calculated by two methods; the split-half calculation gave a value of .824, whilst the KR20 formula gave .820.

2. Mean Scores and Standard Deviations

Each item on the test was given one mark if answered correctly. There was no penalty for guessing. The charts below give the raw score (maximum is 40) and the standard deviation for various groups. N is the number of students in each group

By Grade

Grade	N	Mean	St. Dev.
9	179	20.6	6.41
10	220	22.3	6.37

By Phase

Phase	N	Mean	St. Dev.
4	166	18.5	5.28
5	234	23.6	6.28

By Grade and Phase

Grade	Phase	N	Mean	St. Dev.
9	4	67	17.4	5.11
	5	112	22.6	6.22
10	4	99	19.4	5.21
	5	121	24.6	6.26

By Age

Age	N	Mean	St. Dev.
14	110	21.6	5.58
15	200	22.1	6.7
16	62	19.4	6.16

For the whole sample

N	Mean	St. Dev.
402	21.5	6.4

In the total sample the highest score was 39 and the lowest 6.

There were a number of 17 and 18 years olds who wrote the test but there were too few to include the results.

Of the 402 students who wrote the test, 48 scored 30 or more, 192 scored between 20 and 29, 152 between 10 and 19 and 10 students scored less than 10.

3. Difficulty of Items

These difficulties are based on the response of the whole group. The index of difficulty can range from 0 to 1. It indicates the percentage of students getting the item correct (e.g. a difficulty of .17 means 17% of the responses were correct). Lack of space makes it impossible to give the difficulties of all the items. Only the easiest and most difficult items are shown.

The Most Difficult Items (less than 40% correct responses)

Item	Difficulty	Topic
7	.15	annual interest
10	.38	calculation of cost
14	.29	deductions from pay
16	.27	bank interest
21	.13	perimeter and area
23	.36	perimeter of border
37	.22	gas used per mile
38	.23	time-distance problem
39	.36	Pythagorean Theorem
40	.28	Angle Sum of a triangle

Easiest Items (more than 70% correct response)

Item	Difficulty	Topic
1	.91	change from $10 bill
4	.85	find error in receipt
17	.77	use of metric units
18	.78	
19	.88	
22	.79	ratio
31	.79	reading a graph
32	.91	
33	.75	

Comments

Reliabilities

The reliabilities obtained are acceptable for this kind of test. They give a standard error of 2.7 (for the whole sample).

Mean and Standard Deviations

Since the items used in the test were of a fairly simple nature, the mean score of 21.5 was disappointing and indicative of a lack of ability to handle practical mathematics. Students in phase 5 mathematics courses did considerably better than their counterparts in pase 4 (a difference of just over 5 marks). This is perhaps to be expected but it is nevertheless disturbing, since it is the general level student who will need practical skills soonest and most often.

The difference in scores, by grade (1.7 marks) is too small to be significant. The improvement between grade 9 and 10 is not supported by an analysis of the scores by age. The 15 years old did better than either the 14 or 16 years old. The difference, however, is small and is most likely explained by the fact that the 15 year old group contained the highest percentage of phase 5 students.

The standard deviation of the various groups consistently indicates that there is a greater variety of ability (with respect to this test) amongst phase 5 students.

Item Difficulty

Several items were used for each topic. The table below shows the average difficulty of items for the various topics.

Topic	Number of Items	Average Difficulty
Metrication	4	.75
Shopping	4	.70
Reading charts and graphs	7	.65
Pay Cheques	3	.44
Maths in the home	5	.41
Automobiles (speed, time, distance, gas consumption)	4	.36
Finance (banking, credit buying)	6	.35

Conclusion

It is difficult (and probably dangerous) to draw conclusions from surveys of this nature. Statistics are often used (or abused) to suit a particular person's viewpoint. Having given this advice, we will now ignore it and state; The scores obtained are less than adequate. Whether this has always been the case or whether it is due to a lack of emphasis on practical mathematics, it is not possible to determine. It should be noted, however, that where a topic has received attention in the classroom (e.g. metrication) students seemed to perform well on related items.

2. Phase 5 students achieved much better scores thanphase4 students.
3. Differences in grade or age seemed to have little effect on the results.
4. A place should be found for the teaching of practical skills within the secondary school curriculum, particularly in grades 9 and 10, and especially in the 4 year phase.

Appendix

1. The following people were responsible for constructing and administering the test: Prof. Shmuel Avital, University of Toronto; Tom May, Opeongo D.H.S.; Douglas George Gamble, Guelph C.V.I.; John Leightell, Pickering College, Newmarket; Art Prudham, Forest Heights C.I., Kitchener; Tony Macaluso, St. Josephs, Aurora; Bob Vanstone, T. A. Blakelock H.S., Oakville.

2. Sample Test Items. Only 19 of the 40 items are given here. These are the easiest and most difficult items referred to earlier in the article. Copies of the entire test can be obtained by contacting the author. The wording of many of the items has been modified in order to shorten them.

1. What change would you receive from a $10 bill for groceries costing $8.49?

4. Below is a bill from your local dairy. There is an error in the bill.

Quantity	Description	Price
27 qts.	Milk @ 55¢	$14.85
16 qts.	Cream @ 53¢	8.48
7 pts.	Cheese @ 43¢	2.81
8 qts.	Ice Cream @ 72¢	5.76
	Total	$31.90

(a) the bill is added incorrectly
(b) $1.00 too much is charged for milk
(c) 10 cents too little for cream
(d) 20 cents too little for cheese
(e) 40 cents too much for ice cream

7. You borrow $10.00 and are asked to pay $.50 per week in interest. The annual rate of interest is (a) 5% (b) 25% (c) 50% (d) 120% (e) 260%.

10. Some car dealers sell at 'cost plus 10%', i.e., they add 10% to the cost of the car. If a dealer sells a car for $4400 what did the car cost the dealer? (a) $3960 (b) $4000 (c) $4300 (d) $4356 (e) more than $4400.

14. Use the pay cheque shown to answer this question

Regular Hours	Hourly Rate	Overtime Hours	Overtime Rate	Overtime Pay	Gross Income	Tax	Pension	Total Deductions
37.0	$3.90	12	hourly rate × 1.5		$214.50	$38.47	$8.53	$47.00

Net Income _____

What percentage of your gross income do you lose in deductions? (a) less than 8% (b) 8-15% (c) 16-24% (d) 25-32% (e) over 32%. NOTE: items 12 and 13 required the student to calculate overtime pay and Net Income.

16. A bank pays $\frac{1}{2}$% interest per month on the minimum monthly balance. What amount of interest would be payable on the account shown below?

DATE	DEPOSITS	WITHDRAWALS	BALANCE
May 1			426.00
May 5	256.00		
May 17		165.00	
June 4		43.00	

(a) $ .85
(b) 1.68
(c) 2.13
(d) 2.59
(e) 3.41

For questions 17, 18 and 19 choose the correct response from the list on the right.

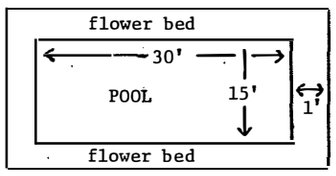
17. John ran 4 ? to school.
18. What units would most likely be used to measure weight?
19. 2000 lbs. is equivalent to one ?

(a) liter
(b) cm
(c) km
(d) kg
(e) ton

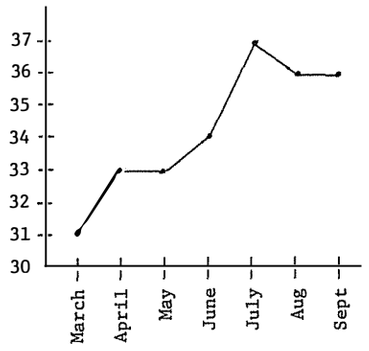
21. You wish to fence an area of 100 square feet. Which statement is true?
(a) A square needs more fence; (b) A rectangle needs more fence; (c) Both require the same; (d) There is insufficient data; (e) None of the above are true.

22. A recipe for 4 people requires 2 cups of cake mix and one cup of water. To use the same recipe for 10 people would you use: (a) 3 cups of cake mix and $1\frac{1}{2}$ cups of water; (b) 4 cups and 2 of water; (c) 5 and $2\frac{1}{2}$ of water; (d) 6 and 3 of water; (e) 10 and 5 of water?

23. A flower bed one foot wide is to be dug around the pool. What is the length of the outer perimeter of the flower bed?
(a) less than 85 feet; (b) 85-89; (c) 90-94; (d) 95-100; (e) more than 100'.



Items 31, 32 and 33 refer to the graph below



31. What was weekly cost in June ?
32. In which month was the cost of food $37 ?
33. Which two month period had the largest increase in food costs?
(a) March to May; (b) April to June; (c) May to July; (d) June to August; (e) July to September.

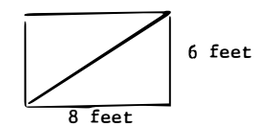
37. A car travels 330 miles and uses 15 gallons of gas. The gas costs 71.9¢ a gallon. Which of the following is the amount of gas used per mile? (a) 0.5 gal.; (b) 0.2; (c) 0.08; (d) 0.05; (e) 0.01.

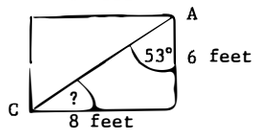
38. You have to continue driving for another 60 miles and you can average 50 miles per hour. How long will it take to reach your destination? (a) 1h., 5m.; (b) 1h., 8m.; (c) 1h.,10m.; (d) 1h., 12m.; (e) 1h.;15m.

Items 39 and 40 refer to the following: A carpenter is checking the wall shown in the diagram to see if it is "true", i.e., the angles at the corners are right angles.

39. If the wall is true the length of the diagonal will be: (a) 2'; (b) 10'; (c) 14'; (d) 28'; (e) more than 30'.

40. A different carpenter measures the angles. The angle at A measures 53°. If the wall is true the angle at C (shown by the question mark) should measure: (a) less than 30°; (b) 37°; (c) 47°; (d) 53°; (e) 127°.





Editors Note: A copy of the complete test may be obtained by writing to the author.

VECTOR PROOFS IN GEOMETRY

Bob Alexander, Faculty of Education, University of Toronto

Part 1 of a two-part article

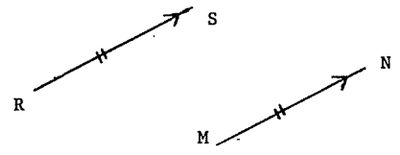
Vector solutions of problems in plane geometry seem to be of the following types:

(a) those which are so short that they are finished before we realize it.
(b) those which are so complicated that we wonder how the solver thought of the solution.

These problems frequently cause difficulties for students because they often do not know how to begin, and because they do not have a plan or strategy to follow. They must also learn to recognize the kinds of problems for which vector methods are suitable. The purpose of this article is to present some ideas that might help.

1. Some Useful Facts. When solving problems using vector methods we usually look for the following things:

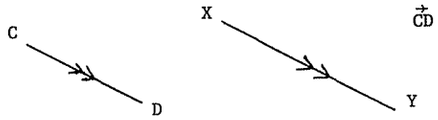
Parallel and equal segments



$\vec{RS} = \vec{MN}$ which means $\begin{cases} RS = MN \\ RS \parallel MN \end{cases}$

Notice that one vector statement contains two pieces of information. This is why some vector proofs are so short.

Parallel and unequal segments



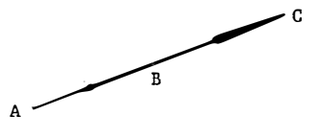
$\vec{CD} = t\,\vec{XY}$ which means $\begin{cases} CD = t\ XY \\ CD \parallel XY \end{cases}$

Equal segments along a line

$\vec{AB} = \vec{BC} = \frac{1}{2}\vec{AC}$

$\vec{BA} + \vec{BC} = \vec{0}$

$\vec{AB} + \vec{CB} = \vec{0}$



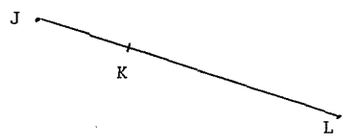
Unequal segments along a line

$\vec{JK} = t\,\vec{JL}$

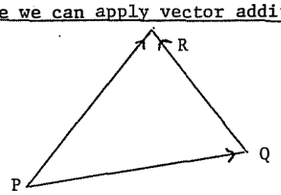
$\vec{KL} = \vec{JL} - \vec{JK}$

$= \vec{JL} - t\,\vec{JL}$

$= (1 - t)\,\vec{JL}$



Triangles where we can apply vector addition and subtraction



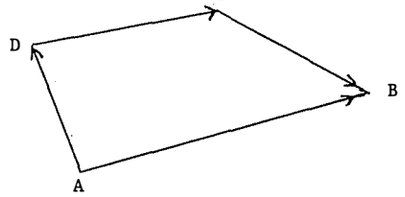
$\vec{PR} = \vec{PQ} + \vec{QR}$

$\vec{QR} = \vec{PR} - \vec{PQ}$

Quadrilaterals and polygons

$\vec{AB} = \vec{AD} + \vec{DC} + \vec{CB}$

$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DA} = \vec{0}$



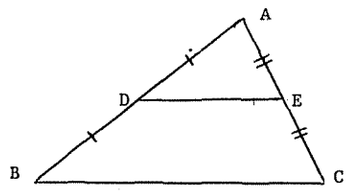
2. Some Examples. We are now ready to apply the above properties to a problem. When doing this, it is important to have a basic strategy or plan which helps us know how to begin.

STRATEGY: Look for two triangles (or other figures) where the laws of addition can be used twice.

Example 1

Given: AD = DB
AE = EC

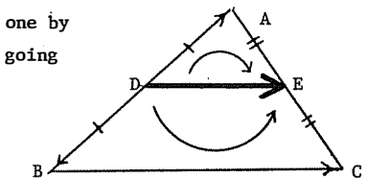
Prove: $DE = \frac{1}{2}BC$
$DE \parallel BC$



Strategy:

1) We note that the two statements to be proved are equivalent to the single statement $\vec{DE} = \frac{1}{2}\vec{BC}$. This starts us thinking about the vector $\vec{DE}$.

2) We form two expressions for $\vec{DE}$, one by going around ΔDAE, the other by going around quadrilateral DBCE.



$\therefore \quad \vec{DE} = \vec{DA} + \vec{AE}$

and $\quad \vec{DE} = \vec{DB} + \vec{BC} + \vec{CE}$

3) Looking at the vectors on the right sides we see that some of them occur along lines AB and AC on the diagram. Those that do add to zero:

$$\vec{DA} + \vec{DB} = \vec{0}$$

and $\quad \vec{AE} + \vec{CE} = \vec{0}.$

This suggests adding the two equations for $\vec{DE}$, and if we do, we should see how to finish the problem.

Solution:

$$\vec{DE} = \vec{DA} + \vec{AE}$$
$$\underline{\vec{DE} = \vec{DB} + \vec{BC} + \vec{CE}}$$

Add $\quad 2\vec{DE} = \vec{BC}$

$$\therefore \quad \vec{DE} = \frac{1}{2}\vec{BC}$$

$$\therefore \quad DE = \frac{1}{2}BC \quad \text{and} \quad DE \parallel BC. \qquad \text{q.e.d.}$$

Alternate Strategy:

1) We apply the triangle law to the two overlapping triangles, ΔADE and ΔABC. Since we need to prove that $\vec{DE} = \frac{1}{2}\vec{BC}$, we will write expressions for $\vec{DE}$ and $\vec{BC}$.

$$\vec{DE} = \vec{DA} + \vec{AE}$$
$$\vec{BC} = \vec{BA} + \vec{AC}$$

2) Since the vectors on the right sides occur along lines AB and AC, we can write:

$$\vec{DE} = \vec{DA} + \vec{AE}$$
$$= \frac{1}{2}\vec{BA} + \frac{1}{2}\vec{AC}.$$

The solution should now be apparent.

Alternate Solution:

$$\vec{DE} = \vec{DA} + \vec{AE}$$
$$= \frac{1}{2}\vec{BA} + \frac{1}{2}\vec{AC}$$
$$= \frac{1}{2}(\vec{BA} + \vec{AC})$$
$$= \frac{1}{2}\vec{BC}$$

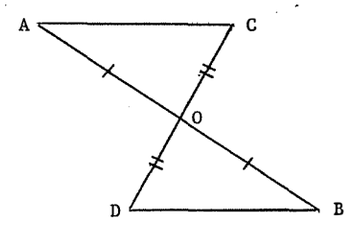
$$\therefore \quad DE = \frac{1}{2}BC \quad \text{and} \quad DE \parallel BC. \qquad \text{q.e.d.}$$

Here are other examples where similar solutions can be constructed.

Example 2.

Given: $AO = OB$
$CO = OD$

Prove: $AC = DB$
$AC \parallel DB$

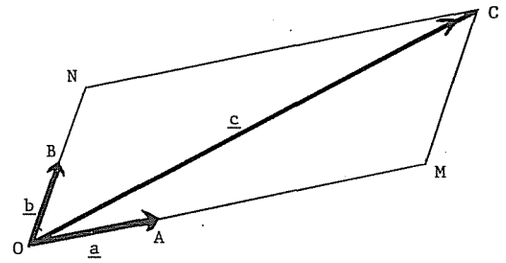


Example 3.

Given any quadrilateral, prove that the midpoints of its sides are the vertices of a parallelogram.

3. More Vector Facts. More difficult vector proofs will involve the following.

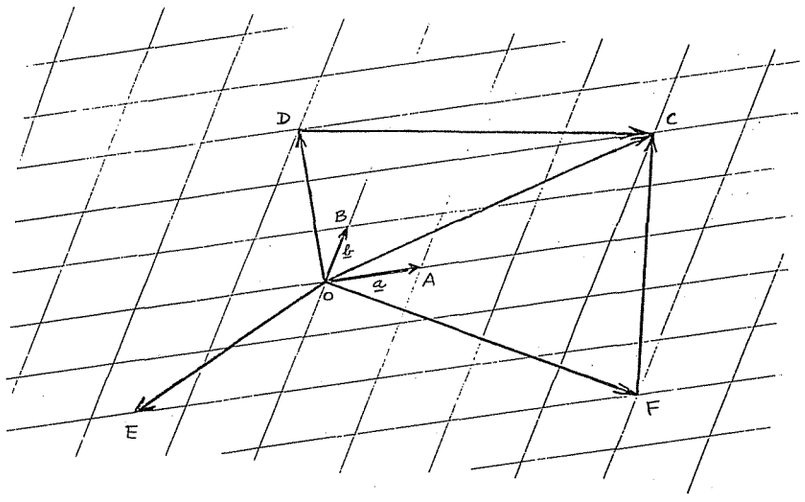
Coplanar vectors



If $\vec{OA}$ and $\vec{OB}$ (also represented by $\underline{a}$ and $\underline{b}$) are any two non-parallel vectors, then any vector $\vec{OC}$ in the plan can be uniquely expressed as a linear combination of them.

$$\vec{OC} = \vec{OM} + \vec{ON}$$
$$\therefore \quad \vec{OC} = s\,\vec{OA} + t\,\vec{OB} \qquad s,t \in R$$
$$\text{or} \quad c = s\,\underline{a} + t\,\underline{b}$$

To remember this, it is helpful to visualize an oblique coordinate system with centre O and axes determined by $\vec{OA}$ and $\vec{OB}$. Then the numbers s and t represent the oblique coordinates of C. The following diagram shows examples of four vectors expressed as linear combinations of $\underline{a}$ and $\underline{b}$, one in each "quadrant".



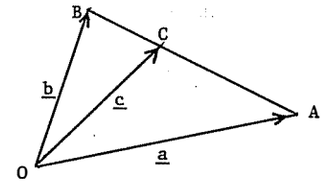
$$\vec{OC} = 3\,\underline{a} + 2\,\underline{b}$$
$$\vec{OD} = -1\,\underline{a} + 3\,\underline{b}$$
$$\vec{OE} = -1.5\,\underline{a} - 2\,\underline{b}$$
$$\vec{OF} = 4\,\underline{a} - 3\,\underline{b}.$$

Using vector addition or subtraction we can express any other vector as a linear combination of $\underline{a}$ and $\underline{b}$, for example:

$$\vec{DC} = \vec{OC} - \vec{OD}$$
$$= (3\,\underline{a} + 2\,\underline{b}) - (-1\,\underline{a} + 3\,\underline{b})$$
$$= 4\,\underline{a} - 1\,\underline{b}$$
$$\vec{FC} = \vec{OC} - \vec{OF}$$
$$= (3\,\underline{a} + 2\,\underline{b}) - (4\,\underline{a} - 3\,\underline{b})$$
$$= -1\,\underline{a} + 5\,\underline{b}\ .$$

Collinear points

Many geometry problems involve a line drawn from one vertex of a triangle to the opposite side. This configuration frequently occurs in overlapping parts of a diagram. Instead of treating this situation from first principles each time it arises, it is better to develop a basic strategy for dealing with it. To begin, we know that $\underline{c}$ is a linear combination of $\underline{a}$ and $\underline{b}$.



$$\therefore \quad \underline{c} = s\,\underline{a} + \underline{b} \ .$$

However, this is true for any position of the point C in the plane. The fact that C is collinear with A and B should impose some restriction on s and t. This can be found by expressing $\vec{AC}$ and $\vec{AB}$ as linear combinations of $\underline{a}$ and $\underline{b}$ and forcing them to be collinear.

$$\begin{aligned} \therefore \quad \vec{AC} &= \underline{c} - \underline{a} \\ &= (s\,\underline{a} + t\,\underline{b}) - \underline{a} \\ &= t\,\underline{b} + (s - 1)\,\underline{a} \ . \end{aligned}$$

$$\vec{AB} = \underline{b} - \underline{a} \ .$$

Since $\vec{AC}$ is a multiple of $\vec{AB}$, the coefficients of $\underline{b}$ and $\underline{a}$ must be in proportion.

$$\therefore \quad \frac{t}{1} = \frac{s - 1}{-1}$$

$$s - 1 = -t$$

$$\therefore \quad s + t = 1 \ .$$

This is the condition that C is collinear with A and B—the oblique coordinates of C must add to 1. (To remember this, it helps to recall the equation of the line through (1,0) and (0,1) in Cartesian coordinates: $x + y = 1$.)

Division of a line segment

On the previous diagram, the ratio in which C divides AB may be found by expressing $\vec{AC}$ and $\vec{CB}$ as multiples of $\vec{AB}$.

$$\begin{aligned} \vec{AC} &= t\,\underline{b} + (s - 1)\,\underline{a} \\ &= t\,\underline{b} + (1 - t - 1)\,\underline{a} \quad \text{(since } s + t = 1\text{)} \\ &= t\,(\underline{b} - \underline{a}) \end{aligned}$$

$$\therefore \quad \vec{AC} = t\,\vec{AB}.$$

$$\begin{aligned} \vec{CB} &= \vec{AB} - \vec{AC} \\ &= \vec{AB} - t\,\vec{AB} \\ &= (1 - t)\,\vec{AB} \end{aligned}$$

$$\therefore \quad \vec{CB} = s\,\vec{AB} \ .$$

$$AC : CB = t : s$$

Therefore, the ratio in which C divids AB is given by the coordinates of C, taken in reverse order.

The above results were developed using a diagram showing the point C dividng the segment AB internally. Since we did not use the fact that C was between A and B, the analysis also applies to external division. However, it is not necessary to be aware of this when solving most geometry problems.

Students should practise using the preceding results in various simple situations before attempting to use them in vector proofs. It is particularly important that they recognize coefficients that add to 1 when points are collinear.

4. More Examples. When solving more difficult problems, the following strategy is always helpful.

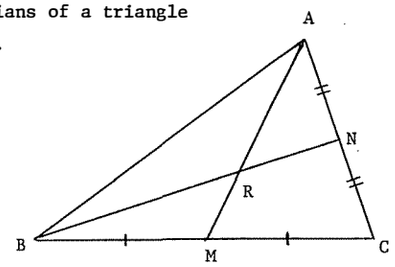
STRATEGY:	Choose two appropriate non-parallel vectors and express other vectors as linear combinations of them.

Many problems involve the configuration of a line drawn from one vertex of a triangle to the opposite side. The two side vectors which start at that vertex are usually good choices for the "appropriate non-parallel vectors". For convenience, we will refer to these as basis vectors.

Example 4: Prove that the medians of a triangle trisect each other.

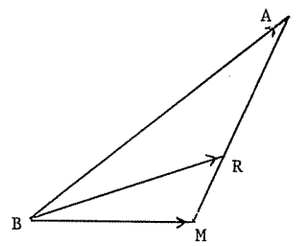
Given: AN = NC
BM = MC

Prove: MR : RA = 1 : 2



Strategy

1) We see the configuration of a line drawn from a vertex to the opposite side in ΔABC and also in ΔABM. Since we want to prove something about MR and RA, and since these segments occur on side AM of ΔABM, we select this triangle and choose sides $\vec{BM}$ and $\vec{BA}$ as our two "basis vectors". This means that we will try to express $\vec{BR}$ as a linear combination of $\vec{BM}$ and $\vec{BA}$.



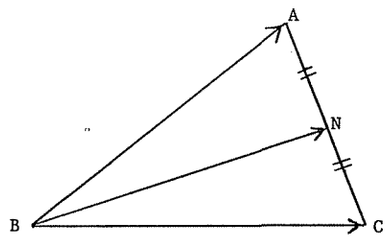
2) The only thing we know about $\vec{BR}$ is that it is a multiple of $\vec{BN}$, but we do not know which multiple. Therefore we must let $\vec{BR} = k\,\vec{BN}$.

3) This introduces $\vec{BN}$, which goes from vertex B of ΔABC to the midpoint of the opposite side. Therefore, $\vec{BN} = \frac{1}{2}\vec{BC} + \frac{1}{2}\vec{BA}$.

4) Substituting in $\vec{BR} = k\,\vec{BN}$, and remembering that we are trying to express $\vec{BR}$ in terms of $\vec{BM}$ and $\vec{BA}$, we have $\vec{BR} = k\,(\frac{1}{2}\vec{BC} + \frac{1}{2}\vec{BA})$

$= k\,(\vec{BM} + \frac{1}{2}\vec{BA})$.

$\therefore\ \vec{BR} = k\,\vec{BM} + \frac{1}{2}k\,\vec{BA}$.



5) This is where we must recognize that the coefficients add to 1, since M, R, A are collinear.

$\therefore\ k + \frac{1}{2}k = 1$.

Solving for k, the solution is now apparent.

Solution:

Let $\vec{BR} = k\,\vec{BN}$ (1)

$= k(\frac{1}{2}\vec{BC} + \frac{1}{2}\vec{BA})$.

$\therefore\ BR = k\,BM + \frac{1}{2}k\,BA$. (2)

Since M, R, A are collinear,

$k + \frac{1}{2}k = 1$

$\therefore\ k = \frac{2}{3}$.

Substituting in (2), we have

$$\vec{BR} = \frac{2}{3}\vec{BM} + \frac{1}{3}\vec{BA}$$

$\therefore$ MR : RA = $\frac{1}{3} : \frac{2}{3}$

= 1 : 2

$\therefore$ R trisects AM. q.e.d.

Alternatively, substitute in (1)

$$\vec{BR} = \frac{2}{3}\vec{BN}$$

$\therefore$ R trisects BN.

Alternate Strategy:

Express $\vec{BR}$ as a linear combination of $\vec{BM}$ and $\vec{BA}$ in two different ways, and then equate coefficients of $\vec{BM}$ and $\vec{BA}$.

Alternate Solution:

Let $\vec{BR} = k\,\vec{BN}$

$= k\,(\frac{1}{2}\,\vec{BC} + \frac{1}{2}\,\vec{BA})$

$\therefore$ $\vec{BR} = k\,\vec{BM} + \frac{1}{2}k\,\vec{BA}$ (1)

Also, $\vec{BR} = s\,\vec{BM} + t\,\vec{BA}$ (2)

where $s + t = 1$.

In (1) and (2) the coefficients are equal.

$\therefore$ $k + \frac{1}{2}k = 1$

The solution concludes as before

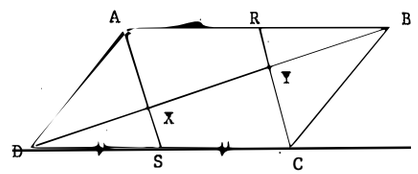
Similar solutions can be constructed for the following problems.

Example 5: Prove that the diagonals of a parallelogram bisect each other.

Example 6:

Given: Parallelogram ABCD
AR = RB
DS = SC

Prove: X and Y trisect DB
X trisects AS
Y trisects CR



Part II will appear in the March 1978 issue of the Gazette.

Editors Note: Professor Alexander has provided numerous additional examples which will be included in Part II.

THE CIRCLING CAR

The outside wheels of a car, running on a circular track, are going twice as fast as the inside ones. What is the length of the circumference described by the outer wheels? The wheels are five feet apart on the two axles. (Answer on page 44).

Henry E. Dudeney: **536 Puzzles and Curious Problems** No. 290 (Scribner's 1967)

SOME COMMENTS ON THE LEARNING OF CALCULUS

Fred G.B. Maskell, Algonquin College, Ottawa

MENTAL SET

How one first encounters a subject sometimes leaves a lasting and definitive impression. When I first encountered the differential calculus, I found it fascinating, and not difficult. Details have disappeared from memory, but what has remained is a mental set. "The way the twig is bent, so the tree grows"

In my case, I was handed algebraic expressions - many of them, such as I had developed skill in manipulating - and challenged to differentiate them. An expression such as

$$y = \frac{(x + 2)^2}{x + 1}$$

could be dealt with in several ways. (Incidentally, the y = was always there.) One could use the quotient rule, $du/v = (vdu - udv)/v^2$, $y = (x + 2)^2 (x + 1)^{-1}$

$$duv^{-1} = v^{-1}du + u(-v^{-2})dv$$

$$\ln y = 2\ln(x + 2) - \ln(x + 1)$$

$$y = x + 3 + \frac{1}{x + 1}$$

and so find $\frac{dy}{dx}$.

I had no notion at all of the meaning of function as a set of ordered pairs. There was no distinction for me between an algebraic expression and a function. To this day, there is a natural set to my thinking which tends to confine it within the bounds of my original horizons, even though I have since travelled far beyond them.

CURRENT (AND TRADITIONAL) APPROACH

Commonly, textbooks follow the sequence: functions - limits - differentiation - applications - integration - applications - transcendental functions - techniques of integration. My comments on this sequence relate to its learning by students at the technology level. generally below degree level, but definitely post-secondary.

Note the postponement of transcendental functions (exponential - logarithmic - trigonometric - inverse trigonometric), so that the students' first encounter with the concept and process and application of differentiation and its inverse is limited to algebraic functions. For our students, the general functional notation f(x) in isolation does not have much meaning. As a symbol for something they know, understand, and have experience of, it is useful to them, and so, for them, it stands for a simple algebraic function, such as $(x + 2)^2(x + 1)^{-1}$.

There are some obvious apparent advantages: early introduction of the concept of integration; and availability of elementary applications. But what is hidden is the mental set induced by this approach, which, to my observation can severely handicap less-than-bright students who need to go further in the subject. It is that differentiation and integration are accomplished quite simply by a formula which is reversible.

One learns how easy differentiation is - just one simple rule: u^n becomes nu^{n-1}, together with a procedure for functions of functions ("chain" rule) and products of functions (product rule); and antidifferentiation appears to be just as simple - u^n becomes $u^{n+1}/(n+1)$.

Unfortunately, with only the power function available, the opportunities to practice the "chain" rule and the product rule are quite limited. There the subject is left

to incubate during the summer vacation.

Good students with keen minds can cope with the reality that the power function is only one of several elementary functions, that the majority of the processes of differentiation are not reversible, and that integration is less a matter of plugging into formulae than it is like looking for the criminal given a set of clues. Others at best are disillusioned to find how confusing the subject now is, and at worst are simply unable to open their minds to reality.

AN ALTERNATIVE APPROACH

As a result of such observations, I have developed a preference for the sequence:

(a) Functions (power, with $n = \pm 1, \pm 2, \pm\frac{1}{2}$; exponential and logarithmic; trigonometric and inverse trigonometric);

(b) Differentiation (of sine and logarithmic functions; of multiples, sums, and functions of functions; of cosine functions; of tangent and cotangent functions by logarithmic differentiation; of inverses of functions; of exponential functions and of inverse tangent and sine functions; of power functions; and of products of functions);

(c) Simple anti-differentiation as the solution to a differential equation. Some applications such as curve sketching may be introduced during the sequence.

The critical features in the sequence are the postponement of the product rule, to allow the "chain" rule to take hold, and the postponement of the power function, to prevent a formula mentality setting up.

I do not know how much better such a sequence might be or what peculiar disadvantages it might have. But I am sure of the difficulties that students have in overcoming a formula mentality, and I regard it as a paedogogical principle to be on the look-out for what in an initial presentation might develop in students' minds into a troublesome mental set at a later stage.

For the same reason, I try to rely as little as possible on geometric models of differentiation and integration, so that students are not inhibited at a later date by a "slope" mental set.

It is clearly an advantage to have available from the beginning different types of function to practice the "chain" rule with, and subsequently the product rule, and to broaden experience with curve sketching.

RELEVANCE TO TECHNOLOGY

The power function, of course, is relevant, particularly those for which $n = 1, 2, -1$; but no more so than the transcendental functions. The exponential function is directly applicable to growth and decay (exemplified at the present time by inflation and rusty cars), and the sine function to the many periodic phenomena in nature and in engineering.

To say the least, these functions should have at least as much attention as the power function. I would expect that similar considerations would apply to students' courses in engineering and science programmes at university, so perhaps there is a general case for reforming the traditional (and current) sequence of presentation to one which may be more suited to the learning process and more relevant to the applications for which elementary calculus provides models.

FACTORIAL SUMMATIONS

Dominic A. Di Felice, Overlea Secondary School, Don Mills

Most senior high school texts deal very briefly with the concept of factorial summations and then only in their simplest forms. This article will illustrate some interesting techniques which are available in developing factorial summation formulae using only the concepts of the binomial theorem, complex numbers and simple calculus.

Let us consider the binomial expansion of $(1+x)^n$, $n \in N$

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \qquad \text{where } \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

and since $\binom{n}{r} = 0$ for $r > n$, this can be rewritten as

$$(1+x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r \quad . \tag{1}$$

Since equation (1) is true $\forall\ x \in C$, let $x = 1$ in the equation to obtain

$$2^n = \sum_{r=0}^{\infty} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r} + \cdots \tag{2}$$

also substituting $x = -1$ in equation (1) we obtain

$$0 = \sum_{r=0}^{\infty} (-1)^r \binom{n}{r} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^r \binom{n}{r} + \cdots \tag{3}$$

adding (2) and (3) and simplifying gives

$$2^{n-1} = \sum_{r=0}^{\infty} \binom{n}{2r} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2r} + \cdots \quad . \tag{4}$$

This is the extent that most textbooks deal with this fascinating topic. Let us briefly elaborate on these ideas.

If we differentiate equation (1) we obtain

$$n(1+x)^{n-1} = \sum_{r=1}^{\infty} \binom{n}{r} r x^{r-1} \tag{5}$$

and letting $x = \pm 1$ gives respectively

$$n2^{n-1} = \sum_{r=1}^{\infty} \binom{n}{r} r = \binom{n}{1} + 2\binom{n}{2} + \cdots + r\binom{n}{r} + \cdots \tag{6}$$

and

$$0 = \sum_{r=1}^{\infty} (-1)^{r-1} \binom{n}{r} r = \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots \tag{7}$$

adding (6) and (7) we obtain after simplification

$$n2^{n-2} = \sum_{r=1}^{\infty} (2r-1) \binom{n}{2r-1} = \binom{n}{1} + 3\binom{n}{3} + 5\binom{n}{5} + \cdots \quad . \tag{8}$$

Further differentiation of (5) will lead to even more interesting results.

If we integrate equation (1) we obtain

$$\int_0^x (1+x)^n \, dx = \int_0^x \sum_{r=0}^{\infty} \binom{n}{r} x^r \, dx \tag{9}$$

performing the integration

$$\frac{(1+x)^{n+1}}{(n+1)} - \frac{1}{(n+1)} = \sum_{r=0}^{\infty} \frac{x^{r+1}}{r+1} \binom{n}{r} \tag{9a}$$

now setting $x = \pm 1$ gives

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \sum_{r=0}^{\infty} \frac{1}{r+1} \binom{n}{r} = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \cdots + \frac{1}{r+1} \binom{n}{r} + \cdots \tag{10}$$

and

$$-\frac{1}{n+1} = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r+1} \binom{n}{r} = -\binom{n}{0} + \frac{1}{2}\binom{n}{1} - \frac{1}{3}\binom{n}{2} + \cdots \quad . \tag{11}$$

Taking linear combinations of these sums produce an endless variety of new factorial formulae.

Finally, a new vista is opened to the student if he does not restrict himself to the real domain for the variable x. If we consider equation (1) and substitute $x = \pm i$ where $i = \sqrt{-1}$ we obtain

$$(1+i)^n = \sum_{r=0}^{\infty} \binom{n}{r} i^r = \binom{n}{0} + i\binom{n}{1} - \binom{n}{2} - i\binom{n}{3} + \cdots \tag{12}$$

and

$$(1-i)^n = \sum_{r=0}^{\infty} \binom{n}{r} (-i)^r = \binom{n}{0} - i\binom{n}{1} - \binom{n}{2} + i\binom{n}{3} + \cdots \tag{13}$$

If we now add equations (2), (3), (12) and (13) we obtain

$$2^n + (1+i)^n + (1-i)^n = 4\{\binom{n}{0} + \binom{n}{4} + \cdots + \binom{n}{4k} + \cdots\} \quad . \tag{14}$$

Now since $(1+i) = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$ we can use De Moivre's theorem to obtain $(1+i)^n = \sqrt{2^n} \operatorname{cis} \frac{n\pi}{4}$ and similarily $(1-i)^n = 2^n \operatorname{cis}(-\frac{n}{4})$.

Equation (14) now becomes

$$2^n + \sqrt{2^n} \operatorname{cis} \frac{n\pi}{4} + \sqrt{2^n} \operatorname{cis} (\frac{-n\pi}{4}) = 4\{\binom{n}{0} + \binom{n}{4} + \cdots\} \tag{14a}$$

however, since $\cos \frac{n\pi}{4} = \cos (-\frac{n\pi}{4})$ and $\sin (\frac{n\pi}{4}) = -\sin (-\frac{n\pi}{4})$ equation (14a) becomes upon simplification

$$2^{n-2} + \frac{\sqrt{2^n}}{2} \cos \frac{n\pi}{4} = \sum_{r=0}^{\infty} \binom{n}{4r} = \binom{n}{0} + \binom{n}{4} + \cdots + \binom{n}{4r} + \cdots \tag{14b}$$

We leave it to the imagination of the student to develop new and more interesting factorial sum formulae using the techniques illustrated in this article or perhaps new techniques which the student can discover for himself.

Editors Note: In response to my letter of inquiry about one or two points, Mr. Di Felice wrote:

Dear Arn:

The basic ideas in the paper are a result of problems and solutions which appear in the "Ontario Senior Math Problems Competition: Solutions and Discussion", sponsored by the Junior Math Contest Committee from the University of Waterloo.

I have made several extensions on these ideas mainly the integration techniques which can be applied to these series.

To justify $\binom{n}{r} = 0$ if $r > n$, I quote from the "Ontario Senior Math Problems Competition 1971".

"There are 3 common ways of defining $\binom{n}{k}$. These are as follows:

(a) $\binom{n}{k}$ is the coefficient of x^k in the expansion $(1+x)^n = 1 + \binom{n}{1}x + \cdots + \binom{n}{k}x^k + \cdots$

(b) $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$

(c) $\binom{n}{k}$ is the number of different selections of k objects from n distinct objects.

Thus, any of these definitions shows that $\binom{n}{k}$ is quite meaningful for $k > n$; indeed; $\binom{n}{k} = 0$ for $k > n$".

I hope this will clarify your concern.

Sincerely yours,
D. Di Felice
Pure Science Department

SECONDARY SCHOOL STATISTICS: PROBLEMS & PROSPECTS

Ian B. MacNeill, Statistics and Actuarial Science Group
University of Western Ontario, London, Canada

1. INTRODUCTION

Valid reasons can be given for the inclusion of almost any topic in the high school syllabus; obviously choices must be made among the myriad contenders to avoid overloading the students. In this competition for space in the syllabus where does statistics stand? The answer depends a little upon which educational jurisdiction one is considering. Parts of the British school system permit the teaching of substantial statistics programs, and descriptive statistics is often regularly taught even in the elementary schools[11]. Furthermore, there exist in Britain well-supported programs to develop teaching materials to improve the quality of statistics courses and to make the teaching of statistics more wide-spread[7]. Other countries, such as Sweden, have less well-developed programs; many jurisdictions, including several provinces in Canada, permit experimentation; still others continue to ignore the subject completely. The overall trend seems to be toward an increase in the number of school systems teaching statistics and an increase in the amount of time devoted to statistics instruction.

The criteria that are used in making curriculum choices reflect what a society considers important; some may emphasize cultural and traditional values, others may emphasize utilitarian values. The case for statistics is usually based upon the usefulness of the subject.

II. APPLICATIONS OF STATISTICS

It is only a slight exaggeration to say that everyone is regularly confronted with the need to interpret descriptive statistical summaries of data and to judge inferences made on the basis of opinion polls, surveys and experiments. In addition, statistics is a very widely used technical tool. It has a prominent place in the methodology of such social sciences as: anthropology, economics, political science, psychology and sociology. It is an important tool for the management sciences. Many of the most widely used statistical tools were developed to solve problems in the biological and agricultural sciences. High standards of health care are maintained using statistical tools; probability models for the spread of epidemics are used in designing immunization procedures to control the spread of communicable diseases; statistical procedures are used to monitor morbidity and mortality rates; probabilistic studies are used to schedule efficiently the use of expensive hospital facilities. New drugs produced by the pharmaceutical industry require careful screening before being allowed into general use; designed experiments are used in the screening procedure, and a statistical analysis of these experiments decides whether or not these drugs will be used.

Market research and opinion polling make use of the statistical methodology associated with the design and analysis of surveys. A producer must be aware of: the size of the market for his product, what share of the market is his, what changes are taking place in market shares, and how loyal are consumers to his brand and to his competitors' brands. The only reasonable way to obtain this information is by means of market surveys. Politicians are heavy users of opinion polls. They may implement, or promise to implement, programs if polls indicate that the general public wishes them. The prime minister may call an early election if opinion polls indicate a good chance for re-election. Opinions of only a small fraction of the electorate are sufficient to provide him with the information he wants. Popularity of television programming is monitored by means of regularly conducted sample surveys. Again, the viewing habits of only a handful of people determine the future television fare for the whole nation. Enormous amounts of power and money can be gained or lost on the basis of these market surveys and opinion polls.

Still other areas making non-trivial use of statistics and probability theory are: meteorology, oceanography, demography, criminology, education, actuarial science, industrial engineering, and traffic engineering.

The wide use of statistics in Canada has been noted and documented in several publications emanating from the Mathematics Study of the Science Council of Canada. Statistics, computing and mathematical modelling were found by the study to be the areas of mathematics most widely used. See[1] , pp. 64-65. The Mathematics Study organized eight seminars which brought together representatives from business, industry, government and the university community. The seminar proceedings, including **Mathematics in Today's World**[6], give accounts by users of the uses to which the mathematical sciences are put in Canada. Statistics is found to be one of the most widely used of these sciences.

It is unnecessary to expand the preceding list to convince most people of the value of statistics. In fact, thoughtful people have been aware for some time that statistics is important and that everyone should be exposed to its precepts. The quotation, attributed to H.G. Wells, that "Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write", made this point, perhaps somewhat extravagantly, several decades ago. Why, then, is statistics not more widely taught? One can point to several valid reasons.

III. REASONS WHY STATISTICS IS NOT TAUGHT WIDELY AT THE SECONDARY LEVEL

To begin, one needs to be aware that most of the methodology of statistics is of recent vintage, developed in the past half-century and widely applied only in the last twenty-five years. The fact that statistics has not made it into the secondary school curriculum in many places is not surprising; the surprising thing is that statistics has proliferated so far so fast. A large fraction of college students are now exposed to the ideas of statistical inference during their university careers, whether through a course from a statistics or mathematics department, or by means of a quantitative methods course from a non-mathematics department. Curriculum chang takes time, and statistics, until recently, did not possess a profile high enough to attract the attention of those who design curricula.

Another impediment has been the low status that statistics has had within the mathematical community. Statistics has been regarded as the shady lady of mathematics, hardly fit to be housed with such pristine beauties as geometry, algebra, analysis, and topology. She has been known to consort openly with many who have made use of her ample, yet youthful, charms. Efforts have been made to bring statistics off the streets, rehabilitating her, and clothe her in the respectable garments of measure theory with the hope that such habits would cool the ardour of offensive suitors who "used" statistics. These efforts have not been wholly successful and now, with the encroachment of hard times and the descent of genteel poverty upon the mathematics scene it has become difficult to keep statistics at home, pure but poor. In fact, the Queen of the Sciences herself has recently been found furtively and awkwardly soliciting business! Future mathematics curriculum revision will not likely see statistics being ignored because of low status within the mathematical community.

A question that must be answered before statistics is given a place in the sun is: What, if anything, is to be removed from the course of study to make way for statistics? The physical scientists and mathematicians have traditionally provided the input from the tertiary level into secondary school mathematics curriculum design. They are likely to be reluctant to see changes that result in statistics replacing topics that have been included at their behest. The argument is made that statistics is too specialized to be taught to high school students; rather they should be taught "basic" subject matter. But, basic to what? In Canadian universities more students now study the social sciences than the natural sciences, and statistics is the mathematics of the social sciences. A survey of social science departments in Ontario universities has been conducted[10] with a view to obtaining their opinions concerning the secondary school mathematics curriculum. One set of questions in this survey called for a rating of the usefulness to their disciplines of various possible topics in high school mathematics. Statistics, basic algebra, and probability were top-rated, and such traditional topics as analytic geometry, euclidean geometry and trigonometry were rated near the bottom. See 10, pp. 169-179. Also see Figure 1. Another set of questions asked for predictions of the future usefulness (growth potential) to their disciplines of the same possible topics for high school mathematics. Computer science, probability, statistics and matrix algebra were top-rated, and analytic geometry, euclidean geometry and trigonometry were rated near the bottom. See 10, pp. 169-179. Also see Figure 2. Does this data suggest that it may be necessary to tilt secondary mathematics curricula in a direction that does not serve almost exclusively the special interests of the physical sciences? A solution that has been suggested is to leave things as they are since most college students study statistics sometimes during their university careers. This of course ignores the interests of the vast majority who do not go to college. Statistics may be the only mathematics that these people will use regularly.

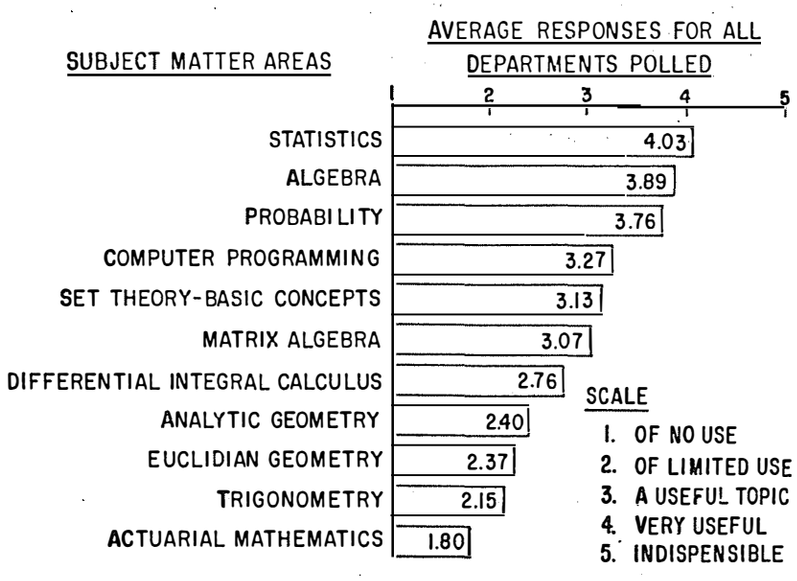


FIGURE 1: ORDERED PLOTS OF AVERAGE RATINGS GIVEN BY SOCIAL SCIENTISTS TO THE USEFULNESS OF VARIOUS TOPICS IN MATHEMATICS

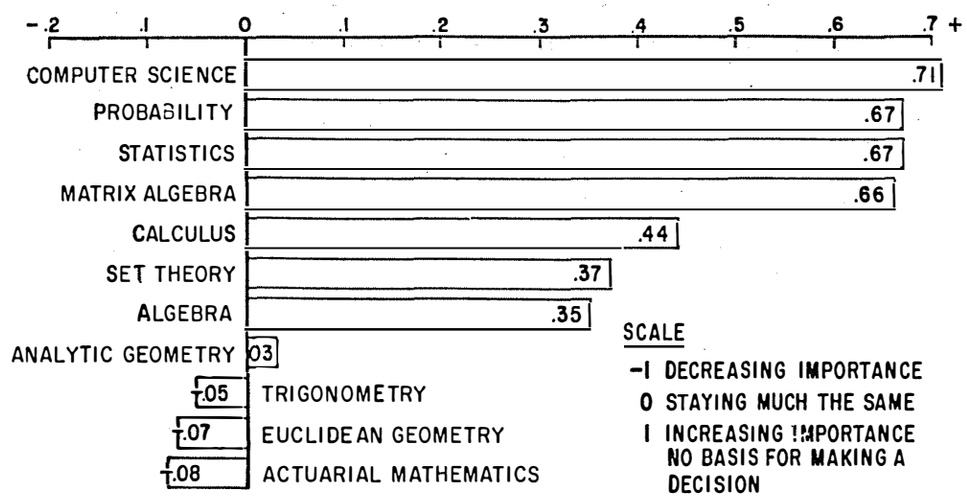


FIGURE 2: ORDERED PLOT OF AVERAGE PREDICTIONS GIVEN BY SOCIAL SCIENTISTS TO THE FUTURE USEFULNESS OF VARIOUS TOPICS IN MATHEMATICS

The notion that statistics cannot be taught to students until they have completed several years of honors mathematics (and preferably a term of real analysis) is fast fading. This once was an argument used against the inclusion of statistics in the secondary mathematics programs. Universities have not accumulated much experience teaching statistics to students with very modest mathematics backgrounds. The fact is that the basic ideas of statistics can be taught with little use of mathematics. Social scientists need to do it all the time. (This raises the possibility that statistics will be taught by teachers in other disciplines, a contingency that should weight heavily on secondary school mathematics departments).

A genuine problem that secondary school statistics must contend with is the lack of statistical training on the part of teachers. A recent survey[5] of mathematics heads in Ontario schools indicates that mathematics teachers, on average, have credit in one university "statistics" course. See Table 1. There are few who have majored in statistics, but one can be hopeful that most large departments will have at least one teacher with sufficient interest and background in statistics to consider instituting a statistics program. It is plain that in-service training programs will be required in the beginning. Hopefully, more potential high school teachers will consider statistics as a major.

Number of University Statistics Courses Taken	Proportion of Teachers
0	0.362
1	0.345
2	0.186
3	0.082
4	0.009
5	0.005
6 or more	0.011

TABLE I: NUMBER OF UNIVERSITY STATISTICS COURSES TAKEN BY ONTARIO SECONDARY SCHOOL TEACHERS

The mathematics curricula in Canadian schools have come through a period of substantial change after having been relatively static for a long period. The changes, which brought about the "new" math, have developed some uneasiness in the minds of parents and principals. The Secondary School Statistics Project at The University of Western Ontario polled the principals of Ontario secondary schools[4] asking them how ready they were for further change to their programs; the change suggested was the introduction of statistics. They have indicated that changes coming late would be preferred to changes coming soon, a not unexpected bureaucratic reaction. A complaint

repeatedly raised by principals concerned the seeming loss of computational and manipulative ability on the part of their students. Another point brought out by the survey was that, although mathematics is no longer a compulsory subject, enrolments have held up well.

Lack of adequate teaching material is probably the major obstacle to the teaching of secondary school statistics. Well-written textbooks, containing examples of interest to secondary school students, will have to be available before significant progress can be made. A point raised by Professor J.M. Gani[3] is that such textbooks will likely be written only with the collaboration of secondary school teachers after they have gained some experience teaching statistics; university professors are not sufficiently attuned to the secondary school scene to write proper textbooks without this collaboration. Some excellent material has been prepared under the auspices of the Joint Committee of the American Statistical Association and the National Council of Teachers of Mathematics (ASA-NCTM). A publication of this committee entitled, **Statistics: A Guide to the Unknown**[8], is a collection of forty-four essays by professional statisticians about interesting applications of statistics. The essays, written at a non-technical level, have been warmly received by a broad readership. A sescond publication emanating from the Joint Committee is a series of four pamphlets under the generic title, **Statistics by Example**[9]. The four subtitles are: **Exploring Data, Detecting Patterns, Weighing Chances and Finding Models.** These superb pamphlets discuss interesting examples of statistics and can be used in the classroom.

The Secondary School Statistics Project at The University of Western Ontario commissioned a number of essays on statistical topics with the hope that they would prove useful in the preparation of suitable teaching materials. The essays were to emphasize the local scene and the present time; it was felt that this was the best way to stimulate students' interest. Some of the titles are: "Vitamin C and the Common Cold", "Olympic Lottery", "NHL Hockey Statistics", and "Harness Racing at Western Fair Raceway". A danger in emphasizing the present is that the present rapidly becomes the past. The Olympic Lottery, an exciting concept in 1974, rapidly became of little or no interest to secondary school students. The great interest in megavitamin therapy also rapidly diminished. The danger in emphasizing the local scene is obvious: such examples may not be portable; how great is the geographic spread of interest in Western Fair Raceway?

IV. THE FUTURE FOR SECONDARY SCHOOL STATISTICS

Obstacles and problems exist; yet there is enthusiasm for the teaching of statistics to secondary school students. Several international conferences have been held on this topic. In 1969 Lennard Rade, a Swedish statistician, organized a Conference on the Teaching of Probability and Statistics, the proceedings of which appeared in 1970[11]. The International Statistical Institute has held conferences on non-tertiary statistics education in: Oisterwijk, Holland (1970), Los Angeles (1972), and Vienna (1974).

The joint ASA-NCTM effort continued on a dual front: preparation of materials suitable for a high school course in statistics, and design of training programs for teachers[2]. In Britain, where secondary school statistics is as advanced as anywhere, the Schools Council Project on Statistical Education (POSE) has prepared teaching materials which have been used in a number of schools in Britain. In addition, the Royal Statistical Society and the Institute of Statisticians have set up a committee to investigate the teaching of statistics in the non-tertiary sector. Teaching materials for use in the secondary schools of Australia have been prepared[13]. A conference on "Teaching Statistics and Mathematics for the Social Sciences" was held in London, Canada in 1974 under the auspices of the Secondary School Statistics Project[3]. The large and enthusiastic group of secondary school teachers from all parts of Ontario who convened for this meeting were addressed by an international panel of experts in the field of statistics education.

Enthusiastic support for secondary school statistics exists in many places and at many levels. This enthusiasm springs from the importance of statistics in everyday life. Deficiencies in teaching materials and in teacher training programs are holding back the spread of secondary school statistics. However, the time and effort of the people presently working on these deficiencies will no doubt overcome these deficiencies. The future for secondary school statistics looks bright.

* The author would like to acknowledge the assistance of the Canada Council in the preparation of this paper.

Math 555

Sets; Subsets; Permutations and Combinations; Binomial Theorem; Proof by Induction; Probability; Conics; Transformation; Math of Investment.

Math 556

Trigonometry; Vector Analysis; Complex Numbers and Polar Coordinates; Differential Calculus of the Trig. Functions; Distance, Velocity, Acceleration Problems (Differentiation and Integration); Volumes of Revolution; Mini-max problems with Trig. Functions; Related Rates.

Areas of Application in Math 554

Calculus

Business & Economics - Revenue & Supply Functions
- Cost Equations
- Demand Equations
- Marginal Analysis
- Advertising
- Inventory Control

Human Kinetics	Biology
Agriculture	Medicine
Fuel Consumption	Public Health
Education	Resource Management (Forestry, Fossil Fuels)
Psychology	Forensic Sciences
Sociology	Archaeology
Ecology	Traffic Control

Matrix Algebra

Psychology (Stimulus - Response)
Sociology (Communication, Dominance)
Urban Planning (Movement of Population)
Encoding, decoding messages
Marketing research (switching of brand preference)

Some Worked Examples

Example 1: (Public Health)

Based on data collected in the study of communicable deases, the medical health unit of a certain community approximated the proportion P of people who would be infected t days after onset by

$$P = \frac{1}{600}(15t^2 - t^3) .$$

Predict when the deasease will peak and find out what proportion of the population will be infected then.

Solution: For a peak (Maximum)

$$\frac{dP}{dt} = 0 \quad \text{and} \quad \frac{d^2P}{dt^2} < 0$$

$$\therefore \quad \frac{1}{600}(30t - 3t^2) = 0 \quad \text{and} \quad \frac{1}{600}(30 - 6t) < 0$$

$$\therefore \quad t = 0 \quad \text{or} \quad t = 10 \quad \text{and} \quad t > 5.$$

Reject $t = 0$ as not satisfying $t > 5$

$$\therefore \quad t = 10$$

i.e., After 10 days from onset, the disease will peak.

When $t = 10$, $P = \frac{1}{600}(15t^2 - t^3)$

$= \frac{1}{600}(1500 - 1000) = \frac{5}{6}$

i.e., A maximum of $\frac{5}{6}$ of the population will be infected.

NOTE: Due to the answer being relatively large, this gave rise to discussion about third world countries; we could also relate the problem to the encephalitis outbreaks in Ontario in 1975.

Example 2: (A Business Problem)

A furniture manufacturer produces a chair that costs on the average \$7.00. From the market-research department the demand for the chair is approximated by $q = 150 - 10p$ where q is the number of chairs that can be sold at \$p each. Find the price at which the chair should be sold to maximize the profit.

Solution: Since the revenue $R = pq$ and

the profit $P = R - \text{Cost}$

$P = R - C$ where $C = 7q$.

$\therefore P = pq - 7q$.

But, from $q = 150 - 10p$

$p = 15 - 0.1q$.

$\therefore P = (15 - 0.1q)\,q - 7q$

$P = 15q - 0.1q^2 - 7q$

$= 8q - 0.1q^2$

For P (Max), $\frac{dP}{dq} = 0$ and $\frac{d^2P}{dq^2} < 0$

$\therefore 8 - .2q = 0$ and $-.2 < 0$ (which is true always)

$\therefore q = 40$ for a maximum profit, and the price would be

$p = 15 - 0.1 \times (40)$

$p = 11$ i.e., a price of \$11 would maximize the pro

Example 3: (Urban Planning)

The population of a certain city in 1975 was 100,000 divided as follows: 60,000 in the core of the city and 40,000 in the suburbs. From past records it has been observed that in any given year, about 20% of those in the core moved to the suburbs and 10% of those in the suburbs moved back to the centre. Assuming the trend continues, how will the population be distributed in 1976? 1977?

Solution: We let $P_0 = (60{,}000 \quad 40{,}000)$ be the initial population vector.

Then we let $T = \begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix}$ be the transition matrix which gives the shift in population. Note that:

a) the diagonal terms give the population of people who stay, the off diagonal terms show the proportion which moves.

b) the sum of each row is 1.

It is a simple calculation to show that the population vector P_1 for 1976 is given by $P_1 = P_0T$, i.e., $P_1 = (60{,}000)(40{,}000)\begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix} = (52{,}000 \quad 48{,}000)$, so that there are

52,000 left in the core and 48,000 in the suburbs after one year. Also

$P_2 = P_1T = (P_0T)\,T = P_0T^2$ is the population vector for 1977. Now $T^2 = \begin{pmatrix} .66 & .34 \\ .17 & .83 \end{pmatrix}$

so that $P_2 = (60{,}000 \quad 40{,}000) \begin{pmatrix} .66 & .34 \\ .17 & .83 \end{pmatrix}$

$= (46{,}400 \quad 53{,}600).$

Thus after two years there are 46,400 still living in the core, while the suburban population has grown to 53,600.

Our Experience with the Course After Three Years

Some general observations:

1. The topics are extremely interesting both for the teacher and the student.
2. The examples are easily understood.
3. The teachers find the material very "teachable".
4. The science-oriented students still get the traditional applications of Calculus, and their experience is broadened by the applications in the non-physical sciences.
5. The teachers liked the idea of having a text that has an abundance and variety of problems available so that they do not have to supplement the text with ditto-problem sheets.
6. We are finding more students taking Math 555 and Math 450 concurrently (at level 4) than we ever had in the past. As a result their grade 13 mathematics program was spread over two years.
7. The students were exposed to other possible career opportunities as a result of the broad variety of applications (i.e. They could see where Mathematics could be used in Agriculture, Biology, etc.)
8. The textbook was used for a course taught to year 1 students at the University of Windsor for the past 3 years, so we have that experience too.
9. Students who transfer into this program or those who want to do Math in summer school have a problem since the topics don't line up with the traditional program.
10. As yet, we have not been able to get a commercial organization to publish our text. Consequently, getting a copy of the text into the students hands has been difficult but not unsurmountable. The Windsor Board of Education has been very supportive in printing the material and supplying the students at a very reasonable price.

Future Plans

We plan to continue this program as long as possible due to the very favourable experience we have had with it. The comments made by students, teachers and university professors have been most encouraging. The text is continually being revised and updated as we obtain more experience in the classroom. We are attempting to interest more people in our approach, and to encourage a widening of the applications of Calculus and Matrix Algebra.

Please send any comments, suggestions, questions to:

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Chairman
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University of Windsor
Windsor, Ontario N9B 3P4

or

A.N. MacEwan
Head of Mathematics
W.F. Herman Secondary School

Better to remain silent and be thought a fool
than to speak out and remove all doubt.
Abraham Lincoln.

BOOK REVIEWS AND RESEARCH

RESEARCH AND THE CLASSROOM TEACHER 2
THE CLINICAL PARADIGM

Dr. W. Higginson, Queen's University

In the first part of this article (September 1977) the claim was made that the traditional methods of research in education, involving large numbers of subjects and sophisticated quantitative analyses, had produced very little in the way of clear-cut results for influencing the day-to-day practice of the classroom teacher of mathematics. This viewpoint, while iconoclastic to some, has been put forward by many researchers in recent years.

In his comprehensive report to the section on 'Research Related to the Mathematical Learning Process' at the Third International Congress on Mathematical Education, Dr. Heinrich Bauersfeld (1976), chairman of the international working group, wrote that "the reality of mathematics instruction and mathematics learning particularly has been influenced by research outcomes on a very small scale only" (p. 2). At another point in this significant paper we have, "the majority of contributors to research in mathematics education do this as an initiation rite only ... thus, in general, these studies are more rigid in methods 'collecting the most data in the shortest time' and are 'of little value to developing educational theory' " (p. 8).

If one accepts the validity of the argument which contends that the classical research paradigm has not proven to be a productive one for mathematics education, a question immediately arises. Are there any types of research model which show promise? Are there broadly-based, sustained, and intellectually-sound inquiries being made by any research group which would seem to be yielding results which have distinct implications for improving the teaching and learning of mathematics? Fortunately for mathematics education, it would seem possible to answer these questions, in at least one area, in the affirmative.

The method which seems to show considerable promise could be called the clinical paradigm. This method can be seen as a natural extension of Piagetian interview techniques. The leading practitioner of this research style is Dr. Herbert Ginsburg, formerly of Cornell and now of the University of Maryland. Whereas the Genevan school has not been very concerned with questions of instruction and learning, Ginsburg and his colleagues have been particularly interested in children's 'natural' mathematical knowledge and the way in which it relates to their 'school' knowledge of mathematics. In the clinical method (or the method of 'critical exploration' as it is called by some Piagetians) individual children are engaged in a dialogue with an interviewer about a problem. The interviewer attempts to make inferences about the child's conception of the situation by listening carefully to the child and asking questions which will illuminate the child's reasoning processes.

There are two main sources for reading about the applications of this method. The first is **The Journal of Children's Mathematical Behavior** and the second is Ginsburg's (1977) **Children's Arithmetic**. The JCMB is by no means a traditional academic journal; for one thing, it has taken over five years for the first numbers of volume one to appear. Nevertheless it has published a number of items of considerable interest. (Part of the problem of trying to publish clinical material in the traditional journals is that of space; unlike the tables and graphs of the classical paradigm the interviews of the clinical paradigm require a large number of pages). An outstanding example of the clinical method is Erlwanger's (1973) paper 'Benny's Conception of Rules and Answers in IPI Mathematics'. In this paper, Erlwanger reports on some aspects of his dialogues with Benny, a twelve-year-old who was regarded by his teacher as one of the best mathematics pupils in her Grade Six class. The curriculum followed by this class was 'Individually Prescribed Instruction (IPI)' and Benny's revelations would seem to raise doubts about the efficacy and value of such curricula. For what appears to emerge from Erlwanger's interview questions is a picture of a bright young pupil who has constructed a complex and ingenious set of rules for coping quite successfully with many standard grade six mathematical problems. The difficulty is that these rules in many cases will not generalize and it would seem almost certain that Benny is due for large doses of 'cognitive dissonance' in secondary school mathematics. Furthermore, Benny now sees mathematics as an illogical and arbitrary activity where one answer is just as good as another.

E: It (i.e., finding answers) seems to be like a game.
B: (Emotionally) Yes! It's like a wild goose chase.
E: So you're chasing answers the teacher wants?
B: Ya, ya.
E: Which answers would you like to put down?
B: (Shouting) Any! As long as I knew it could be the right answer. You see, I am used to check my own work; and I am used to the key. So I just put down ½ because I don't want to get it wrong.
E: Mm ...
B: Because if I put ¼ and ¼, they'll mark it wrong. But it would be right. You agree with me there, o.k. If I put 2/4, you agree there. If I put ½, you agree there too. They're all right!'' (p. 16).

The appearance of Ginsburg's book in the early part of 1977 can be seen as a landmark in the relation between research and practice in elementary mathematics. For here, for the first time, one has a well-written, jargon-free, publication by an able scholar (Ginsburg is a highly-regarded cognitive psychologist) which speaks directly to the classroom teacher. Ginsburg had two aims in writing the book: ''The first is to show how children learn, do and understand elementary mathematics especially arithmetic. The second is to demonstrate how such knowledge can be used to improve mathematics education and to resolve children's difficulties in learning'' (p. iii). These are ambitious aims and one cannot really expect any author to be completely successful at them within the span of two hundred pages. This be as it may, Ginsburg has at least taken a major first step.

The first part of the book deals with the child's pre-school experience with mathematics. Ginsburg argues strongly that, far from being mathematical ''blank slates'' when they arrive at school, children have an extensive reservoir of natural mathematical experience on which the sensitive kindergarten or grade one teacher might begin to build school mathematics. The paramount feature of this pre-school experience is counting. Making use of not only his own work and the work of his students and colleagues, but also of previously published material, Ginsburg paints a moving picture of children struggling to impose order and sense on what is often a most confusing jumble of verbal and written symbolism. The book is filled with examples of children's responses to standard arithmetic questions and hypotheses as to why, from a child's viewpoint $87+39=26$, $52+\ 123+4=17$, or $12+7=3$. In some ways these sections make difficult reading for teachers, because one cannot help wincing at the memory of numerous pupils who have been curtly told their answer was ''wrong'' and intentionally or not been given a look which clearly transmitted the message of 'you dummy, you'.

The final chapter of the book consists of ten recommendations to teachers. Although much is lost by simply extracting the bare statements from the chapter where each is elaborated on in some detail, the statements themselves give an idea of the position Ginsburg would press for. These statements are:

1. Go beyond standard tests.
2. Get to know the individual child in an informal manner.
3. Look for the deeper meaning behind children's errors.
4. Encourage whatever form of learning seems appropriate for children.
5. Show children how school arithmetic relates to their own.
6. Relate written symbolism to what children already know.
7. Encourage nonstandard means of computation.
8. Exploit unsuspected strengths to reduce the gap between informal and formal knowledge.
9. Encourage self-directed learning when you see it.
10. Help children to perceive the world of mathematics.

One important feature of clinical research is that in some ways those best qualified to undertake it are classroom teachers, for they are the people who have the opportunity to come to know and observe children as they grow and learn in school settings. It should not be thought that doing clinical research is in any sense a soft option, for it requires sensitivity, insight, and considerable presence of mind. As an important starting point, however, teachers should think about beginning to collect instances of children's 'unorthodox' methods in arithmetic as Ginsburg has done. There is no reason to think that student at other levels or in other subject areas operate in an essentially different manner and the first steps in these areas probably remain to be taken.

Bauersfeld, Heinrich. Research Related to the Mathematical Learning Process: Report for the Third International Congress on Mathematical Education. Karlsruhe, Germany: 1976. Mimeo. 26 pp. A revised form of this report is to appear as a chapter in UNESCO's **New Trends in the Teaching of Mathematics IV.** Forthcoming.

Erlwanger, S.H. 'Benny's Conception of Rules and Answers in IPI Mathematics.' **The Journal of Children's Mathematical Behavior 1** (2) Autumn 1973, (7-26).

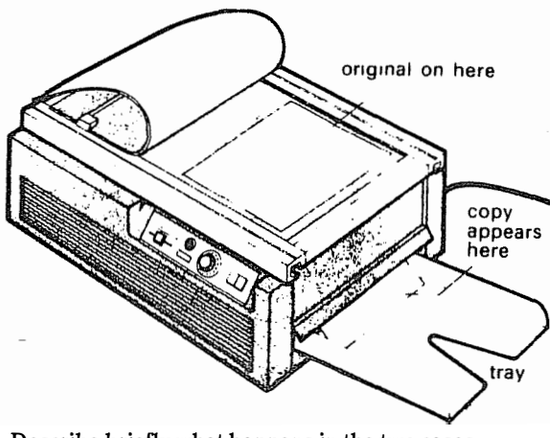
Ginsburg, Herbert. **Children's Arithmetic: The Learning Process.** New York: Van Nostrand, 1977. pb. 197 pp. ISBN 0442-22695. $6.00

The Journal of Children's Mathematical Behavior. (H. Ginsburg and R.B. Davis ed.). Volume 1, Number 4, Spring, 1977. 172 pp. Available from: University of Illinois, The Curriculum Laboratory, 1210 West Springfield Avenue, Urbana, Illinois 61801.
The following issues are still in print: I (1) Winter 1971-72, I (2) Autumn 1973, I (3) Summer 1975, I (4) Spring, 1977, and Special Supplement Number 1, Summer 1976. Individual numbers are $4.75, and a subscription to the four issues of a single volume costs $19.00. Cheques should be made payable to The Study Group for Mathematical Behavior.

Schools Council Sixth Form Mathematics Project, **Mathematics Applicable.** London: Heinemann Educational, 1975.

One of the major roadblocks in attempting to improve secondary school mathematics curricula has always been the dearth of good textual material on elementary applied mathematics. The publications of the **Mathematics Applicable Project** are, therefore, particularly welcome, for they clearly provide the senior secondary school mathematics teacher with the most comprehensive and best thought-out collection of examples of real-world mathematics currently available. The project, under the direction of Mr. Christopher Ormell of the University of Reading, ran from 1969 to 1976. Its aim was to produce materials for a non-specialist, pre-university mathematics course. After a period of extensive field testing and revision of the original materials the work of the project has begun to appear if book form. In the first wave of publication the 'Starter Units' on **Understanding Indices, Geometry from Coordinates,** and **Introductory Probability,** the 'Continuation Units' on **Mathematics Changes Gear,** have appeared. Forthcoming Student Units will be on **Polynomial Models, Vector Models, Expanding Binomials, Sequences and Series, Taking Limits, Introductory Dynamics,** and **Probability and Statistics.** An introductory book for teachers entitled **Presenting Mathematics from the Applicable Point of View** is also to appear.

A few representative problems will illustrate the approach taken in the series.



Understanding Indices p. 24

In a proposed copying machine the total graphite deposit left on each copy of a document by the machine will be $6+k(k-4.5)$ times the previous one: $6+k(k-4.5)$ is called the *graphite factor.* Show that with $k=2.5$, the copier can be made to work efficiently. What is the value of the graphite factor in this case?

A 160 H 114

Find another value of k for which the copier would work efficiently. **A 722 H 244**

If the firm making the copier fail to achieve the correct value of the graphite factor by 10 per cent (above or below), what practical effect will this have?

Describe briefly what happens in the two cases. **A 439**

The 'standard of acceptability' used by the firm is that a good copy ought not to have more than twice as much graphite on it as the correct amount, or less than half as much. Which will have the more serious effect: a value of the graphite factor which is 10 per cent above the correct value, or a value which is 10 per cent below? (Give the number of good copies producible in each case).

A 301 H 567

Calculus Applicable p. 69

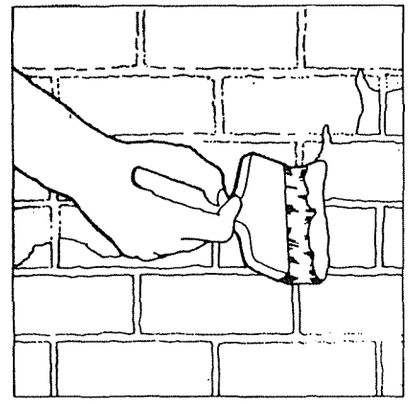
One way of preventing damp rising in the solid brick wall of an old house is to 'paint' the inside and outside of the wall (up to the level where a modern wall would have a damp course) with a special waterproofing solution. If the solution is applied liberally, it should soak through the porous brick until it joins up in the middle of the wall and forms a waterproof barrier.

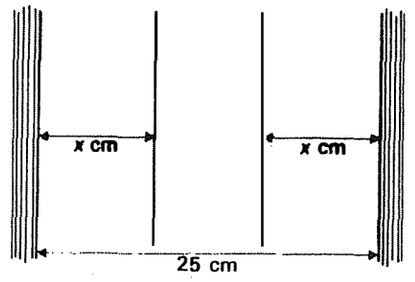
After t days, the solution has penetrated x cm from both sides. (See diagram below). The manufacturer of the special solution claims that the solution penetrates the old brickwork at a rate given by

$$dx/dt = 1/200(225 - t^2), \quad 0 \leq t \leq 15,$$

but it is known that the solution congeals after 15 days. Could it penetrate a wall that is 25 cm thick?

A 789 H 587





The format of the books is somewhat unusual and takes a bit of time to adjust to it. Extensive sections in each book are given over to Background and Mathematical Notes, and Answers and Hints. This means that the books could be used independently by superior students. The diagrams and illustrations which are an integral part of the books are generally clear and attractive. For the price these well-produced paperbound books would seem to be good value for money and high priority items for the library of the math staff-room. In Canada, these publications are available from The Book Society of Canada, 4386 Sheppard Avenue East, Box 200, Agincourt, M1S 3B6.

Introductory Probability, 56 pp. $3.60
Logarithmic/Exponential, 169 pp., $6.35
Calculus Applicable, 124 pp. $6.35
Understanding Indices, 130 pp. $5.85

Crosswhite, F. Joe (ed.) **Organizing for Mathematics Instruction.** 1977 Yearbook of the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Virginia, 22091: N.C.T.M., 1977. 240 pp. $10.00

The second 'thematic' yearbook of the NCTM focuses on organizational alternatives for mathematics instruction. Twelve of the thirteen papers in this volume outline aspects of different ways of approaching this activity. The editor has obviously worked hard to obtain a range of suggested alternatives which span the primary to tertiary levels. From this viewpoint, he has been quite successful. There is, as one might expect, considerable emphasis on individualized instruction but there are also papers on 'Survival Groups' and Alternative Schools.

There is obviously a danger when looking at a range of papers on a theme that one will regard them as a set of term papers from a class, or that one will tend to overestimate the quality of those papers which share one's own biases. Despite sensitivity to this tendency, however, it would still seem that the paper by Rising, Brown and Meyerson on 'The Teacher-centred Classroom' is in a class of its own in this collection. Brief but mathematically sound and quite provocative it packs a punch which, with the possible exception of Fremont on 'Survival Groups', none of the other papers do. With all too many of the other contributions one's overall impression is of mush and flowcharts.

Despite Peterson's valiant struggles, for instance, one is tempted to conclude that he could have been rather more succinct in attempting to answer the question, "What Simulations are Available?" (Answer: Not many really, at least of any value).

Having been somewhat less than charitable to this point, it should in fairness be said that this publication would be a very valuable starting point for any teacher or administrator who wishes a concise survey of possible alternative methods or organization. The odd-chapter-out, the non-thematic essay, is on "Hand-held Calculators: Past, Present, and Future" by Bell, Esty, Payne and Suydam. This is a brief but important statement of this critical aspect of any future mathematics curriculum.

Walter, Marion I. **Boxes, Squares and Other Things.** Reston, Virginia: N.C.T.M., 1970. 88 pp. Bibliography. $2.20

This book is subtitled A Teacher's Guide for a Unit in Informal Geometry - and that is exactly what it is! It is written for teachers and is intended for use primarily in elementary schools (grades 4 to 7), but modifications are possible - some are suggested in the book - for lower and higher grades. (Suitable modifications have been successfully used with university students!)

Two reasons why this book has so much potential are the following: (i) the unit encourages students to "visualize" geometric shapes in two and three dimensions, and at the same time emphasizes the importance of the use of concrete materials to check out these visualizations, and (ii) the mathematics that results from a fairly simple "starting point" includes a great many important ideas suitable for study to varying depths by students at every grade level.

The starting point consists of asking students (or the reader) to

"Close your eyes.
Visualize a box with sides that are squares.
How many sides does it have?
If the box has no top, how many sides does it have?
Imagine it flattened out.
. . . . . .
Open your eyes!
Make a drawing of how the box looks flattened out."

From this, the students are led to consider the number of different ways that five identical squares can be placed on a flat surface so that entire edges are in contact. The ability to see that some suggested arrangements are in fact the same as others involves the notion of congruence and the related congruent transformations known to students in elementary school as slides, flips, and turns. A consideration of which of these transformations will leave a shape apparently unchanged, leads to a discussion of symmetry and an informal look at an example of a group.

One of the chapters contains numerous suggestions for related problems and for extensions of the work to triangles, patterns, three-dimensional geometry, mirror geometry, transformations, perimeter-area relationships, and tessellations.

There is a well-organized and easy-to-read chapter dealing with the mathematics involved in the unit. There is also an extremely useful and well-annotated six page bibliography of related readings.

This is an excellent book and is highly recommended as a superb source of ideas for exciting work in geometry for students at all levels.

Hugh Allen

Note: Single copies of **Boxes, Squares and Other Things** are still available free of charge to OAME members while the supply lasts. Requests should be sent to the Secretary Treasurer of O.A.M.E., Mr. Morley McGregor

BRIEFLY NOTED

In the last few years there would appear to have been a considerable increase in the number of publications on the history of mathematics. Among the most attractive of these is the book by Lakatos (1976) **Proofs and Refutations: The Logic of Mathematics** which attempts to "elaborate the point that informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitable established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations". In attempting to make his case Lakatos uses as a vehicle the familiar result of Euler relating the number of faces, vertices, and edges of a polyhedron.

Biggs et al (1976) have done a very good job of collecting and presenting the seminal papers in Graph Theory from Euler to Whitney. Professor G. de B. Robinson of the University of Toronto, the only PhD. student of Alfred Young, the Cambridge mathematician, has edited his mentor's papers. In this task he has been admirable supported by the University of Toronto Press which has produced an attractive volume at a very reasonable price.

Hilbert's famous list of problems from the 1900 International Congress in Paris is the starting point for a recent American Mathematical Society publication edited by Felix Browder (1976). A number of the world's best contemporary mathematicians outline the progress that has been made in attempting to resolve these problems over a period of three-quarters of a century. Three quite readable biographies of eminent mathematicians are Constance Reid's efforts on Hilbert (1970) and Courant (1976) and the autobiography of S.M. Ulam. Leo Rogers has updated his 1970 bibliography of the History of Mathematics which appeared in **Mathematics Teaching 51**; this recent version is very substantial (44 pp.) and is quite a useful document for any teacher interested in the History of Mathematics.

Biggs, N.L., Lloyd, E.K. and Wilson, R.I. **Graph Theory: 1736-1936.** London: Oxford University Press, 1976 $22.50

Browder, Felix E. (ed.). **Mathematical Developments Arising from Hilbert Problems.** Two volumes. Providence, R.I.: American Mathematical Society, 1976. pb. $20.00

Lakatos, Imre. **Proofs and Refutations: The Logic of Mathematical Discovery.** Cambridge: Cambridge University Press, 1976. pb. $5.00

Reid, Constance. **Hilbert.** New York: Springer, 1970, $10.00

Reid, Constance. **Courant in Gottingen and New York: The Story of an Improbable Mathematician.** New York: Springer, 1976. $12.00

Rogers, Leo. 'Finding Out in the History of Mathematics. Mimeo. Available from 'Leapfrogs', Coldharbour, Newton St. Cyres, Exeter, England. $2.00

Ulam, S.M. **Adventures of a Mathematician.** New York: Scribner's, 1976. $15.00

Young, Alfred. **The Collected Papers of Alfred Young.** Edited by G. de B. Robinson. Toronto: University of Toronto Press, 1977 $10.00

Note: Given the inflation rate and changes in international currency, the prices listed for publications should be taken as rough approximations.

Answer to **The Circling Car** puzzle: 62.832 ft.

INTERVIEW

Taped Talks with Bob Smith

• • • DR. PETER HILTON

Dr. Peter Hilton is a Fellow of the Battelle Seminars and Studies Program in Seattle, Washington, and holds the Louis D. Beaumont University Chair at Case Western Reserve University. It was my privilege to interview him on May 17, at the Ottawa convention. We sat outside Fauteux Hall in the courtyard and watched as delegates to the convention mixed with student players competing in the volleyball tournament on that sunny Saturday.

R.S. How did you come to leave Birmingham University where you held the Mason Chair of pure mathematics?

P.H. I was one of the lucky ones who was promoted to a Chair in England at a good university at a relatively early age. However, I found myself devoting a huge amount of my time to committee work and administration, and I found it depressing to think of doing this for another 30 years. So after much hesitation, I decided to accept an offer from Cornell University and crossed the ocean. In an interview on the BBC about my decision, I made it clear that I had no complaint to make about my University, but that this change would allow me to return to full-time mathematics. I expressed myself, I recall, by saying that "I felt it was about time I made way for an older man".

R.S. Your concern now seems to have turned to teacher education.

P.H. I should make it clear that I remain primarily a research mathematician and a teacher of undergraduate and graduate students. However, it is true that I have become increasingly involved in problems of mathematics education and the training of teachers. I have become very much disturbed by what I have seen of the education that, in particular, the potential elementary teacher receives, and of the text material which the elementary student uses. I thus find myself devoting more and more of my time to try to do something about the teacher training and the content of curricula.

R.S. What are some of the principles you keep in mind?

P.H. The mathematics taught must be good and useful. I believe very firmly that no self-respecting teacher should be asked to teach children anything of which he or she does not see the point. For example, to teach a sterile piece of technique such as certain manipulations of fractions, without seeing any reason why one should do that sort of thing, is, I think, bound to fail and to lead to the demoralization and mystification of the student. Pupils will very often sense that the teacher does not believe in the validity of such material herself. In this way, the whole of the mathematical experience, both of the teacher and of the student, is corrupted.

Next, I would say that the teacher must know how to approximate an answer, and must be able to explain such methods to the student, so that both can see whether an answer arrived at by some rather formal manipulation is reasonable. Such approximation can often be done if the problem comes from a real situation. But even if the problem is a piece of rather formal drill, you can often do it. For example, if you are adding decimals, the sum will be approximated if you cut off the decimal part. It is important to point out that with the addition of fractions such approximations cannot be done conveniently, and that is one reason why I strongly object to such formal manipulation with fractions. I should add that I think we have to get away from the idea -- which is valid for mathematicians but not for anybody else -- that decimals and fractions are the same thing. In the introductory stage, where the concepts are being formed, they play totally different psychological and scientific roles. The concept of fractions has to do with taking parts of a whole, whereas the role which decimals play has to do with measurement and estimation. When we start saying that ⅓ is .33333..., I think we are again indulging in meaningless mystification.

R.S. Yes, these are two points which you stressed in your speech this morning. If I could turn to another area about which you have been writing, I would like to discuss your attempt to involve practising teachers in research on problems of mathematics education.

P.H. Yes, let me tell you something about an initiative I have taken at the local level in Seattle. In conjunction with colleagues at neighboring universities, I am forming a clearing house for the dissemination of information about what is going on in mathematics education, e.g., innovative courses, experiments, special courses available at particular institutions, etc. I believe that such a clearing house has an absolutely vital function to perform. Naturally, I want teachers involved in that activity as in other activities connected with mathematics education, including the assessment, for funding purposes, of the value of a research project in the field. For there is a very important distinction to be made between research in mathematics and research in mathematics education as at present conducted. In mathematics, a result is proved and, provided I follow and accept the argument, I can build on it to do further research. Now what I look forward to is the time when, in the same way, I can say in mathematics education, right, this work has been done, this experiment has been performed, I am aware of the conditions and the controls, and I accept the conclusion. Thus, I will build from here. At present, this sort of certainty and confidence is lacking. Thus I hope that better information will lead eventually to research in mathematics education itself having the structure it should have. Then certain things will be solidly established, others would still be in the realm of conjecture, and areas of research could be designated as being profitable for further investigation. In this way we would eventually have much more of a discipline. So I do hope our efforts in the Pacific Northwest take root and spread throughout Canada and the United States.

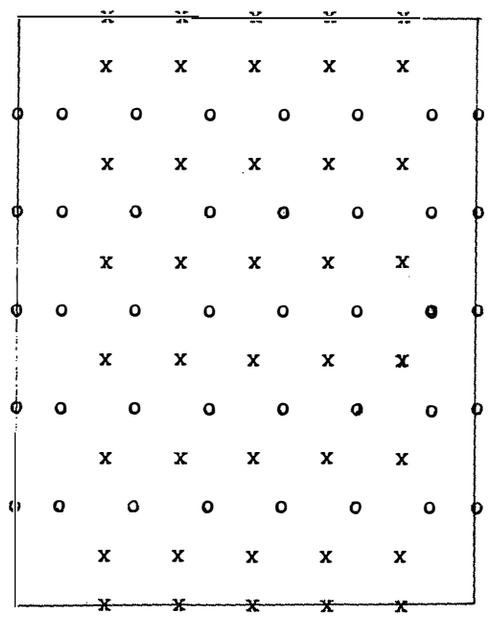
R.S. Well, I wish you success and may we be among the first to know of the progress of your venture.

DIVERSIONS

by: T.G. Friday

As promised the diversions section this issue is dedicated to material suitable for our elementary school based readers. However, if any of you out there in high school land like some of the ideas, please feel free to use them.

Game Bridge-it



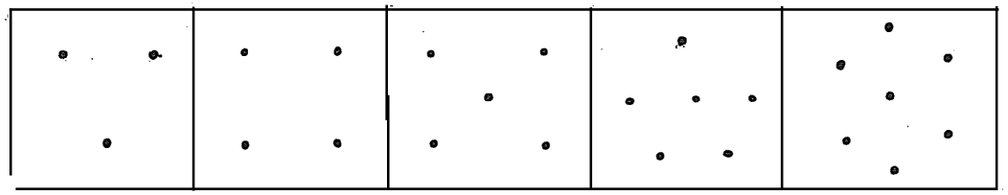
This is a two player game.
One player plays dots and the other crosses.
The two players use writing instruments which make distinguishable marks, e.g. pencil and ball pen.
The objective is to be the first to complete a chain joining opposite sides of the "board" labeled with your mark.
The players take alternate turns in joining their marks, horizontally or vertically but not diagonally, in such a way that no line crosses another.

Miscellaneous Puzzles

1. Determine the total number of squares on an n by n checker board, counting all squares of all sizes and positions. Start with n=1 and work up to a standard 8 x 8 checker board.

2. You are a camper ordered to go to the river with an 8 gallon can and a 5 gallon can. You are to return exactly two gallons. How would you do this?

3. In the following diagrams join as many points together without any line crossing another line. (Joins must be between the 2 points)



For the different numbers of points complete the following table:

No. of points	3	4	5	6	7	8	20
Maximum no. of joins without crossings.							

Can you spot a pattern?

What would be the maximum number of joins without crossings
for 100 points _____
for 500 points _____
for n points _____

PAIRING OFF WITH THE RUSSIAN PEASANT MULTIPLICATION ALGORITHM

**H. Laurence Ridge, Faculty of Education,
University of Toronto**

In the May 1975 issue of ABACUS (Vol. 2, #4) William J. Bruce[1] gave an algebraic explanation of the so-called Russian Peasant Multiplication Algorithm and a proof by mathematical induction that the algorithm holds for any two positive integers.

An alternative explanation involves the binary numeration system and, by implication, suggests consideration of the standard algorithm for conversion of numeration base from the decimal system to the binary system (and to any other base for that matter).

1. Bruce, William J. "On the Russian Peasant Multiplication Algorithm" **Abacus**, Vol 2, #4, May 1975.

To recall the 'Russian algorithm' we'll use Bruce's example (226x35) and TABLE 1 at the right.

STEP 1. Double the A-number (entry in Column A).

STEP 2. a) If the corresponding B-number is even, divide by 2.
b) If the corresponding B-number is odd, subtract 1 (to yield the next lower even number); divide by 2.

STEP 3. Repeat STEPS 1 and 2 until the B-number becomes 1.

STEP 4. Add the A-numbers corresponding to odd B-numbers (*).

In this case $226 \times 35 = 7232 + 452 + 226 = 7910$.

TABLE 1.

A	B	C
226*	35	1
452*	17	1
~~904~~	~~8~~	0
~~1808~~	~~4~~	0
~~3616~~	~~2~~	0
7232*	1	1

Referring back to TABLE 1, each C-number is the amount subtracted from the corresponding B-number before division by 2 (in STEP 2 of the algorithm). The C-numbers identify which A-numbers are to be added (1's) and which A-numbers are not to be included (0's). A more fruitful interpretation is that each C-number is the remainder upon division by 2 of the corresponding B-number.

Columns B and C are really an abbreviation of the following division operations:

$$2\overline{)35} - 1, \quad 2\overline{)17} - 1, \quad 2\overline{)8} - 0, \quad 2\overline{)4} - 0, \quad 2\overline{)2} - 0, \quad 2\overline{)1} - 1$$
$$17 \qquad\quad 8 \qquad\quad 4 \qquad\quad 2 \qquad\quad 1 \qquad\quad 0$$

which can all be combined as

$$\left.\begin{array}{l} 2\overline{)35} - 1 \\ 2\overline{)17} - 1 \\ 2\overline{)\ 8} - 0 \\ 2\overline{)\ 4} - 0 \\ 2\overline{)\ 2} - 0 \\ 2\overline{)\ 1} - 1 \\ \quad 0 \end{array}\right\} \#$$

Readers may recall that the binary numeral for thirty-five can be read from the column of remainders in process (#) above starting at the bottom.

$$35_{(ten)} = 100011_{two}$$

Check: $100011_{two} = 2^5 + 2^1 + 2^0$
$= 32 + 2 + 1$
$= 35$

TABLE 2. Below is an extension of TABLE 1. Column D shows the binary place value associated with each C-number while Column D' is the standard form of Column D. D-numbers also indicate which multiples of 226 lie in Column A (as set out in Column A').

TABLE 2.

A	A'	B	C	D	D'
226	226×1	35	1	2^0	$= 1$
452	226×2	17	1	2^1	$= 2$
904	226×4	8	0	2^2	$= 4$
1808	226×8	4	0	2^3	$= 8$
3616	226×16	2	0	2^4	$= 16$
7232	226×32	1	1	2^5	$= 32$

From TABLE 2 and the preceding paragraphs it should be evident why the Russian Peasant Multiplication Algorithm works. Thirty-five has been considered (in binary form)

as a sum of powers of 2. That is, $226 \times 35 = 226 \times (32 + 2 + 1)$

$= 226 \times 32 + 226 \times 2 + 226 \times 1$

$= 7232 + 452 + 226$

$= 7910$

Clearly the method applies to any pair of positive integers since either one can be written (in binary form) as a sum of powers of 2.

A Change of Base Algorithm

Process (#), reviewed at the right, was essential to the previous discussion. Why should this process generate the binary numeral for thirty-five? A pictorial approach may be illustrative for an intuitive explanation.

$$\left.\begin{array}{l} 2)\underline{35} - 1 \\ 2)\underline{17} - 1 \\ 2)\underline{\ 8} - 0 \\ 2)\underline{\ 4} - 0 \\ 2)\underline{\ 2} - 0 \\ 2)\underline{\ 1} - 1 \\ \quad\ 0 \end{array}\right\} \#$$

If 35 ÷ 2 indicates the pairing of 35 (dots)*, the remainder 1 represents the one remaining unpaired dot. This accounts for the 1's digit of 1 in the binary numeral.

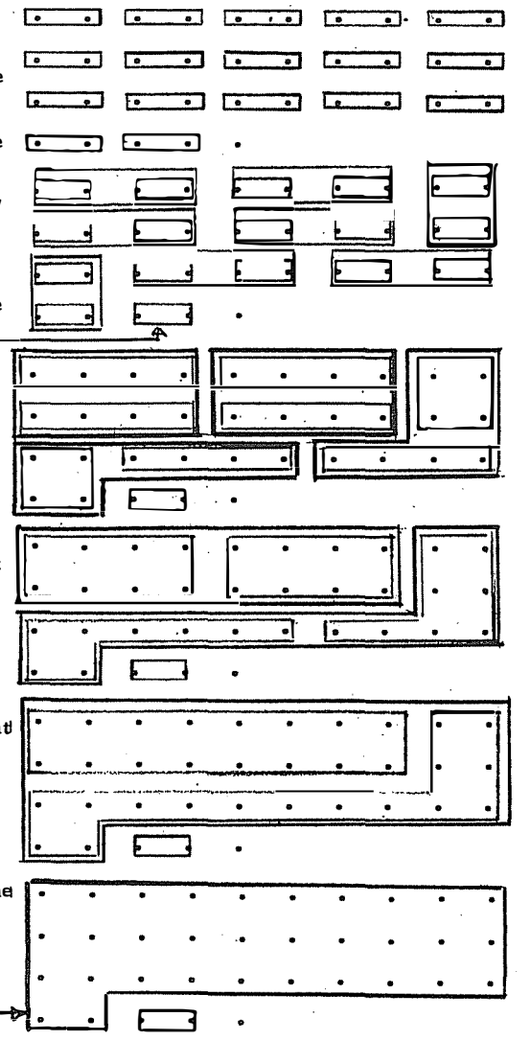
17 ÷ 2 now indicates the pairing of 17 2-groupings. The remainder 1 represents the one remaining unpaired 2-grouping. This accounts for the 2's digit of 1 in the binary numeral.

8 ÷ 2 indicates the pairing of 8 4-groupings. The zero remainder shows that there is no unpaired 4-grouping remaining. This accounts for the 2^2-digit of 0 in the binary numeral.

4 ÷ 2 indicates the pairing of 4 8-groupings. The zero remainder shows that there is no unpaired 8-grouping remaining. This accounts for the 2^3-digit of 0 in the binary numeral.

2 ÷ 2 indicates the pairing of 2 16-groupings. The zero remainder shows that there is no unpaired 16-grouping remaining. This accounts for the 2^4-digit of 0 in the binary numeral.

1 ÷ 2 indicates the pairing of 1 32-grouping. The remainder 1 represents the one remaining unpaired 32-grouping. This accounts for the 2^5-digit of 1 in the binary numeral.



1. The 'measurement' interpretation of division.

Hence, each remainder represents the number of unpaired groupings of a particular power of 2 and is thereby a numeral in the binary representation of the number under consideration. A similar explanation holds for the conversion of a decimal numeral to any other base.

Numerical analysis provides a more sophisticated explanation of the role of the remainders.

$$2)\underline{35} \; - 1 \;\Rightarrow\; 35 = 2 \times 17 + 1$$
$$17$$

$$2)\underline{17} \; - 1 \;\Rightarrow\; 17 = 2 \times 8 + 1 \;\Rightarrow\; 35 = 2(2 \times 8 + 1) + 1$$
$$8 \qquad\qquad\qquad\qquad\qquad\qquad = 2^2 \times 8 + 2 \times 1 + 1$$

$$2)\underline{\;8} \; - 0 \;\Rightarrow\; 8 = 2 \times 4 + 0 \;\Rightarrow\; 35 = 2^2(2 \times 4 + 0) + 2 \times 1 + 1$$
$$4 \qquad\qquad\qquad\qquad\qquad\qquad = 2^3 \times 4 + 2 \times 1 + 1$$

$$2)\underline{\;4} \; - 0 \;\Rightarrow\; 4 = 2 \times 2 + 0 \;\Rightarrow\; 35 = 2^3(2 \times 2 + 0) + 2 \times 1 + 1$$
$$2 \qquad\qquad\qquad\qquad\qquad\qquad = 2^4 \times 2 + 2 \times 1 + 1$$

$$2)\underline{\;2} \; - 0 \;\Rightarrow\; 2 = 2 \times 1 + 0 \;\Rightarrow\; 35 = 2^4(2 \times 1 + 0) + 2 \times 1 + 1$$
$$1 \qquad\qquad\qquad\qquad\qquad\qquad = 2^5 \times 1 + 2 \times 1 + 1$$

$$2)\underline{\;1} \; - 1 \;\Rightarrow\; 1 = 2 \times 0 + 1 \;\Rightarrow\; 35 = 2^5(2 \times 0 + 1) + 2 \times 1 + 1$$
$$0 \qquad\qquad\qquad\qquad\qquad\qquad = 2^5 \times 1 + 2 \times 1 + 1$$

Conversion of any positive integer, a, from decimal notation to base b numeration system (where b is a positive integer such that $2 \le b < a$), is a topic suitable for senior level consideration.

$$b)\underline{\;a\;} - r_1 \;\Rightarrow\; a = bd_1 + r_1, \quad r_1, d_1 \text{ positive integers, } r_1 < b$$
$$d_1$$

$$b)\underline{d_1} - r_2 \;\Rightarrow\; d_1 = bd_2 + r_2 \;\Rightarrow\; a = b(bd_2 + r_2) + r_1$$
$$d_2 \qquad\qquad\qquad\qquad\qquad\qquad = b^2 d_2 + br_2 + r_1$$

After k divisions by b, assume $\quad a = b^k d_k + b^{k-1} r_k + \cdots + br_2 + r_1$

At the (k + 1)st division by b

$$b)\underline{d_k} - r_{k+1} \;\Rightarrow\; d_k = bd_{k+1} + r_{k+1} \;\Rightarrow\; a = b^k(bd_{k+1} + r_{k+1}) + b^{k-1} r_k + \cdots + br_2 + r_1$$
$$d_{k+1} \qquad\qquad\qquad\qquad\qquad\qquad = b^{k+1} d_{k+1} + b^k r_{k+1} + b^{k-1} r_k + \cdots + br_2 + r_1 .$$

Hence, for all positive integers, n, after n divisions by b

$$b)\underline{d_{n-1}} - r_n \;\Rightarrow\; d_{n-1} = bd_n + r_n \;\Rightarrow\; a = b^n d_n + b^{n-1} r_n + \cdots + br_2 + r_1$$
$$d_n$$

If on the (n + 1)st division by b, $d_{n+1} = 0$, then

$$b)\underline{d_n} - r_{n+1} \;\Rightarrow\; d_n = b \times 0 + r_{n+1} \;\Rightarrow\; a = b^n r_{n+1} + b^{n-1} r_n + \cdots + br_2 + r_1$$
$$0$$

and the process terminates. The base b representation of the positive integer a is $r_{n+1} r_n r_{n-1} \cdots r_3 r_2 r_1$ where the n^{th} division by b yields the first quotient, d_n, less than b.

Studying the basic arithmetic operations in numeration systems other than base ten as a supposed basis for better understanding of and facility with the decimal numeration system has not proved to be a particularly sound rationale[1] . However, a consideration of such systems as an outgrowth of explanations for algorithms, sometimes encountered in a recreational or enrichment mode, might well be a more worthwhile enterprise.

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1. Callahan, L.G., & Glennon, V.J. **Elementary school mathematics: a guide to current research.** (4th ed.) Washington: Association for Supervision and Curriculum Development, 1975.

Letter from Mr. J.C. Asbury

Recently I have been having some pleasure with a Geometry deduction or theorem which I dreamed up. At least I cannot recall having seen it anywhere. Maybe you will find it of some interest.

Given that X, Y, Z are points on the sides AB, BC, CA respectively, of the triangle ABC, such that the ratios $\frac{AX}{AB}, \frac{BY}{BC}, \frac{CZ}{CA}$ are equal to 1/n, determine the ratio in terms of n, of the areas of triangles HKL, ABC, where H, K, L are the points of intersection of the transversals AY, BZ, CX.

PROBLEMS

Readers will have read the letter from Dr. G.H.M. Thomas, chairman of the 1977 Olympiad Committee. We have taken the liberty of printing the questions of the 1977 paper. Solutions to selected questions will be given in the next issue.

The first five winners are given below with their sponsoring teachers.

Igor E. Rivin, Vincent Massey S.S., Windsor (Dennis Leach)
Pui Wing Wong, Overlea S.S., Don Mills (D. Di Felice)
Robert Alexander Ewan, Loyalist C.V.I., Kingston (Jim Boyd)
Thomas C. Box, Port Hope H.S. (Mrs. Caroline MacNamara)
Paul W. Beame, Hamilton C.I. (D. Otter)

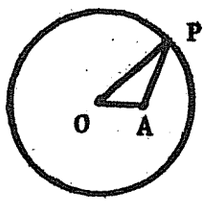
CANADIAN MATHEMATICAL CONGRESS

NINTH CANADIAN MATHEMATICS OLYMPIAD 1977

NEUVIEME OLYMPIADE CANADIENNE DE MATHEMATIQUES 1977

1. If $f(x) = x^2 + x$, prove that the equation $4f(a) = f(b)$ has no solutions in positive integers a and b.
   Soit $f(x) = x^2 + x$. Montrer que l'équation $4f(a) = f(b)$ n'a aucune solution en entiers positifs a et b.

2. Let O be the centre of a circle and A a fixed interior point of the circle different from O. Determine all points P on the circumference of the circle such that the angle OPA is a maximum.



Soit O le centre d'un cercle. Soit A un point intérieur au même cercle, mais différent de O. Quels sont les points P de ce cercle pour lesquels l'angle OPA est maximal?

3. N is an integer whose representation in base b is 777. Find the smallest positive integer b for which N is the fourth power of an integer.

Soit N un entier dont la représentation en base b est 777. Trouver le plus petit entier positif b pour lequel N est la quatrième puissance d'un entier.

4. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

and $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$

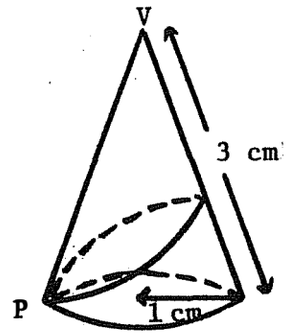
be two polynomials with integer coefficients. Suppose that all the coefficients of the product $p(x) \circ q(x)$ are even but not all of them are divisible by 4. Show that one of $p(x)$ and $q(x)$ has all even coefficients and the other has at least one odd coefficient.

Soient $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

et $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$

deux polynômes à coefficients entiers tels que les coefficients de leur produit sont tous pairs, mais pas tous divisibles par quatre. Montrer que l'un de ces polynômes n'a que des coefficients pairs, tandis que l'autre a au moins un coefficient impair.

5. A right circular cone of base radius 1 cm and slant height 3 cm is given. P is a point on the circumference of the base and the shortest path from P around the cone and back to P is drawn (see diagram). What is the minimum distance from the vertex V to this path?



On considère un cône circulaire droit dont la base a un rayon de 1 cm et dont la hauteur, measurèe le long d'une génératrice, est de 3 cm. Soint P un point du cercle de la base. On indique sur la figure le chemin le plus court qui part de P, fait le tour du cône et retourne à P. Quelle est la distance la plus courte d'un point de ce chemin au sommet V du cône?

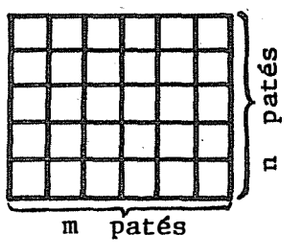
6. Let $0 < u < 1$ and define $u_1 = 1 + u$, $u_2 = \frac{1}{u_1} + u$, ..., $u_{n+1} = \frac{1}{u_n} + u$, $n \geq 1$.

Show that $u_n > 1$ for all values of $n = 1, 2, 3, \ldots$.

Soit $0 < u < 1$. On dèfinit $u_1 = 1 + u$, $u_2 = \frac{1}{u_1} + u$, ..., $u_{n+1} = \frac{1}{u_n} + u$, $n \geq 1$.

Montrer que $u_n > 1$ pour tout $n = 1, 2, 3, \ldots$.

7. A rectangular city is exactly m blocks long and n blocks wide (see diagram). A woman lives in the southwest corner of the city and works in the northeast corner. She walks to work each day but, on any given trip, she makes sure that her path does not include any intersection twice. Show that the number f(m,n) of different paths she can take to work satisfies $f(m,n) \leq 2^{mn}$.



Une citè de forme rectangulaire comport m × n patès de maison (voir figure). Une dame habite dans le coin sud-ouest de la cité, mais son lieu de travail est dans le coin nord-est. Elle se rend à son travail à pieds chaque jour et aime essayer différents trajets, évitant seulement de passer deux fois par la même intersection. Montrer que le nombre f(m,n) de trajets différents qu'elle peut emprunter ne peut dépasser 2^{mn} .

OLYMPIADE CANADIENNE DE MATHÉMATIQUES

1977

CANADIAN MATHEMATICS OLYMPIAD

sous les auspices de la
SOCIETE MATHEMATIQUE DU CANADA

Nous certifions que

a participé à
l'Olympiade Canadienne de Mathématiques
et a obtenu

under the auspices of the
CANADIAN MATHEMATICAL CONGRESS

This is to certify that

competed in the
Canadian Mathematics Olympiad
and was awarded

CERTIFICATE OF COMPETITION

PRESIDENT SOCIETE MATHEMATIQUE DU CANADA

CHAIRMAN OLYMPIAD COMMITTEE

ONTARIO ASSOCIATION
FOR
MATHEMATICS EDUCATION
1977-78 EXECUTIVE and COUNCILLORS
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PRESIDENT'S MESSAGE

1977 11 05

The OAME-AOEM have been very active across the province during the fall term.

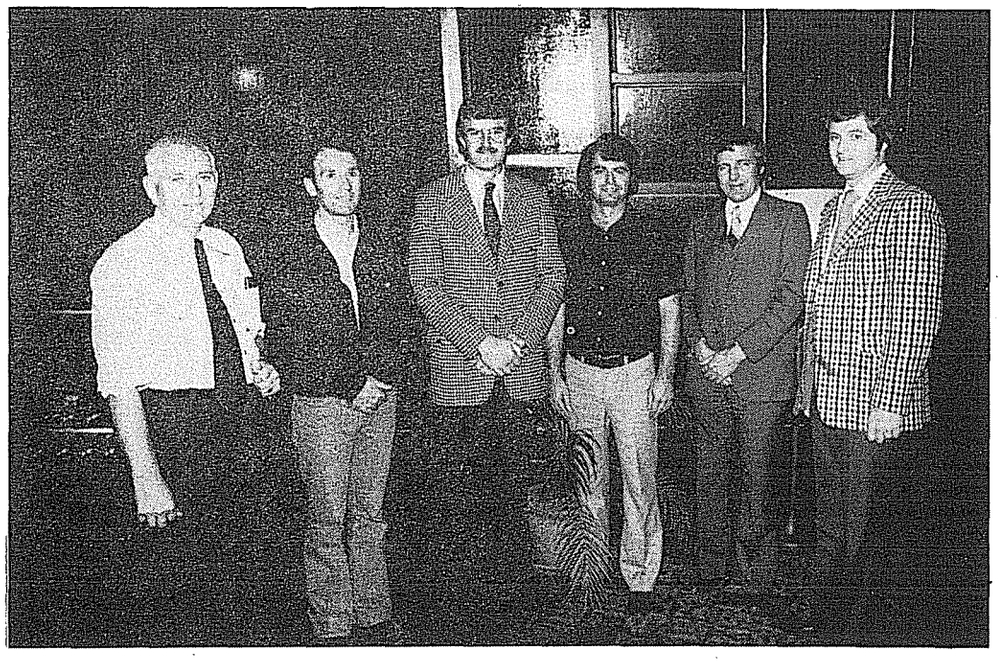
On October 1, the Northern Ontario Mathematics Association (NOMA) sponsored an excellent bilingual conference with about 130 in attendance. I enjoyed returning to Kirkland Lake (my home during the late forties) and was impressed by the energy and enthusiasm of the teachers in this area.

On October 20 through 22, the OAME-AOEM Leadership Seminar was held in Toronto with about 80 in attendance. The participants were presented with a judicious blend of leadership skills, trends in Mathematics and specific program ideas. We thank Morley and Mona McGregor, Bruce Jay and the staff of 13 for an excellent program.

On October 27, the Scarborough-East York Math Association (SEYMA) hosted their second mini-conference with about 200 in attendance. The major theme was Geometry K-13 and the sessions were well received.

On November 3, the Quinte-St. Lawrence Math Association (QSLMA) hosted a meeting in Napanee. Topics included were Piaget, Geometry, Spin Theory, Math at CAATS and Senior Math Exams. About 130 teachers attended and Marie McGregor, president of FWTAO was a special guest.

We understand that SWOAME and COMA are also sponsoring meetings and we on Council are impressed with the leadership being displayed across the province.



The O.A.M.E. Executive at a recent meeting.

Left to Right: Morley MacGregor, Hugh Allen, Neil Williamson, Dave McPhail, Paul Pogue, Bruce Topping.

Your executive have met twice this term and the various committees have been quite active. We are pleased with the continuance of the high standards in our publications. Thanks to the editors and those who contribute articles.

OAME-AOEM membership to date is about 1350 and we are pleased to welcome more elementary teachers and former students to our organization. We wish to remind all who have not renewed or joined to do so prior to January 1/78 as the dues will increase to $13.00 at that time.

We sadly report the passing of two of our members, Jack McKnight and Gaye Queen. We are grateful for their contributions through the years and we share the sorrow of their families at this time.

We are pleased to report that Don Roberts of Grimsby is recovering well from a recent heart attack.

Finally, I urge all members to plan to attend the Annual Meeting in Hamilton on May 5, 6, 7, 1978. Reserve these dates now.

Neil J. Williamson
President OAME-AOEM

P.S. My wife, Jackie, and I are pleased to announce the birth of our first child, Scott J. Williamson, 4080 grams (9 pounds), on 1977 10 25 at 19:55

P.P.S. Lorna Morrow, Dave McPhail and I attended the FWTAO Math Conference at the Hyatt Regency Hotel on November 4. We led a sing-song of Lorna's songs and they were well received. About 350 elementary women teachers were in attendance, and many of our OAME members were making presentations.

Future Meetings

Meeting Dates effective May 1978 following Hamilton Conference.

Executive Meeting
First Friday after May 24th weekend holiday
Council Meeting
Second Saturday after May 24th weekend holiday
Executive Meeting
Second Friday after Labour Day
Council Meeting
First Saturday after Thanksgiving
Executive Meeting
First Friday after November 11
Council Meeting
Second Saturday of February
Executive Meeting
First Friday of April unless it is Good Friday - then one week later

PUBLICATIONS AVAILABLE

TO O.A.M.E. MEMBERS

	Regular Price	O.A.M.E. Member Price
Handbook of Trigonometry (1975)	$2.00	N/C
A Preparedness Test for Grade Nine Students (1973)	2.00	N/C
Sample Grade 13 Examinations	1.00	N/C
Grade 11 (Level 5) Sample Examinations	1.00	N/C
Reprints of 1974 & 1975 Gauss Contest Papers (Elementary)	.10	N/C

To receive these please contact:

Mr. R. G. Scoins,
Faculty of Mathematics,
University of Waterloo,
WATERLOO, Ontario.

Boxes, Squares and Other Things (Elementary) A leather guide for a unit in informal geometry	1.80	N/C

Available from: Mr. M. MacGregor, Secretary-Treasurer O.A.M.E.

NINE CHAPTERS OF OAME/AOEM

AMEO/EOMA (Eastern Ontario)

President - P. Brule
Representative - L. Lefebvre

COMA (Carleton - Ottawa)

Chairman - Maurice Poirier
Ont. Representative - Christine Hubbard

Golden Section

Representative - George Knill

G.V.M.A (Grand Valley)

President - Ed. Baumgart
Representative - David Davidson, Collingwood C.I., Collingwood

N.O.M.A (Northern Ontario)

President & Representative - Neil Truman

P:R.M.A. (Pine Ridge)

Representative - Bill Jeffers

Quinte - St. Lawrence

President - Elaine Harvey, Sydenham High School, Sydenham, Ontario
Representative - James Whitley, Dept. of Mathematics, Queen's University, Kingston, Ontario

S.E.Y.M.A. (Scarborough - East York)

President - Paul Sherk
Representative - Paul Zolis, Wolburn C.I.

S.W.O.A.M.E (Southwestern Ontario)

President - Barbara Palmer. Laurier S.S.
Representative - Gary Peters

CHAPTER ITEMS

COMA: continues to sponsor the excellent journal, EUREKA. Bound copies of Volumes I and II are now available for $10.00.

GOLDEN SECTION: The insert tells the story of the work they are doing.

G.V.M.A. We understand the booklet on Geometry as well as the Recreational Math booklet is now ready. These may be obtained from R. Scoins, Faculty of Mathematics, University of Waterloo.

N.O.M.A. Upcoming Conference - A.M. "Statistics at the Intermediate Level" - 1977 12 13
P.M. Various topics

From a letter re: the fall conference:

> "L'intérêt et la participation d'un grande nombre de professeurs francophones encourage le comité de l'an prochain à continuer à présenter un programme en français.

QUINTE-ST. LAWRENCE: Note the logo. A worthy addition to letterheads. This chapter managed to obtain 86 new members for OAME/AOEM at their annual meeting November 3 and challenges any other chapter to better that mark. Who will take up the challenge?

S.E.Y.M.A. This new chapter has 155 members and a newsletter. Congratulations.

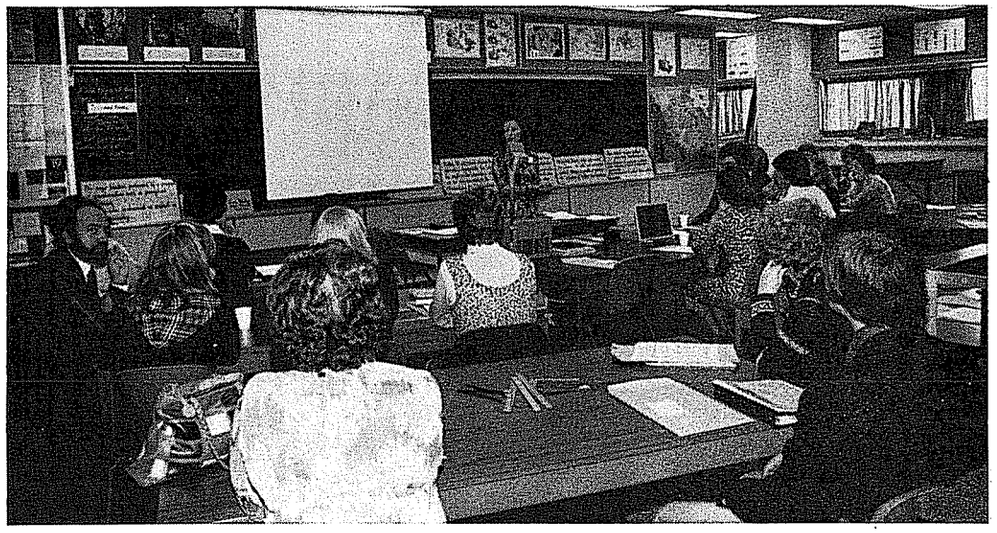
We regret to note the passing of Jack McKnight, a highly respected teacher of mathematics. He was a lifetime member of OAME, a Descartes Medallist, a devout member of the Anglican church serving in the choir. He had served as a department head, a chief marker of the old trigonometry Departmental examinations, then as Mathematics Coordinator in Scarborough and during his retirement as a consultant with the Gage Publishing Company.

Our sympathy goes out to his wife Jean and to sons Don and Jim.

A scholarship in honour of Jack was established some years ago. A campaign to augment the fund is being mounted by friends. Donations may be forwarded to:

Paul M. Sherk, Treasurer
J.R. McKnight Scholarship Committee,
86 Banmoor Blvd.,
Scarborough, Ontario
M1J 2Z6

S.W.O.A.M.E: Annual Conference was held November 19 with the main topic being the Intermediate Division guidelines - a continuing theme with this chapter.



Anna Pataky discusses Transformation Geometry at the SEYMA Conference.

INTERNATIONAL MATHEMATICS EDUCATION

OAME/AOEM 5th annual meeting Hamilton, May 5, 6, 7, 1978.

N.C.T.M. 56th annual meeting San Diego, April 12-15, 1978

NCTM 57th annual meeting Boston, April 18-21, 1979.

Groupe International de Recherce in Pèdagogie de la Mathématique 7th annual meeting, Glasgow, Scotland, July 15-23, 1978.

YOUR ASSISTANCE IS REQUIRED

At the Annual Meeting of of OAME in Hamilton in May 1978 we shall elect the following: 1 President-elect, 3 Executive Members (Directors), about 10 Councillors. If you would like to fill one of these positions or wish to nominate someone else please copy the following.

I nominate ..... (name)

..... (professional address)

for the position of ..... (president-elect, executive member, councillor)

Name of Nominator .....

Professional address .....

Nominations Chairman Mr. T.F. Griffiths

Note: 1. This nomination should be accompanied by information supporting the nomination.
Note: 2. A summary of the qualifications of those members who are allowing their names to stand in nomination will be published with the March Gazette.
3. Return to Ont. Ass. for Mathematics Education Office N7T 4E9 by Feb. 1, 1978.

MEMBERSHIP IN THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

At the founding meeting of the Ontario Association for Mathematics Education in 1973, the fee for membership was set at $10.00. The Council of the Association has continued to operate on that fee. It believes that it has been of service to the teachers and students of mathematics in the province through its two publications the Gazette (3 times a year) and the Abacus (4 times a year). It has returned to the various chapters in the province a portion of the membership fee where a refund is applicable. The cost of our publications has increased substantially in the past four years due to increased cost of materials, labour and postage.

The council studied the matter of an increased fee and after great hesitation has agreed that beginning January 1/1978, the fee will be $13.00. This will apply to both new memberships and any 76/77 memberships that have not been renewed by that date.

The council trusts that all present members of OAME will encourage their fellow teachers to join the Association and that those purchasing new or renewal memberships will understand the reasons for the increased fee.

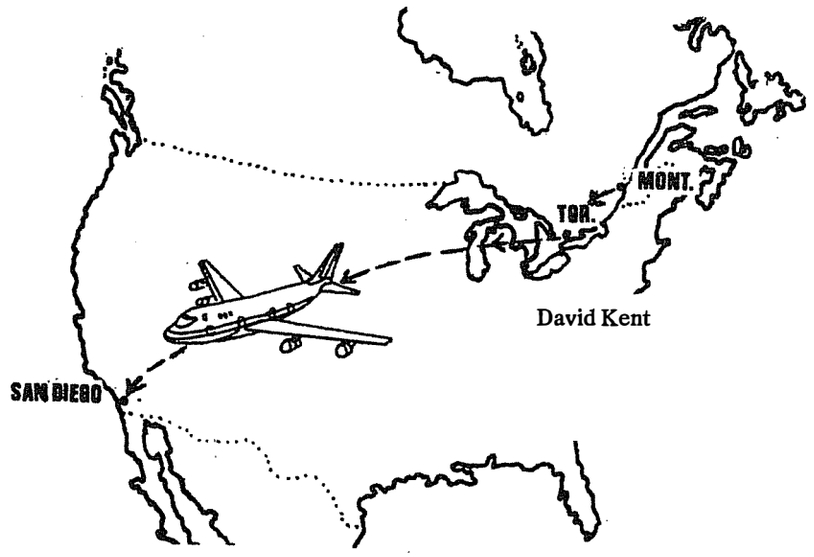
The council asks that professors in teacher training colleges encourage the student teacher to take

advantage of the $5.00 student membership. Membership in the Ontario Association for Mathematics Education stands at 1350 as of November 15, 1977. In addition, we have 443 schools or individuals who purchase the Gazette at a subscription fee of $6.00 per year.

To obtain a membership simply send a $13.00 cheque made out to OAME to 247 Bright St., Sarnia, Ont. N7T 4E9 with your name and complete address.

NCTM Annual Meeting - San Diego, California Apr. 12-15, 1978

David Kent of Lower Canada College, Montreal, is attempting to organize a charter flight to the 56th annual meeting. Dave is the QAMT-NCTM representative. Please contact him immediately if you are interested as he must determine numbers and departure date.



David Kent

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

MEMBERSHIP APPLICATION

DATE .....

Dr. ☐ **Mrs.** ☐ .....
Mr. ☐ **Miss** ☐ (please print) NEW MEMBER ☐
APARTMENT No. ..... RENEWAL ☐

STREET or R.R. .....

TOWN or CITY .....POSTAL CODE .....

SCHOOL .....
Chapter Affiliation
(if applicable) .....

PLEASE ENCLOSE CHEQUE OR MONEY ORDER FOR $13.00 PAYABLE TO OAME

Hooray for the Calculator
by Lorna Morrow (Sung at the FWTAO Math Conference, Nov. 4)
(Santa Claus is Coming to Town)

So I cannot add,
Nor yet multiply?
No need to fret,
I'm telling you why ---
My cal-cu-la-tor can do it all.

No need to remember
Seven times four.
Eight plus eleven
Matters no more ---
My cal-cu-la-tor can do it all.

It even finds me cosines,
And roots and powers, too.
My brain can take a rest, now;
It hasn't much to do.

My fingers can do
Calculations with ease
Till the batt'ry runs down.
Now who'll help me, please?
My cal-cu-la-tor won't work at all!

THE ROLE OF MATHEMATICS IN THE DEVELOPMENT OF SCIENCE

Glen S. Aikenhead, Dept. of Curriculum Studies, University of Saskatchewan, Saskatoon, Sask.
and Ranjana Kiran, Teacher at St. Clements School, Toronto, Ontario.

Reprinted with permission from the Saskatchewan Mathematics Teachers' Society News Journal, Fall 1975, James E. Beamer, Editor.

Introduction

Mathematics and science have interacted through the ages. Mathematics, as a means of articulation and theorization in science, now spans the universe all the way from the largest galaxy to the smallest elementary particle. The present-day relationship between mathematics and science is by no means static. It has evolved from the past and will continue to evolve in the years ahead. Out of past associations new ones emerge, then with a further change in the intellectual and cultural climate, new interactions develop.

Today's mathematical involvement in the physical and social sciences can be traced to its historical routes. By doing so, one gains an over view of the various roles of mathematics in the development of science. From such a perspective one can understand the ways in which mathematics interacts with science today; but more, one can bring greater clarity into speculations of future relationships.

Mathematics itself is not a part of the realm of science. Both have different subject matters, differences which cannot be bridged by anything but superficial similarities. Basically, natural science deals with objects and events in the "external" world while mathematics concerns itself with the objects in its own "aesthetic" perception. These objects are internally conceived and inwardly structured.

Through the ages, and indeed today, people have had diverse ideas regarding the relationship between mathematics and science. Some would have us believe that mathematics is the "queen of the sciences", while others insist that it is simply a servant to science:

Pythagoreans (**ca.** 530 B.C.) maintained that in nature "all is number".
Aristotle (**ca.** 350 B.C.) defined mathematics as an abstraction from nature.
Kepler (**ca.** 1600 A.D.) thought that just as ears are made for sound and eyes for color, the mind of the man is meant "to consider quantity".
Galileo (**ca.** 1600 A.D.) said that the book of the universe was written in mathematical language, and its alphabet consisted of geometrical figures.
Bacon (**ca.** 1600 A.D.) regarded mathematics merely as a servant to physics, and actually complained of the dominion which it was beginning to exercise in science.

Feynman (1965) writes, "mathematics can help physics, but they are two quite different activities--mathematics deals with the abstract world, and physics deals with the real world".

Eric Rogers (1960) suggests that we might describe mathematics as a "master architect designing the building in which science can grow at its best."

Just as there is a danger to science in over glorifying mathematics and attempting to subordinate all of science to it, so is there a danger in calling mathematics the "handmaiden of science". Accelerations and retardations in the development of science can be traced, in large measure, to such attitudes.

This paper, in investigating the role of mathematics in the rise of science, chronologically examines various epochs of our past.

PRE-GREEK AND GREEK ERA

The mathematics of Egypt and Babylon preponderantly served a practical function. Pre-Greek mathematics was integral to astonomy, taxation, and the construction of moats and temples.

The most successful product of the Greek mind was the deductive quality of geometry. The first Greeks to grasp this possibility of abstraction in geometry were probably Thales (600-550 B.C.), and Pythagorus (**ca.** 530 B.C.). Then around 300 B.C. one of the most famous masters of geometry, Euclid, set out to collect the theorems of his predecessors and to arrange them as a single self-contained work entitled **Elements.**

The next century produced two or more gifted mathematicians. Apollonius (**ca.** 200 B.C.) discovered the so-called conic sections which later contributed directly to astronomy. In addition, there was Archimedes (**ca.** 250 B.C.) whose brilliance at mathematics was matched by his genius for mechanics.

In Greek astronomy, the first mathematically conceived system was that of Eudoxus (**ca.** 370 B.C.). Spurred on by Plato's notion of reality, Eudoxus reduced the irregular movements of celestial bodies of uniform circular motion. His system consisted of 27 concentric spheres, one inside the other. The whole system was a purely geometrical hypothesis, calculated to represent the apparent paths of the planets. Later, Ptolemy (90-168 A.D.) wrote his famous astronomical treatise **De Algamest** in which he strove for a mathematical model of the universe, known today as the Ptolemaic system.

Greek physics did in no way bequeath a book comparable to the mathematical works of Euclid, Archimedes, or Apollonius. Yet, Greek physics culminated in the system of Aristotle and held the stage of physics for almost 2000 years.

The Greeks created a general concept of mathematics and a general concept of physics. Yet, if one assesses the pragmatic outcomes of the Greek mathematics and physics, and if one confronts the mathematics and physics of Greek antiquity with 20th century knowledge, then one may find that Greek physics as a whole **never** developed into a mathematical system. The system of Greek mathematics had severe limitations and shortcomings. For example, they were incapable of "founding" functions. Their symbolization did not advance beyond an elementary stage, the stage of abstraction from "direct actuality". (Full scale symbolization involves abstraction from abstraction.) These inadequacies made Greek mathematics unsuited for promoting the rise of theoretical physics as we know it today. In other words, these shortcomings prevented a type of scientific thinking from developing. Perhaps the Greeks did not introduce mathematics as a technique for mastering problems that arose in man's mind. Although Greek astronomy was mathematical and its mathematization made it successful, the Greeks never had the insight (necessary in pursuing mechanics, physics, and other sciences) to articulate qualitative attributes by quantitative magnitudes. They never realized that writing physical laws as mathematical formulae and applying mathematical procedures to the formulae can sometimes lead to further explications and developments. The Greeks never arrived at such an insight inspite of the dramatic first steps taken by Archimedes. There was no determination to solve "scientific puzzles" through mathematical manipulation and empirical verification. This was so fateful that eventually even Greek mathematics ended to wither away.

Nevertheless, the Greeks did have areas of knowledge in which science and mathematics overlapped. Their astronomy was clearly mathematical in Eudoxus and Ptolemy. Pythagoreans envisioned some sort of mathematization of physics, although the extent and depth of their insights is not easy to appraise. Again, the laws of Archimedes on balancing the lever and on floating bodies clearly pertain to mathematical physics and were the first of their kind. And yet they did not have the effect of initiating a mathematical physics at that time. Even atomists like Epicurus and Democritus did not show any tendency to initiate a mechanics of the stature of the 17th or 18th century.

THE RENAISSANCE

The Greeks wrote hundreds of books on mathematics, treating it for the first time as worthy of study for its own sake. In the centuries of darkness that followed, much of this mathematical treasury was lost. But enough remained so that the scholars of the late Middle Ages once again launched a search for knowledge, giving the study of mathematics an impetus that kept on accelerating.

Men in the 16th and 17th centuries were already looking beyond arithmetic into the vistas of algebra. It was the French mathematician Rene Descartes (1596-1650 A.D.) who first started writing algebraic equations. In his ambition to "remake the world" he developed a new branch of mathematics, analytic geometry, a technique for visualizing numbers as points on a graph, equations as geometric shapes, and shapes as equations. Trigonometry and logarithms also emerged from the Cartesian system of Descartes.

Then in 1665, England's Isaac Newton produced calculus, which for the first time permitted the mathematical analysis of all movement and change. Meanwhile, a German mathematician named Leibnitz independently invented his versiou of calculus and in 1684 published his account of it. (Today the symbols derived by Leibnitz: $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ are more generally used than those derived by Newton: $\dot{x}$, $\dot{y}$, $\dot{z}$.)

In Rene Descartes' work, we meet a system of thought much more intensive, concentrated and intricately interlocked than the Greek system. In his system, everything was to be accounted for mathematically -- by configuration or by number. He regarded physics as reducible to mechanism, and even considered the human body as being analogous to a machine. The mechanization of his highly concentrated deductive system became the template for the structure of physical science.

Men of the seventeenth century were extremely conscious of the importance of mathematics to scientific development. Therefore, it is not surprising that the development of science after 1600 A.D. began with the establishment of this so-called "Rational Mechanics" which held the stage of science during the 17th, 18th and 19th centuries. Mechanics was a problem which only became manageable when, in a certain sense, it had been "geometrized". Motion became envisaged as occurring in the emptiness of Euclidean space. (The Aristotelian system had discouraged the idea of the composition of motion, and was uncongenial to any mathematical treatment of it.) Although Galileo was one of the first persons to treat motion quantitively, he failed to achieve the perfect formulation of the modern law of inertia because he could not imagine a purely geometrical body sailing off into an utterly empty and directionless Euclidean space. The law of inertia had to wait for Descartes.

The 17th century produced the greatest single statement on the relation between mathematics and physics. It was Galileo's dictum that mathematics is a language of science. He went so far as to say that the mind was to be constantly directed only to those things, and to apply itself to only those problems, which were amendable to measurement and calculation. Descartes, Torricelli, Kepler, Huygens, Newton and others who succeeded Galileo, clarified this scientific value by their geometrizing problems concerning natural phenomena.

The mechanized study of motion may well have been the high point of 17th century science. The century witnessed one attempt after another not only to explain motion and other natural phenomena, but to interpret all changes of the physical universe in terms of a purely mechanistic universe. Kepler (1571-1630) inaugurated the scientist's quest for a mechanistic universe. To generations of astronomers, the basic celestial figure was a circle. Kepler broke away from this supposition by introducing ellipses. He did not find his ellipses in the tables of Tycho Brahe or

the writings of Copernicus. He found them by searching untiringly in the work of Apollonius. Guided by astronomical observations, he was the first to grasp the true meaning of foci of conics. Conics and their theory were in no way Kepler's private mathematical invention. They had existed for nearly 2,000 years for anybody to find and use.

The determination to formulate all explanations in mechanistic terms had important effects upon the biological sciences. Harvey (1567-1650), in his enquiries into the circulation of blood had a purely mechanical approach. G.A. Borelli (1608-1679), in his book **The Motion of Animals,** wrote a chapter on the "Mechanical Propositions Useful for the More Exact Determination of the Motive Power of Muscles." This tendency to glorify mere mechanization led to the ubiquitous view that the animal body was nothing more than a piece of clockwork.

The effects of the new mechanistic outlook are vividly illustrated in the works of Robert Boyle (1627-1691). He is quoted to have said that he did not expect to "see any principles proposed more comprehensible and intelligible than the corpuscularian". This philosophical position is often called "mechanical philosophy", since it tends to give a mechanical explanation of the physical universe. One of Boyle's works includes a discourse on the mechanical origin of heat and magnetism. According to him, chemistry itself could be reduced to micro-mechanics.

Later in the 17th century, a culminating event took place when Newton and Leibnitz introduced "derivatives" and laid the foundation for calculus and mechanics. The ultimate triumph came in publishing the **Principia** in 1886. The **Principia** is important, but so much because of its laws, definitions, concepts of time, space, and gravitational force, but because Newton constructed and deduced, by mathematical reasoning, what Kepler had only divined and postulated.

Outwardly in the **Principia**, there is hardly any mathematics invoked or presupposed which should not have been quickly accessible to Archimedes and Apollonius. No attempt had been made in the **Principia** to introduce Descartes' innovation of analytically using symbols and functions. Newton was quite skilled in the use of symbols and functions as he was very familiar with Descartes' work. In fact, Newton mastered the method of Descartes much better than Descartes himself. However, in the **Principia**, Newton's definitions of limit and derivative ("ultimate ratio") seem to be such that a personal disciple of Archimedes should have been able to compose them in principle. Why did this not occur? Greek thinking, in general, did not formulate such logical abstractions as: a relation of a relation, a property of properties, or an aggregate of aggregates (for example a rate of change of a rate of change -- acceleration). Second derivatives were at the centre of Newton's mechanics. It was this kind of limitation to Greek rationality that separated Archimedes from Newton; a hiatus which Archimedes could never succeed in crossing.

Inwardly, there is a difference between the Euclidean space that underlies the **Principia** and the Euclidean space that underlies Greek mathematics and physics. The Euclidean space of the **Principia** continues to emphasize the Greek congruencies and similarities between figures. However, it does something new. Several significant physical entities found in the **Principia**, velocities, momenta and forces, are vectors. Vectorial composition and decomposition of these entities constitutes an innermost scheme of the entire theory. In the course of the 18th century, the vectorial statements of Newton and others were gradually transferred and reinterpreted into analytical statements. The 20th century widened the concept of a vector into the broader concept of a tensor.

To summarize the events in the 17th century one could say that there was considerable scientific development where geometrical and mathematical methods could be easily and directly applied. Therefore, not only did the sciences make a remarkable development in the 17th century, but mathematics also progressed to a great extent. This is because the sciences, specially physics and dynamics, were pressing upon the frontiers of mathematics all the time. The sciences created a need for mathematics, and therefore, mathematics flourished. The relationship between science and mathematics has never quite been the same since then. Today, both tend to go their separate ways, and draw upon one another when the need arises. But it was not so in the 17th century. Science depended on mathematics, and mathematics depended on science. Without the achievements of mathematics, the scientific revolution as we know it, would have been impossible.

18TH - 19TH CENTURIES

Let us first look at the achievements of some of the great mathematicians and scientists in the 18th and 19th centuries. Then, we shall analyze the role mathematics played in the rise of science at this point in history.

Because of Newton's success, mathematical theorists of the 18th and 19th centuries held fast to a philosophy of "mechanistic determination". The French mathematician, Pierre Simon de Laplace (1749-1827), perfected Newtonian analysis of the solar system in a great work entitled **Mechanique Celeste.** He also used calculus to explore and advance probability theory. The most celebrated partial differential equation

$$\frac{\partial^2 V}{\partial^2 x} + \frac{\partial^2 V}{\partial^2 y} + \frac{\partial^2 V}{\partial^2 z} = 0$$

was devised by Laplace. (This equation has been used to describe the stability of the solar system, the electric field around a charge, a steady distribution of heat, and many other phenomena).

Euler (1707-1783), of Swiss fame, created a host of new uses for calculus as it applies to curves and surfaces. He has been called the father of modern topology.

Lagrange's (1716-1813) hallmarks were his famous works **Mechanique Analytique** and **Theorie des Fractions Analytiques** -- master textbooks in its subject. But by far his greatest achievement was the space of "generalized coordinates" of our mechanics of today.

The genius who dominated 19th century mathematics and physics was Carl Friedrick Gauss (1776-1855). He gave direction to the new movement toward generality in mathematics by imposing on it his own stern standard -- a demand for absolute rigourous thinking. As a 17-year old, he audaciously questioned certain rules of Euclid's geometry that generations of mathematicians had taken for granted, pointing out that many of them did not hold true on curved surfaces.

But it remained for Gauss's pupil Riemann (1827-1866) to shatter the boundaries of traditional geometry by postulating not only curved spaces of 3 dimensions, but spaces made up of 4 and more dimensions. Fifty years later, the physicist Albert Einstein brought the process to a stunning climax by borrowing these abstractions and using them in his theory of relativity to describe the real universe.

Out of the 17th and into 18th century, preference continued to be given to Rational Mechanics (mathematical analysis of everything possible). In this context several so-called 'principles of mechanics' were produced. Meanwhile, the theories of light, heat, electricity and magnetism were not forgotten but they advanced at a slower pace until their turn for full attention came in the 19th century.

The mechanics of the 18th century and the first decades of the 19th century was virtually inseparable from the mathematics. Almost all the leading architects of the various parts of mechanics were eminent mathematicians -- James Bernoulli (1667-1748), d'Alembert (1717-1783), Euler (1707-1783), Cauchy (1789-1857), Lagrange (1736-1813), Poisson (1791-1840), Laplace (1749-1827), Gauss (1777-1855), and Jacobi (1804-1851), to name a few. Correspondingly, most of their theorizing emanated from "pure thinking" with very marginal entanglements in direct experimentation.

The need for 17th century mathematics continued to grow. Scientific progress was now even more strongly dependent on mathematics than it was before. In this period, the amount of mathematics which was created for, and because of, mechanics (theoretical and applied) was enormous -- specially in the area of analysis. The calculus of variation was instigated largely by mechanics of particles (finite systems), while other mathematical theories were instigated largely by mechanics of continua (hydro-dynamics, acoustics, general theory of elasticity). Virtually all of partial differential equations were created this way. Indeed, the mathematical theory of waves, which eventually became the hallmark of theoretical physics in all its parts, emerged from mechanics of continua. Fourier analysis was the result of the mechanics of continua and the

theory of heat. The concept of potential energy originated in the Lagrangian theory of finite particles. Finally, it appears that the mechanics of continua had a share in the emergence of tensor theory.

Beginning with the 19th century, the relationship between mathematics and mechanics changed. Mathematics became more or less independent of mechanics and physics. It assumed a philosophical nature, and began to develop for its own sake. Yet another kind of relationship between mathematics and theoretical physics developed. It was a rapport built more on parallelisms of pursuits rather than on identities of aims. Mathematical functions were no longer created for a particular purpose.

From time to time in this century, theoretical physics was able to seize upon an unfamiliar ready-made piece of mathematics and use it instantly. It would have appeared as if the mathematics had been prefabricated especially for the theoretical physicist. For example, in the second half of the 19th century, statistical mechanics of the kinetic theory of matter was able to draw upon the mathematical theory of probability (initiated in the Renaissance age and developed by Laplace). Another example is the theory of relativity. It utilized the non-Euclidean geometry of the 19th century.

For all of physics, and gradually for other sciences as well, mechanics became a model of mathematization in the 19th century. Most of the development in electricity, magnetism, optics and heat conduction was mathematically modeled on paradigms from mechanics of continua. Therefore in many parts of physics, the mathematics was uniformly the same, not only in technique but in the manner in which mathematical and physical conceptions were correlated with each other. However, there was one part of physics which did not conform to this general pattern: the theory of thermodynamics. It was mathematically linked to a novel kind of mechanics--statistical mechanics.

It would seem that the relationship between mathematics and science, from the Greek times to the 19th century took a full circle. Generally speaking, the Greeks regarded mathematics and science as two separate entities. Scientists in the Renaissance had a totally mechanistic outlook towards all knowledge. This resulted in a remarkable development of mathematics. In the 18th century, the scientific community's need for new mathematics continued to grow. The amount of mathematics that was created for and because of mechanics was enormous. Then in the 19th century, the relationship between mathematics and science changed. Mathematics began to be independent of science. There developed from then on, a limited collaboration between physicists and mathematicians, that remains unbroken today.

20TH CENTURY

It is intriguing that every so often it is possible to apply an almost forgotten mathematical development of yesterday to a scientific problem of today. The 20th century has some fine examples of this.

The power of mathematics has rarely been proven more effective than in relativity theory--a brilliant application of the geometry of curved surfaces to the treatment of space, time and motion. In his theory of general relativity, Einstein applied the 19th century ideas of Gauss and Reimann in suggesting the existence of a curved universe of 4 dimensions.

Another example of science drawing upon prefabricated mathematics is Boolean Algebra. Contemporary studies in network and information theory, mechanical and human, had to fall back on the work of George Boole (1815-1864). Boole developed symbolic logic to clarify difficult Aristotelian logic. Today, his system is widely used as a tool to augment sound reasoning and has practical uses in designing parts of telephone circuits and electronic computers.

In quantum physics, it happened that a scientific setting was fashioned out of a mathematics created 20 years previously. The original disparate quantum physics versions of Heisenberg and Schrodinger were merged into one by Schrodinger. The union of the two was mathematically brought about in the precincts of so-called Hilbert space. Since entering physics, this theory of operators has developed the concept of an operator from a tool in physics to a reality in nature, and it has raised the mathematization of physics to new levels. There is hardly a purely

mathematical statement on operators in Hilbert space which some physicist would not interpret as an event, or as a property of an event in nature. In fact, it has become a general belief that mathematics and science have correspondence rules: if a purely unexpected mathematical formula arises, then a corresponding unknown occurance in nature exists. Maxwell's prediction that light is an electromagnetic wave is a good example.

Meanwhile, pure mathematicians are climbing to new levels of abstraction. How their work will relate to future scientific knowledge no one really knows. It may be decades before science gets a chance to draw upon the mathematics of today.

CONCLUSION

We have made a cursory historical review of the role of mathematics in the rise of science. Some general observations are suggested:

(1) As seen from history, any area of inquiry capable of mathematization developed the earliest and fastest. This is why physics developed before chemistry, chemistry before biology, and biology before any of the social sciences. The characteristics of physical science are such that a vast range of phenomena can be handled by linear algebra or differential equations. On the other hand, the inexact sciences are less amenable to mathematical treatment, and therefore have not developed so fast. Apparently, the mathematization of a science affects the role and nature of revolutions that may and do occur in it.

(2) Much of the newly created mathematics has, at the time of its creation, no overt bearing on applied science or even on theoretical science. There is today a warehouse of mathematical knowledge of which scientists have not yet taken advantage.

(3) It has become a part of the celebrated scientific methodology that if a purely unexpected mathematical conclusion arises then a corresponding unknown occurance in nature should be detectable.

(4) Mathematical formulation of scientific statements bestows a peculiar kind of lucidity and precision upon them, and establishes logical and cognitive relations among them. It also introduces challenging analogies and unifications. For instance, we have seen that most wave propagation phenomena, whether in acoustics, electricity or optics, are assumed to be governed by virtually the same set of differential equations.

Mathematics is not part of, or subordinate to, science. Mathematics is a unique realm of knowledge from which science borrows in order to develop (a) a set of tools for inquiry into natural phenomena, and (b) a language for the articulation of subsequent explanations.

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